

## Chapter 2

# General Results for Differential Equations with Involutions

As mentioned in the Introduction, this chapter is devoted to those results related to differential equations with reflection not directly associated with Green's functions. The proofs of the results can be found in the bibliography cited for each case. We will not enter into detail with these results, but we summarize their nature for the convenience of the reader. The reader may consult as well the book by Wiener [1] as a good starting point for general results in this direction. It is interesting to observe the progression and different kinds of results collected in this chapter with those related to Green's functions that we will show latter on.

### 2.1 The Bases of the Study

As was pointed out in the introduction, the study of differential equations with reflection starts with the solving of the Silberstein equation in 1940 [2].

**Theorem 2.1.1** *The equation*

$$x'(t) = x\left(\frac{1}{t}\right), \quad t \in \mathbb{R}^+,$$

*has exactly the following solutions:*

$$x(t) = c\sqrt{t} \cos\left(\frac{\sqrt{3}}{2} \ln t - \frac{\pi}{6}\right), \quad c \in \mathbb{R}.$$

In Silberstein's article it was written  $\frac{\pi}{3}$  instead of  $\frac{\pi}{6}$ , which appears corrected in [1, 3]. Wiener provides a more general result in this line.

**Theorem 2.1.2** ([1]) *Let  $n \in \mathbb{R}$ . The equation*

$$t^n x'(t) = x\left(\frac{1}{t}\right), \quad t \in \mathbb{R}^+,$$

*has exactly the following solutions:*

$$x(t) = \begin{cases} ct, & n = -1, \\ ct(1 - 2 \ln t), & n = 3, \\ c(t^{\lambda_1} + \lambda_1 t^{\lambda_2}), & n < -1 \text{ or } n > 3, \\ ct \frac{1-n}{2} \left[ \cos(\alpha \ln t) + \sqrt{\frac{n+1}{3-n}} \sin(\alpha \ln t) \right], & n \in (-1, 3), \end{cases}$$

where  $c \in \mathbb{R}$ ,  $\lambda_1$  and  $\lambda_2$  are the roots of the polynomial  $\lambda^2 + (n-1)\lambda + 1$  and

$$\alpha = \frac{\sqrt{(n+1)(3-n)}}{2}.$$

It is also Wiener [1, 3] who formalizes the concept of differential equation with involutions.

**Definition 2.1.3** ([3]) An expression of the form

$$f(t, x(\varphi_1(t)), \dots, x(\varphi_k(t)), \dots, x^n(\varphi_1(t)), \dots, x^n(\varphi_k(t))) = 0, \quad t \in \mathbb{R}$$

where  $\varphi_1, \dots, \varphi_k$  are involutions and  $f$  is a real function of  $n k + 1$  real variables is called *differential equation with involutions*.

The first objective in the research concerning this kind of equations was to find a way of reducing them to ordinary differential equations of systems of ODEs. In this sense, we have the following reduction results for the existence of solutions [1, 3].

**Theorem 2.1.4** *Consider the equation*

$$x'(t) = f(t, x(t), x(\varphi(t))), \quad t \in \mathbb{R} \tag{2.1}$$

*and assume the following hypotheses are satisfied:*

- *The function  $\varphi$  is a continuously differentiable strong involution with fixed point  $t_0$ .*
- *The function  $f(t, y, z)$  is defined and is continuously differentiable in the space where its arguments take values.*
- *The Eq. (2.1) is uniquely solvable with respect to  $x(\varphi(t))$ , i.e. there exists a unique function  $g(t, x(t), x'(t))$  such that*

$$x(\varphi(t)) = g(t, x(t), x'(t)).$$

Then, the solution of the ODE

$$x''(t) = \left[ \frac{\partial f}{\partial t} + x'(t) \frac{\partial f}{\partial y} + \varphi'(t) f(\varphi(t), g(t, x(t), x'(t)), x(t)) \frac{\partial f}{\partial z} \right] (t, x(t), g(t, x(t), x'(t))),$$

with initial conditions

$$x(t_0) = x_0, \quad x'(t_0) = f(t_0, x_0, x_0),$$

is a solution of the equation (2.1) with initial conditions  $x(t_0) = x_0$ .

**Corollary 2.1.5** ([3]) *Let us assume that in the equation*

$$x'(t) = f(x(\varphi(t))) \tag{2.2}$$

*the function  $\varphi$  is a continuously differentiable function with a fixed point  $t_0$  and the function  $f$  is monotone and continuously differentiable in  $\mathbb{R}$ . Then, the solution of the equations*

$$\begin{aligned} x''(t) &= f'(f^{-1}(x'(t))) f(x(t)) \varphi'(t), \\ x(\varphi(t)) &= f^{-1}(x'(t)), \end{aligned}$$

with initial conditions

$$x(t_0) = x_0, \quad x'(t_0) = f(x_0),$$

is a solution of the equation (2.2) with initial condition  $x(t_0) = x_0$ .

In Lemma 3.1.1 (p. 27) we prove a result more general than Corollary 2.1.5. There we show the equivalence of  $x'(t) = f(x(\varphi(t)))$  and

$$x''(t) = f'(f^{-1}(x'(t))) f(x(t)) \varphi'(t).$$

Lučić has extended these results to more general ones which include higher order derivatives or different involutions. We refer the reader to [1, 4, 5].

On the other hand, Šarkovskii [6] studies the equation  $x'(t) = f(x(t), x(-t))$  and, noting  $y(t) := x(-t)$ , arrives to the conclusion that the solutions of such equation are solutions of the system

$$\begin{aligned} x'(t) &= f(x, y), \\ y'(t) &= -f(y, x), \end{aligned}$$

with the condition  $x(0) = y(0)$ . Then he applies this expression to the stability of differential-difference equations. We will arrive to this expression by other means in Proposition 3.1.6 (see p. 31).

The traditional study of differential equations with involutions has been done for the case of connected domains. Watkins [7] extends these results (in particular Theorem 2.1.4) to the case of non-connected domains, as it is the case of the inversion  $1/t$  in  $\mathbb{R} \setminus \{0\}$ .

The asymptotic behavior of equations with involutions has also been studied.

**Theorem 2.1.6** ([8]) *Let  $a > 0$ . Assume  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  is a continuously differentiable involution such that*

$$\varphi(x) - \varphi(b) < \frac{1}{x} - \frac{1}{b}, \text{ for all } x, b \in (a, +\infty), x > b.$$

*Then the equation  $y'(t) = y(\varphi(t))$  has an oscillatory solution.*

Related to this oscillatory behavior is the fact, pointed out by Zampieri [9], that involutions are related to a potential of some second order differential equations.

**Definition 2.1.7** An equilibrium point of a planar vector field is called a (local) *center* if all orbits in a neighborhood are periodic and enclose it. The center is called *isochronous* if all periodic orbits have the same period in a neighborhood of the center.

**Theorem 2.1.8** ([9]) *Let  $\varphi \in \mathcal{C}^1(J)$  be an involution,  $\omega > 0$ , and define*

$$V(x) = \frac{\omega^2}{8}(x - \varphi(x))^2, \quad x \in J.$$

*Then the origin is an isochronous center for  $x''(t) = -V'(x(t))$ . Namely, all orbits which intersect  $J$  and the interval of the  $x$ -axis in the  $x, x'$ -plane, are periodic and have the same period  $2\pi/\omega$ .*

*On the other hand, if  $g$  is a continuous function defined on a neighborhood of  $0 \in \mathbb{R}$ , such that  $g(0) = 0$ , there exists  $g'(0) > 0$  and the origin is an isochronous center for  $x''(t) = g(x(t))$ , then there exist an open interval  $J$ ,  $0 \in J$ , which is a subset of the domain of  $g$ , and an involution  $\varphi : J \rightarrow J$  such that*

$$\int_0^x g(y)dy = \frac{\omega^2}{8}(x - \varphi(x))^2, \quad x \in J,$$

*where  $\omega = \sqrt{g'(0)}$ .*

## 2.2 Differential Equations with Reflection

The particular field of differential equations with reflection has been subject to much study motivated by the simplicity of this particular involution and its good algebraic properties.

O'Regan [10] studies the existence of solutions for problems of the form

$$y^{(k)}(t) = f(t, y(t), y(-t), \dots, y^{(k-1)}(t), y^{(k-1)}(-t)), \quad -T \leq t \leq T, \quad y \in \mathcal{B},$$

where  $\mathcal{B}$  represents some initial or boundary value conditions, using a nonlinear alternative result.

On the same line, existence and uniqueness results are proven by Hai [11] for problems of the kind

$$x''(t) + c x'(t) + g(t, x(t), x(-t)) = h(t), \quad t \in [-1, 1],$$

$$x(-1) = a x'(-1), \quad x(1) = -b x'(1),$$

with  $c \in \mathbb{R}$ ,  $a, b \geq 0$ .

Wiener and Watkins study in [12] the solution of the equation  $x'(t) - a x(-t) = 0$  with initial conditions. Equation  $x'(t) + a x(t) + b x(-t) = g(t)$  has been treated by Piao in [13, 14]. For the equation

$$x'(t) + ax(t) + bx(-t) = f(t, x(t), x(-t)), \quad b \neq 0, \quad t \in \mathbb{R},$$

Piao [14] obtains existence results concerning periodic and almost periodic solutions using topological degree techniques (in particular Leray–Schauder Theorem). In [1, 7, 12, 15, 16] some results are introduced to transform this kind of problems with involutions and initial conditions into second order ordinary differential equations with initial conditions or first order two dimensional systems, granting that the solution of the last will be a solution to the first.

Beyond existence, in all its particular forms, the spectral properties of equations with reflection have also been studied. In [17], the focus is set on the eigenvalue problem

$$u'(-t) + \alpha u(t) = \lambda u(t), \quad t \in [-1, 1], \quad u(-1) = \gamma u(1).$$

If  $\alpha^2 \in (-1, 1)$  and  $\gamma \neq \alpha \pm \sqrt{1 - \alpha^2}$ , the eigenvalues are given by

$$\lambda_k = \sqrt{1 - \alpha^2} \left[ k\pi + \arctan \left( \frac{1 - \gamma}{1 + \gamma} \sqrt{\frac{1 + \alpha}{1 - \alpha}} \right) \right], \quad k \in \mathbb{Z},$$

and the related eigenfunctions by

$$\begin{aligned} u_k(t) := & \sqrt{1 + \alpha} \cos \left[ k\pi + \arctan \left( \frac{1 - \gamma}{1 + \gamma} \sqrt{\frac{1 + \alpha}{1 - \alpha}} \right) \right] t \\ & + \sqrt{1 - \alpha} \sin \left[ k\pi + \arctan \left( \frac{1 - \gamma}{1 + \gamma} \sqrt{\frac{1 + \alpha}{1 - \alpha}} \right) \right] t, \quad k \in \mathbb{Z}. \end{aligned}$$

The study of equations with reflection extends also to partial differential equations. See for instance [1, 18].

Furthermore, asymptotic properties and boundedness of the solutions of initial first order problems are studied in [8] and [19] respectively. Second order boundary value problems have been considered in [1, 20–22] for Dirichlet and Sturm–Liouville boundary value conditions, higher order equations has been studied in [10]. Other techniques applied to problems with reflection of the argument can be found in [23–25].

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