

Preface

Involutions have been an interesting subject of research at least since Rothe first computed the number of different involutions on finite sets in 1800 [1]. After that, Babbage published in 1815 [2] the foundational paper in which functional equations are first considered, in particular those of the form $f(f(t)) = t$ which are called involution equations.¹

Despite the progresses on the theory of functional equations, we have to wait for Silberstein, who in 1940 [3] solved the first functional differential equation with an involution. The interest on differential equations with involutions is retaken by Wiener in 1969 [8]. Wiener, together with Watkins, will lead the discoveries in this direction in the following decades [4–11]. Quite a lot of work has been done ever since by several authors. We make a brief review of this in Chap. 2. In 2013 the first Green's function for a differential equation with an involution was computed [12] and the field rapidly expanded [13–16].

This monograph goes through those discoveries related to Green's functions. In order to do that, first we recall some general results concerning involutions which will help us understand their remarkable analytic and algebraic properties. Chapter 1 deals about this subject while Chap. 2 gives a brief overview of differential equations with involutions to set the reader in the appropriate research framework. We recommend the reader to go through the monograph [9] which has a whole chapter on the subject and, although it was written more than twenty years ago, it contains most of what is worth knowing on the matter.

The second part of the monograph is about Green's functions. In Chap. 3 we start working on the theory of Green's functions for functional differential equations with involutions in the simplest cases: order one problems with constant coefficients and reflection. Here we solve the problem with different boundary conditions,

¹Babbage, in the Preface to his work [2], described very well the importance of involutions: '*Many of the calculations with which we are familiar, consist of two parts, a direct, and an inverse; thus, when we consider an exponent of a quantity: to raise any number to a given power, is the direct operation: to extract a given root of any number, is the inverse method [...]* In all these cases the inverse method is by far the most difficult, and it might perhaps be added, the most useful'.

studying the specific characteristics which appear when considering periodic, anti-periodic, initial, or arbitrary boundary conditions. We also apply some very well-known techniques (lower and upper solutions method or Krasnosel'skiĭ's Fixed Point Theorem, for instance) in order to further derive results.

Computing explicitly the Green's function for a problem with nonconstant coefficients is not simple, not even in the case of ordinary differential equations. We face these obstacles in Chap. 4, where we reduce a new, more general problem containing nonconstant coefficients and arbitrary differentiable involutions, to the one studied in Chap. 3. In order to do this we use a double trick. First, we reduce the case of a general involution to the case of the reflection using some of the knowledge gathered in Chap. 1 and then we use a special change of variable (only valid in some cases) that allows obtaining of the Green's function of problems with nonconstant coefficients from the Green's functions of constant-coefficient analogs.

To end this part of the work, we have Chap. 5, in which we deepen in the algebraic nature of reflections and extrapolate these properties to other algebras. In this way, we not only generalize the results of Chap. 3 to the case of n -th order problems and general two-point boundary conditions, but also solve functional differential problems in which the Hilbert transform or other adequate operators are involved.

Part III is about applying the results we have proved so far to some related problems. In Chap. 6 we apply the knowledge obtained in Chap. 3 to a nonlocal Hammerstein integral problem with reflection and provide some necessary calculations and examples.

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