Chapter 2
Second Order Problem with Nonlinear Boundary Conditions

Abstract The chapter is devoted to the impulsive nonlinear boundary value problem

$$u''(t) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [a, b] \subset \mathbb{R},$$

$$u(t_i+) = J_i(u(t_i-)), \quad u'(t_i+) = M_i(u'(t_i-)), \quad i = 1, \dots, p,$$

$$g_1(u(a), u(b)) = 0, \quad g_2(u'(a), u'(b)) = 0,$$

where $p \in \mathbb{N}$, $f \in \mathrm{Car}([a, b] \times \mathbb{R}^2)$, $g_1, g_2 \in \mathbb{C}(\mathbb{R}^2)$, $J_i, M_i \in \mathbb{C}(\mathbb{R})$, $i = 1, \dots, p$. Impulses are considered at the fixed points $t_1, \dots, t_p$, $a < t_1 < \dots < t_p < b$. We prove the solvability of the problem under the assumption that there exists a well-ordered pair of lower and upper functions associated with the problem. No growth restrictions are imposed on the functions f, g_1, g_2, J_i, M_i, $i = 1, \dots, p$.

2.1 Introduction

The chapter deals with boundary value problems having nonlinear boundary conditions and impulses at fixed points $t_1, \dots, t_p$, where $a = t_0 < t_1 < \dots < t_p < t_{p+1} = b$, $[a, b] \subset \mathbb{R}$, $p \in \mathbb{N}$. More precisely, we consider the problem

$$u''(t) = f(t, u(t), u'(t)) \quad \text{for a.e. } t \in [a, b] \subset \mathbb{R}, \tag{2.1}$$

$$u(t_i+) = J_i(u(t_i-)), \quad u'(t_i+) = M_i(u'(t_i-)), \quad i = 1, \dots, p, \tag{2.2}$$

$$g_1(u(a), u(b)) = 0, \quad g_2(u'(a), u'(b)) = 0, \tag{2.3}$$

where $f \in \mathrm{Car}([a, b] \times \mathbb{R}^2)$, $g_1, g_2 \in \mathbb{C}(\mathbb{R}^2)$, $J_i, M_i \in \mathbb{C}(\mathbb{R})$, $i = 1, \dots, p$.
Since the function f fulfils the Carathéodory conditions on the whole set where we search for solutions, we say that Eq. (2.1) is regular, in contrast to Chaps. 3 and 4, where we investigate singular equations.

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Definition 2.1 A function $u \in \mathbb{AC}^1_{\mathscr{D}}([a,b])$ that satisfies differential equation (2.1) for a.e. $t \in [a,b]$ and fulfils conditions (2.2) and (2.3) is called a *solution* of problem (2.1)–(2.3).

Our main tool is a well-ordered pair of *lower and upper functions* σ_1 and σ_2 of problem (2.1)–(2.3).

Definition 2.2 A function $\sigma_k \in \mathbb{AC}^1_{\mathscr{D}}([a,b])$ is called a *lower (upper) function* of problem (2.1)–(2.3) provided the conditions

$$[\sigma_k''(t) - f(t, \sigma_k(t), \sigma_k'(t))](-1)^k \le 0 \quad \text{for a.e. } t \in [a,b], \tag{2.4}$$

$$\sigma_k(t_i+) = J_i(\sigma_k(t_i)), \quad \left[\sigma_k'(t_i+) - M_i(\sigma_k'(t_i))\right](-1)^k \le 0, \quad i = 1, \dots, p, \tag{2.5}$$

$$g_1(\sigma_k(a), \sigma_k(b)) = 0, \quad g_2(\sigma_k'(a), \sigma_k'(b))(-1)^k \le 0, \tag{2.6}$$

where $k = 1$ ($k = 2$), are satisfied.

Throughout the chapter we assume:

$$\left.\begin{array}{l}\sigma_1 \text{ and } \sigma_2 \text{ are respectively lower and upper functions}\\ \text{of problem (2.1)–(2.3) and } \sigma_1(t) \le \sigma_2(t) \text{ for } t \in [a,b],\end{array}\right\} \tag{2.7}$$

$$\left.\begin{array}{ll}g_1(\sigma_1(a), \sigma_1(b)) \ne g_1(x, \sigma_1(b)) & \text{if } x > \sigma_1(a),\\ g_1(\sigma_2(a), \sigma_2(b)) \ne g_1(x, \sigma_2(b)) & \text{if } x < \sigma_2(a),\end{array}\right\} \tag{2.8}$$

$$\left.\begin{array}{ll}g_1(\sigma_1(a), \sigma_1(b)) \le g_1(\sigma_1(a), y) & \text{if } \sigma_1(b) \le y,\\ g_1(\sigma_2(a), \sigma_2(b)) \ge g_1(\sigma_2(a), y) & \text{if } \sigma_2(b) \ge y,\end{array}\right\} \tag{2.9}$$

$$\left.\begin{array}{l}g_2(\sigma_1'(a), \sigma_1'(b)) \le g_2(x,y) \ \text{ if } \ x \ge \sigma_1'(a), \quad y \le \sigma_1'(b),\\ g_2(\sigma_2'(a), \sigma_2'(b)) \ge g_2(x,y) \ \text{ if } \ x \le \sigma_2'(a), \quad y \ge \sigma_2'(b),\end{array}\right\} \tag{2.10}$$

$$J_i(\sigma_1(t_i)) \le J_i(x) \le J_i(\sigma_2(t_i)) \quad \text{if } \sigma_1(t_i) \le x \le \sigma_2(t_i) \tag{2.11}$$

for $i = 1, \dots, p$,

$$\left.\begin{array}{l}\exists\, \varphi_1, \varphi_2 \in \mathbb{AC}_{\mathscr{D}}([a,b]) : \quad \varphi_1(t) \le \sigma_k'(t) \le \varphi_2(t),\ t \in [a,b],\\ \varphi_1'(t) > f(t, x, \varphi_1(t)),\ \varphi_2'(t) < f(t, x, \varphi_2(t)),\\ \text{for a.e. } t \in [a,b], \quad \text{all } x \in [\sigma_1(t), \sigma_2(t)],\end{array}\right\} \tag{2.12}$$

$$g_2(x, \varphi_1(b)) > 0,\ g_2(x, \varphi_2(b)) < 0, \quad x \in [\varphi_1(a), \varphi_2(a)], \tag{2.13}$$

$$M_i(\varphi_1(t_i)) \le \varphi_1(t_i+), \quad M_i(\varphi_2(t_i)) \ge \varphi_2(t_i+), \tag{2.14}$$

and

$$M_i(x) \text{ is nondecreasing for } x \in [\varphi_1(t_i), \varphi_2(t_i)] \tag{2.15}$$

for $i = 1, \dots, p$.

Remark 2.3 If we put for $x, y \in \mathbb{R}$

$$g_1(x, y) = y - x, \quad g_2(x, y) = x - y, \tag{2.16}$$

then (2.3) reduces to the periodic conditions

$$u(a) = u(b), \quad u'(a) = u'(b). \tag{2.17}$$

By virtue of (2.16) we see that g_1 is one-to-one in x, which implies that g_1 satisfies (2.8). Moreover, g_1 fulfils (2.9) because g_1 is increasing in y. Similarly, since g_2 is increasing in x and decreasing in y, we have that g_2 satisfies (2.10). If $\varphi_1(a) > \varphi_1(b)$ and $\varphi_2(a) < \varphi_2(b)$, then g_2 fulfils (2.13), as well.

Remark 2.4 The simplest case of assumptions (2.4)–(2.15) is the one with constant functions $\sigma_1, \sigma_2, \varphi_1, \varphi_2$. Let us put

$$\sigma_1(t) = r_1, \quad \sigma_2(t) = r_2, \quad t \in [a, b],$$

where $r_1, r_2 \in \mathbb{R}$, $r_1 \le r_2$. Then (2.4)–(2.11) reduce to

$$\begin{aligned}
&f(t, r_1, 0) \le 0, \quad f(t, r_2, 0) \ge 0 \quad \text{for a.e. } t \in [a, b],\\
&g_1(r_1, r_1) = 0, \quad g_1(r_2, r_2) = 0, \quad g_2(0, 0) = 0,\\
&J_i(r_1) = r_1, \quad J_i(r_2) = r_2, \quad J_i(x) \in (r_1, r_2) \quad \text{for } x \in (r_1, r_2),\ i = 1, \dots, p,\\
&M_i(0) = 0, \quad i = 1, \dots, p,\\
&g_1(r_1, r_1) \ne g_1(x, r_1) \quad \text{if } x > r_1,\\
&g_1(r_2, r_2) \ne g_1(x, r_2) \quad \text{if } x < r_2,\\
&g_1(r_1, r_1) \le g_1(r_1, y) \quad \text{if } r_1 \le y,\\
&g_1(r_2, r_2) \ge g_1(r_2, y) \quad \text{if } r_2 \ge y,\\
&g_2(0, 0) \le g_2(x, y) \quad \text{if } x \ge 0,\ y \le 0,\\
&g_2(0, 0) \ge g_2(x, y) \quad \text{if } x \le 0,\ y \ge 0.
\end{aligned}$$

Clearly, g_1 and g_2 given by (2.16) fulfil the above conditions. Now, in addition, let us put

$$\varphi_1(t) = c_1, \quad \varphi_2(t) = c_2, \quad t \in [a, b],$$

where $c_1, c_2 \in \mathbb{R}$. Then (2.12)–(2.15) have the form

$$c_1 < 0 < c_2,\ f(t, x, c_1) < 0,\ f(t, x, c_2) > 0 \ \text{ for a.e. } t \in [a, b],\ \text{all } x \in [r_1, r_2],$$

$$g_2(x, c_1) > 0, \quad g_2(x, c_2) < 0 \ \text{ for } x \in [c_1, c_2], \tag{2.18}$$

$$M_i(c_1) \le c_1, \quad M_i(c_2) \ge c_2, \quad M_i \text{ is nondecreasing for } x \in [c_1, c_2], \quad i = 1, \dots, p.$$

We see that if g_1 and g_2 are given by (2.16), assumption (2.18) is not fulfilled. Therefore assumption (2.13) cannot be used for periodic problems having constant functions φ_1, φ_2. Such case is covered by Theorem 2.11, where the more general nonstrict condition (2.58) is used.

In the literature we can find a lot of papers dealing with fixed-time impulsive BVPs but the existence results in most of them are proved under some growth conditions for f. The present chapter provides existence results for problem (2.1)–(2.3) with f satisfying conditions of the sign type with respect to the third variable of f (cf. conditions (2.12)), which means that we impose no growth restrictions on f. Moreover, we do not require the monotonicity of the impulse functions J_i, $i = 1, \dots, p$, and use the weaker conditions (2.11). No growth restrictions are imposed on g_1, g_2, J_i, M_i, $i = 1, \dots, p$, as well, see Theorem 2.10 containing the first existence result. Its proof is based on the method of lower and upper functions providing the construction of an appropriate auxiliary problem (cf. problem (2.29)–(2.31)) and on the method of a priori estimates for solutions of the auxiliary problem (cf. Proposition 2.9).

As was mentioned in Remark 2.4, the conditions (2.13) fail to be satisfied for periodic boundary value problems taking σ_1, σ_2, φ_1, φ_2 as constant functions. Therefore we provide another existence result in Theorem 2.11. Both the theorems have been published in [12].

Let us note that other nonlinear boundary conditions for the second order impulsive problem have been studied in [7, 13, 14] under the Nagumo type growth restrictions. Functional second order differential equations with nonlinear functional boundary conditions and fixed-time impulses are discussed in [2–5]. Relative first order impulsive problems can be found in [1, 6, 9, 11] for the scalar case and in [10] for the vector case.

2.2 Auxiliary Problem

The section is devoted to one auxiliary problem (cf. (2.29)–(2.31)), which will be exploited in the proof of our main existence result in Sect. 2.3. In its construction we use functions $\sigma_1, \sigma_2 \in \mathbb{AC}^1_{\mathscr{D}}([a, b])$ and

$$\omega_k(t, \varepsilon) = \sup\{|f(t, \sigma_k(t), \sigma_k'(t)) - f(t, \sigma_k(t), y)| : |\sigma_k'(t) - y| \le \varepsilon\} \tag{2.19}$$

for a.e. $t \in [a, b]$, and for $\varepsilon \in [0, 1]$, $k = 1, 2$. Functions ω_1 and ω_2 are nondecreasing in their second variable and fulfil the Carathéodory conditions on $[a, b] \times [0, 1]$, which results from the following three lemmas.

Lemma 2.5 *Let* $h \in \mathrm{Car}([a, b] \times S)$*,* $S \subset \mathbb{R}^m$*,* $m \in \mathbb{N}$*. Then for every compact set* $B \subset S$ *the function*

$$\psi_B(t) = \sup_{x \in B} |h(t, x)|$$

is Lebesgue integrable on $[a, b]$.

Proof Let $B \subset S$ be a compact set. First, we will prove that ψ_B is measurable on $[a, b]$. There exists a countable set $B_0 \subset B$ such that

$$\overline{B_0} = B. \tag{2.20}$$

We write $B_0 = \{q_n\}$, where $\{q_n\}$ is a sequence in $\mathbb{R}^m$, and get the sequence of measurable functions

$$\{|h(\cdot, q_n)|\}.$$

Let us define a function

$$\psi_{B_0}(t) = \sup_{x \in B_0} |h(t, x)| = \sup_{n \in \mathbb{N}} |h(t, q_n)| \quad \text{for a.e. } t \in [a, b].$$

From the third Carathéodory condition for the function h we get that there is $h_B \in \mathbb{L}^1([a, b])$ such that $0 \le \psi_{B_0} \le h_B$ a.e. on $[a, b]$, and so ψ_{B_0} is measurable and finite a.e. on $[a, b]$ (cf. [8], Sect. 20, Theorem A, p. 84). It remains to prove that

$$\psi_B = \psi_{B_0} \quad \text{a.e. on } [a, b]. \tag{2.21}$$

Let us take $t \in [a, b]$ for which $h(t, \cdot)$ is continuous on S. Then there exists $x_0 \in B$ such that

$$|h(t, x_0)| = \max_{x \in B} |h(t, x)| = \sup_{x \in B} |h(t, x)| = \psi_B(t).$$

From (2.20) it follows that there exists $\{x_n\}$ such that

$$\{x_n\} \subset B_0 \qquad \text{and} \qquad \lim_{n \to \infty} x_n = x_0.$$

Since $h(t, \cdot)$ is continuous on B, it follows that

$$\lim_{n \to \infty} |h(t, x_n)| = |h(t, x_0)| = \psi_B(t).$$

Obviously, $\psi_{B_0}(t) \ge \lim_{n \to \infty} |h(t, x_n)|$ for a.e. $t \in [a, b]$, i.e. $\psi_{B_0} \ge \psi_B$ a.e. on $[a, b]$. Since $B_0 \subset B$, we have $\psi_{B_0} \le \psi_B$ a.e. on $[a, b]$. Thus, (2.21) is valid. □

Lemma 2.6 *Let* $h \in \mathbb{C}([0, \eta])$, *where* $\eta > 0$. *Then the function*

$$g(y) = \max_{0 \le x \le y} h(x), \qquad y \in [0, \eta]$$

is continuous on $[0, \eta]$.

Proof Let $\varepsilon > 0$ be an arbitrary real number.
(a) Let us prove that g is continuous from the right at $q \in [0, \eta)$. Since $h \in \mathbb{C}([0, \eta])$, it follows that there exists $\delta_1 > 0$ such that $(q, q + \delta_1) \subset (0, \eta)$ and

$$|h(x) - h(q)| < \varepsilon \tag{2.22}$$

for every $x \in (q, q + \delta_1)$. Let $y \in (q, q + \delta_1)$. Then we can write

$$g(y) = \max\left(g(q), \max_{q \le x \le y} h(x)\right).$$

Obviously, if $g(y) = g(q)$, then $|g(y) - g(q)| = 0 < \varepsilon$. If $g(y) > g(q)$, then $g(y) = \max_{q \le x \le y} h(x)$ and there exists $\xi \in [q, y]$ such that $g(y) = h(\xi)$. Consequently, we get from (2.22) that

$$0 < g(y) - g(q) = h(\xi) - g(q) \le h(\xi) - h(q) < \varepsilon.$$

(b) Now, let us prove that g is continuous from the left at $q \in (0, \eta]$. There exists $\delta_2 > 0$ such that (2.22) is valid for $x \in (q - \delta_2, q)$. Let $y \in (q - \delta_2, q)$. We can write

$$g(q) = \max\left(g(y), \max_{y \le x \le q} h(x)\right).$$

Obviously, if $g(y) = g(q)$, then $|g(y) - g(q)| = 0 < \varepsilon$. If $g(q) > g(y)$, then $g(q) = h(\theta)$ for some $\theta \in [y, q]$. Therefore

$$0 < g(q) - g(y) \le h(\theta) - h(y) = h(\theta) - h(q) + h(q) - h(y) < 2\varepsilon. \qquad \square$$

Lemma 2.7 *Let* ω_k, $k = 1, 2$, *be defined by (2.19). Then* $\omega_k \in \mathrm{Car}([a, b] \times [0, 1])$ *for* $k = 1, 2$.

Proof Choose $k \in \{1, 2\}$ and denote

$$\tilde{f}_k(t, y) = f(t, \sigma_k(t), \sigma_k'(t) - y) - f(t, \sigma_k(t), \sigma_k'(t)) \tag{2.23}$$

for a.e. $t \in [a, b]$ and all $y \in [-1, 1]$. Let $\varepsilon \in [0, 1]$. Obviously, $\tilde{f}_k(t, y) \in \mathrm{Car}([a, b] \times [-\varepsilon, \varepsilon])$ and $\omega_k(t, \varepsilon) = \sup\{|\tilde{f}_k(t, y)| : |y| \le \varepsilon\}$. Lemma 2.5 implies that $\omega_k(\cdot, \varepsilon)$ is measurable on $[a, b]$. Since

$$\omega_k(t, \varepsilon) \le \omega_k(t, 1) \quad \text{for a.e. } t \in [a, b], \quad \text{all } \varepsilon \in [0, 1],$$

and $\omega_k(\cdot, 1)$ is Lebesgue integrable on $[a, b]$ it follows that ω_k fulfils the third Carathéodory condition.

It remains to prove the continuity of the function $\omega_k(t, \cdot)$ for a.e. $t \in [a, b]$. Let us take $t \in [a, b]$ such that $t \neq t_i$ for $i = 0, \dots, p + 1$ and such that $f(t, \cdot)$ is continuous on $\mathbb{R}^2$. According to (2.23), we have

$$\omega_k(t, \varepsilon) = \max\left(\max_{0 \le y \le \varepsilon} |\tilde{f}_k(t, y)|, \max_{0 \le y \le \varepsilon} |\tilde{f}_k(t, -y)|\right) \quad \text{for each } \varepsilon \in [0, 1].$$

In view of Lemma 2.6, the proof is complete. □

We are ready to construct an auxiliary impulsive problem as follows.
First, let us put

$$\tilde{f}(t, x, y) = \begin{cases} f(t, \sigma_1(t), y) - \omega_1\left(t, \dfrac{\sigma_1(t) - x}{\sigma_1(t) - x + 1}\right) & \\ \quad -\dfrac{\sigma_1(t) - x}{\sigma_1(t) - x + 1} & \text{for } x < \sigma_1(t), \\ f(t, x, y) & \text{for } \sigma_1(t) \le x \le \sigma_2(t), \\ f(t, \sigma_2(t), y) + \omega_2\left(t, \dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1}\right) & \\ \quad +\dfrac{x - \sigma_2(t)}{x - \sigma_2(t) + 1} & \text{for } \sigma_2(t) < x, \end{cases} \tag{2.24}$$

for a.e. $t \in [a, b]$ and all $x, y \in \mathbb{R}$,

$$\varphi(t, y) = \begin{cases} \varphi_1(t) & \text{for } y < \varphi_1(t), \\ y & \text{for } \varphi_1(t) \le y \le \varphi_2(t), \\ \varphi_2(t) & \text{for } \varphi_2(t) < y, \end{cases} \tag{2.25}$$

for all $t \in [a, b]$, $y \in \mathbb{R}$,

$$f^*(t, x, y) = \tilde{f}(t, x, \varphi(t, y)) \qquad \text{for a.e. } t \in [a, b] \text{ and all } x, y \in \mathbb{R}, \tag{2.26}$$

and

$$g_2^*(x, y) = g_2(\varphi(a, x), \varphi(b, y)) \qquad \text{for all } x, y \in \mathbb{R}. \tag{2.27}$$

Obviously, φ is nondecreasing and Lipschitz-continuous (with the Lipschitz constant equal to 1) in the second variable. By virtue of Lemma 2.7 we have $f^* \in \text{Car}([a, b] \times \mathbb{R}^2)$.

Finally, put

$$\sigma(t,x) = \begin{cases} \sigma_1(t) & \text{for } x < \sigma_1(t), \\ x & \text{for } \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_2(t) & \text{for } \sigma_2(t) < x, \end{cases} \tag{2.28}$$

for all $t \in [a,b]$, $x \in \mathbb{R}$.

Now, define the problem

$$u''(t) = f^*(t, u(t), u'(t)) \quad \text{for a.e. } t \in [a,b] \tag{2.29}$$

$$\left.\begin{aligned} u(t_i+) - u(t_i) &= J_i(\sigma(t_i, u(t_i))) - \sigma(t_i, u(t_i)), \\ u'(t_i+) - u'(t_i) &= M_i(\varphi(t_i, u'(t_i))) - \varphi(t_i, u'(t_i)), \quad i = 1, \dots, p, \end{aligned}\right\} \tag{2.30}$$

$$\left.\begin{aligned} u(a) &= \sigma(a, u(a) + g_1(u(a), u(b))), \\ u(b) &= \sigma(b, u(b) + g_2^*(u'(a), u'(b))). \end{aligned}\right\} \tag{2.31}$$

Definition 2.8 A function $u \in \mathbb{AC}^1_{\mathcal{D}}([a,b])$ that satisfies differential equation (2.29) for a.e. $t \in [a,b]$ and fulfils conditions (2.30), (2.31) is called a *solution* of problem (2.29)–(2.31).

Proposition 2.9 *Let conditions (2.7)–(2.15) and (2.24)–(2.28) hold. Let u be a solution of problem (2.29)–(2.31). Then*

$$\sigma_1(t) \le u(t) \le \sigma_2(t), \quad t \in [a,b], \tag{2.32}$$

$$\varphi_1(t) \le u'(t) \le \varphi_2(t), \quad t \in [a,b], \tag{2.33}$$

and u is a solution of problem (2.1)–(2.3).

Proof STEP 1. We will prove inequality (2.32). Let us consider the function

$$v(t) = u(t) - \sigma_2(t), \quad t \in [a,b].$$

Suppose that there exist $i \in \{0, \dots, p\}$ and $\tau \in (t_i, t_{i+1})$ such that

$$\max_{t \in (t_i, t_{i+1}]} v(t) = v(\tau) > 0. \tag{2.34}$$

Then

$$v'(\tau) = 0,$$

which together with (2.34) implies that there exists $\gamma > 0$ such that

$$v(t) > 0 \quad \text{and} \quad |v'(t)| < \frac{v(t)}{v(t)+1} < 1 \tag{2.35}$$

for $t \in (\tau, \tau+\gamma) \subset (t_i, t_{i+1})$. Then, by (2.24)–(2.26),

$$\begin{aligned} v''(t) &= u''(t) - \sigma_2''(t) \geq \tilde{f}(t, u(t), \varphi(t, u'(t))) - f(t, \sigma_2(t), \sigma_2'(t)) \\ &= f(t, \sigma_2(t), \varphi(t, u'(t))) - f(t, \sigma_2(t), \sigma_2'(t)) + \omega_2\left(t, \frac{v(t)}{v(t)+1}\right) + \frac{v(t)}{v(t)+1} \end{aligned}$$

for a.e. $t \in (\tau, \tau+\gamma)$. Note that, due to the properties of function φ (cf. (2.25) and the note below (2.27)), for each $t \in [a, b]$, $x, y \in \mathbb{R}$, satisfying $\varphi_1(t) \leq y \leq \varphi_2(t)$ the inequality

$$|\varphi(t, x) - y| = |\varphi(t, x) - \varphi(t, y)| \leq |x - y|$$

holds. From this fact and (2.12) we have

$$|\varphi(t, u'(t)) - \sigma_2'(t)| \leq |u'(t) - \sigma_2'(t)| = |v'(t)|, \quad t \in (t_i, t_{i+1}). \tag{2.36}$$

By virtue of (2.19), (2.35) and (2.36), we get

$$v''(t) \geq -\omega_2(t, |v'(t)|) + \omega_2\left(t, \frac{v(t)}{v(t)+1}\right) + \frac{v(t)}{v(t)+1} \geq \frac{v(t)}{v(t)+1} > 0 \tag{2.37}$$

for a.e. $t \in (\tau, \tau+\gamma)$. Thus

$$0 < \int_\tau^t v''(s)\,\mathrm{d}s = v'(t) - v'(\tau) = v'(t), \quad t \in (\tau, \tau+\gamma),$$

which contradicts (2.34). So, we have proved that

$$\left.\begin{array}{l}\text{the function } v \text{ has no positive maximum inside the} \\ \text{interval } (t_i, t_{i+1}),\ i = 0, \dots, p.\end{array}\right\} \tag{2.38}$$

In addition, (2.28) and (2.31) yield $v(a) \leq 0$. Now, suppose that there exists $q \in (a, t_1)$ satisfying $v(q) > 0$. Then, according to (2.38),

$$\max_{t \in [a, t_1]} v(t) = v(t_1) > 0, \tag{2.39}$$

i.e. $u(t_1) > \sigma_2(t_1)$. We get $\sigma(t_1, u(t_1)) = \sigma_2(t_1)$ and from the first equality in (2.30) it follows that

$$u(t_1+) = J_1(\sigma_2(t_1)) - \sigma_2(t_1) + u(t_1) > J_1(\sigma_2(t_1)).$$

Using (2.5) we get $u(t_1+) > \sigma_2(t_1+)$, which means $v(t_1+) > 0$. Further, (2.39) implies

$$v'(t_1) \geq 0. \tag{2.40}$$

Suppose that

$$v'(t_1+) < 0. \tag{2.41}$$

In view of (2.40), (2.15), (2.12), (2.25) and (2.5), we derive

$$M_1(\varphi(t_1, u'(t_1))) \geq M_1(\varphi(t_1, \sigma_2'(t_1))) = M_1(\sigma_2'(t_1)) \geq \sigma_2'(t_1+),$$

and applying it to (2.30) we obtain the inequality

$$u'(t_1+) - \sigma_2'(t_1+) \geq u'(t_1) - \varphi(t_1, u'(t_1)).$$

Due to (2.41), we get $u'(t_1) < \varphi(t_1, u'(t_1))$, i.e. $u'(t_1) < \varphi_1(t_1)$. Using this and (2.40), we see that $\sigma_2'(t_1) < \varphi_1(t_1)$, which contradicts (2.12).

Therefore $v'(t_1+) \geq 0$. If $v'(t_1+) = 0$ and v is nonincreasing on some interval $(t_1, t_1 + \gamma) \subset (t_1, t_2)$ where $\gamma > 0$, then (2.35) is valid for all $t \in (t_1, t_1 + \gamma_1)$, $0 < \gamma_1 \leq \gamma$. Hence, (2.37) is satisfied for a.e. $t \in (t_1, t_1 + \gamma_1)$. Consequently, we get

$$0 < \int_{t_1}^{t} v''(s)\,\mathrm{d}s = v'(t) - v'(t_1+) = v'(t), \quad t \in (t_1, t_1 + \gamma_1),$$

which contradicts the assumption of monotony of the function v. Thus $v'(t_1+) > 0$. By virtue of (2.38), the inequalities

$$0 < v(t_1+) < v(t_2) \quad \text{and} \quad v'(t_2) \geq 0$$

hold in all other cases. Then we use the preceding procedure and deduce by induction that

$$v(t_i) > 0, \quad i = 1, \dots, p+1,$$

i.e. $v(b) > 0$, contrary to (2.31). This means that (2.39) is not valid, which together with (2.38) gives $v \leq 0$ on $[a, t_1]$, i.e. $u(t) \leq \sigma_2(t)$ for $t \in [a, t_1]$.

To prove that $u(t) \geq \sigma_1(t)$ for $t \in [a, t_1]$, we argue similarly. Therefore we get $\sigma_1(t) \leq u(t) \leq \sigma_2(t)$ for $t \in [a, t_1]$. Using

$$\sigma_1(t_1) \leq u(t_1) \leq \sigma_2(t_1)$$

and (2.11), we obtain

$$J_1(\sigma_1(t_1)) \leq J_1(u(t_1)) \leq J_1(\sigma_2(t_1)). \tag{2.42}$$

Further, the first equality in (2.30) leads to

$$u(t_1+) = J_1(u(t_1)).$$

Therefore, according to (2.5) and (2.42), the estimate

$$\sigma_1(t_1+) \le u(t_1+) \le \sigma_2(t_1+)$$

is valid. We continue with such argument on each interval $[t_i, t_{i+1}]$ for $i = 1, \dots, p$, and get (2.32).
STEP 2. We will prove that

$$g_1(u(a), u(b)) = 0, \quad g_2^*(u'(a), u'(b)) = 0. \tag{2.43}$$

To this aim we will show that

$$\sigma_1(a) \le u(a) + g_1(u(a), u(b)) \le \sigma_2(a) \tag{2.44}$$

and

$$\sigma_1(b) \le u(b) + g_2^*(u'(a), u'(b)) \le \sigma_2(b). \tag{2.45}$$

Suppose that the first inequality in (2.44) is not true. Then

$$\sigma_1(a) > u(a) + g_1(u(a), u(b)).$$

In view of (2.31), we have $u(a) = \sigma_1(a)$, thus it follows from (2.9) and (2.32) that

$$0 > g_1(\sigma_1(a), u(b)) \ge g_1(\sigma_1(a), \sigma_1(b)),$$

which contradicts (2.6). The second inequality in (2.44) can be proved similarly. Suppose that the first inequality in (2.45) is not valid, i.e. let

$$\sigma_1(b) > u(b) + g_2^*(u'(a), u'(b)). \tag{2.46}$$

Then, by (2.31), we see that

$$u(b) = \sigma_1(b), \tag{2.47}$$

and $0 > g_2^*(u'(a), u'(b))$. By virtue of (2.6), (2.31) and (2.44), we have

$$g_1(\sigma_1(a), \sigma_1(b)) = 0 = g_1(u(a), u(b)) = g_1(u(a), \sigma_1(b)),$$

which by (2.8) gives

$$u(a) = \sigma_1(a). \tag{2.48}$$

Further, relations (2.32), (2.47) and (2.48) imply that $\sigma_1'(b) \ge u'(b)$ and $u'(a) \ge \sigma_1'(a)$. Finally, by (2.10), we get the inequalities

$$0 > g_2^*(u'(a), u'(b)) \ge g_2(\sigma_1'(a), \sigma_1'(b)),$$

contrary to (2.6). The second inequality in (2.45) can be proved by a similar argument. Due to (2.31), conditions (2.44) and (2.45) imply (2.43).
STEP 3. We will prove (2.33). According to (2.32), we have

$$f^*(t, u(t), u'(t)) = \tilde{f}(t, u(t), \varphi(t, u'(t))) = f(t, u(t), \varphi(t, u'(t))) \tag{2.49}$$

for a.e. $t \in [a, b]$. Define $z = u' - \varphi_2$ on $[a, b]$ and suppose that there exists $q \in [a, t_1)$ satisfying

$$\max_{t \in [a, t_1]} z(t) = z(q) > 0. \tag{2.50}$$

Then we can find $\delta > 0$ such that $z(t) > 0$, i.e. $u'(t) > \varphi_2(t)$ for $t \in (q, q + \delta)$. Using (2.12) we deduce that

$$z'(t) = u''(t) - \varphi_2'(t) = f(t, u(t), \varphi(t, u'(t))) - \varphi_2'(t) > 0$$

for a.e. $t \in (q, q + \delta)$. This implies that

$$0 < \int_q^t z'(s)\,\mathrm{d}s = z(t) - z(q), \quad t \in (q, q + \delta),$$

which contradicts (2.50). Suppose that (2.50) is valid for $q = t_1$. According to (2.25), we see that $\varphi(t_1, u'(t_1)) = \varphi_2(t_1)$, and hence (2.30) yields

$$u'(t_1+) - M_1(\varphi_2(t_1)) = u'(t_1) - \varphi_2(t_1).$$

Having in mind (2.50) with $q = t_1$, we get by means of (2.14)

$$u'(t_1+) > M_1(\varphi_2(t_1)) \geq \varphi_2(t_1+),$$

i.e. $z(t_1+) > 0$. We can apply the preceding procedure on $(t_i, t_{i+1}]$ for $i = 1, \dots, p$, and get $z(t_2) > 0, \dots, z(b) > 0$. The last inequality yields $\varphi(b, u'(b)) = \varphi_2(b)$, and therefore (2.13) and (2.27) lead to

$$g_2^*(u'(a), u'(b)) = g_2(\varphi(a, u'(a)), \varphi_2(b)) < 0.$$

According to (2.43) we get a contradiction. The second inequality in (2.33) can be derived similarly.
STEP 4. To summarize, we have proved that an arbitrary solution u of problem (2.29)–(2.31) satisfies (2.32), (2.33) and (2.43). This implies, by (2.24)–(2.26) and (2.28), that u satisfies (2.2), and in addition u fulfils (2.1) for a.e. $t \in [a, b]$. Moreover, due to (2.27), u satisfies (2.3). This completes the proof. □

2.3 Main Results

We are ready to formulate our first existence result.

Theorem 2.10 *Let conditions (2.7)–(2.15) hold. Then there exists a solution u of problem (2.1)–(2.3) such that*

$$\sigma_1(t) \le u(t) \le \sigma_2(t), \qquad \varphi_1(t) \le u'(t) \le \varphi_2(t), \quad t \in [a,b]. \tag{2.51}$$

Proof Let f^* be defined by (2.24) and (2.26). Since $f^* \in \mathrm{Car}([a,b] \times \mathbb{R}^2)$, there exists $h \in \mathbb{L}^1([a,b])$ such that

$$|f^*(t,x,y)| \le h(t) \quad \text{for a.e. } t \in [a,b] \text{ and all } x, y \in \mathbb{R}. \tag{2.52}$$

Consider the Green's function

$$G(t,s) = \begin{cases} \dfrac{(a-s)(b-t)}{b-a} & \text{for } a \le s < t \le b, \\ \dfrac{(a-t)(b-s)}{b-a} & \text{for } a \le t \le s \le b, \end{cases}$$

and a function $G_1 : [a,b] \times [a,b] \to \mathbb{R}$ defined by

$$G_1(t,s) = \begin{cases} \dfrac{b-t}{b-a} & \text{for } a \le s < t \le b, \\ \dfrac{a-t}{b-a} & \text{for } a \le t \le s \le b. \end{cases}$$

Denote

$$g_0 = \operatorname*{sup\,ess}_{(t,s) \in [a,b]^2} \left(|G(t,s)| + |G_1(t,s)| + \left| \frac{\partial G(t,s)}{\partial t} \right| + \left| \frac{\partial G_1(t,s)}{\partial t} \right| \right)$$

and

$$\begin{aligned} K = {} & 2 \max \left\{ 1, \frac{1}{b-a} \right\} (\|\sigma_1\|_\infty + \|\sigma_2\|_\infty) \\ & + g_0 \Bigg[\int_a^b h(s)\,\mathrm{d}s + p\,(\|\varphi_1\|_\infty + \|\varphi_2\|_\infty + \|\sigma_1\|_\infty + \|\sigma_2\|_\infty) \\ & + \sum_{i=1}^p \left(\max_{\varphi_1(t_i) \le x \le \varphi_2(t_i)} |M_i(x)| + \max_{\sigma_1(t_i) \le x \le \sigma_2(t_i)} |J_i(x)| \right) \Bigg]. \end{aligned} \tag{2.53}$$

In order to prove the existence of a solution to problem (2.29)–(2.31) we consider an operator $\mathscr{F} : \Omega \subset \mathbb{C}^1_{\mathscr{D}}([a,b]) \to \mathbb{C}^1_{\mathscr{D}}([a,b])$, where

$$\Omega = \{u \in \mathbb{C}^1_{\mathscr{D}}([a,b]) : \|u\|_{1,\infty} \le K\}. \tag{2.54}$$

The operator $\mathscr{F}$ has the form

$$\mathscr{F} = \mathscr{F}_1 + \mathscr{F}_2,$$

where

$$(\mathscr{F}_1 u)(t) = \int_a^b G(t,s) f^*(s, u(s), u'(s))\,\mathrm{d}s, \tag{2.55}$$

$$\begin{aligned}(\mathscr{F}_2 u)(t) = {} & \frac{b-t}{b-a}\sigma(a, u(a) + g_1(u(a), u(b))) \\ & + \frac{t-a}{b-a}\sigma(b, u(b) + g_2^*(u'(a), u'(b))) \\ & + \sum_{i=1}^{p} G(t,t_i)[M_i(\varphi(t_i, u'(t_i))) - \varphi(t_i, u'(t_i))] \\ & + \sum_{i=1}^{p} G_1(t,t_i)[J_i(\sigma(t_i, u(t_i))) - \sigma(t_i, u(t_i))],\end{aligned} \tag{2.56}$$

for each $u \in \mathbb{C}^1_{\mathscr{D}}([a,b])$ and each $t \in [a,b]$. Here φ, g_2^* and σ are given by (2.25), (2.27) and (2.28), respectively. According to (2.53) and (2.54) we see that $\mathscr{F}(\Omega) \subset \Omega$. To prove the existence of a fixed point of the operator $\mathscr{F}$ we use the Schauder Fixed Point Theorem. Since Ω is a nonempty, closed, convex and bounded subset in $\mathbb{C}^1_{\mathscr{D}}([a,b])$, the only thing left to prove is the absolute continuity of $\mathscr{F}$. Using the Lebesgue Dominated Convergence Theorem and (2.25), (2.28), (2.11) with the continuity of the functions $\sigma_1, \sigma_2, g_1, g_2, J_i, M_i$, for $i = 1, \ldots, p$, we can show by standard arguments that $\mathscr{F}_1$ and $\mathscr{F}_2$ are continuous. The Arzelà–Ascoli Theorem and (2.52) guarantee that the operator $\mathscr{F}_1 : \Omega \to \mathbb{C}^1([a,b])$ is absolutely continuous. Since $\mathscr{F}_2$ maps Ω into a finite dimensional subspace of $\mathbb{C}^1_{\mathscr{D}}([a,b])$ with the basis $\{1, t, G(t,t_i), G_1(t,t_i) : i = 1, \ldots, p\}$ and $\mathscr{F}_2$ is a bounded, continuous operator, it follows that $\mathscr{F}_2$ is also absolutely continuous. Thus, there exists a fixed point u of $\mathscr{F}$, i.e.

$$u = \mathscr{F}u = \mathscr{F}_1 u + \mathscr{F}_2 u.$$

The definition (2.55) implies that $\mathscr{F}_1 u \in \mathbb{AC}^1([a,b])$ and, by (2.56), we have $\mathscr{F}_2 u \in \mathbb{AC}^1_{\mathscr{D}}([a,b])$. Therefore $u \in \mathbb{AC}^1_{\mathscr{D}}([a,b])$.

By a direct computation we get from (2.55)

$$(\mathscr{F}_1 u)''(t) = f^*(t, u(t), u'(t)) \quad \text{and} \quad (\mathscr{F}_2 u)''(t) = 0 \quad \text{for a.e. } t \in [a,b],$$

which means that u satisfies (2.29). Further,

$$
\begin{aligned}
&(\mathscr{F}_1 u)(a) = (\mathscr{F}_1 u)(b) = 0,\\
&(\mathscr{F}_2 u)(a) = \sigma(a, u(a) + g_1(u(a), u(b))),\\
&(\mathscr{F}_2 u)(b) = \sigma(b, u(b) + g_2^*(u'(a), u'(b))),
\end{aligned}
$$

hence (2.31) is valid. Finally,

$$
(\mathscr{F}_1 u)^{(j)}(t_i+) = (\mathscr{F}_1 u)^{(j)}(t_i), \qquad j = 0, 1,
$$

and

$$
\begin{aligned}
(\mathscr{F}_2 u)(t_i+) - (\mathscr{F}_2 u)(t_i) &= J_i(\sigma(t_i, u(t_i))) - \sigma(t_i, u(t_i)),\\
(\mathscr{F}_2 u)'(t_i+) - (\mathscr{F}_2 u)'(t_i) &= M_i(\varphi(t_i, u'(t_i))) - \varphi(t_i, u'(t_i))
\end{aligned}
$$

for $i = 1, \dots, p$. Thus, u is a solution of problem (2.29)–(2.31), and in view of Proposition 2.9, it is a solution of problem (2.1)–(2.3). □

According to Remark 2.4, we replace the strict inequalities (2.13) by the nonstrict inequalities (2.58) in the next theorem. Note that conditions (2.6), (2.12) and (2.13) imply

$$
\varphi_1(b) < \sigma_2'(b) \quad \text{and} \quad \sigma_1'(b) < \varphi_2(b). \tag{2.57}
$$

For instance, the first inequality can be obtained by way of contradiction. If $\varphi_1(b) \geq \sigma_2'(b)$, then by (2.12) we get $\varphi_1(b) = \sigma_2'(b)$. From (2.12) we get $\varphi_1(a) \leq \sigma_2'(a) \leq \varphi_2(a)$ as well. This together with (2.13) yields

$$
g_2(\sigma_2'(a), \sigma_2'(b)) = g_2(\sigma_2'(a), \varphi_1(b)) > 0,
$$

which contradicts (2.6). Similarly for the second inequality. Using condition (2.57) we state our second existence result which is applicable also to periodic problems with constant functions φ_1 and φ_2.

Theorem 2.11 *Let conditions (2.7)–(2.12), (2.14), (2.15), (2.57) and*

$$
g_2(x, \varphi_k(b))(-1)^k \leq 0 \quad \textit{for } x \in [\varphi_1(a), \varphi_2(a)],\ k = 1, 2, \tag{2.58}
$$

hold. Then there exists a solution u of problem (2.1)–(2.3) satisfying (2.51).

Proof Define a function $\psi : \mathbb{R} \to [-1, 1]$ by

$$
\psi(y) = \begin{cases}
1 & \text{for } y \leq \varphi_1(b),\\
\dfrac{\lambda - y}{\lambda - \varphi_1(b)} & \text{for } \varphi_1(b) < y < \lambda,\\
0 & \text{for } \lambda \leq y \leq \mu,\\
\dfrac{\mu - y}{\varphi_2(b) - \mu} & \text{for } \mu < y < \varphi_2(b),\\
-1 & \text{for } \varphi_2(b) \leq y,
\end{cases}
$$

where $\varphi_1(b) < \lambda \le \mu < \varphi_2(b)$. If $\varphi_1(b) < \sigma_1'(b)$ and $\sigma_2'(b) < \varphi_2(b)$, then we put $\lambda = \min(\sigma_1'(b), \sigma_2'(b))$ and $\mu = \max(\sigma_1'(b), \sigma_2'(b))$. In the case when $\varphi_1(b) = \sigma_1'(b)$ and $\sigma_2'(b) < \varphi_2(b)$ we put $\lambda = \mu = \sigma_2'(b)$ and similarly, if $\varphi_1(b) < \sigma_1'(b)$ and $\sigma_2'(b) = \varphi_2(b)$, then $\lambda = \mu = \sigma_1'(b)$. Otherwise, we can take λ and μ arbitrarily. Choose $n \in \mathbb{N}$ and define a function $g_{2,n}$ by

$$g_{2,n}(x, y) = g_2(x, y) + \frac{1}{n}\psi(y) \tag{2.59}$$

for $x, y \in \mathbb{R}$, and consider the problem (2.1), (2.2),

$$\left.\begin{aligned} g_1(u(a), u(b)) = 0, \\ g_{2,n}(u'(a), u'(b)) = 0. \end{aligned}\right\} \tag{2.60}$$

We can check that for each $n \in \mathbb{N}$, problem (2.1), (2.2), (2.60) fulfils conditions (2.7)–(2.15), and by Theorem 2.10, we get its solution u_n. According to the proof of Theorem 2.10, u_n satisfies

$$u_n = \mathscr{F}_n u_n,$$

where $\mathscr{F}_n = \mathscr{F}_1 + \mathscr{F}_{2,n}$. Here $\mathscr{F}_1$ is defined by (2.55) and $\mathscr{F}_{2,n}$ is defined in (2.56), with $g^*_{2,n}$ instead of g^*_2, where

$$g^*_{2,n}(x, y) = g_{2,n}(\varphi(a, x), \varphi(b, y)), \quad x, y \in \mathbb{R},$$

and $g_{2,n}$ is from (2.59). From (2.59) it follows that for each $x, y \in \mathbb{R}$ we have

$$\left.\begin{aligned} g^*_{2,n}(x, y) &= g_{2,n}(\varphi(a, x), \varphi(b, y)) = g_2(\varphi(a, x), \varphi(b, y)) + \frac{1}{n}\psi(\varphi(b, y)) \\ &= g^*_2(x, y) + \frac{1}{n}\psi(\varphi(b, y)). \end{aligned}\right\} \tag{2.61}$$

Since $\mathscr{F}_1$ is a compact operator and $\{\mathscr{F}_{2,n}u_n\}$ is a bounded sequence in a subspace of finite dimension, there exists a convergent subsequence of $\{u_n\}$. Without any loss of generality we can assume that $\{u_n\}$ is such a sequence and $u \in \mathbb{C}^1_{\mathscr{D}}([a, b])$ is its limit. We will show that u is a solution of problem (2.1)–(2.3). Consider the operator $\mathscr{F} = \mathscr{F}_1 + \mathscr{F}_2$, where $\mathscr{F}_2$ is defined by (2.56). We have

$$\|\mathscr{F}u - u\|_{1,\infty} \le \|\mathscr{F}u - \mathscr{F}u_n\|_{1,\infty} + \|\mathscr{F}u_n - \mathscr{F}_n u_n\|_{1,\infty} + \|u_n - u\|_{1,\infty}.$$

The first and third term tend to zero as $n \to \infty$. We deal with the second term:

$$\begin{aligned}
&\|\mathscr{F}u_n - \mathscr{F}_n u_n\|_{1,\infty} \\
&\quad = \left\| \frac{t-a}{b-a} \Big(\sigma(b, u_n(b) + g_2^*(u_n'(a), u_n'(b))) - \sigma(b, u_n(b) + g_{2,n}^*(u_n'(a), u_n'(b))) \Big) \right\|_{1,\infty} \\
&\quad = \left\| \frac{t-a}{b-a} \right\|_{1,\infty} \left| \sigma(b, u_n(b) + g_2^*(u_n'(a), u_n'(b))) - \sigma(b, u_n(b) + g_{2,n}^*(u_n'(a), u_n'(b))) \right| \\
&\quad \le \left(1 + \frac{1}{b-a}\right) \left| g_2^*(u_n'(a), u_n'(b))) - g_{2,n}^*(u_n'(a), u_n'(b)) \right| \\
&\quad = \left(1 + \frac{1}{b-a}\right) \frac{1}{n} \left| \psi(\varphi(b, u_n'(b))) \right| \le \frac{1}{n(b-a)} + \frac{1}{n},
\end{aligned}$$

where we used the fact that σ is Lipschitz continuous in its second variable with the constant 1. Since the right hand side of this inequality approaches zero as $n \to \infty$, it follows that u is a fixed point of $\mathscr{F}$ and consequently $u \in \mathbb{AC}^1_{\mathscr{D}}([a, b])$. From the uniform convergence of $\{u_n\}$, $\{u_n'\}$ and $\{g_{2,n}\}$, we get (2.3) and (2.2). It remains to prove that u satisfies the differential equation (2.1). We have

$$u_n''(t) = f(t, u_n(t), u_n'(t)) \quad \text{for a.e. } t \in [a, b].$$

Let $i \in \{0, \dots, p\}$ and $t \in (t_i, t_{i+1})$. Then

$$u_n'(t) - u_n'(t_i) = \int_{t_i}^{t} f(s, u_n(s), u_n'(s))\, \mathrm{d}s$$

for all $n \in \mathbb{N}$. From the fact that $f \in \mathrm{Car}([a, b] \times \mathbb{R}^2)$, $u_n \to u$ in $\mathbb{C}^1_{\mathscr{D}}([a, b])$ and from the Lebesgue Theorem we have

$$u'(t) - u'(t_i) = \int_{t_i}^{t} f(s, u(s), u'(s))\, \mathrm{d}s$$

for each $t \in (t_i, t_{i+1})$. The proof is complete. □

Remark 2.12 In Remark 2.3 we have shown that if g_1 and g_2 are defined by (2.16), they fulfil (2.8)–(2.10). For the validity of (2.58) it suffices to assume that $\varphi_1(a) \ge \varphi_1(b)$ and $\varphi_2(a) \le \varphi_2(b)$ instead of the strict inequalities which are necessary for (2.13) (cf. the end of Remark 2.3). Then φ_1 and φ_2 can be constant functions in Theorem 2.11. The existence result for constant lower and upper functions $\sigma_1(t) = r_1$, $\sigma_2(t) = r_2$ for $t \in [a, b]$ and constant functions $\varphi_1(t) = c_2$, $\varphi_2(t) = c_2$ for $t \in [a, b]$ follows from Theorem 2.11 and is presented in the next corollary.

Corollary 2.13 *Let r_1, $r_2 \in \mathbb{R}$ be such that $r_1 \le r_2$,*

$$f(t, r_1, 0) \le 0, \quad f(t, r_2, 0) \ge 0 \quad \textit{for a.e. } t \in [a, b],$$

and let

$$J_i(r_k) = r_k, \qquad J_i(x) \in (r_1, r_2), \quad x \in (r_1, r_2)$$

for $i = 1, \ldots, p$, $k = 1, 2$. Further, let c_1, $c_2 \in \mathbb{R}$ be such that $c_1 < 0 < c_2$,

$$f(t, x, c_1) < 0, \quad f(t, x, c_2) > 0 \quad \textit{for a.e. } t \in [a, b] \textit{ and for } x \in [r_1, r_2],$$

and let

$$M_i(0) = 0, \quad M_i(c_k) = c_k, \quad M_i(x) \textit{ be nondecreasing on } [c_1, c_2]$$

for $i = 1, \ldots, p$, $k = 1, 2$. Then the periodic impulsive problem (2.1), (2.2), (2.17) has a solution u satisfying

$$r_1 \le u(t) \le r_2, \qquad c_1 \le u'(t) \le c_2, \quad t \in [a, b].$$

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