# Chapter 2
# Time Relaxization

Different models can share common stationary states. In the previous chapter we described the quantized blowup mechanics of Smoluchowski-Poisson equation in two-space dimensions. This chapter is devoted to the full system of chemotaxis and its relatives. Among them are the tumor growth model and the geometric flow. The quantized blowup mechanism is realized in some models, but is violated in the other cases.

## 2.1 Full System of Chemotaxis

A typical form of the full system of chemotaxis is

$$u_t = \nabla \cdot (\nabla u - u\nabla v),\ \tau v_t = \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega \times (0,T)$$
$$\left(\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}, \frac{\partial v}{\partial \nu}\right)\Big|_{\partial\Omega} = 0, \quad \int_\Omega v = 0 \tag{2.1}$$

where $\tau > 0$ in the left-hand side of the second equation stands for the relaxization time which describes a chemical process of the formation of the field $v$. The fundamental structure is the existence of the Lagrangian,

$$L(u,v) = \int_\Omega u(\log u - 1) + \frac{1}{2}\|\nabla v\|_2^2 - \int_\Omega vu \tag{2.2}$$

defined for $\int_\Omega v = 0$. Then, (2.1) is equivalent to

$$u_t = \nabla \cdot u\nabla L_u(u,v),\ \tau v_t = -L_v(u,v) \quad \text{in } \Omega \times (0,T)$$
$$\left(u\frac{\partial}{\partial \nu}L_u(u,v), \frac{\partial v}{\partial \nu}\right)\Big|_{\partial\Omega} = 0, \quad \int_\Omega v = 0.$$

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Thus it holds that

$$\frac{d}{dt}L(u,v) = -\int_\Omega u\,|\nabla L_u|^2 + \tau^{-1}v_t^2\,dx \le 0$$

for the solution $(u,v) = (u(\cdot,t), v(\cdot,t))$. This formulation to (2.1) is summarized as *dual variation*, where the stationary state is described by both components equivalently. Provided with the variational structures individually, their linearly stable critical points are dynamically stable, see Chap. 3.

Well-posedness local-in-time and the standard blowup criterion are valid also to this system. More precisely, we have always $0 < T_{\max} \le +\infty$. If $T = T_{\max} < +\infty$, furthermore, then $\lim_{t\uparrow T}\|u(\cdot,t)\|_\infty = +\infty$ holds, and the blowup set $\mathcal{S}$ of $u$, defined by (1.25), is not empty. Similarly to Theorem 1.5.1, furthermore, $\lambda = \|u_0\|_1 < 4\pi$ implies $T_{\max} = +\infty$ for the general case, while $\lambda < 8\pi$ implies $T_{\max} = +\infty$ in the radially symmetric case [26, 126, 251].

The local blowup criterion, or $\varepsilon$-regularity, also holds and we obtain

$$\limsup_{t\uparrow T}\|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \ge m_*(x_0)$$

for each $x_0 \in \mathcal{S}$, where $R > 0$ is arbitrary [252]. The bounded variation in time of the local $L^1$-norm, however, is not known, and, consequently, the finiteness of the blowup points, or the blowup threshold as in Theorem 1.5.2, is not valid.

## 2.2 Non-local Parabolic Equation

Replacing $u_t$ by $\varepsilon u_t$ in (2.1) and making $\varepsilon \downarrow 0$, we obtain (1.37) which induces another simplified system of chemotaxis,

$$\begin{aligned} &\tau v_t = \Delta v + \lambda\left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|}\right) \quad \text{in } \Omega\times(0,T)\\ &\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = 0, \quad \int_\Omega v = 0 \end{aligned} \tag{2.3}$$

formulated by Wolansky [420, 421]. This problem is provided with the Lyapunov function

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda\log\int_\Omega e^v + \lambda(\log\lambda - 1), \quad \int_\Omega v = 0,$$

and it holds that

$$\frac{d}{dt}J_\lambda(v) = -\tau^{-1}\|v_t\|_2^2 \le 0.$$

Not so much is known to (2.3) also, but the following theorems [192, 420] show the *dis-quantized blowup mechanism* of the relative model,

$$\tau v_t = \Delta v + \frac{\lambda e^v}{\int_\Omega e^v} \quad \text{in } \Omega \times (0, T), \quad v|_{\partial\Omega} = 0. \tag{2.4}$$

**Theorem 2.2.1** *If $\Omega = B(0, R)$, $v_0 = v_0(|x|)$, and $\lambda \geq 8\pi$, then*

$$u \equiv \frac{\lambda e^v}{\int_\Omega e^v} \rightharpoonup \lambda\delta_0 \tag{2.5}$$

*as $t \uparrow T = T_{\max} \in (0, +\infty]$ in $C(\overline{\Omega})$, where $v = v(\cdot, t)$ is the solution to (2.4) with $v(\cdot, 0) = v_0$.*

**Theorem 2.2.2** *If $\lambda > 8\pi$ in the previous theorem, it holds that $T = T_{\max} < +\infty$, and, therefore, we have the formation of collapse (2.5) in finite time with the dis-quantized mass $\lambda > 8\pi$.*

Blowup in infinite time is expected at the threshold case $\lambda = 8\pi$.

**Non-local ODE**

The spatially homogeneous part of (2.3) or (2.4) is formulated by the *non-local ODE*. Adopting the normalization $\tau = 1$, $\lambda = |\Omega|$, we obtain

$$v_t = \frac{|\Omega| e^v}{\int_\Omega e^v} - 1 \quad \text{in } \Omega \times (0, T) \tag{2.6}$$

or

$$v_t = \frac{|\Omega| e^v}{\int_\Omega e^v} \quad \text{in } \Omega \times (0, T). \tag{2.7}$$

Equation (2.6) is reduced to (2.7) by $\tilde{v} = v + t$. In (2.7) it holds that

$$\frac{1}{|\Omega|} \frac{d}{dt} \int_\Omega v = 1$$

and hence

$$\frac{1}{|\Omega|} \int_\Omega v = t + \frac{1}{|\Omega|} \int_\Omega v_0. \tag{2.8}$$

Then we use $-(e^{-v})_t = a(t) \equiv \dfrac{|\Omega|}{\int_\Omega e^v}$ to derive

$$e^{-v_0(x)} - e^{-v(x,t)} = A(t) \equiv \int_0^t a(t')dt', \tag{2.9}$$

where $v_0 = v(\cdot, 0)$. Equality (2.9) implies

$$e^{-v_0(x_1)} - e^{-v(x_1,t)} = e^{-v_0(x_2)} - e^{-v(x_2,t)},$$

and, hence, the *order preserving property*

$$v_0(x_1) \le v_0(x_2) \quad \Rightarrow \quad v(x_1, t) \le v(x_2, t).$$

In particular, it holds that

$$\|v_0\|_\infty = v_0(x_*) \quad \Rightarrow \quad \|v(\cdot, t)\|_\infty = v(x_*, t).$$

Since $v_t > 0$, we have $\lim\limits_{t\uparrow T} v(x, t) = v(x, T) \in (-\infty, +\infty]$ for each $x \in \overline{\Omega}$, and, therefore,

$$\lim_{t\uparrow T} \|v(\cdot, t)\|_\infty = +\infty \quad \Rightarrow \quad v(x_*, T) = +\infty.$$

It thus follows that

$$e^{-v_0(x_*)} = e^{-\|v_0\|_\infty} = A(T) = e^{-v_0(x)} - e^{-v(x,T)} = A(T)$$

from (2.9), so that

$$v(x, T) = -\log\{e^{-v_0(x)} - e^{-\|v_0\|_\infty}\}.$$

Pluging this relation into (2.8), we obtain

$$-\frac{1}{|\Omega|}\int_\Omega \log\{e^{-v_0} - e^{-\|v_0\|_\infty}\} = T + \frac{1}{|\Omega|}\int_\Omega v_0,$$

and hence the following theorem.

**Theorem 2.2.3** *The solution $v = v(x, t)$ to the non-local ODE (2.7) blows-up at the blowup time*

$$T = -\frac{1}{|\Omega|}\int_\Omega \log\{e^{-v_0} - e^{-\|v_0\|_\infty}\} - \frac{1}{|\Omega|}\int_\Omega v_0. \tag{2.10}$$

The case $T = +\infty$ is admitted in (2.10), while the right-hand side is always positive by

$$-\frac{1}{|\Omega|}\int_\Omega \log\{e^{-v_0}\} - \frac{1}{|\Omega|}\int_\Omega v_0 = 0.$$

## 2.3 Smoluchowski-ODE System

Several mathematical models are proposed to describe the movement of living things attracted by non-diffusive chemical factors using re-inforced random walk which results in parabolic-ODE systems in the limit state, that is

$$p_t = \nabla \cdot (D\nabla p - p\chi(w)\nabla w), \quad w_t = g(p, w), \tag{2.11}$$

see [291]. Here, $p$ and $w$ are due to the conditional probability of the decision of the walkers and the density of the control species, respectively, $D > 0$ the diffusion constant, $\chi$ the chemotactic sensitivity, and $g$ the chemical growth rate.

Angiogenesis is the formation of blood vessels from pre-existing vasculature. It is a process whereby capillary sprouts are formed in response to externally supplied stimuli and provides with a drastic stage to the tumor growth. A parabolic-ODE system modelling tumor-induced angiogenesis is proposed in this connection [9], using the endotherial-cell density per unit area $n$, the TAF (tumor angiogenic factors) concentration $f$, and the matrix macromolecule fibronectin concentration $c$, that is,

$$\begin{aligned} n_t &= D\Delta n - \nabla \cdot (\chi(c)n\nabla c) - \rho_0 \nabla \cdot (n\nabla f) \\ f_t &= \beta n - \mu n f, \quad c_t = -\gamma n c \qquad \text{in } \Omega \times (0, T), \end{aligned} \tag{2.12}$$

where

$$\chi(c) = \frac{\chi_0}{1 + \alpha c}$$

denotes the chemotactic sensitivity and $D$, $\rho_0$, $\beta$, $\mu$, $\gamma$, $\chi_0$, $\alpha$ are positive constants. System (2.11) is formulated as an evolution equation with strong dissipation [205, 212, 430]. There is also an approach from the comparison theorem [122]. Here we use the calculus of variation.

We formulate the problem as the parabolic-ODE system

$$\begin{aligned} &q_t = \nabla \cdot (\nabla q - q\nabla\varphi(v)), \ v_t = q \quad \text{in } \Omega \times (0, T) \\ &\left.\frac{\partial q}{\partial \nu}\right|_{\partial\Omega} = 0, \quad (q, v)|_{t=0} = (q_0(x), v_0(x)), \end{aligned} \tag{2.13}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, $\nu$ is the outer unit normal vector, $q_0 > 0$ and $v_0$ are smooth function on $\overline{\Omega}$, and $\varphi : \mathbf{R} \to \mathbf{R}$ is a smooth function [378]. We impose the compatibility condition

$$\left.\frac{\partial v_0}{\partial \nu}\right|_{\partial\Omega} = 0.$$

Then we obtain the null flux boundary condition

$$\left.\frac{\partial q}{\partial \nu} - q\frac{\partial \varphi(v)}{\partial \nu}\right|_{\partial\Omega} = 0$$

by a simple calculation using the ODE part. Although the form (2.13) is restrictive, several important cases of (2.11) are reduced to it. First, if $g(p, w) = (p - \mu)w$, $w > 0$, then we obtain (2.13) with

$$v = \log w, \quad q = p - \mu, \quad \varphi(v) = A(e^v), \quad A' = \chi.$$

Next, if $g(p, w) = p(\mu - w)$, $w < \mu$, then (2.13) follows from

$$v = -\log(\mu - w), \quad q = p, \quad \varphi(v) = A(\mu - e^{-v}), \quad A' = \chi.$$

Finally, if $g(p, w) = -pw$, $w > 0$, then (2.13) holds for

$$v = -\log w, \quad q = p, \quad \varphi(v) = A(e^{-v}), \quad A' = \chi.$$

System (2.12) is also transformed into a similar form,

$$q_t = \nabla \cdot (\nabla q - q\nabla\varphi(v, w)), \ v_t = q, \ w_t = q \quad \text{in } \Omega \times (0, T). \tag{2.14}$$

In 1979, Rascle [312] studied (2.13) for $\varphi(v) = -v$. This system looks like the case of (2.1) where the diffusion of the second equation is neglected. Furthermore, the chmotaxis term on the right-hand side of the first equation has the negative sign, with the sensitivity $\varphi(v) = -v$ regarded as the self-impulsive factor. In the actual interpretation, however, we replace $-v$ by $v$. Thus $u$ is attractive to the material $v$ and this $v$ is consumed by $u$ itself. In [312], global-in-time solution is obtained using the Lyapunov function in the case of one-space dimension, and a related system of angiogenesis is studied by [119]. This method is applicable to (2.13) under the assumption

$$\begin{aligned} &\varphi \in C^3(\mathbf{R}), \quad \varphi'' \geq 0 \geq \varphi', \\ &q_0, v_0 \in C^{2+\alpha}(\overline{\Omega}), \quad \left.\frac{\partial}{\partial \nu}(q_0, v_0)\right|_{\partial\Omega} = 0 \end{aligned} \tag{2.15}$$

for $0 < \alpha < 1$. System (2.13) is equivalent to the one studied by [74],

$$n_t = \nabla \cdot (\nabla n - n\chi(c)\nabla c),\ c_t = -cn \quad \text{in } \Omega \times (0, T)$$
$$\frac{\partial n}{\partial \nu} - \chi(c)\frac{\partial c}{\partial \nu}\Big|_{\partial\Omega} = 0,$$

where $c, n > 0$, and $\chi = \chi(c)$ is a $C^1$—function satisfying

$$\chi(c) \geq 0, \quad c\chi'(c) + \chi(c) \geq 0.$$

Global-in-time existence of the weak solution with the convergence

$$q(\cdot, t) \to \overline{q}_0 \equiv \frac{1}{|\Omega|}\int_\Omega q_0 \tag{2.16}$$

as $t \uparrow +\infty$ is derived formally, see also [75]. Similar to [312], this property arises in the classical sense when the space-dimension is one, using the continuous embedding, see J.L. Lions [219],

$$L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \hookrightarrow L^4(0, T; L^\infty(\Omega)).$$

This conclusion is a counter part of the result [430], that is if $\varphi(v) = v$, we have both global and blowup in finite time solutions depending on their initial data. We note that $\varphi(v) = v$ does not satisfy $\varphi'(v) \leq 0$. If $\varphi(v) = v$, system (2.13) is a relative to (2.1) with the diffusion term $-\Delta v$ in the second equation neglected:

$$u_t = \nabla \cdot (\nabla u - u\nabla v),\ \tau v_t = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega \times (0, T)$$
$$\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

The key structure of (2.13) is

$$\frac{dL}{dt} = -\int_\Omega q^{-1}|\nabla q|^2 + \frac{1}{2}\varphi''(v)q|\nabla q|^2\, dx$$
$$L = \int_\Omega q(\log q - 1) + 1 - \frac{1}{2}\varphi'(v)|\nabla v|^2\, dx, \tag{2.17}$$

and thus $L$ can be a Lyapunov function. We can show the following theorems [378].

**Theorem 2.3.1** *If (2.15) holds, then there exists a unique solution to (2.13) such that $q, v \in C^{2+\alpha, 1+\alpha/2}(\overline{Q_T})$ with $q = q(x, t) > 0$, provided that $T$ is sufficiently small.*

**Theorem 2.3.2** *If (2.15) holds and the space dimension $n = 1$, the solution in the previous theorem exists for all $T > 0$. Given $t_k \uparrow +\infty$ and $\delta > 0$, furthermore, we have $t_k' \in (t_k - \delta, t_k + \delta)$ such that $q(\cdot, t_k') \to \overline{q}_0$ for $\overline{q}_0$ defined by (2.16) uniformly on $\overline{\Omega}$.*

Similar results to Theorems 2.3.1 and 2.3.2 are valid to (2.14). Actually, we obtain the following theorem [369, 378].

**Theorem 2.3.3** *If $0 < n_0, f_0, c_0$ are $C^{2+\alpha}$ on $\overline{\Omega}$,*

$$f_0 > \frac{\beta}{\mu} \quad \frac{\partial}{\partial \nu}(n_0, f_0, c_0)\Big|_{\partial\Omega} = 0, \tag{2.18}$$

*and the space dimension $n = 1$, then there is a unique solution global-in-time to (2.12) with the initial boundary condition*

$$D\frac{\partial n}{\partial \nu} - \chi(c)n\frac{\partial c}{\partial \nu} - \rho_0 n\frac{\partial f}{\partial \nu}\Big|_{\partial\Omega} = 0, \quad (n, f, c)|_{t=0} = (n_0(x), f_0(x), c_0(x)).$$

*Any $t_k \uparrow +\infty$ and $\delta > 0$, furthermore, admits $t_k' \in (t_k - \delta, t_k + \delta)$ such that*

$$n(\cdot, t_k') \to \overline{n}_0 \equiv \frac{1}{|\Omega|}\int_\Omega n_0$$

*uniformly on $\overline{\Omega}$.*

In the other case of the first Assumption of (2.18),

$$0 < f_0 < \frac{\beta}{\mu}.$$

we obtain a priori bounds of the solution for any space dimension under the assumption of

$$(\beta - \mu f_0)^{\gamma/\beta} \ll c_0,$$

and this provides the global in time solution converging to the stationary solution, see [122].

Differently from the elliptic-parabolic system [324], the possibility of the oscillation of $q(\cdot, t)$ as $t \uparrow +\infty$ has not been excluded even for $n = 1$ because of the lack of the dissipation of the ODE part. Here we show the following.

**Theorem 2.3.4** *If (2.15) holds with*

$$(\varphi')^2 \le C\varphi'' \tag{2.19}$$

*and the space dimension $n = 1$, then we have*

$$\lim_{t\uparrow+\infty} \|q(\cdot,t) - \overline{q}_0\|_4 = 0. \tag{2.20}$$

First, from $\Phi(s) = s(\log s - 1) + 1 \geq 0$, $s > 0$, and $\varphi'(v) \leq 0$ we obtain

$$\begin{aligned} 0 \ &\leq \int_\Omega \Phi(q) - \frac{1}{2}\varphi'(v)\,|\nabla v|^2 \ dx \\ &\quad + \int_0^t dt \cdot \int_\Omega \frac{1}{2}\varphi''(v) q |\nabla v|^2 + q^{-1}\,|\nabla q|^2 \ dx = L(0), \end{aligned} \tag{2.21}$$

and, then, it follows that

$$\|\nabla q^{1/2}\|_{L^2(0,t;L^2(\Omega))} \leq L(0)^{1/2}/2.$$

Next, we have $\dfrac{d}{dt}\displaystyle\int_\Omega q = 0$ which implies

$$\|q(t)\|_1 = \|q_0\|_1 . \tag{2.22}$$

We thus get

$$\sup_{s\in(0,t)} \|q(s)^{1/2}\|_2 + \left\{ \int_0^t \|\nabla q(s)^{1/2}\|_2^2 \, ds \right\}^{1/2} \leq C. \tag{2.23}$$

Now we show the following lemma.

**Lemma 2.3.1** *Assume (2.15) and let*

$$Q_\delta(x,t) = \frac{1}{2\delta}\int_{t-\delta}^{t+\delta} q(x,t')dt', \quad \delta > 0.$$

*Then it holds that*

$$\lim_{t\uparrow+\infty} \|Q_\delta(t) - \overline{q}_0\|_\infty = 0. \tag{2.24}$$

*Proof* By (2.22) and (2.21) we have

$$\|h(t)\|_2 = \|h_0\|_2, \quad \int_0^\infty \|\nabla h(t)\|_2^2 \, dt < +\infty \tag{2.25}$$

where $h(t) = q(t)^{1/2}$ and $h_0 = q_0^{1/2}$. Then it holds that

$$\|\nabla Q_\delta(\cdot, t)\|_1 \le \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} 2\|h\|_2 \|\nabla h\|_2 \, dt' = \frac{2\|h_0\|_2}{2\delta} \cdot \int_{t-\delta}^{t+\delta} \|\nabla h\|_2 \, dt'$$

which implies

$$\|\nabla Q_\delta(\cdot, t)\|_1^2 \le 4\|h_0\|_2^2 \cdot \frac{1}{2\delta} \int_{t-\delta}^{t+\delta} \|\nabla h(\cdot, t)\|_2^2 \, dt \to 0, \quad t \uparrow +\infty \tag{2.26}$$

by (2.25). The equality

$$\frac{1}{|\Omega|} \int_\Omega Q_\delta(\cdot, t) = \overline{q}_0 \tag{2.27}$$

implies (2.24) by $n = 1$. □

The second relation of (2.25) implies Theorem 2.3.2. In fact, given $t_k \uparrow +\infty$, we have

$$\lim_{k\to\infty} \int_{t_k-\delta}^{t_k+\delta} \|\nabla h(t)\|_2^2 \, dt = 0,$$

and, therefore,

$$\lim_{k\to\infty} \|\nabla h(t_k')\|_2 = 0, \quad \exists t_k' \in (t_k - \delta, t_k + \delta),$$

which implies the convergence of $\{h(\cdot, t_k')\}$ to a constant. Then we obtain $\lim_{k\to\infty} q(\cdot, t_k') = \overline{q}_0$ by (2.25).

Turning to Theorem 2.3.4, we use

$$\|p\|_{L^4(0,t;L^\infty(\Omega))} \le C_3 \|p\|_{L^\infty(0,T;L^2(\Omega))}^{1/2} \cdot \|\nabla p\|_{L^2(0,T;L^2(\Omega))}^{1/2} \tag{2.28}$$

valid to $n = 1$ and $\int_\Omega p = 0$. Here, the constant $C$ on the right-hand side independent of $T$, see [210] p. 74 and p. 63. We apply (2.28) for

$$p = q^{1/2} - \frac{1}{|\Omega|} \int_\Omega q^{1/2}.$$

Since

$$\|p\|_2^2 = \|q\|_1 - \left( \frac{1}{|\Omega|} \int_\Omega q^{1/2} \right)^2 \le \|q\|_1, \quad \nabla p = \nabla q^{1/2}$$

we obtain

$$\int_0^T \left\| q^{1/2} - \frac{1}{|\Omega|} \int_\Omega q^{1/2} \right\|_\infty^4 dt \le C. \tag{2.29}$$

Let

$$\overline{q} \equiv \frac{1}{|\Omega|} \int_\Omega q = \overline{q}_0. \tag{2.30}$$

Since

$$\begin{aligned} \left( q^{1/2} - \frac{1}{|\Omega|} \int_\Omega q^{1/2} \right)^2 &= q - \overline{q} + h \\ h = \overline{q} - \frac{2q^{1/2}}{|\Omega|} \int_\Omega q^{1/2} &+ \left( \frac{1}{|\Omega|} \int_\Omega q^{1/2} \right)^2 \end{aligned} \tag{2.31}$$

inequality (2.29) means

$$\int_0^T \|q - \overline{q} + h\|_\infty^2 dt \le C. \tag{2.32}$$

Here we have

$$\begin{aligned} \|\nabla h\|_2 &= \frac{2}{|\Omega|} \int_\Omega q^{1/2} \cdot \|\nabla q^{1/2}\|_2 \\ &\le 2 \left( \frac{1}{|\Omega|} \int_\Omega q \right)^{1/2} \|\nabla q^{1/2}\|_2 = 2\overline{q}^{1/2} \|\nabla q^{1/2}\|_2 \end{aligned}$$

by (2.31), and, therefore,

$$\int_0^T \|\nabla h\|_2^2 dt \le C. \tag{2.33}$$

By $n = 1$ inequality (2.33) implies

$$\int_0^T \|h - \overline{h}\|_\infty^2 \le C \tag{2.34}$$

where

$$\overline{h} = \frac{1}{|\Omega|} \int_\Omega h = \overline{q} - \left( \frac{1}{|\Omega|} \int_\Omega q^{1/2} \right)^2,$$

and, therefore,

$$\int_0^T \|q(\cdot,t)-a(t)\|_\infty^2\,dt \le C, \quad a(t)=\left(\frac{1}{|\Omega|}\int_\Omega q(\cdot,t)^{1/2}\right)^2. \tag{2.35}$$

Now we show the following lemma.

**Lemma 2.3.2** *Under the Assumptions of (2.15) and (2.19) it holds that*

$$\frac{d}{dt}\|q\|_2^2+\|\nabla q\|_2^2 \le b(t), \quad \int_0^T b(t)dt \le C. \tag{2.36}$$

*Proof* Inequality $q(x,t)\ge 0$ implies

$$v(x,t)\ge v_0(x) \qquad \text{in } Q_T=\Omega\times(0,T) \tag{2.37}$$

while $\varphi'(v(x,t))$ is uniformly bounded by $\varphi'\le 0\le(\varphi')'$:

$$\left|\varphi'(v(x,t))\right| \le C. \tag{2.38}$$

Here we use

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|q\|_2^2 &= \int_\Omega q\cdot q_t = -\int_\Omega \nabla q\cdot(\nabla q - q\nabla\varphi(v))\\ &= -\|\nabla q\|_2^2 + a(t)\int_\Omega \varphi'(v)\nabla q\cdot\nabla v + \int_\Omega (q-a(t))\varphi'(v)\nabla q\cdot\nabla v\end{aligned}$$

to derive

$$\begin{aligned}\left|\int_\Omega (q-a(t))\varphi'(v)\nabla q\cdot\nabla v\right| &\le \|q-a(t)\|_\infty\cdot C\|\nabla q\|_2\\ &\le \frac{1}{4}\|\nabla q\|_2^2 + C^2\|q-a(t)\|_\infty^2\end{aligned} \tag{2.39}$$

by (2.21) and (2.38). We have, on the other hand,

$$\begin{aligned}\left|a(t)\int_\Omega \varphi'(v)\nabla q\cdot\nabla v\right| &= \left|2a(t)\int_\Omega \varphi'(v)q^{1/2}\nabla v\cdot\nabla q^{1/2}\right|\\ &\le 2C\int_\Omega \varphi''(v)^{1/2}q^{1/2}|\nabla v\cdot\nabla q^{1/2}|\\ &\le C\int_\Omega \varphi''(v)q\,|\nabla v|^2+|\nabla q^{1/2}|^2\,dx\end{aligned} \tag{2.40}$$

by (2.19). Now we apply (2.35) and (2.23), (2.21) to (2.39) and (2.40), respectively, and then, obtain the result. □

Inequality (2.36) implies

$$\sup_{t\in(0,T)} \|q(t)\|_2 + \int_0^T \|\nabla q\|_2^2 \, dt \le C \tag{2.41}$$

and then it follows that

$$\int_0^T \|q - \overline{q}\|_\infty^2 dt \le C \tag{2.42}$$

from $n = 1$. Now we apply (2.28) to $p = q - \overline{q}$ which results in

$$\|q - \overline{q}\|_{L^4(0,t;L^\infty(\Omega))} \le C. \tag{2.43}$$

Now we show the following lemma.

**Lemma 2.3.3** *Under the Assumptions of (2.15) and (2.19) it holds that*

$$\frac{d}{dt}\|q - \overline{q}\|_4^4 \le C(\|q - \overline{q}\|_\infty^4 + \|q - \overline{q}\|_\infty^2). \tag{2.44}$$

*Proof* Letting $A(q) = (q - \overline{q})^4$ and $a(q) = A''(q) = 12(q - \overline{q})^2$, we obtain

$$\frac{d}{dt}\int_\Omega A(q) = \int_\Omega A'(q) q_t = -\int_\Omega A''(q)\nabla q \cdot (\nabla q - q\nabla\varphi(v))$$

which means

$$\begin{aligned}\frac{d}{dt}\int_\Omega A(q) + \int_\Omega a(q)|\nabla q|^2 &= \int_\Omega a(q) q \nabla q \cdot \nabla\varphi(v) \\ &= \int_\Omega a(q)(q - \overline{q})\nabla q \cdot \nabla\varphi(v) + \overline{q}\int_\Omega a(q)\nabla q \cdot \nabla\varphi(v) = I + II.\end{aligned}$$

Here, inequality (2.38) implies

$$\begin{aligned}|I| &= \left|\int_\Omega a(q)(q - \overline{q})(-\varphi'(v))^{1/2}\nabla q \cdot (-\varphi'(v))^{1/2}\nabla v\right| \\ &\le (2C)^{1/2}\left\{\int_\Omega a(q)^2|q - \overline{q}|^2(-\varphi'(v))|\nabla q|^2\right\}^{1/2} \\ &\le C'\|a(q)^{1/2}(q - \overline{q})\|_\infty \cdot \left\{\int_\Omega a(q)|\nabla q|^2\right\}^{1/2} \\ &\le \frac{1}{4}\int_\Omega a(q)|\nabla q|^2 + (C')^2\|a(q)^{1/2}(q - \overline{q})\|_\infty^2\end{aligned}$$

and

$$\begin{aligned} |II| &\le \overline{q} \cdot \int_\Omega a(q)(-\varphi'(v))^{1/2}\nabla q \cdot (-\varphi'(v))^{1/2}\nabla v \\ &\le \frac{1}{4}\int_\Omega a(q)|\nabla q|^2 + C\|a(q)^{1/2}\|_\infty^2. \end{aligned}$$

Then it holds that

$$\begin{aligned} &\frac{d}{dt}\int_\Omega A(q) + \frac{1}{2}\int_\Omega a(q)|\nabla q|^2 \\ &\quad \le C\left\{\|a(q)^{1/2}(q-\overline{q})\|_\infty^2 + \|a(q)^{1/2}\|_\infty^2\right\}, \end{aligned} \tag{2.45}$$

and, hence, (2.44). □

*Proof of Theorem* 2.3.4: We have

$$\int_0^\infty \|q(t)-\overline{q}\|_\infty^4 + \|q(t)-\overline{q}\|_\infty^2\, dt < +\infty \tag{2.46}$$

by (2.42) and (2.43). Here we take $t_k' \uparrow +\infty$ in Lemma 2.3.1 to apply (2.44) and (2.46). It holds for $t > t_k$ that

$$\|q(t)-\overline{q}\|_4^4 \le \|q(t_k')-\overline{q}\|_4^4 + C\int_{t_k'}^\infty \|q(t')-\overline{q}\|_\infty^4 + \|q(t')-\overline{q}\|_\infty^2\, dt'.$$

We obtain (2.20) by making $t \uparrow +\infty$ and then $k \to \infty$. □

## 2.4 Harmonic Heat Flow

Several geometric flows are known to have the quantized blowup mechanism [352]. In the harmonic heat flow case, the total energy acts as the Lyapunov function. Differently from (1.8), this Lyapunov function is non-negative, and then type (II) blowup rate arises from this non-negativity. In this connection, the solution constructed by [207] to (1.8) has the bounded total free energy. This property holds in the case of $\lambda \notin 4\pi\mathbf{N}$ as is shown in the proof of Theorem 1.2.2.

If we take the flat torus $\Omega = \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z}$, $a, b > 0$ and the $(n-1)$-dimensional sphere $S^{n-1} = \{x \in \mathbf{R}^n \mid |x| = 1\}$ as the domain and the target, respectively, this flow is described by

$$u = u(x,t) : \Omega \times [0,T) \to S^{n-1} \subset \mathbf{R}^n$$

satisfying

$$u_t - \Delta u = u\,|\nabla u|^2\,, \quad |u| = 1 \quad \text{in } \Omega \times (0, T). \tag{2.47}$$

In this case, it holds that

$$\|u_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 = \int_\Omega u \cdot u_t |\nabla u|^2 = \frac{1}{2}\int_\Omega \left(\frac{\partial}{\partial t}|u|^2\right)|\nabla u|^2 = 0$$

and hence

$$\frac{dE}{dt} = -\,\|u_t\|_2^2 \le 0, \quad E = \frac{1}{2}\,\|\nabla u\|_2^2\,. \tag{2.48}$$

Thus, the above $E$ casts the Lyapunov function to (2.47).

The stationary solution to (2.47) is called the *harmonic map*. In the general setting, we take the $m$-dimensional compact Riemannian manifold $(\Omega, g)$ and the compact Riemannian manifold $N$ without boundaries. By Nash's theorem this $N$ is isometrically imbedded in $\mathbf{R}^n$ for large $n$. We define the Sobolev space composed of a class of the mappings from $\Omega$ to $N$ provided with the finite energy as in the previous section, that is

$$\begin{aligned} H^1(\Omega, N) &= \left\{u \in H^1(\Omega, \mathbf{R}^n) \mid u \in N, \ \text{ a.e. on } \Omega\right\} \\ E(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\, dv_\Omega, \end{aligned}$$

where $dv_\Omega$ is a volume element of $(\Omega, g)$. We call a mapping $u \in H^1(\Omega, N)$ (weakly) harmonic if

$$\frac{d}{d\varepsilon} E(\Pi(u + \varepsilon\phi))\Big|_{\varepsilon=0} = 0$$

for any $\phi \in C_0^\infty(\Omega, \mathbf{R}^n)$, where $\Pi : U \to N$ is a smooth nearest point projection from some tubular neighborhood $U$ of $N$ to $N$. This relation is equivalent to saying that $u$ is a weak solution of the Euler-Lagrange equation

$$-\Delta u = A(u)(\nabla u, \nabla u) \qquad \text{on } \Omega, \tag{2.49}$$

sometimes called the *harmonic map equation*, where $\Delta$ and $A(u)(\cdot, \cdot)$ denote the Laplace-Beltrami operator on $(M, g)$ and the second fundamental form of the imbedding $N \hookrightarrow \mathbf{R}^n$ at $y \in N$, respectively. Harmonic heat flow is the mapping $u = u(x, t) : \Omega \times [0, T) \to N \subset \mathbf{R}^n$ satisfying

$$u_t = -\delta E(u)$$

and, therefore, the harmonic map is regarded as a stationary state of the harmonic heat flow.

First, we have the energy quantization for the sequence of harmonic maps [89, 184, 247, 307–309, 352, 408].

**Theorem 2.4.1** *Let $\{u_k\}_k$ be a harmonic map sequence satisfying*

$$\sup_k E(u_k) < +\infty.$$

*Then, passing to a subsequence we assume $u_k \rightharpoonup u$ in $H^1(\Omega, N)$ weakly to some map $u \in H^1(\Omega, N)$. This $u$ is a harmonic map, and there exist*

- *p-sequences of points $\{x_k^1\}, \ldots, \{x_k^p\}$ in $\Omega$*
- *p-sequences of positive numbers $\{\delta_k^1\}, \ldots, \{\delta_k^p\}$ converging to 0*
- *p-non-constant harmonic maps $\{\omega^1\}, \ldots, \{\omega^p\} : S^2 \to N$*

*satisfying*

$$\begin{aligned}
&\lim_{k\to\infty} E(u_k) = E(u) + \sum_{j=1}^{p} E_0(\omega^j) \\
&\lim_{k\to\infty} \max_{i\neq j} \left\{ \frac{\delta_k^i}{\delta_k^j}, \frac{\delta_k^j}{\delta_k^i}, \frac{\left|x_k^i - x_k^j\right|}{\delta_k^i + \delta_k^j} \right\} = +\infty \\
&\lim_{k\to\infty} \left\| u_k - u - \sum_{j=1}^{p} \left\{ \omega^j\left( \frac{\cdot - x_k^j}{\delta_k^j} \right) - \omega^j(\infty) \right\} \right\|_{H^1(\Omega,N)} = 0 \qquad (2.50)
\end{aligned}$$

*where*

$$E_0(\omega) = \frac{1}{2} \int_{S^2} |\nabla \omega|^2 \, dv_{S^2}.$$

The first equality of (2.50) is an energy identity which says that there is no unaccounted energy loss during the iterated rescaling process near the point of singularity, sometimes referred to as the bubbling process, and that the only reason for failure of strong convergence to the weak limit is the formation of several bubbles due to the non-constant harmonic maps $\omega^j : S^2 \to N$, $j = 1, \ldots, p$. Differently from (1.38), there is a possibility that some of $\left\{x_k^j\right\}_k$, $j = 1, \ldots, p$, converge to the same point, and this process is classified into two cases—the separated bubbles and the bubbles on bubbles [376].

The $\varepsilon$-regularity and the monotonicity formula are known to the harmonic heat flow, similarly to the simplified system of chemotaxis (1.8). In the case of (2.47) for $\Omega = \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z}$, first, there is $\varepsilon_0 > 0$ such that $u = u(x, t)$ is smooth in $B_{R/2} \times [0, T]$, provided that

$$\sup_{t\in[0,T]} E(u(\cdot, t), B_R) < \varepsilon_0,$$

where

$$E(u, B_R) = \frac{1}{2}\int_{B_R} |\nabla u|^2, \quad B_R = B(0, R).$$

Second,

$$E(u(\cdot, T), B_R) \le E(u_0, B_{2R}) + CE_0T/R^2, \quad E_0 = \frac{1}{2}\|\nabla u_0\|_2^2$$

holds with $C > 0$. These analytic structures guarantee that there is a weak solution global-in-time with a finite number of sigular points in $\Omega \times [0, +\infty)$, and then we will obtain a homotopy between the initial state and the expected ultimate state, see [351, 352].

Similarly to the chemotaxis system, (1.8) with (1.132), the non-negativity of $E$ implies

$$\begin{aligned}\frac{1}{2}\sup_{t\in[0,T)}\|\nabla u(\cdot, t)\|_2^2 + \int_0^T \|u_t(\cdot, s)\|_2^2\, ds &\le \frac{1}{2}\|\nabla u_0\|_2^2 \\ &= E_0 \end{aligned} \tag{2.51}$$

by (2.48). Then, there is $t_k \uparrow T$ satisfying

$$(T - t_k)\,\|u_t(t_k)\|_2^2 \to 0 \tag{2.52}$$

because otherwise it holds that

$$\int_0^T \|u_t(\cdot, s)\|_2^2 ds = +\infty,$$

a contradiction. For this $u_k = u(\cdot, t_k)$, it is known, see [401], that the conclusion of Theorem 2.4.1 arises. The hyper-parabola also works, and thus there are similarities and differences between (1.8) and (2.47).

More precisely, we have

$$\begin{aligned}\int_\Omega |u_t|^2\varphi^2 + \int_\Omega \nabla u \cdot \nabla(u_t\varphi^2) &= \int_\Omega u_t \cdot u|\nabla u|^2\varphi^2 \\ &= \frac{1}{2}\int_\Omega \left[\frac{\partial}{\partial t}|u|^2\right]|\nabla u|^2\varphi^2 = 0\end{aligned}$$

with

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla(u_t\varphi^2) &= \int_\Omega (\nabla u \cdot \nabla u_t)\varphi^2 + \int_\Omega [(u_t \cdot \nabla)u] \cdot \nabla\varphi^2 \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\varphi^2 + \int_\Omega [(u_t \cdot \nabla)u] \cdot \nabla\varphi^2\end{aligned}$$

from (2.47), and, therefore,

$$\int_\Omega |u_t|^2\varphi^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\varphi^2 + \int_\Omega [(u_t\cdot\nabla)u]\cdot\nabla\varphi^2 = 0$$

for each $\varphi \in C^1(\Omega)$ which implies that

$$\begin{aligned}\left|\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\varphi^2\right| &\le \|u_t\|_2^2\|\varphi\|_\infty^2 + \|u_t\|_2\cdot\left\{\int_\Omega |\nabla u|^2|\nabla\varphi^2|^2\right\}^{1/2}\\ &\le \|u_t\|_2^2\|\varphi\|_\infty^2 + \sqrt{2}\|u_t\|_2\cdot E_0^{1/2}\|\nabla\varphi^2\|_\infty, \end{aligned} \tag{2.53}$$

and then it follows that

$$\int_0^T \left|\frac{d}{dt}\int_\Omega |\nabla u|^2\varphi^2\right| dt < +\infty.$$

Thus

$$\mu(dx,t) = |\nabla u(x,t)|^2 dx \in C_*([0,T],\mathcal{M}(\Omega))$$

follows again from (2.51). Using $t_k \uparrow T$ satisfying (2.52), we obtain

$$\mu(dx,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \tag{2.54}$$

with a finite set $\mathcal{S}$ and $0 \le f = f(x) \in L^1(\Omega)$. The above described result [401] guarantees the energy quantization. More precisely, $m(x_0) > 0$ is a finite sum of the energies of non-constant harmonic maps: $S^2 \to N$.

Inequality (2.53) implies also

$$\begin{aligned}&\left|\int_\Omega |\nabla u(\cdot,t)|^2\varphi_{x_0,R}^2 - \int_\Omega f(x)\varphi_{x_0,R}^2 - m(x_0)\right|\\ &\le 2C\int_t^T \|u_t(\cdot,s)\|_2^2 ds + 2\sqrt{2}E_0^{1/2}\left\{\int_t^T \|u_t(\cdot,s)\|_2^2 ds\right\}^{1/2}\cdot C(T-t)^{1/2}/R\end{aligned}$$

for $x_0 \in \mathcal{S}$ with $C > 0$ independent of $0 < R \ll 1$. Then it follows that

$$\lim_{t\uparrow T}\int_\Omega |\nabla u(\cdot,t)|^2\varphi_{x_0,bR(t)}^2 = m(x_0)$$

from (2.51) again, where $b > 0$ is arbitrary and $R(t) = (T-t)^{1/2}$. The hyperparabola thus arises and we obtain type (II) blowup rate at each $x_0 \in \mathcal{S}$ similarly to (1.8) provided with (1.132).

## 2.5 Normalized Ricci Flow

The normalized Ricci flow describes an evolution in time of the metric $g = g(t)$ on a compact Riemannian manifold. If $\Omega$ is a compact Riemannian surface without boundary, this flow is given by

$$\frac{\partial g}{\partial t} = (r - R)\, g, \qquad t > 0 \tag{2.55}$$

where $R = R(\cdot, t)$ stands for the scalar curvature of $(\Omega, g(t))$ and $r = r(t)$ represents the average scalar curvature given by

$$r = \frac{\int_\Omega R(\cdot, t)\, d\mu_t}{\int_\Omega d\mu_t}$$

with the volume element $\mu = \mu_t$. R. Hamilton [154] introduced the above flow to approach the Poincaré conjecture, and this idea was realized later [298–300]. In the case that $\Omega$ is a compact Riemannian surface is described in [155], and it is shown that the solution to (2.55) exists globally in time, converges in $C^\infty$—topology as $t \uparrow +\infty$, and the scalar curvature of the limit metric is constant. Here, we describe an arguement using the analytic form of (2.55).

First, we obtain

$$\frac{\partial R}{\partial t} = \Delta_t R + R(R - r)$$

and, therefore, $R(\cdot, t) > 0$ everywhere on $\Omega$ follows from $R(\cdot, 0) > 0$ everywhere on $\Omega$, where $\Delta_t$ denotes the Laplace-Beltrami operator associated with $g = g_t$. Henceforth, we deal with this case. Then, from Gauss-Bonnet's theorem there follows

$$\int_\Omega R(\cdot, t)\, d\mu_t = 4\pi\chi(\Omega). \tag{2.56}$$

Here, $\chi(\Omega) = 2 - 2k(\Omega)$ stands for the Euler characteristic of $\Omega$, and hence $k(\Omega)$ is the genus of $\Omega$. Since $R(\cdot, t) > 0$ in $\Omega$, this formula gives $k(\Omega) = 0$, and then the uniformization theorem reduces the problem to the case

$$\Omega = S^2, \quad g(t) = e^{w(\cdot,t)} g_0,$$

where $S^2$ is the two dimensional sphere, $g_0$ is its standard metric, and $w = w(\cdot, t)$ is a smooth function.

In this case, the scalar curvature $R_0$ corresponding to the metric $g_0$ is a constant, and it is related to $R = R(\cdot, t)$ through

$$R = e^{-w}(-\Delta w + R_0) \tag{2.57}$$

with $\Delta = \Delta_{g_0}$. Here, we obtain

$$\int_{S^2} R(\cdot, t)\, d\mu_t = 8\pi \tag{2.58}$$

by (2.56) and hence

$$r = \frac{8\,\pi}{\int_{S^2} d\mu_t} = \frac{8\,\pi}{\int_{S^2} e^w dx}, \tag{2.59}$$

setting $dx = d\mu_{g_0}$. Finally, we have

$$|S^2| R_0 = 8\,\pi \tag{2.60}$$

because (2.58) holds and $R_0$ is a constant.

By pluging (2.57) into (2.55) and using (2.59)–(2.60), we end up with

$$\frac{\partial e^w}{\partial t} = \Delta w + 8\pi \left( \frac{e^w}{\int_{S^2} e^w} - \frac{1}{|S^2|} \right) \quad \text{in } S^2 \times (0, T). \tag{2.61}$$

The result in [155] thus reads as follows. The solution to (2.61) exists globally in time and $w(\cdot, t) \to w_\infty$ as $t \uparrow +\infty$ in $C^\infty$—topology, with $w_\infty$ standing for a stationary solution:

$$-\Delta w_\infty = 8\pi \left( \frac{e^{w_\infty}}{\int_{S^2} e^{w_\infty}} - \frac{1}{|S^2|} \right) \quad \text{in } S^2. \tag{2.62}$$

From the viewpoint of dynamical systems, the proof of [155] consists of three ingredients: extension of the solution globally in time, compactness of the orbit, and uniqueness of the $\omega$-limit set. All the steps are based on the geometric structure of (2.55), involving Harnack's inequality for the scalar curvature, monotonicity of an awkward geometric quantity called "entropy", soliton solutions of the Ricci flow, the modified Ricci flow, and so forth. There are, however, several complementary analytic arguments.

First, the third step originally achieved by modifying (2.55) via a transformation group may be replaced by the uniqueness of the solution to

$$-\Delta w = 8\pi \left( \frac{e^w}{\int_{S^2} e^w} - \frac{1}{|S^2|} \right) \quad \text{in } S^2, \qquad \int_{S^2} w = 0. \tag{2.63}$$

In fact, since

$$\frac{d}{dt} \int_{S^2} e^w = 0$$

follows from (2.61), the stationary solution $w = w_\infty$ to (2.61) constitutes of (2.62) with

$$\int_{S^2} e^{w_\infty} = \int_{S^2} e^{w_0}, \quad w|_{t=0} = w_0(x).$$

This $w_\infty$, on the other hand, must be a constant, provided that only the trivial solution $w = 0$ is admitted to (2.63). The uniqueness of the steady state of (2.61), thererfore, means that (2.63) implies $w = 0$. This property is actually the case because the metric on $S^2$ with constant Gaussian curvature is the standard one. Thus the steady state of (2.61) is unique:

$$w_\infty = \frac{1}{|\Omega|} \int_{S^2} e^{w_0}.$$

Next, a gradient estimate of the form

$$|\nabla_{S^2} w| \le C \tag{2.64}$$

is obtained using the symmetry of $S^2$, that is the moving sphere method, with $C$ depending only on $w_0 = w(\cdot, 0)$, see [20]. This estimate, combined with the argument in [431] based on Harnack's inequality, induces also global-in-time existence of the solution to (2.61) and also the compactness of the orbit.

Then, what happens to

$$\frac{\partial e^w}{\partial t} = \Delta w + \lambda \left( \frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|} \right) \quad \text{in } \Omega \times (0, T), \tag{2.65}$$

where $\lambda > 0$ is a constant and $\Omega$ is a compact Riemannian surface without boundary? Unless $\lambda = 8\pi$ and $\Omega = S^2$, it is not the normalized Ricci flow (2.61) any more. An estimate like (2.64) may not be obtained because of the lack of the symmetry of $\Omega$. The arguments of [155] using the geometric structure such as the covariant and Lie derivatives, Bochner-Weitzenböck's formula, and so forth, see [71], are also invalid, and even global in time existence of the solution is not obvious.

Similar to (2.61), however, we have

$$\frac{d}{dt} \int_\Omega e^w = 0,$$

and, therefore,

$$r = \frac{\lambda}{\int_\Omega e^w}$$

is a constant. Under the change of variables $u = re^w$ and $t = r^{-1}\tau$, and writing $t$ for $\tau$, problem (2.65) is transformed into

$$u_t = \Delta \log u + u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0, T) \tag{2.66}$$

with

$$\int_\Omega u(\cdot, t) = \lambda. \tag{2.67}$$

This form, (2.66), is a non-local perturbation of the logarithmic diffusion equation

$$u_t = \Delta \log u \qquad \text{in } \Omega \times (0, T) \tag{2.68}$$

which also describes the evolution of surfaces by Ricci flow [155]. There are also some other physical problems which could be described by (2.68); the spread over a thin colloidal film at a flat surface [41, 82, 414], the modelling of the expansion of a thermalized cloud of electrons [225], and the central limit approximation to the Calerman's model of the Boltzmann equation [209, 224]. Due to a variety of applications logarithmic diffusion equation (2.68) has attracted the interests, see for instance [403].

The spatially homogeneous part of (2.66) is formulated by the linear non-local ODE,

$$u_t = u - \frac{1}{|\Omega|}\int_\Omega u \qquad \text{in } \Omega \times (0, T)$$

but the non-local term is removable by (2.67). The regions

$$\Omega_+(t) = \{x \in \Omega \mid u(x, t) > \lambda/|\Omega|\}$$
$$\Omega_-(t) = \{x \in \Omega \mid u(x, t) < \lambda/|\Omega|\}$$

are invariant in $t$, and, therefore, the *extinction* $\lim_{t\uparrow T} \inf_\Omega u(\cdot, t) = 0$ arises before the blowup $\lim_{t\uparrow T} \sup_\Omega u(\cdot, t) = +\infty$ comes. This fact is also known to (2.68), and we obtain

$$\|u(\cdot, t)\|_1 = \|u_0\|_1 - 4\pi t$$

in the case of $\Omega = \mathbf{R}^2$, see [404] and the references therein.

Equation (2.66), however, may be written as

$$u_t = \Delta(\log u - v), \ -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega \times (0, T)$$
$$\int_\Omega v = 0 \tag{2.69}$$

which is to be compared with the Smoluchowski-Poisson Eq. (1.8) formulated by (1.32),

$$u_t = \nabla \cdot (u\nabla(\log u - v)), \ -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega \times (0,T)$$
$$\int_\Omega v = 0. \tag{2.70}$$

Actually, (2.66) is again a model (B) equation formulated by Helmholtz' free energy

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\langle(-\Delta)^{-1}u, u\rangle$$
$$v = (-\Delta)^{-1}u \quad \Leftrightarrow \quad -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u, \ \int_\Omega v = 0. \tag{2.71}$$

From this fact, then, it is easy to derive the total mass conservation and the decrease of the free energy, that is

$$\frac{d}{dt}\int_\Omega u = 0, \quad \frac{d}{dt}\mathcal{F}(u) = -\int_\Omega |\nabla(\log u - v)|^2 \le 0. \tag{2.72}$$

Concerning the Smoluchowski-Poisson Eq. (2.70), there is a fundamental property obtained by the method of symmetrization that is (1.49) which results in the formation of collapse with the quantized mass. This control in time of the local $L^1$—norm of the solution $u = u(\cdot, t)$ to (2.69) is not available. The equivalent form (2.66), however, is provided with the comparison principle thanks to (2.67). An important consequence of this property is the monotonicity formula of the Benilan-Crandall type,

$$\frac{\partial}{\partial t}\left(\frac{u}{e^t - 1}\right) \le 0, \tag{2.73}$$

which guarantees the point-wise convergence of

$$u(x,T) = \lim_{t \uparrow T} u(x,t)$$

in the case of $T = T_{\max} < +\infty$, where $T_{\max} \in (0, +\infty]$ denotes the existence time of the solution. The following theorem [193, 194] is obtained by this structure and Trudinger-Moser-Fontana's inequality [118],

$$\inf\left\{ J_{8\pi}(v) \mid v \in H^1(\Omega), \ \int_\Omega v = 0 \right\} > -\infty$$
$$J_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda(\log\lambda - 1). \tag{2.74}$$

**Theorem 2.5.1** *If $0 < \lambda \le 8\pi$, then the solution $u = u(\cdot, t)$ to (2.66) satisfies the uniform estimates*

$$0 < u(x,t),\ u(x,t)^{-1} \le C \quad \text{in } \Omega \times (0,T). \tag{2.75}$$

The solution $w = w(\cdot, t)$ to (2.65) with $0 < \lambda \le 8\pi$, therefore, exists global-in-time, and the orbit $\mathcal{O} = \{w(\cdot, t)\}_{t \ge 0}$ is compact in $C(\Omega)$. The $\omega$-limit set of $\mathcal{O}$ is thus non-empty, connected, comapct, and contained in the set of stationary solutions, and in particular, any $t_k \uparrow +\infty$ admits $\{t_k'\} \subset \{t_k\}$ and $w_\infty = w_\infty(x)$ satsifying

$$-\Delta w = \lambda \left( \frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|} \right) \quad \text{in } \Omega \tag{2.76}$$

with

$$\int_\Omega e^w = \int_\Omega e^{w_0}$$

such that $w(\cdot, t_k') \to w_\infty$ uniformly on $\Omega$. If only $w = 0$ satisfies (2.76) with

$$\int_\Omega w = 0,$$

therefore, $w_\infty$ is a constant and it holds that $w(\cdot, t) \to w_\infty$ uniformly on $\Omega$ as $t \uparrow +\infty$. The asymptotic behavior $w = w(\cdot, t)$ to $\lambda \le 8\pi$ is thus controlled by the uniqueness of the stationary solution. This property holds for $0 < \lambda \le 8\pi$ if either $\Omega = S^2$ or $\Omega = \mathbf{R^2}/a\mathbf{Z} \times b\mathbf{Z}$, $\dfrac{\pi}{4} \le \dfrac{b}{a} \le 1$, see [55, 68, 217, 218].

The proof of Theorem 2.5.1 for the critical case $\lambda = 8\pi$ relies on the following lemma [45, 274, 276] of which proof is given later. Here we recall (2.71).

**Lemma 2.5.1** *Let $\Omega$ be a compact Riemannian surface without boundary isometrically embedded in $\mathbf{R}^N$, and suppose that $\{u_k\}$ is a family of positive measurable functions on $\Omega$ satisfying*

$$\|u_k\|_1 = 8\pi, \quad \mathcal{F}(u_k) \le C, \quad \lim_{k \to \infty} \left\langle (-\Delta)^{-1} u_k, u_k \right\rangle = +\infty$$
$$\lim_{k \to \infty} \int_\Omega x u_k = 8\pi x_\infty \in \mathbf{R}^N.$$

*Then it holds that $x_\infty \in \Omega$ and $u_k(x)dx \rightharpoonup 8\pi \delta_{x_\infty}(dx)$ in $\mathcal{M}(\Omega)$.*

Now we show the following lemma.

**Lemma 2.5.2** *In (2.65), $0 < \lambda \le 8\pi$, it holds that $\liminf_{t \uparrow \infty} \overline{w}(t) > -\infty$.*

*Proof* Rewriting $w = \log u$ in (2.66) and (2.67), we reach

$$\frac{\partial e^w}{\partial t} = \Delta w + e^w - \frac{\lambda}{|\Omega|}, \quad \int_\Omega e^w = \lambda. \tag{2.77}$$

For the moment, we work with (2.77).

First, $p = w_t$ satisfies

$$p_t = e^{-w}\Delta p + p - p^2 \quad \text{in } \Omega \times (0, T)$$

for $0 < t < T = T_{\max}$ which implies

$$p = w_t(\cdot, t) \le \frac{e^t}{e^t - 1} \quad \text{in } \Omega, \tag{2.78}$$

equivalent to (2.73). Using $J_\lambda(v)$ in (2.74) for $v = w - \overline{w}$ with

$$\overline{w} = \frac{1}{|\Omega|}\int_\Omega w,$$

next, we obtain the field functional

$$\begin{aligned} \tilde{J}_\lambda(w) &= J_\lambda(w - \overline{w}) - \lambda(\log\lambda - 1) \\ &= \frac{1}{2}\|\nabla w\|_2^2 - \lambda\left\{\log\left(\int_\Omega e^w\right) - \overline{w}\right\} \end{aligned} \tag{2.79}$$

which casts the Lyapunov function to (2.77). The solution $w = w(\cdot, t)$ to (2.65), thus, satisfies

$$\frac{d}{dt}\tilde{J}_\lambda(w) = \int_\Omega \nabla w \cdot \nabla w_t - \lambda\left(\frac{e^w}{\int_\Omega e^w} - \frac{1}{|\Omega|}\right) w_t \, dx = -\int_\Omega e^w w_t^2, \tag{2.80}$$

where Fontana's inequality (2.74) is applicable as

$$\tilde{J}_{8\pi}(w(\cdot, t)) \ge -C, \quad 0 \le t < T. \tag{2.81}$$

Since we obtain

$$\begin{aligned} \tilde{J}_\lambda(w) &= (8\pi - \lambda)\log\left(\int_\Omega e^{w - \overline{w}}\right) + \tilde{J}_{8\pi}(w) \\ &= (8\pi - \lambda)(\log\lambda - \overline{w}) + \tilde{J}_{8\pi}(w) \end{aligned}$$

in (2.79), the lemma is obvious for $0 < \lambda < 8\pi$ by (2.80) and (2.81).

Let $\lambda = 8\pi$. The elementary inequality $e^w w \geq -e^{-1}$, $w \in \mathbf{R}$, now, implies

$$H(t) \equiv \int_\Omega e^w w \geq -e^{-1} |\Omega| .$$

By (2.65) and (2.81), next, we have

$$\begin{aligned} \frac{dH}{dt} &= \int_\Omega e^w w_t + e^w w w_t \, dx = \frac{d}{dt} \int_\Omega e^w + \int_\Omega \frac{\partial e^w}{\partial t} \cdot w \\ &= \int_\Omega \left[ \Delta w + \left( e^w - \frac{8\pi}{|\Omega|} \right) \right] w = - \|\nabla w\|_2^2 + \int_\Omega e^w w - 8\pi \overline{w} \\ &\leq \int_\Omega e^w w + 8\pi \overline{w} + C = H + 8\pi \overline{w} + C. \end{aligned} \tag{2.82}$$

If

$$\liminf_{t \uparrow \infty} \overline{w}(t) = -\infty$$

is the case, inequality (2.78) implies the existence of $t_k \uparrow +\infty$ in $t_{k+1} > t_k + \delta$ with $\delta > 0$ such that

$$8\pi \overline{w}(t) \leq -C - k, \quad k = 1, 2, \cdots, \ t_k - \delta < t < t_k$$

in (2.82), and, therefore,

$$\frac{d}{dt} \left( e^{-t} H \right) \leq -k e^{-t}, \quad t_k - \delta < t < t_k. \tag{2.83}$$

Operating $\int_t^{t_k} \cdot \, dt$, $t \in (t_k - \delta, t_k - \delta/2)$, to (2.83) implies

$$e^{-t_k} H(t_k) \leq e^{-t} H(t) + k(e^{-t_k} - e^{-t}).$$

Then it holds that

$$\begin{aligned} H(t) &\geq e^{t - t_k} H(t_k) + k(1 - e^{t - t_k}) \\ &\geq -e^{-\delta - 1} |\Omega| + k(1 - e^{-\delta/2}), \quad t_k - \delta < t < t_k - \delta/2, \end{aligned}$$

and, in particular,

$$\lim_{k \to \infty} \inf_{t \in (t_k - \delta, t_k - \delta/2)} \int_\Omega (e^w w)(\cdot, t) = +\infty. \tag{2.84}$$

We have, on the other hand, (2.79) with $\lambda = 8\pi$ and (2.81), which implies

$$\sum_{k=1}^{\infty} \int_{t_k-\delta}^{t_k-\delta/2} dt \int_\Omega (e^w w_t^2)(\cdot, t) \le \int_0^\infty dt \int_\Omega (e^w w_t^2)(\cdot, t) < +\infty. \tag{2.85}$$

It holds also that

$$\|e^w w_t\|_1^2 \le \int_\Omega e^w \cdot \int_\Omega e^w w_t^2 = 8\pi \int_\Omega e^w w_t^2,$$

and, therefore,

$$\lim_{k\to\infty} \int_{t_k-\delta}^{t_k-\delta/2} \|(e^w w_t)(\cdot, t)\|_1^2 \, dt = 0 \tag{2.86}$$

by (2.85).

From (2.84) and (2.86) there is $t_k' \in (t_k - \delta, t_k - \delta/2)$ such that

$$\lim_{k\to\infty} \int_\Omega e^w w(\cdot, t_k') = +\infty, \quad \lim_{k\to\infty} \left\| \frac{\partial e^w}{\partial t}(\cdot, t_k') \right\|_1 = 0. \tag{2.87}$$

Using $u = e^w$ and $u_k = u(\cdot, t_k')$, we rewrite the first relation of (2.87) as

$$\lim_{k\to\infty} \int_\Omega u_k \log u_k = +\infty, \tag{2.88}$$

while (2.72) implies

$$\mathcal{F}(u_k) \le \mathcal{F}(u_0). \tag{2.89}$$

We thus obtain

$$\lim_{k\to\infty} \langle (-\Delta)^{-1} u_k, u_k \rangle = +\infty$$

by (2.88)–(2.89). We have also $\|u_k\|_1 = 8\pi$ by (2.77) with $\lambda = 8\pi$, and, furthermore, may assume $x_\infty \in \mathbf{R}^N$ such that

$$\lim_{k\to\infty} \int_\Omega x u_k = 8\pi x_\infty \in \mathbf{R}^N$$

up to a subsequence. By Lemma 2.5.1 we have $x_\infty \in \Omega$ and it holds that

$$e^{w(\cdot, t_k')} = u_k \rightharpoonup 8\pi \delta_{x_\infty} \quad \text{in } \mathcal{M}(\Omega) \tag{2.90}$$

by a subsequence.

Here we apply the second relation of (2.87) and (2.90) to both sides of (2.65) for $t = t_k'$. From the elliptic $L^1$ estiamte [38] it holds that

$$w(\cdot, t_k') \to 8\pi G(\cdot, x_\infty) \qquad \text{in } W^{1,q}(\Omega), \quad 1 \le q < 2 \tag{2.91}$$

where $G = G(x, x')$ stands for the Green's function to $v = (-\Delta)^{-1}u$ defined by (2.71). Using the asymptotics of $G(x, x_\infty)$ as $x \to x_\infty$, we can derive

$$\lim_{k\to\infty} \int_\Omega e^{w(\cdot,t_k')} = +\infty$$

from (2.91), see [37], a contradiction by (2.77), where $\lambda = 8\pi$. □

To complete the proof of Theorem 2.5.1, first, we show

$$\|\nabla w\|_2 \le C, \quad 0 \le t < T \tag{2.92}$$

which is obvioius for $0 < \lambda < 8\pi$ by (2.80), (2.81), and

$$\tilde{J}_\lambda(w) = \frac{1}{2}\left(1 - \frac{\lambda}{8\pi}\right)\|\nabla w\|_2^2 + \frac{\lambda}{8\pi}\tilde{J}_{8\pi}(w).$$

Even in the case of $\lambda = 8\pi$ we obtain

$$\tilde{J}_{8\pi}(w) = \frac{1}{2}\|\nabla w\|_2^2 - 8\pi \log\left(\int_\Omega e^{w-\overline{w}}\right) = \frac{1}{2}\|\nabla w\|_2^2 + 8\pi\overline{w} - 8\pi\log(8\pi).$$

Then, (2.92) follows from (2.80) and Lemma 2.5.2.

Jensen's inequality, next, assures

$$\exp\left(\frac{1}{|\Omega|}\int_\Omega w\right) \le \frac{1}{|\Omega|}\int_\Omega e^w = \frac{\lambda}{|\Omega|},$$

and, therefore, $|\overline{w}(t)| \le C$ by Lemma 2.5.2. Poincaré-Wirtinger's inequality then implies

$$\|\log u\|_{H^1(\Omega)} = \|w\|_{H^1(\Omega)} \le C,$$

and hence

$$\left\|e^{p\log u}\right\|_1, \ \left\|e^{-p\log u}\right\|_1 \le C_p, \quad p > 0$$

by Fontana's inequality. In particular, it holds that

$$\|u(\cdot,t)\|_p, \ \|u^{-1}(\cdot,t)\|_p \le C_p, \quad p \ge 1, \tag{2.93}$$

and, then, Moser's iteration scheme implies (2.75), see [193]. □

## 2.6 Concentration of Probability Measures

This paragraph is devoted to the proof of Lemma 2.5.1. First, we show the following lemma [63].

**Lemma 2.6.1** *Let $\Omega \hookrightarrow \mathbf{R}^N$ be a compact Riemannian surface without boundary, and $\delta > 0, \gamma_0 \in (0, 1/2)$, and $\varepsilon \in (0, 1)$ are given. Then, there is $K = K(\varepsilon, \delta_0, \gamma_0) > 0$ such that for $v \in H_0^1(\Omega)$ satisfying*

$$S_1, S_2 \subset \overline{\Omega}\text{: closed, } \ dist\,(S_1, S_2) \geq \delta_0, \quad \frac{\int_{S_i} e^v}{\int_\Omega e^v} \geq \gamma_0, \ i = 1, 2 \tag{2.94}$$

*it holds that*

$$\log \int_\Omega e^v \leq \frac{1 + 2\varepsilon}{32\pi} \|\nabla v\|_2^2 + K. \tag{2.95}$$

*Proof* Letting $g_1, g_2 \in C_0^\infty(\mathbf{R}^2)$ be such that

$$\text{supp } g_1 \cap \text{supp } g_2 = \emptyset, \quad g_i = 1 \text{ on } S_i, i = 1, 2,$$

we obtain

$$\int_\Omega e^v \leq \frac{1}{\gamma_0} \int_{S_i} e^v \leq \frac{1}{\gamma_0} \int_\Omega e^{g_i v}, \quad i = 1, 2.$$

Then, Fontana's inequality implies

$$\log \left( \frac{1}{|\Omega|} \int_\Omega e^{g_i v} \right) \leq \frac{1}{16\pi} \|\nabla(g_i v)\|_2^2 + C, \quad i = 1, 2. \tag{2.96}$$

Here we assume $\|\nabla(g_1 v)\|_2 \leq \|\nabla(g_2 v)\|_2$, $g = g_1 + g_2$, without loss of generality, to get

$$\begin{aligned} \|\nabla(g_1 v)\|_2^2 &\leq \frac{1}{2} \left( \|\nabla(g_1 v)\|_2^2 + \|\nabla(g_2 v)\|_2^2 \right) = \frac{1}{2} \|\nabla(g_1 + g_2) v\|_2^2 \\ &= \frac{1}{2} \int_\Omega g^2 |\nabla v|^2 + 2\nabla g \cdot \nabla v + v^2 |\nabla g|^2 \, dx \\ &\leq \frac{1 + \varepsilon}{2} \|\nabla v\|_2^2 + C \|v\|_2^2 + C_\varepsilon \end{aligned} \tag{2.97}$$

From (2.96) and (2.97) it follows that

$$\log\left(\frac{1}{|\Omega|}\int_\Omega e^v\right) \le \frac{1+\varepsilon}{32\pi}\|\nabla v\|_2^2 + C\|v\|_2^2 + C \tag{2.98}$$

in the case (2.94). Now we apply Rellich's theorem to the second term on the right-hand side on (2.98), to obtain (2.95). □

Let $P(\Omega) = \left\{\rho \in L^1(\Omega) \mid \rho \ge 0, \|\rho\|_1 = 1\right\}$ be the space of absolutely continuous probability measures. We put

$$\mathcal{I}(\rho) = \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')\rho\otimes\rho - \frac{1}{8\pi}\int_\Omega \rho\log\rho, \quad \rho \in P(\Omega),$$

recalling the Green's function $G = G(x,x')$ to $v = (-\Delta)^{-1}u$ defined by (2.71). For $0 \le u \in L^1(\Omega)$, $\|u\|_1 = 8\pi$, it holds that $\rho = u/8\pi \in P(\Omega)$ with the relation

$$\begin{aligned}\mathcal{I}(\rho) &= -\frac{1}{64\pi^2}\left\{\int_\Omega u(\log u - 1) - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u\otimes u\right\}\\ &\quad -\frac{1}{64\pi^2}\{1-\log(8\pi)\} = -\frac{1}{64\pi^2}\mathcal{F}(u) + \text{constant}.\end{aligned}$$

The dual form of Fontana's inequality arises as

$$\sup\{\mathcal{I}(\rho) \mid \rho \in P(\Omega)\} < +\infty,$$

see the next section, while we have $\mathcal{I} = \mathcal{K} + \mathcal{E}/(8\pi)$ for

$$\mathcal{K}(\rho) = \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')\rho\otimes\rho, \quad \mathcal{E}(\rho) = -\int_\Omega \rho(\log\rho - 1).$$

With this $\mathcal{I}_\beta = \mathcal{K} + \mathcal{E}/\beta$, $\beta > 8\pi$, Lemma 2.6.1 is stated as follows.

**Lemma 2.6.2** *Each $d > 0$ admits $C = C(d) > 0$ provided with the following property. Namely, given $m > 0$, we have $\beta = \beta(m) > 8\pi$ such that if $\rho \in P(\Omega)$ satisfies*

$$dist(A_1, A_2) \ge d, \ \int_{A_1}\rho \ge m, \ \int_{A_2}\rho \ge m$$

*for some measurable sets $A_1$, $A_2 \subset \Omega$, then it holds that $\mathcal{I}_\beta(\rho) \le C(d)$.*

Using Lemma 2.6.2, now we show the following lemma, which is equaivalent to Lemma 2.5.1.

**Lemma 2.6.3** *If* $\{\rho_k\} \subset P(\Omega)$ *assumes*

$$\lim_{k\to\infty} \mathcal{K}(\rho_k) = +\infty, \ \lim_{k\to\infty} \mathcal{I}(\rho_k) = I_\infty > -\infty, \ \lim_{k\to\infty} \int_\Omega x\rho_k = x_\infty \in \mathbf{R}^N$$

*then it holds that* $x_\infty \in \Omega$ *and* $\rho_k(x)dx \rightharpoonup \delta_{x_\infty}(dx)$ *in* $\mathcal{M}(\Omega)$.

*Proof* We take the concentration function of $\rho_k = \rho_k(x) \in P(\Omega)$ denote by

$$Q_k(r) = \sup_{y\in\Omega} \int_{\Omega\cap B(y,r)} \rho_k,$$

to show

$$\lim_{k\to\infty} \{1 - Q_k(r)\} = 0, \quad 0 < r \ll 1. \tag{2.99}$$

For this purpose we take

$$x_k \in \Omega, \quad \int_{\Omega\cap B(x_k,r/2)} \rho_k = Q_k(r/2),$$

regarding the compactness of $\Omega$. Since

$$1 - Q_k(r) \le 1 - \int_{\Omega\cap B(x_k,r)} \rho_k = \int_{\Omega\setminus B(x_k,r)} \rho_k$$

it holds that

$$\min\{Q_k(r/2), 1 - Q_k(r)\} \le \min\left\{\int_{\Omega\cap B(x_k,r/2)} \rho_k, \ \int_{\Omega\setminus B(x_k,r)} \rho_k\right\}.$$

Applying Lemma 2.6.2 for $d = r/2$, we have $C > 0$. Hence each $m > 0$ admits $\beta = \beta(m) > 8\pi$ such that

$$m \le \min\{Q_k(r/2), 1 - Q_k(r)\} \quad \Rightarrow \quad \mathcal{I}_\beta(\rho_k) \le C,$$

while we have

$$\mathcal{I}_\beta(\rho_k) = \frac{8\pi}{\beta}\left\{\left(\frac{\beta}{8\pi} - 1\right)\mathcal{K}(\rho_k) + \mathcal{I}(\rho_k)\right\} \to +\infty$$

from the assumption. Therefore, it holds that

$$\lim_{k\to\infty} \min\{Q_k(r/2), 1 - Q_k(r)\} = 0.$$

We obtain, on the other hand, $c_0 > 0$ such that

$$Q_k(r) \geq c_0 r^2, \quad k = 1, 2, \dots, \ 0 < r \ll 1$$

from the covering theorem of Vitali, and, hence (2.99).

For the proof of $x_\infty \in \Omega$, first, each $0 < r \ll 1$ admits $k \gg 1$ such that $1 - Q_k(r/2) \leq r$ by (2.99). This property implies for $\overline{x_k} = \displaystyle\int_\Omega x \rho_k$ that

$$\begin{aligned} |\overline{x_k} - x_k| \ &= \left| \int_\Omega (x - x_k) \rho_k \right| \leq \int_{\Omega \cap B(x_k, r)} |x - x_k| \, \rho_k \\ &\quad + \int_{\Omega \setminus B(x_k, r)} |x - x_k| \, \rho_k \leq r + \operatorname{diam} \Omega \cdot \int_{\Omega \setminus B(x_k, r/2)} \rho_k \\ &= r + \operatorname{diam} \Omega \cdot (1 - Q_k(r/2)) \leq (1 + \operatorname{diam} \Omega) \, r \end{aligned}$$

and hence $\lim_{k \to \infty} |\overline{x_k} - x_k| = 0$. Thus we obtain $x_\infty \in \Omega$.

Similarly, each $\zeta = \zeta(x) \in C(\Omega)$ admits $k \gg 1$ such that

$$\begin{aligned} \left| \zeta(x_k) - \int_\Omega \zeta \rho_k \right| \ &\leq \int_{\Omega \cap B(x_k, r)} |\zeta(x_k) - \zeta(x)| \, \rho_k \\ &\quad + \int_{\Omega \setminus B(x_k, r)} |\zeta(x_k) - \zeta(x)| \, \rho_k \leq \| \zeta - \zeta(x_k) \|_{L^\infty(B(x_k, r))} \\ &\quad + 2 \cdot \operatorname{diam} \Omega \cdot \| \zeta \|_\infty \int_{\Omega \setminus B(x_k, r/2)} \rho_k \\ &\leq \| \zeta - \zeta(x_k) \|_{L^\infty(B(x_k, r))} + 2 \cdot \operatorname{diam} \Omega \cdot \| \zeta \|_\infty r. \end{aligned}$$

Since $\zeta \in C(\Omega)$ is uniformly continuous, we get

$$\lim_{k \to \infty} \left| \zeta(x_k) - \int_\Omega \zeta \rho_k \right| = 0$$

with $r \downarrow 0$, and, therefore, $\rho_k(x) dx \rightharpoonup \delta_{x_\infty}(dx)$. □

## 2.7 Summary

We studied several models related to the Smoluchowski-Poisson equation, in accordance with the stationary and ODE parts.

1. Some non-stationary problems sharing the same stationary problem arise in biology and geometry which, however, obey different features from those of the Smoluchowski-Poisson equation.

2. Non-local parabolic equation arises as the other limit of the full system of chemotaxis, which is subject to the dis-quantized blowup mechanism.
3. Type (II) blowup rate, formation of sub-collapses, and possible collision of collapses are observed when the free energies are bounded in the energy quantization, such as harmonic heat flow and semilinear parabolic equation with critical Sobolev exponent.
4. In the harmonic heat flow, sub-collapses suffer fast collisions inside the hyperparabola.
5. There is a global-in-time compact orbit for the normalized Ricci flow even at the critical level of the total mass.

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