Chapter 2
My involvement in Walsh and Dyadic Analysis

Franz Pichler

The following description of involvement of F. Pichler in Walsh and dyadic analysis is an excerpt from *Reprints from the Early Days of Information Sciences Reminiscences of the Early Work in Walsh Functions Interviews with Franz Pichler, William R. Wade, Ferenc Schipp*, Radomir S. Stanković, Jaakko T. Astola, (eds.), TICSP Series # 58, ISBN 978-952-15-2598-8, ISSN 1456-2744.

In the above interview, Prof. Pichler said the following.

It is so that the inventor of the Walsh functions for Innsbruck was Roman Liedl [1], he is still there a Professor, maybe he retired already, a mathematician, and he invented as many other researchers also on his own the Walsh functions. Later he found out that the concept exist already, but Liedl already saw also the group relations, group theoretical relations and topological group relations [1], [2], [3]. Then research started and I think that in Innsbruck about twenty PhD theses on Walsh functions were made. Many theses were defended. For example, Peter Weiss, he is still at Linz, was one of the first, and they were mainly devoted to generalized Walsh functions. At that time we looked at the work of Lèvy [4], and Rice and Selfdrige [10] and others. Selfridge, these were names that passed, and also Vilenkin, the Russian important Walsh function researcher.

Then Harmuth discovered that there was a Walsh researcher in Innsbruck and he contacted the Innsbruck people. He finally wanted to develop some theoretical framework for his meander functions, which were essentially what have been called *cal* and *sal* functions [5].

Since I was already starting a PhD work, and they knew, the mathematicians there, Roman Liedl knew that I had a communication background, I was the right one to get interested in that, and so it started, and it was interesting. But of course,

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at that time we had no overview, for example, I had to discover also the concept of the dyadic filter and this is dyadic convolution, I did not know that before. So this is all separate, you are a student, you do not know, and so it started.

How I came to Maryland is interesting, because Harmuth was not an easy man, and he is until today not easy, but he is a devoted scientist and so on, so I had some struggle with him in Innsbruck already. But he was so concentrated to push forward Walsh functions and research, and then, I think we split. We had no much contacts, but I continued to make my work, especially the PhD thesis, and also papers. Yes, and in 1968 I think I published my first paper in the *AEU* (*Archiv der Elektrischen Übertragung* abbreviated as *Archiv eiektr. Übertragung*), this is the journal where Hansi Piesch was an editor, or co-editor [6]. You see, it was not so easy at that time to publish about Walsh functions. AEU had already published the papers of Harmuth, and I was still a student, and not experienced. So I had my doubts if they would accept it, but I had a promoter in the East Germany, in the DDR. This was Franz Heinrich Lange, a Professor of communication engineering in Rostock. He was very well known, and he was fan of Harmuth and of Walsh functions, and so on, and he knew about my work, and when I wrote and sent my paper to the AEU, I do not remember the main editor there, the secretary so to say, I mentioned that if they would not publish it, I could publish it in the DDR, because Lange would have liked to take it. I was already clever, I think, to mention this and they finally reviewed the paper and one of the reviewers was Hansi Piesch. And so I really brought the final version of the paper to Vienna. I drove with my Fiat 600 from Innsbruck, with a friend of mine, we drove to Vienna, and I went to her apartment, yes, at the Gürtel near Sud Bahnhof, and was friendly welcomed and I gave the paper for publication.

I continued publishing in this area, let me mention just first papers of mine [7], [8], [9].

I went to Linz in 1968, and Harmuth again contacted me, and needed me for this first conference as a mathematician, because he was always criticized that he could not define exactly the Walsh function in the continuous case of sequency as he called it. Yes, so he needed me, Harmuth was able to define Walsh functions just as a limes, yes, if n goes to infinity then this is the function, yes, so he was picked by some people when they said *Tell me, how does the Walsh function with sequency π, yes, 3.14 and so on, does look like?*, and he could not answer. He could not really answer, he was, and really is a gifted intuitive working scientist. A kind of engineer with mathematic intellect, Harmuth, there is no doubt yes. Like also Gibbs, they would make formulas without knowing how they can derive these formulas. I was just the opposite. I was, say, educated as a step-wise, going further, operating mathematician, so he needed me in Washington (for the conference on Walsh functions in 1970), and I got the invitation as a visiting research assistant professor.

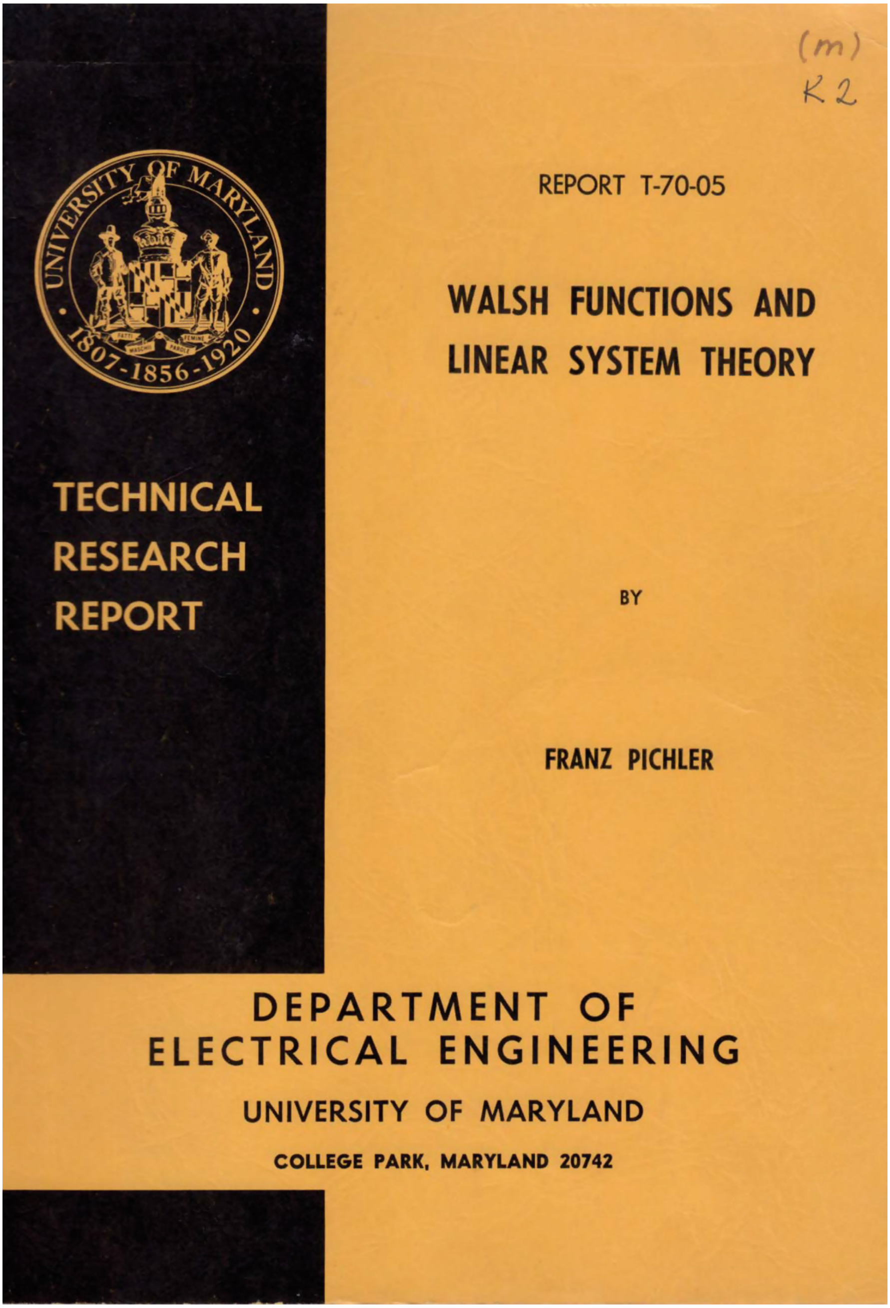
My stay at the Laboratory of Professor Harmuth at Maryland University resulted in two reports that are reprinted in this book [2]

[2] *Comment by Editors* More on the early work of Prof. Franz Pichler in this area and his cooperation with Dr. J.E. Gibbs, can be found in F. Pichler, "Remembering J. Edmund Gibbs", in *Walsh and Dyadic Analysis*, R.S. Stanković, (ed.), *Proc. Workshop on Walsh and Dyadic Analysis*, October 2007, Faculty of Electronic Engineering, Niš, Serbia, XXI-XXVI, 2008.

References

1. Liedl, R., *Vollständige orthonormierte Furiktionen-folgen des Hilbertraumes L_2, deren Elemente bezglich der Multiplikation eine Gruppe bilden*, Dissertation, Philosophische Fakultät Universität Innsbruck, 1964.
2. Liedl, R., *Eine algebraische Herleitung und eine Verallgemeinerung des Satzes von Fine ber Gruppen von orthonormalen Funktionen und eine Beschreibung der vielfait-erhaltenden Transformationen des Intervalles* $[0,1]$ *audsich selbst*, Habilitationsschrift, Universität, Innsbruck, Austria.
3. Liedl, R., ”Über eine spezielle Klasse von stark multiplikativ orthogonalen Funktionensystemen”, *Monatshefte für Mathematik*, 68, 1964, 130-137.
4. Levy, P., ”Sur une generalization de fonctions orthogonales de M. Rademacher”, *Comment. Math. Helv.*, 16, 1944, 146-152.
5. Ohnsorg, F. R., ”Application of Walsh functions to complex signals”, *Proc. of the Walsh Functions Symposium*, Naval Research Laboratory, Washington, DC, USA, 1970, 123-127.
6. Pichler, F., ”Synthese lin earer periodisch zeitvariabler Filter mit vorgschriebenem Sequenzverhalten”, *Archiv elektr. Übertragung*, 22, 1968, 150-161.
7. Pichler, F., ”Walsh-Fourier Synthese optimaler Filter”, *Archiv elektr. Übertragung*, 24, 1970, 350-360.
8. ∗Pichler , F., *Walsh Functions and Linear Systems Theory*, Techn. Report, Department of Electrical Engineering, University of Maryland, Washington D.C., Appendix B-l to 8-1, ”Sampling Theorem with Respect to Walsh-Fourier Analysis”, April 1970.
9. Pichler, F., ”Walsh Functions-Introduction to the Theory”, in *Proceedings Nato Advanced Study Institute on Signal Processing*, Loughbourough, U.K., 1972, 18 pages.
10. Selfridge, R.G., ”Generalized Walsh transforms”, *Pacific J. Math.*, 5, 1955, 451-480.

The ∗ sign in the citations indicates that the paper is reprinted in this book.



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WALSH FUNCTIONS AND
LINEAR SYSTEM THEORY

TECHNICAL
RESEARCH
REPORT

BY

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Walsh Functions and Linear System Theory*

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* Lecture to be presented at the workshop on "Applications of Walsh Functions", Naval Research Laboratory and University of Maryland April 2, 1970.

CONTENTS

1

I. Introduction

In this paper we present some ideas in the applications of Walsh functions to the analysis and synthesis of linear systems.

To do this we first consider the mathematical background of Walsh functions. We define the Walsh functions $\psi(y,\cdot)$, $\psi^*(y,\cdot)$ and cal $(s,\cdot)$, sal $(s,\cdot)$ and consider some of the mathematical results. Next, we define a special class of linear systems, which we call dyadic-invariant systems. During the course of examining the dyadic-invariant systems we outline the synthesis procedure of sequency bandpasses and of optimal-filters. Finally, we shall attempt the first steps of a state-space approach for dyadic-invariant systems.

2. Walsh Functions

The mathematical theory of Walsh functions is well developed. Since the fundamental paper of J. L. Walsh [1] was published in 1923, many additional papers concerning this theory have been published. Among these are Paley [2], Levy [3], Fine [4], [5], Vilenkin [6], [7], Chrestenson [8], Watari [9], Weiss [10] and Liedl [11].

There are two common aspects to these papers: The one is, that they search for a theory similar to the theory of the trigonometric functions. Questions concerning summability and convergence of Walsh-Fourier series and Walsh-Fourier integrals are often the most interesting ones. The other aspect is an attempt to embed the theory of Walsh-functions in a more general one: the theory of abstract harmonic analysis. There the Walsh-functions can be derived from the characters of a certain locally compact topological group: the Dyadic Group F of Fine [5]. Often it happens, that theoretical questions concerning Walsh-functions can be solved by means of the theory of abstract harmonic analysis.

2.1 Definition of the Walsh Functions

There are two different definitions in common use. The first gives us the Walsh-functions wal $(i, \cdot)$. For these functions the parameter i represents a count of the sign changes of the function per unit interval. This principle of ordering the Walsh-functions was also used originally by Walsh [1]. In the application of the functions it is often of practical interest to use particular symbols for even and odd Walsh-functions. For the even Walsh-functions we use the symbol cal $(i, \cdot)$, and for the odd we use sal $(i, \cdot)$. We have:

$$\mathrm{cal}\,(i, \cdot) = \mathrm{wal}\,(2i, \cdot) \quad \text{for } i = 0, 1, 2, \ldots$$

and (1)

$$\mathrm{sal}\,(i, \cdot) = \mathrm{wal}\,(2i-1, \cdot) \quad \text{for } i = 1, 2, 3, \ldots$$

The number i represents the generalized frequency, which has been called "sequency" (Harmuth [12], [13]).

The second method of defining the Walsh-functions was introduced by Paley [2]. This method presents the Walsh-functions $\psi(n, \cdot)$ as products of Rademacher-functions $\phi(k, \cdot)$. If the number n has the dyadic representation.

$$n = \sum_{k=-N}^{0} n_k 2^{-k} \tag{2}$$

then the Walsh-function $\psi(n, \cdot)$ is defined by

$$\psi(n, t) = \prod_{k=-N}^{0} \left[\phi(-k, t)\right]^{n_k} \tag{3}$$

The Rademacher-functions $\phi(-k, \cdot)$, $k \in Z$ (Z denotes the set of integers), can be defined as the functions given by

$$\phi(-k, t) = \exp \pi i t_{1-k} \tag{4}$$

where t is a nonnegative real number given by

$$t = \sum_{k=-\infty}^{\infty} t_k 2^{-k}$$

For negative real numbers -t the Rademacher functions are defined by

$$\phi(-k, -t) = -\phi(-k, t) \tag{5}$$

It should be mentioned that the number V(n), given by

$$V(n) = \sum_{k=-N}^{0} n_k \tag{6}$$

has been called the "Vielfalt" of the Walsh-function $\psi(n, \cdot)$. V(n) is the number of Rademacher-functions $\phi(-k, \cdot)$ which generates $\psi(n, \cdot)$. There are many interesting mathematical results associated with the "Vielfalt" and with the problem of approximating a polynomial with a Walsh-Fourier series [10], [11].

4

The relationship between the Walsh-functions wal $(i, \cdot)$ and $\psi(n, \cdot)$ is given by the formula

$$\mathrm{wal}\,(i, \cdot) = \begin{cases} \psi\,(i/2) \oplus i, \cdot) & i = 0, 2, 4, \ldots \\ \psi\,[(i-1)/2] \oplus i, \cdot) & i = 1, 3, 5, \ldots \end{cases} \tag{7}$$

where $\oplus$ denotes the addition modulo 2 of the integers written as binary numbers.

2.2 Walsh-Fourier Analysis

It is well known that the Walsh functions form a complete orthonormal system of functions for the real Hilbert-space $L_2\,[a, a+1]$ of functions defined on a interval of Length 1. Therefore, we have a theory to analyze and synthesize functions $f \in L_2\,[a, a+1]$. The discrete Walsh-Fourier transform $\hat{f}$ is defined by

$$\hat{f}\,(n) = \langle f, \psi\,(n, \cdot)\rangle \quad n = 0, 1, 2, \ldots \tag{8}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner-product of the space $L_2[a, a+1]$. Using Walsh functions cal $(i, \cdot)$ and sal $(i, \cdot)$, we will denote the discrete Walsh-Fourier transform of $f \in L_2\,[a, a+1]$ by the symbols F_c and F_s:

$$\begin{aligned} F_c(i) &= \langle f, \ \mathrm{cal}\,(i, \cdot)\rangle \quad i = 0, 1, 2, \ldots \\ F_s(i) &= \langle f, \ \mathrm{sal}\,(i, \cdot)\rangle \quad i = 1, 2, 3, \ldots \end{aligned} \tag{9}$$

Next, we must have a theory of Walsh-Fourier integrals to represent nonperiodic functions as superpositions of Walsh functions. To formulate such a theory we must define the Walsh functions for continuous parameters. With respect to the Walsh functions $\psi\,(n, \cdot)$ this work was done in a paper by Fine [5]. According to Fine the Walsh functions $\psi(y,\cdot)$, $y \in R_+$ *), are defined in a very natural way as the functions given by

$$\psi(y,t) = \exp \pi i \sum_{k=-N}^{M+1} y_k \, t_{1-k} \tag{10}$$

where $y, t \in R_+$ have the dyadic representation

$$y = \sum_{k=-N}^{\infty} y_k 2^{-k} \quad \text{and} \quad t = \sum_{k=-M}^{\infty} t_k 2^{-k} \tag{11}$$

If $y \in D_+$ or $t \in D_+$ where D_+ denotes the set of nonnegative dyadic rational numbers, we use for (11) the finite dyadic representation. The Walsh functions $\psi^*(y,\cdot)$, $y \in D_+$, Fine defines as the functions given by

$$\psi^*(y,t) = \exp \pi i \sum_{k=-N}^{M+1} y_k \, t_{1-k} \tag{12}$$

where y is now represented by (11) by an infinite sum. Note that the Walsh functions $\psi(y,\cdot)$ are only defined for $t \in R_+$.

The Walsh functions cal $(s,\cdot)$ and sal $(s,\cdot)$ can be defined for $s \in R_+$ in a similar manner. If $s \in R_+$ and $t \in R_+$ have the dyadic representation

$$s = \sum_{k=-N}^{\infty} s_k 2^{-k} \quad \text{and} \quad t = \sum_{k=-M}^{\infty} t_k 2^{-k} \tag{13}$$

then the functions cal $(s,\cdot)$ and sal $(s,\cdot)$ are given for $t \in R_+$ by

$$\text{cal}\,(s,t) = \exp \pi i \sum_{k=-N}^{M+1} (s_k + s_{k+1}) t_{1-k} \tag{14}$$

*) R denotes the set of real numbers, R_+ denotes the nonnegative real numbers.

and

$$\mathrm{sal}\ (s,t) = \exp \pi i \sum_{k=-N}^{M+1} (s_k + s_{k+1}) t_{1-k} \tag{15}$$

The difference between (14) and (15) is only that if $s \in D_+$ we have to use in (14) the finite dyadic representation of s but in (15) the infinite representation.

To define the Walsh functions cal $(s, \cdot)$ and sal $(s, \cdot)$ on the entire real axis we determine the cal functions to be even and the sal functions to be odd functions of the variable t.

The connection of the cal and sal functions to the functions ψ and ψ^* is given by the formula

$$\begin{aligned} \mathrm{cal}\ (s,t) &= \psi(s,t)\psi(2s,t) \qquad & s,t \in R_+ \\ \mathrm{sal}\ (s,t) &= \psi^*(s,t)\psi^*(2s,t) \qquad & s \in D_+,\ t \in R_+ \end{aligned} \tag{16}$$

Now one can derive a theory of Walsh-Fourier integrals.

If $f \in L_2[0,\infty)$, the Walsh-Fourier transform $\hat{f}$ of f is defined as the function given by the integral

$$\hat{f}(y) = \int_0^\infty f(t)\psi(y,t)dt \tag{17}$$

where the integral converges with respect to norm of the space $L_2\ [o,\infty)$. If $f \in L_2(-\infty,\infty)$, we define the cal transform F_c and the sal transform F_s of f as the functions given by

$$F_c(s) = \int_{-\infty}^{\infty} f(t)\ \mathrm{cal}\ (s,t)dt \tag{18}$$

and

$$F_s(s) = \int_{-\infty}^{\infty} f(t)\ \mathrm{sal}\ (s,t)dt \tag{19}$$

There are theorems concerning the transforms of a function $f \varepsilon L_1[0,\infty)$ and $f \varepsilon L_1(-\infty,\infty)$, respectively. For theorems such as Plancherel theorem and convolution theorem the reader is advised to consult the papers of Fine [5], Vilenkin [7], Selfridge [14] and Pichler [15].

A general transform theory defined on an arbitrary locally compact Abelian group has been presented in a paper by Falb and Friedman [16].

2.3 Dyadic Correlation Analysis

Next, we are concerned with a generalization of correlation methods. A generalization may be obtained by using "addition modulo 2" $\oplus$ instead of the usual addition of real numbers. Let us first define what we mean by "addition modulo 2" of real numbers. Let $u \varepsilon R_+$ and $v \varepsilon R_+$ with, if possible, u and v having a finite dyadic representation

$$u = \sum_{i=-N}^{\infty} u_i 2^{-i}, \quad v = \sum_{i=-M}^{\infty} v_i 2^{-i}, \tag{21}$$

then the real number $u \oplus v$ is given by

$$u \oplus v = \sum_{i=-L}^{\infty} (u_i \oplus v_i) 2^{-i} \tag{22}$$

where $L = \max(N, M)$ and $u_i \oplus v_i$ denotes addition modulo of numbers 0 or 1 in the usual sense ($0\oplus 1 = 1\oplus 0 = 1, 0\oplus 0 = 1\oplus 1 = 0$). For negative real numbers, addition modulo can be defined by

$$u \oplus (-v) = (-u) \oplus v = -(u \oplus v) \tag{23}$$

and

$$(-u) \oplus (-v) = u \oplus v \tag{24}$$

Let $W_2(-\infty, \infty)$ denote the space of functions $f: R \to R$ for which the integral

$$\lim_{t \to \infty} \frac{1}{2T} \int_{-T}^{+T} f^2(t)dt \tag{25}$$

exists and has a finite value. Let $f, g \in W_2(-\infty, \infty)$. The dyadic cross-correlation function $R(f, g, \cdot)$ of f and g then is defined as the function given by

$$R(f, g, \tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} f(t)g(t \oplus \tau)d\tau \tag{26}$$

For g = f we write for $R(f, f, \cdot)$ easier $R(f, \cdot)$. $R(f, \cdot)$ is called the dyadic autocorrelation function of f.

For dyadic correlation there is a theory similar to the classical theory of correlation. The initial development of this theory was presented in a thesis by Weiser [17] and a paper by this author [18]*). The possibility of such a theory was mentioned in a paper of Wiener and Paley [19] as early as 1932, but there have been no reports of further development. There is, however, hope that progress will be made in this direction. Like the generalized harmonic analysis of N. Wiener, one can embed this theory in a more general theory of certain stochastic processes on topological groups and now there are many efforts to complete this theory.

*) See also, J.E. Gibbs and H.A. Gebbie: "Application of Walsh Functions to Transform Spectroscopy", Nature, pp. 1012-1013, Dec. 1969.

3. Linear Dyadic-Invariant Systems

We shall now define a certain class of linear systems, which have an invariant behavior against dyadic translations of the input-functions.

Let I denote the space of input-functions of a scalar continuous linear system S. Each $x \in I$ should be a real-valued stepwise integrable time functions. Let O denote the corresponding space of output-functions of the system S. We define then S as the system, which has an input-output relation $R(S) \subset I x O$ given by the integral

$$y(t) = \int_{-\infty}^{\infty} h(t \oplus \tau)\, x(\tau) d\tau \tag{27}$$

where y denotes an output-function and $h: R \rightarrow R$ is assumed to be absolutely integrable on R. So $(x, y) \in R(S)$ if x and y fulfill the equation (27). The function $h^{*}: RxR \rightarrow R$ given by $(t, \tau) \rightarrow h(t \oplus \tau)$ is the impulse response of S and h is the impulse response for $\tau = 0$.

The analogy of our system S given by (27) to a linear time-invariant system given by its steady-state representation is apparent. Equation (27) also has a convolution form: it defines the dyadic convolution of the functions h and x. It is easy to show that S is invariant against dyadic translations of the input functions. With that we mean that from $(x, y) \in R(S)$ follows that for all $\lambda \in R$, also $(x_\lambda, y_\lambda) \in R(S)$. Here x_λ and y_λ denotes the λ-dyadic translations of x and y defined by

$$x_\lambda(t) = x(t \oplus \lambda)$$

and (28)

$$y_\lambda(t) = y(t \oplus \lambda)$$

If x is a suited function (e. g. if $x \epsilon L_1(-\infty, \infty)$) applying the convolution theorem of Walsh-Fourier transform theory one can obtain from (27) the equations

$$
\begin{aligned}
Y_c(s) &= H_c(s)X_c(s) \\
Y_s(s) &= H_s(s)X_s(s)
\end{aligned}
\tag{29}
$$

where $Y_c, Y_s, H_c, H_s, X_c, X_s$, denotes the sal and cal transforms of y, h and x given by integrals of the form (18) and (19) respectively.

In (29) we have a description of the system S in terms of "sequency". The functions H_c and H_s are called the transfer functions of the system S.

We have assumed that $h \epsilon L_1(-\infty, \infty)$. From that follows, that S is "bounded-input bounded-output stable". For S to be nonanticipative (causal), it is necessary that

$$h(t) = 0 \quad \text{for all } t < 0 \tag{30}$$

But observe that the condition (30) is only necessary and, in general, not sufficient.

3.1 Synthesis of Sequency Bandpasses

We will now outline a synthesis procedure for a certain class of dyadic-invariant system which are bandpasses. A detailed treatment of these has been given in a paper by this author [20].

Let the transfer functions H_c and H_s of a dyadic-invariant linear system S defined by (27) be given by

$$H_c(s) = \begin{cases} 1 & \text{for all } s \in [n2^k, (n+1)2^k) \\ 0 & \text{elsewhere} \end{cases}$$

and

$$H_c(s) = \begin{cases} 1 & \text{for all } s \in (n2^k, (n+1)2^k] \\ 0 & \text{elsewhere} \end{cases}$$

where k is an integer and n a nonnegative integer. The system S, defined by (31), we shall call a sequency-bandpass with a normalized bandwith $\Delta s = 2^k$ and a cutoff-sequency $s = n2^k$. To obtain the impulse response h of this sequency-bandpass, we apply the inverse Walsh-Fourier transform to H_c and H_s and we have then generally

$$h(t) = \int_0^\infty [H_c(s)\mathrm{cal}(s,t) + H_s(s)\mathrm{sal}(s,t)]\,ds \tag{32}$$

With H_c and H_s defined in (31) h becomes

$$h(t) = \begin{cases} 2^{k+1}\mathrm{cal}(n2^k, t) & \text{for } t \in [0, 2^{-k-1}) \\ 0 & \text{elsewhere} \end{cases} \tag{33}$$

We see that h is a Walsh impulse given on the interval $[0, 2^{-k-1})$. It can be shown that the filters we get in this way are b.i.b.o. stable and nonanticipative. To make these filters "technically realizable" we have to allow a constant delay β with $\beta \geq 2^{-k-1}$. It happens that these filters are exactly the same as the sequency-bandpasses of Harmuth [12], [21], derived by a different approach.

3.2 Synthesis of Optimal Filters

The development of a theory of optimal filtering is now near at hand, based on a Walsh-Fourier decomposition of signals and systems. One can formulate such a theory in a manner similar to the classical theory of optimal filtering of Wiener and Kolmogoroff. In a method similar to Wiener, one can do this without the theory of probability. Assuming that the signal u, as also the noise v, is an element of the space $W_2(-\infty, \infty)$ we obtain relations for the transfer functions H_c and H_s of the optimal sequency filter.

Let the input x of a linear dyadic-invariant system be the sum of a signal u and noise v; $x = u+v$. Both u and v are assumed to be elements of the space $W_2(-\infty, \infty)$. We want to find the impulse response h of the system so that the mean-square deviation $\overline{\varepsilon^2}$ defined by

$$\overline{\varepsilon^2} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} (y(t)-u(t))^2 dt \tag{34}$$

is minimal. Here y denotes the outputsignal related with the inputsignal x. A linear dyadic-invariant system with such an impulse response, h, could be called an optimal sequency filter.

A solution to the problem of finding h is given in a paper by the author [18]. It can be shown that the transfer function H_c and H_s of the optimal filter are related to the one-sided Walsh-Fourier transforms $S_{co}(u, x, \cdot)$, $S_{so}(u, x, \cdot)$, $S_{co}(x, \cdot)$ and $S_{so}(x, \cdot)$ of the dyadic correlation functions $R(u, x, \cdot)$ and $R(x, \cdot)$ respectively, in the following expressions:

$$\begin{aligned} S_{co}(u, x, s) &= H_c(s) S_{co}(x, s) \\ S_{so}(u, x, s) &= H_s(s) S_{so}(x, s) \end{aligned} \tag{35}$$

Further it can be shown, that the derived optimal filter is nonanticipative at the time $\tau = 0$. For applications in communciations it would be of interest to find optimal sequency filters which could easily be built up with electronic elements.

With a slight modification of this theory for other classes of signals and noise, one can get the Harmuth sequency bandpasses as optimal filters [18].

4. On a State Space Approach for Linear Dyadic-Invariant Systems

Now we shall deal with some of the concepts of a state-space approach for dyadic-invariant systems. The development of such a theory seems to be of interest. It could give us a deeper insight into the internal working of these systems. Further it could be a bridge to software systems and to applications of dyadic invariant systems in control problems.

4.1 Dyadic Differentiation

First, we are concerned with the concept of a generalized differentiation. The fundamental work for this theory was done in a paper by Gibbs and Millard [22]. There the main interest was that of finite discrete sturctures. Some slight modifications must be made to obtain a theory of generalized differentiation defined for real valued functions of a continuous nonnegative real variable. Let f be such a function. To f we attach, if possible, a function $f^{[1]}$, given by

$$f^{[1]}(t) = \sum_{k=-\infty}^{\infty} [f(t) - f(t \oplus 2^{-k})] 2^{k-2} \tag{36}$$

If the function $f^{[1]}$ exists, we shall call it the first dyadic derivative of f. *) The function f we call in this case dyadic differentiable. The term "dyadic differentiation" is used since this linear operation has certain qualities, which one associates with differentiation. As the first result of dyadic differentiation we have for the Walsh functions $\psi(y, \cdot)$

$$\psi^{[1]}(y,t) = y\psi(y,t) \qquad \forall y, t \in R_+ \tag{37}$$

Equation (37) shows us, that the Walsh functions $\psi(y, \cdot)$ are eigenfunctions of the dyadic differential operator. This is easy to prove:

From (36) we have

$$\psi^{[1]}(y,t) = \sum_{k=-\infty}^{\infty} [\psi(y,t) - \psi(y, t\oplus 2^{-k}] \, 2^{k-2} \tag{38}$$

With the formula

$$\psi(y, t\oplus 2^{-k}) = \psi(y,t)\psi(y, 2^{-k}) \tag{39}$$

we get

$$\psi^{[1]}(y,t) = \sum_{k=-\infty}^{\infty} [1-\psi(y, 2^{-k})] \, 2^{k-2}\psi(y,t) \tag{40}$$

Due to the fact that $\psi(y, 2^{-k}) = \exp \pi i y_{1-k}$

and so $[1-\psi(y, 2^{-k})] \, 2^{-1} = y_{1-k}$ we get from (40)

$$\psi^{[1]}(y,t) = \sum_{j=-\infty}^{\infty} y_{1-k} 2^{k-1}\psi(y,t) \tag{41}$$

*) Gibbs and Millard use the terms "logical differentiation" and "logical derivative".

and with the substitution $1-k \rightarrow j$ we obtain

$$\psi^{[1]}(y,t) = \sum_{j=-\infty}^{\infty} y_j 2^{-j} \psi(y,t) = y\psi(y,t) \qquad (42)$$

So the Walsh functions $\psi(y,\cdot)$ can be seen to be the solutions of the following dyadic differential equation of the first order for different values of $y \in R_+$.

$$f^{[1]} - yf = 0 \qquad (43)$$

To get this property of the Walsh functions originally was the motive of Gibbs to define a "logical differentiation".

Another interesting result concerns the Walsh-Fourier transforms of dyadic derivatives. Let f be an n-time dyadic-differentiable function. Let $\hat{f}$ denote the Walsh-Fourier transform of f given by (17). We have then the following theorem:

$$\widehat{f^{[n]}}(y) = y^n \hat{f}(y) \qquad \forall y \in R_+ \qquad (44)$$

The proof of this theorem is straight forward and may here be omitted. This theorem is analogous to the well-known theorem concerning the Laplace transforms of derivatives. The difference is the absence of initial values.

4.2 Dyadic Differential Equations and Dyadic Invariant Systems

The concept of dyadic differentiation directly leads to the concept of dyadic differential equations. We define an ordinary linear dyadic differential equation with constant coefficients as a relation given by

$$\sum_{k=o}^{n} a_k f^{[k]} = \sum_{k=o}^{m} b_k u^{[k]} \tag{45}$$

where us is assumed to be k-time dyadic differentiable; a_k and b_k should be real numbers; f denotes the general solution of (45). The relation $R(S_d)$ given by (45) can be interpreted as an input-output relation of a system S_d. It is clear that the system S_d is linear.

We get the general solution, f, (the general output function) of equation (45) as the sum of the solution f_h of the homogeneous equation (the zero-input response of the system) and the particular solution, f_p, of the inhomogeneous equation (the zero-state response)

$$f = f_h + f_p \tag{46}$$

f_h is given by a linear combination of Walsh functions, $\psi(\alpha, \cdot)$. If we assume that the characteristic equation of (45) given by

$$\sum_{k=o}^{n} a_k \alpha^k = 0 \tag{47}$$

has n real and distinct solutions, $\alpha_1, \ldots, \alpha_n$, then f_h is given by

$$f_h = \sum_{k=1}^{n} c_k \psi(\alpha_k, \cdot) \tag{48}$$

Hereby the coefficients $c_1, \ldots, c_n$ result from the initial values connected with equation (45). If the initial values of f are given at time 0 by

$$f(0),\ f^{[1]}(0), \ldots, f^{[n-1]}(0), \tag{49}$$

then we can get the coefficients $c_1, \ldots, c_n$, as the solutions of the following regular system of linear equations

$$\sum_{k=1}^{n} c_k \alpha_k^i = f^{[1]}(0) \quad i = 0,1,\ldots,n-1 \tag{50}$$

To get the particular solution of the dyadic differential equation given by (45) we apply the Walsh-Fourier transform on both sides of (45) and get with regard to (44)

$$\sum_{k=o}^{n} a_k y^k \hat{f}_p(y) = \sum_{k=o}^{m} b_k y^k \hat{u}(y) \tag{51}$$

Here we have assumed that both f_p and u have a Walsh-Fourier transform. With the polynomials p and q given by

$$p(y) = \sum_{k=o}^{m} b_k y^k \quad \text{and}$$
$$q(y) = \sum_{k=o}^{n} a_k y^k \tag{52}$$

we get from (51)

$$\hat{f}_p(y) = \frac{p(y)}{q(y)} \hat{u}(y) \tag{53}$$

The function $\hat{h}$ given by

$$\hat{h} = \frac{p}{q} \tag{54}$$

is called the transfer function of the system S_d. We shall see that the system S_d associated with the dyadic differential equation (45) is a linear dyadic invariant system.

From (53), the inverse Walsh-Fourier transform produces the particular solution f_p

$$f_p(t) = \int_0^\infty \hat{h}(y)\hat{u}(y)\psi(y,t)dy \tag{55}$$

Assuming that we can apply a convolution theorem, we get from (55)

$$f_p(t) = \int_0^\infty h(t \oplus \tau)u(\tau)d\tau \tag{56}$$

where h denotes the inverse Walsh-Fourier transform of $\hat{h}$. We see that (56) has a form similar to (27). The difference is only that the lower limit of the integral is 0 rather than $-\infty$. This is the result of assuming that the functions are only defined on R_+. It seems possible to extend this theory to functions defined on the whole real axis R.

So we have the following result: The ordinary linear dyadic differential equation given by

$$\sum_{k=o}^{n} a_k f^{[k]} = \sum_{k=o}^{m} b_k u^{[k]} \tag{57}$$

where u is a k-time dyadic-differentiable function and a_k, $k = o, \ldots, n$, and b_k, $k = o, \ldots, m$, are real numbers, has a general solution f of the following form

$$f(t) = \sum_{k=1}^{n} c_k \psi(\alpha_k, t) + \int_0^\infty h(t \oplus \tau)u(\tau)d\tau \tag{58}$$

if the following assumptions are fulfilled:

a) The characteristic equation

$$\sum_{k=o}^{n} a_k \alpha^k = 0 \tag{59}$$

of (57) has n distinct real roots,

$$\alpha_1, \ldots, \alpha_n.$$

b) The transfer dunction $\hat{h}$ of (57) given by

$$\hat{h}(y) = \frac{\sum_{k=o}^{m} b_k y^k}{\sum_{k=o}^{n} a_k y^k} \quad \forall y \in R_+ \tag{60}$$

has an inverse Walsh-Fourier transform h.

c) The real numbers c_k, $k = 1, \ldots, n$ of (58) are given by following regular system of linear equations

$$\sum_{k=1}^{n} c_k \alpha_k^i = f^{[i]}(0) \quad i = 0, 1, \ldots, n-1 \tag{61}$$

where $f^{[i]}(0)$, $i = 0, 1, \ldots, n-1$, are the initial values of f at time 0.

For the scalar linear system S_d connected with the dyadic differential equation (57) the solution (58) represents an input-output-state relation [23]. Hereby determine the constant numbers $c_1, \ldots, c_n$ the state x(o) at time 0; $x(o)' = (c_1, \ldots, c_n)$. The vector $\psi(\alpha, \cdot) = (\psi(\alpha_1, \cdot), \ldots, \psi(\alpha_n, \cdot))$ is a basis vector whose components $\psi(\alpha_1, \cdot), \ldots, \psi(\alpha_n, \cdot)$ represents the zero-input response of S_d starting in the initial states $(1, 0, \ldots, 0), \ldots, (0, \ldots, 1)$ respectively.

h is the impulse response of S_d, that is, the zero-state response of S_d to the unit impulse $\delta(t)$. h is the inverse Walsh-Fourier transform of $\hat{h}$, the transfer function of S_d.

To get the linear dyadic invariant system S_d nonanticipative we have to assume that

$$h(t \oplus \tau) = 0 \text{ for all } t < \tau \tag{62}$$

With (62) and expressing the sum in (58) as the scalarproduct $<\psi(\alpha, t), x(o)>$ of the vectors $\psi(\alpha, t)$ and $x(o)$ we get for a linear dyadic invariant nonanticipative system S_d the input-output-state relation in the form

$$f(t) = <\psi(\alpha, t), x(o)> + \int_o^t h(t \oplus \tau) u(\tau) d\tau \tag{63}$$

The formula (63) has an analogous counterpart in the theory of linear time-invariant systems.

These are first steps to a general theory. Much work must yet be invested in this theory.

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REFERENCES

[1] Walsh, J. L., "A closed set of normal orthogonal functions", Amer. Math., vol. 45, pp. 5-24, 1923.

[2] Paley, R. E. A. C., "A remarkable series of orthogonal functions", Proc. London Math. Soc., vol. 34, pp. 241-279, 1932.

[3] Levy, P., "Sur une generalisation de fonctions orthogonales de M. Rademacher", Comment. Math. Helv., vol. 16, pp. 146-152, 1944.

[4] Fine, N. J., "On the Walsh functions", Trans. Amer. Math. Soc., vol. 65, pp. 372-414, 1949.

[5] Fine, N. J., "The generalized Walsh functions", Trans. Amer. Math. Soc., vol. 69, pp. 66-77, 1950.

[6] Vilenkin, N. Y., "On a class of complete orthogonal systems", Izv. Akad. Nauk Ser. Math., vol. 11, pp. 363-400, 1947 (in Russian).

[7] Vilenkin, N. Y., "On the theory of Fourier integrals on topological groups", Mat. Sbornik (N. S.), vol. 30 (72), pp. 233-244, 1952 (in Russian).

[8] Chrestenson, H. E., "A class of generalized Walsh functions", Pacific J. Math., vol. 5, pp. 17-31, 1955.

[9] Watari, Ch., "On generalized Walsh-Fourier series", Tohoku Math. J. (2), vol. 10, pp. 211-241, 1958.

[10] Weiss, P., "Zusammenhang von Walsh-Fourierreihen mit Polynomen", Monatshefte für Mathematik, vol. 71, pp. 165-179, 1967.

[11] Liedl, R., "Eine algebraische Herleitung und eine Verallgemeinerung des Satzes von FINE über Gruppen von orthonormalen Funktionen und eine Beschreibung der vielfalterhaltenden Transformationen des Intervalls [0, 1] auf sich selbst, "Monatshefte für Mathematik, vol. 72, pp. 45-80, 1968.

[12] Harmuth, H. F., "Transmission of information by orthogonal functions", Springer-Verlag, New York, 1969.

[13] Harmuth, H. F., "A generalized concept of frequency and some applications", IEEE Trans. Information Theory, vol. IT-14, No. 3, 375-382, May, 1968.

[14] Selfridge, R. G., "Generalized Walsh transform", Pacific J. Math., vol. 5, pp. 451-480, 1955.

[15] Pichler, F., "Das system der sal-und cal-Funktionen als Erweiterung des Systems der Walsh-Funktionen und die Theorie der sal-und cal-Fouriertransformation", Dissertation, Philosophische Fakultät der Universität Innsbruck, 1967.

[16] Falb, B. L. and M. I. Friedmann, "A generalized transform theory for causal operators", Siam J. Control, vol. 7, pp. 452-471, 1969.

[17] Weiser, F. E., "Walsh function analysis of instantaneous nonlinear stochastic processes", Dissertation, Polytechnic Institute of Brooklyn, June, 1964.

[18] Pichler, F., "Walsh-Fourier Synthese optimaler Filter" to appear in A. E. Ü., 1970.

[19] Wiener, N. and R. E. A. C. Paley, "Analytic properties of infinite abelian groups", Verhandlungen des Internationalen Mathematiker-Kongresses Zürich, 1932, II. Band, p. 95.

[20] Pichler, F., "Synthese linearer periodisch zeitvariabler Filter mit vorgeschriebenem Sequenzverhalten", A. E. Ü., vol. 22, pp. 150-161, 1968.

[21] Harmuth, H. F., "Grundzüge einer Filtertheorie für die Mäander-funktionen $A_n(\theta)$," A. E. Ü., vol. 18, pp. 544-554, 1964.

[22] Gibbs, J. E., and Margaret J. Millard, "Some methods of solution of linear ordinary logical differential equations", National Physical Laboratory, DES Report No. 2, December, 1969.

[23] Zadeh, L. A., and Ch. A. Desoer, "Linear System Theory, The State Space Approach", McGraw-Hill Book Company, New York, 1963.

A-1

APPENDIX A

(a)

t

τ

$t \oplus \tau$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

(b)

t

τ

τ^*	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	1	-1	1	-1	1	-1	1	-1
2	2	2	-2	-2	2	2	-2	-2
3	3	1	-1	-3	3	1	-1	-3
4	4	4	4	4	-4	-4	-4	-4
5	5	3	5	3	-3	-5	-3	-5
6	6	6	2	2	-2	-2	-6	-6
7	7	5	3	1	-1	-3	-5	-7

Figure 1: (a) Addition modulo 2 table for the integers between 0 and 7.

(b) Time shift τ^*, given by $t + \tau^* = t \oplus \tau$ for $t, \tau = 0, 1, \ldots, 7$.

A-2

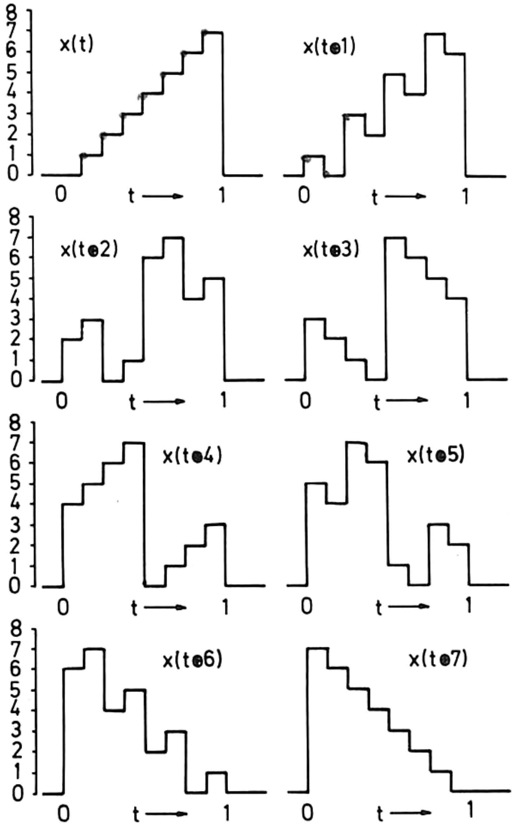


Figure 2: Graph of a stepfunction x given by $x(t) = [8t]$ for $t \in [0,1)$ and $x(t) = 0$ for $t \notin [0,1)$ and its dyadic translations x_λ, $\lambda = 1, 2, \ldots, 7$.

A - 3

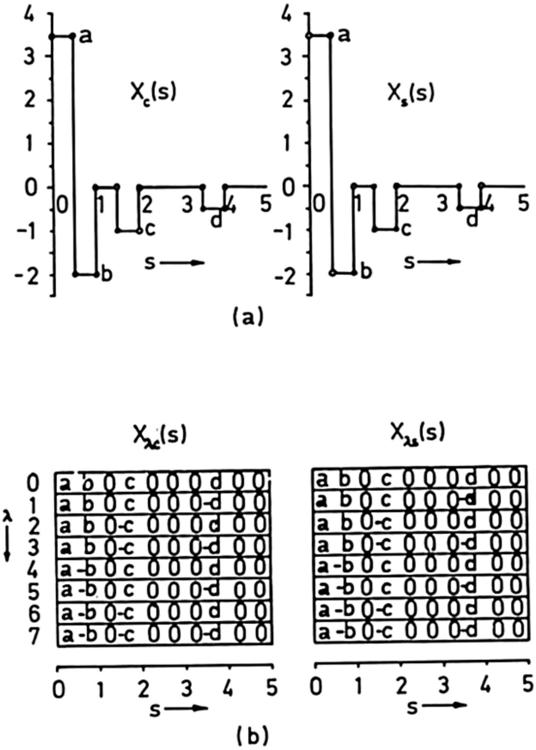


Figure 3: (a) Walsh-Fourier transforms X_c and X_s of the function x given in figure 2

(b) table for the values of the Walsh-Fourier transforms $X_{\lambda c}$ $X_{\lambda s}$ of the functions x_λ of figure 2.

Notice that $X_{\lambda c}^2(s) = X_c^2(s)$ and $X_{\lambda s}^2(s) = X_s^2(s)$ for all λ.

A - 4

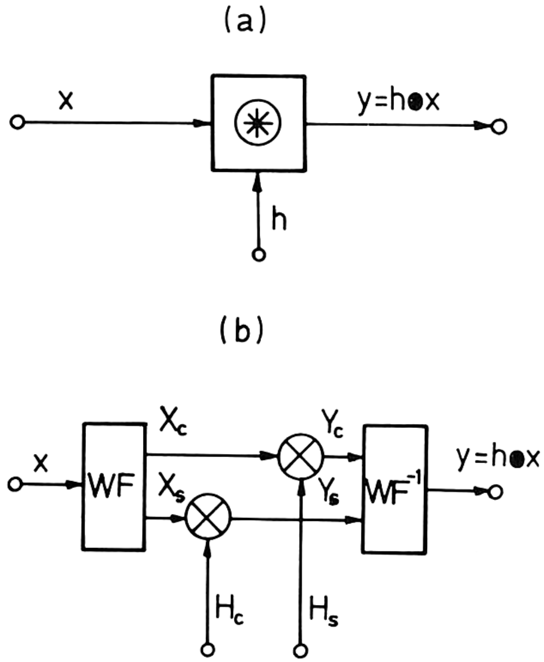


Figure 4: (a) Block-diagram for a relation defined by dyadic convolution. x inputsignal, h inpulse response at time $t = 0$, y outputsignal.

(b) Dyadic convolution using Walsh-Fourier transformation and dyadic convolution theorem.

A-5

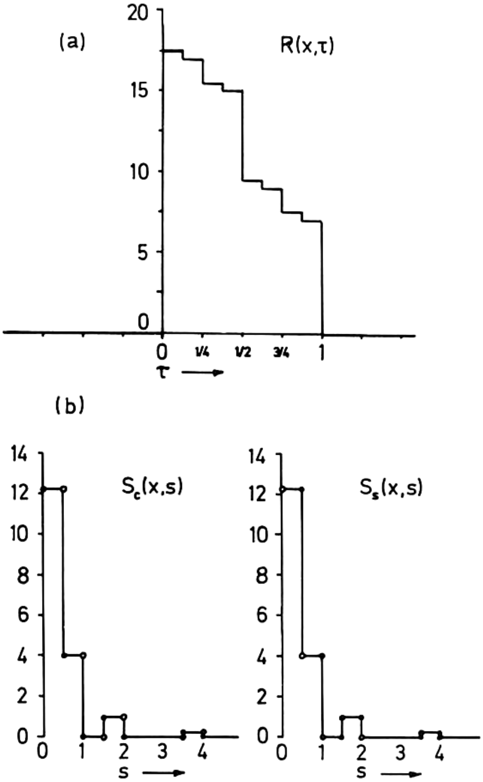


Figure 5: (a) Dyadic autocorrelation function $R(x, \cdot)$ of the functions x_λ of figure 2.

(b) Walsh-Fourier transforms $S_c(x, \cdot)$ and $S_s(x, \cdot)$ of $R(x, \cdot)$

A comparison with $X_{\lambda c}$ and $X_{\lambda s}$ of figure 3 shows, that we have $S_c(x, s) = X_{\lambda c}^2(s)$ and $S_s(x, s) = X_{\lambda s}^2(s)$ for all $\lambda = 0, 1, \ldots, 7$ and all s. We call therefore the functions $S_c(x, \cdot)$ and $S_s(x, \cdot)$ the sequency power spectra of the functions x_λ.

A-6

(a)

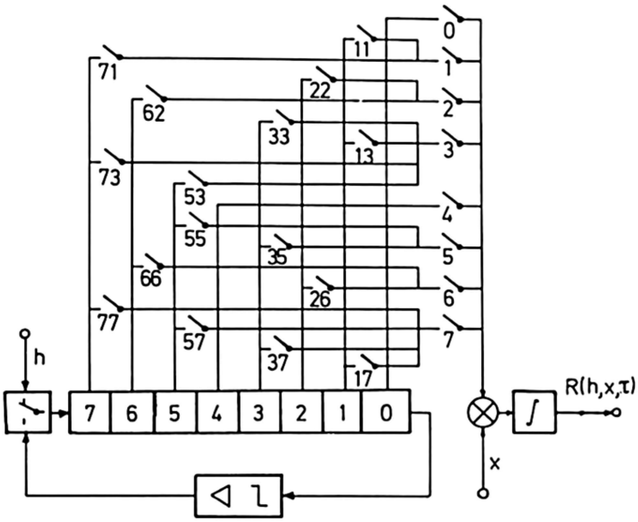


Figure 6: Principles of dyadic correlators.

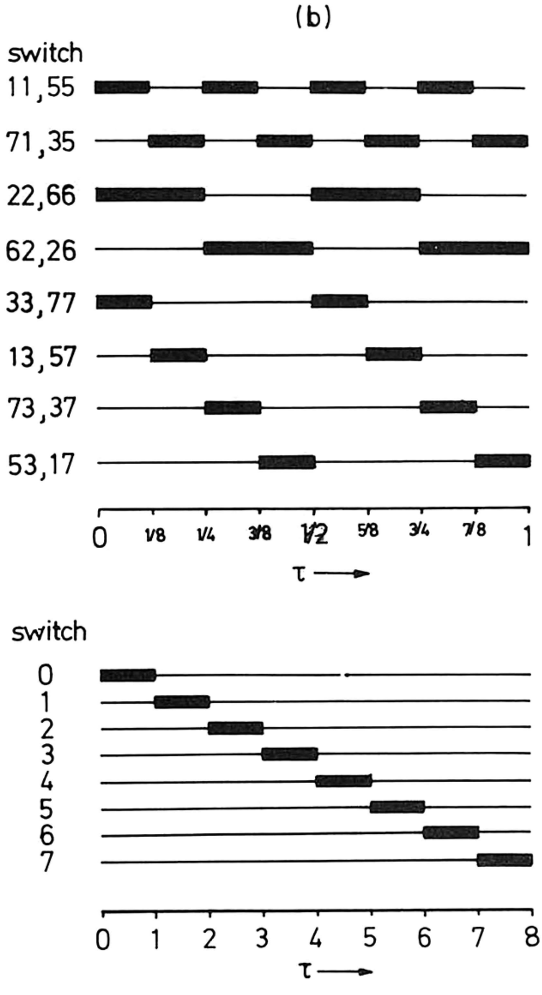
(a) Dyadic correlator using mod 8 shift register for a stepfunction h and a signal x. The switches perform the values $h(t \oplus \tau)$, $t \in [0,1)$ and $\tau = 0,1,\ldots,7$. The integrator computes the values

$$R(h,x,\tau) = \int_0^1 h(t \oplus \tau)x(t)dt$$

where $\tau = 0,1,\ldots,7$.

This principle of a dyadic correlator seems to be interesting for the design of dyadic invariant systems (sequency filters) with prescribed impulse response h.

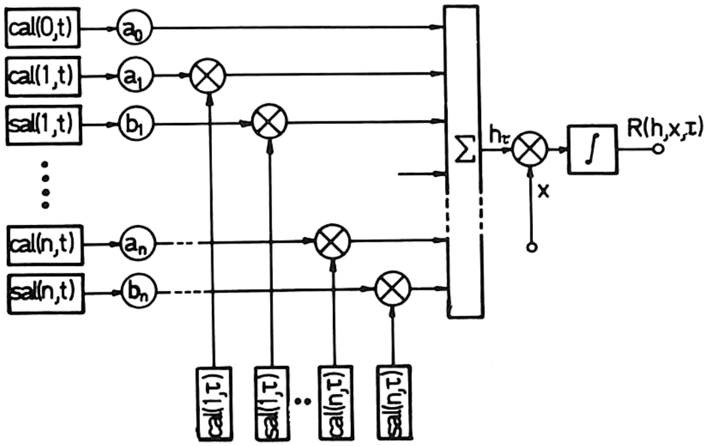
A-7



(b) Time table for the switches of the dyadic correlator of figure 6(a).

A-8

(c)



(c) Dyadic correlator using the Walsh-Fourier representation of h. The signal h is assumed to be given by

$$h(t) = \sum_{i=0}^{n} a_i \mathrm{cal}(i, t) + \sum_{i=1}^{n} b_i \mathrm{sal}(i, t)$$

With the multiplication theorem for Walsh functions we get

$$h_\tau(t) = h(t \oplus \tau) = \sum_{i=0}^{n} a_i \mathrm{cal}(i, t)\mathrm{cal}(i, \tau) + \sum_{i=1}^{n} b_i \mathrm{sal}(i, t)\mathrm{sal}(i, \tau)$$

This signal appears as output of the summator. Multiplication with x and integration of the product $h_\tau x$ gives us the dyadic correlation functions $R(h, x, \cdot)$.

A-9

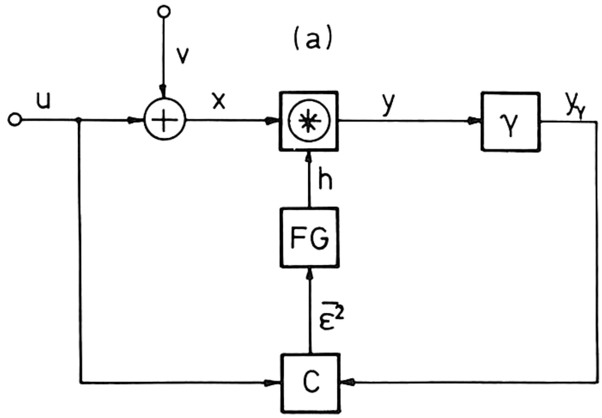


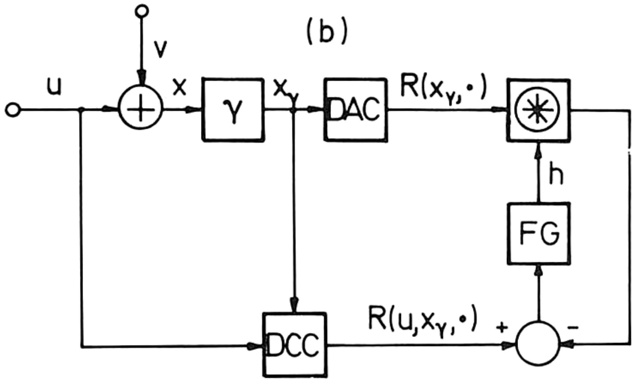
Figure 7: Principles to determine the impulse response h of the optimal linear dyadic invariant system.

(a) minimization of $\overline{\varepsilon^2}$. γ denotes a constant delayor with delay γ. For $\gamma = 0$ we have the case of an optimal filter.

FG function generator which is controlled by $\overline{\varepsilon^2}$.

C correlator performing $\overline{\varepsilon^2}$.

A-10



(b) impulse response h of the optimal system as solution of the generalized Wiener-Hopf integral equation

$$R(u, x_\gamma, \tau) - \int_0^\infty R(x_\gamma, \tau \oplus \lambda) h(\lambda) d\lambda = 0$$

for all $\tau > 0$.

DAC dyadic autocorrelator

DCC dyadic cross correlator

FG function generator

A-11

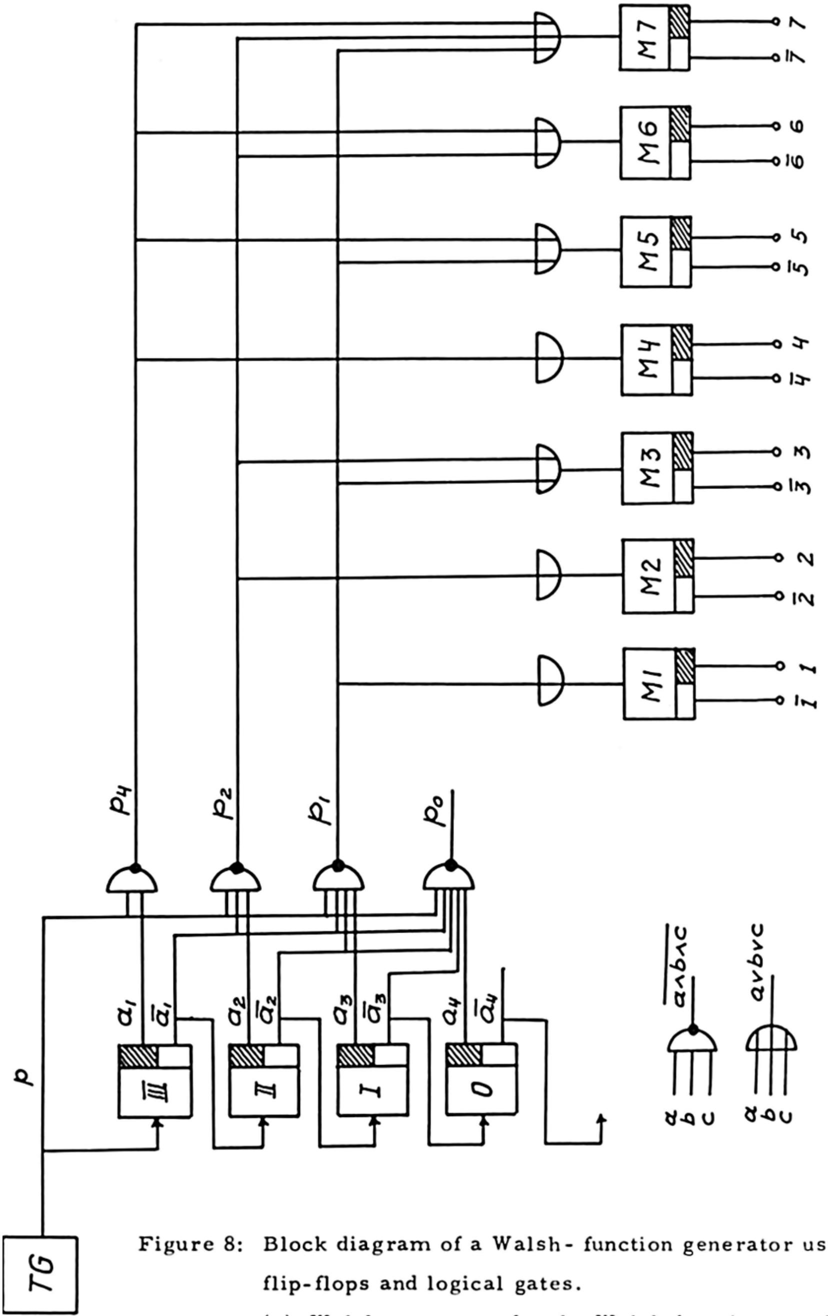
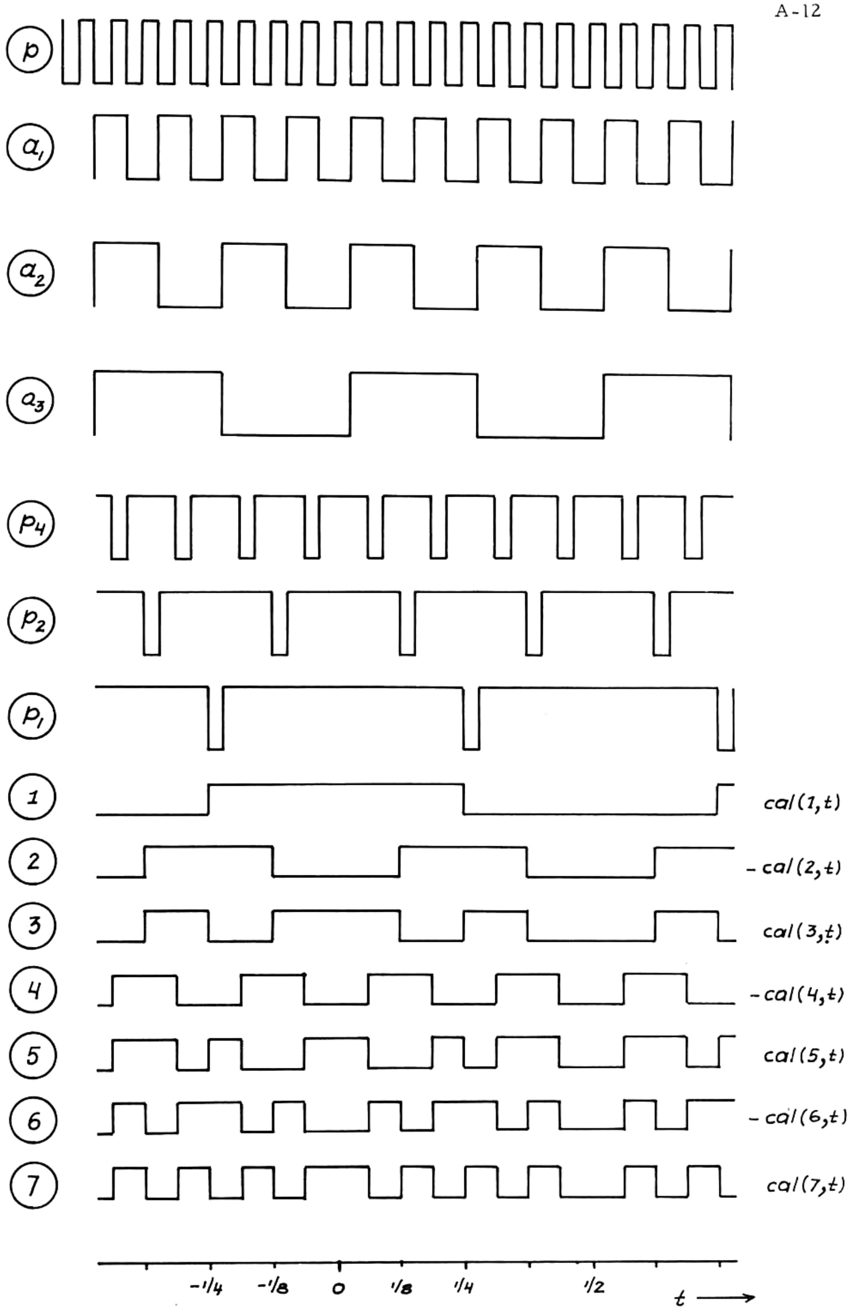


Figure 8: Block diagram of a Walsh- function generator using flip-flops and logical gates.

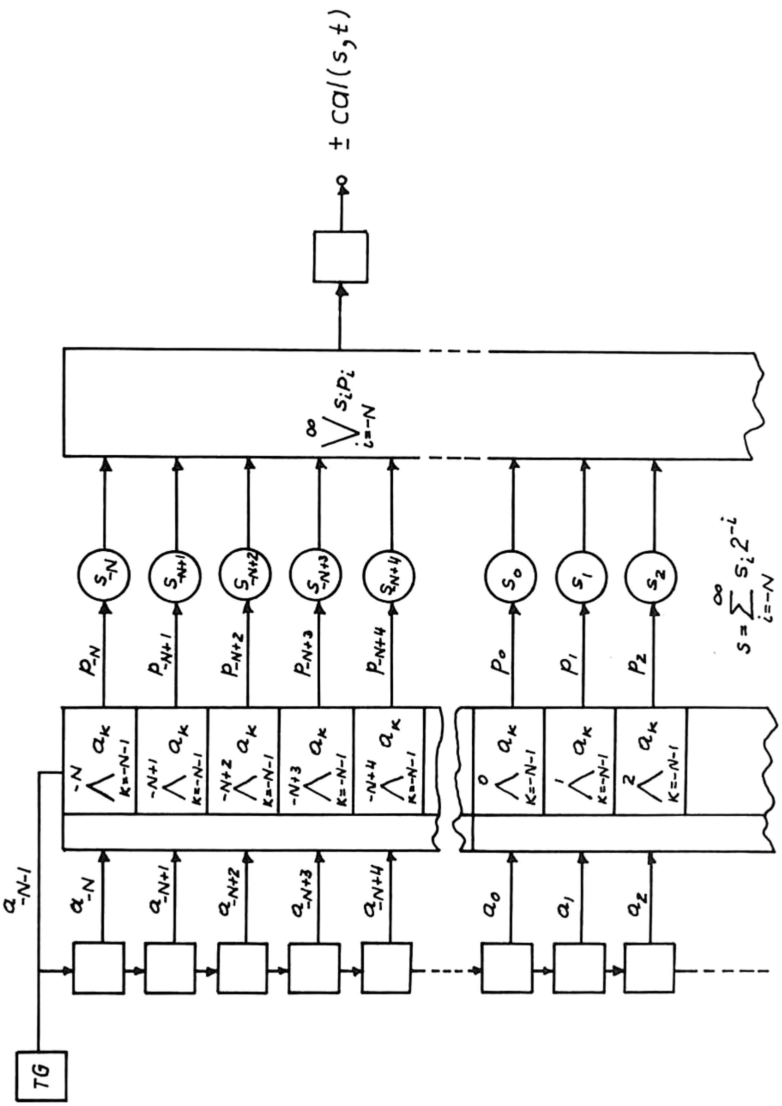
(a) Walsh generator for the Walsh-function ± cal(i, t), i = 1, . . . , 7.

A-12



(b) pulse diagram to the Walsh generator of fig. 8 (a).

A-13



(c) principles of the generation of a arbitrary Walsh function cal(s, t), s > 0.

APPENDIX B

Sampling Theorem with Respect to Walsh-Fourier Analysis

1. Introduction

In the following we are concerned with a generalization of the sampling theorem for band-limited signals. The origin of this classical theorem of communications can hardly be traced. In the mathematical literature it, or some of its analogues, is connected to several authors (e.g. Cauchy [1], Whittaker [2]). The same situation seems to be in the field of communications (e.g. Kotelnikov [3], Shannon [4], Raabe [5]).

Here we shall deal with a formulation of a sampling theorem for the case of representing signals as superpositions of Walsh functions. We shall use results of a paper by Kluvànec [6], in which the sampling theorem has been formulated in terms of the theory of abstract harmonic analysis. We specialize the case of an arbitrary locally compact abelian group to the case of the dyadic group of Fine [7] and get the sampling theorem in terms of Walsh-Fourier analysis. It turns out, that this theorem is a very trivial one if one pays attention to the fact that a function, band limited in the sense of Walsh-harmonic analysis, is equal almost everywhere to a stepfunction. A slight modification of this theorem, depending on the use of the Walsh functions sal(s, t) and cal(s, t), has been presented in a former paper of this author [8].

B-2

2. Sampling Theorem in Abstract Harmonic Analysis

We follow Kluvànek [6]. Let G be a locally compact abelian group (written additively) and $\hat{G}$ the dual group. The value of a character $y \in G$ at a point $x \in G$ will be written as usual as (x, y).

Suppose H be a discrete subgroup of G with discrete annihilator Λ given by $\Lambda = \{ y \in \hat{G}: \ (x, y) = 1 \text{ for all } x \in H \}$. Let $[y]$ denote the coset of Λ which contains the point $y \in \hat{G}$; i.e. $[y] = y + \Lambda$. Let further the set Ω be defined as a measurable subset of $\hat{G}$, $\Omega \subset \hat{G}$, which contains exactly one point from every coset $[y]$ of Λ, i.e. $\Omega \cap [y]$ is a singleton for all $y \in \hat{G}$.

The Haar measure on G is denoted by m, that of $\hat{G}$ by $\hat{m}$. The Haar measure $[\hat{m}]$ of the factorgroup $\hat{G}/\Lambda$ should be normalized, so that $[\hat{m}](\hat{G}/\Lambda) = 1$. The Haar measure on H and Λ respectively have at each point $x \in H$ and $y \in \Lambda$ respectively the value 1. Then $\hat{m}$ can be normalized, so that

$$\int_{\hat{G}} \hat{f}(y)\, d\hat{m}(y) = \int_{\hat{G}/\Lambda} \sum_{z \in \Lambda} \hat{f}(y+z)\, d[\hat{m}]([y]) \tag{1}$$

holds for every integrable function $\hat{f}$ on $\hat{G}$. Finally let the Haar measure m on G be adjusted so that the inversion formulas for Fourier transforms holds, i.e. by the relations

$$\hat{f}(y) = \int_G (-x, y) f(x)\, dm(x) \tag{2}$$

and

$$f(x) = \int_{\hat{G}} (x, y) \hat{f}(y)\, d\hat{m}(y) \tag{3}$$

Let further the function φ be defined by

$$\varphi(x) = \int_\Omega (x, y) d\hat{m}(y) \tag{4}$$

Due to Kluvanek [6] we have the following lemma and theorem:

Lemma: The function φ is defined for all $x \in G$. It is continuous, positive-definite and belongs to $L^2(G)$. Its norm $||\varphi||$ in $L^2(G)$ is equal 1 and $\varphi(0) = 1$. For all $z \in H$ with $z \neq 0$ we have $\varphi(z) = 0$ and

$$\int_G \varphi(x)\overline{\varphi(x-z)}\, dm(x) = 0 \tag{5}$$

Sampling theorem: Suppose $f \in L^2(G)$ and $\hat{f}(y) = 0$ for almost all $y \notin \Omega$. Then f is equal almost everywhere to a continuous function. If f itself is continuous then

$$f(x) = \sum_{z \in H} f(z)\varphi(x-z) \tag{6}$$

uniformly on G and in the sense of the convergence in $L^2(G)$. Furthermore

$$||f||^2 = \sum_{z \in H} |f(z)|^2 \tag{7}$$

If $G = T = (-\infty, \infty)$, the set of real numbers, and H is given by $H = \{mT:\ m = 0, \pm 1, + 2, \ldots\}$ where $T = 1/2f_o$, then we have $\Omega = (-2\pi f_o,\ 2\pi f_o)$ and φ is given by

$$\varphi(x) = \frac{\sin 2\pi f_o(x - mT)}{2\pi f_o(x - mT)} \tag{8}$$

and formula (6) becomes the classical form

$$f(x) = \sum_{m=-\infty}^{\infty} f(mT)\ \frac{\sin 2\pi f_o(x - mT)}{2\pi f_o(x - mT)} \tag{9}$$

3. Sampling Theorem in Dyadic Harmonic Analysis

Our intention is now, to formulate the sampling theorem of above with respect to the case that $G = \mathfrak{F}$, the dyadic group of Fine [7]. According to Fine the dyadic group $\mathfrak{F}$ is given by the set of infinite sequences $\bar{x}$ of the form

$$\bar{x} = (x_i) = (\ldots 00x_{-M}x_{-M+1}\ldots x_0x_1x_2\ldots)$$

the components x_i being 0 or 1 and each $\bar{x} \epsilon \mathfrak{F}$ being 0-periodic to the left side. It is helpful to identify the set $\mathfrak{F}$ with the set of binary representations of nonnegative real numbers. The composition $\oplus$ in $\mathfrak{F}$ is defined via addition modulo 2 of the components. $(\mathfrak{F}, \oplus)$ is then obviously an abelian group, each $x \epsilon \mathfrak{F}$ having order 2; $\bar{x} \oplus \bar{x} = 0$. A topology can be found, such that $(\mathfrak{F}, \oplus)$ becomes a locally compact topological group [7], [9]. The character group $\hat{\mathfrak{F}}$ of $\mathfrak{F}$ is algebraically isomorph to $\mathfrak{F}$. So each character $\bar{y} \epsilon \hat{\mathfrak{F}}$ can be represented by a sequence of the form

$$\bar{y} = (y_i) = (\ldots 00y_{-N}y_{-N+1}\ldots y_0y_1\ldots)$$

The value $(\bar{x}, \bar{y})$ of a character $\bar{y} \epsilon \hat{\mathfrak{F}}$ at a point $\bar{x}$ is given by

$$(\bar{x}, \bar{y}) = \exp \pi i \sum_{i+k=1} y_i x_k \tag{10}$$

From (10) we can see that the characters $\bar{y} \epsilon \hat{\mathfrak{F}}$ are real valued functions which maps $\mathfrak{F}$ onto the set $\{+1, -1\}$. The Haar measure on $\mathfrak{F}$ and $\hat{\mathfrak{F}}$ respectively should be normalized such that the subgroup $\mathfrak{D}$ and $\hat{\mathfrak{D}}$

consisting of all sequences $\bar{x} = (x_i)$ and $\bar{y} = (y_i)$ respectively, where $x_i = y_i = 0$ for all $i \leq 0$ have the measure 1. We can now establish the sampling theorem for the dyadic group $\mathfrak{F}$. The subgroup H should be given by $H = \{\bar{x} \varepsilon G: x_i = 0 \text{ for all } i > k\}$, where k is an integer. Then Λ is given by the subgroup $\Lambda = \{\bar{y} \varepsilon \hat{G}: y_i = 0 \text{ for all } i > -k\}$. The set Ω consists of all sequences $\bar{y} \varepsilon \hat{\mathfrak{F}}$ with $y_i = 0$ for all $i \leq -k$. The functions φ we define (4) by

$$\varphi(\bar{x}) = 2^{-k} \int_{\Omega} (\bar{x}, \bar{y})\, dm(\bar{y}) \tag{11}$$

The difference of (11) to (4) comes from the fact, that we normalized the Haar measure m so that $m(\Omega) = 2^k$ rather than 1. Integration gives us

$$\varphi(\bar{x}) = \chi_{1/\Omega}(\bar{x}) \tag{12}$$

where $\chi_{1/\Omega}$ denotes the characteristic function of the set $1/\Omega$ given by $1/\Omega = \{\bar{x}\varepsilon \mid x_i = 0 \text{ for all } i \leq k\}$. Therefore formula (6) of the sampling theorem gets the form

$$f(\bar{x}) = \sum_{\bar{z} \varepsilon H} f(\bar{z})\, \chi_{1/\Omega}(\bar{x} \oplus \bar{z}) \tag{13}$$

4. Sampling Theorem in Walsh-Fourier Analysis

Walsh-Fourier analysis is a Fourier analysis for functions defined on the nonnegative real line $[0, \infty)$. It can be derived from dyadic Fourier analysis in the following way: The map $\lambda: \mathfrak{F} \rightarrow [0, \infty)$ is according to Fine [7] defined as the map, which takes a point $\bar{x} = (x_i) \varepsilon \mathfrak{F}$

into a point $x \in [0, \infty)$ given by

$$x = \sum_i x_i 2^{-i} \tag{14}$$

For $x \in [0, \infty)$ the inverse mapping μ is defined by (11) choosing the finite dyadic representation if x is a dyadic rational. The map μ neglects only a set $\mathfrak{E}$, consisting of sequences which are 1-periodic to the right side; μ: $[0, \infty) \to \mathfrak{F} \setminus \mathfrak{E}$. The set $\mathfrak{E}$ has Haar measure zero. So we have

$$\lambda(\mu(x) = x \qquad \text{for all } x \in [0, \infty) \tag{15}$$

and

$$\mu(\lambda(\overline{x})) = \overline{x} \qquad \text{for all } \overline{x} \in \mathfrak{F} \setminus \mathfrak{E} \tag{16}$$

he mappings $\hat{\lambda}$: $\hat{\mathfrak{F}} \to [0, \infty)$ and $\hat{\mu}$: $[0, \infty) \to \hat{\mathfrak{F}} \setminus \hat{\mathfrak{E}}$ where $\hat{\mathfrak{E}} \cong \mathfrak{E}$, are efined in a analogous way as the maps λ and μ.

The Walsh functions $\psi(y, \cdot)$: $[0, \infty) \to \{+1, -1\}$ are defined as the functions given for all $x, y \in [0, \infty)$ by

$$\psi(y, x) = (\mu(x), \hat{\mu}(y)) \tag{17}$$

and the transforms formula given by (2) and (3) becomes the form

$$\hat{f}(y) = \int_0^\infty \psi(y, x) f(x)\, dx \tag{18}$$

and

$$f(x) = \int_0^\infty \psi(y, x) \hat{f}(y)\, dy \tag{19}$$

where $f \in L^2[0, \infty)$. This comes from the fact that the map λ^*: $P(\mathfrak{F}) \to P([0, \infty))$, which maps the power set $P(\mathfrak{F})$ of $\mathfrak{F}$ into the power set $P([0, \infty))$ of $[0, \infty)$ and which is induced by λ, is a Haar-Lebesgue measure preserving map. The same is true for $\hat{\lambda}^*$. From that it turns out, that the algebraic and measure theoretic results of the sampling theorem formulated above for $G = \mathfrak{F}$ are invariant against the map λ^* and $\hat{\lambda}^*$. However, the topological results concerning continuity etc. are altered. We have so

$$\lambda^*(\mathfrak{F}) = [0, \infty), \quad \hat{\lambda}^*(\hat{\mathfrak{F}}) = [0, \infty),$$

$$\lambda^*(H) = \left\{m2^{-k}: \; m = 0, 1, 2, \ldots\right\}$$

$$\hat{\lambda}^*(\Lambda) = \left\{n2^{k} : n = 0, 1, 2, \ldots\right\}$$

$$\hat{\lambda}^*(\Omega) = [0, 2^{k}), \; \hat{\lambda}^*(1/\Omega) = [0, 2^{-k})$$

$$dm(\overline{x}) = dx$$

$$dm(\overline{y}) = dy$$

The function $\lambda^* \varphi$: $[0, \infty) \to R$ given for all $x \epsilon [0, \infty)$ by

$$\lambda^*\varphi(x) = 2^{-k} \int_0^{2^k} \psi(y, x)dy \tag{20}$$

is the characteristic function $\chi_{[0, 2^{-k})}$ of the interval $\lambda^*(1/\Omega) = [0, 2^{-k})$. From above the sampling theorem becomes for Walsh-Fourier analysis the form:

<u>Theorem</u>: Suppose f is a real valued function of the space $L^2[0, \infty)$ and $\hat{f}(y) = 0$ for almost all $y \notin [0, 2^k)$. Then f is equal almost everywhere to a stepfunction, continuous from the left which jumps only at points x of the form $x = m2^{-k}$, $m = 0, 1, 2, \ldots$. If f itself is of that kind then

$$f(x) = \sum_{m=0}^{\infty} f(m2^{-k}) \chi_{[0, 2^{-k})} (x - m2^{-k}) \tag{21}$$

uniformly on $[0, \infty)$ and in the sense of the convergence in $L^2[0, \infty)$.

Furthermore

$$2^k ||f||^2 = \sum_{m=0}^{\infty} f^2(m2^{-k}) \tag{22}$$

Knowing that f is equal almost everywhere to a stepfunction of that kind described above, this theorem is a trivial one. It is obvious that such a stepfunction can be generated by characteristic functions as shown in equation (21).

B-9

REFERENCES

[1] Cauchy A., "Memoire sur diverses formules d'analyse", C. R. Acad. Sci. Paris 12 (1841), 283-298.

[2] Whittaker E. T., "On the functions which are represented by the expansions of the interpolation-theory", Proc. Roy. Soc. Edinburgh. Section A. 35 (1915), 181-194.

[3] Kotelnikov, V. A., "The Theory of Optimum Noise Immunity", Doctoral dissertation, 1947, Translation into English by R. A. Silverman, McGraw-Hill, New York, 1959.

[4] Shannon C. E., "Communication in the Presence of Noise", Proc. IRE 37, (1949), 10-21.

] Raabe, H., Untersuchungen an der wechselzeitigen Mehrfachue-bertragung (Multiplexübertragung), Elektrische Nachrichten-Technik 16 (1939), 213-228.

Kluvánek, I., "Sampling Theorem in Abstract Harmonic Analysis", Matematicko-fyzikalny Casopis. Sloven. Akad. Vied., 15 (1965), 43-48.

[7] Fine, N. J., "The generalized Walsh functions", Trans. Amer. Math. Soc. 69, (1950), 66-77.

[8] Pichler, F., "Synthese linearer periodisch zeitvariabler Filter mit vorgeschriebenem Sequenzverhalten", Archiv der Elektr. Übertr. 22 (1968), 150-161.

B-10

[9] Liedl, R., "Eine algebraische Herleitung und eine Verallgemeinerung des Satzes von Fine über Gruppen von orthonormalen Funktionen und eine Beschreibung der vielfalterhaltenden Transformationen des Intervalls [0,1] auf sich selbst", Monatshefte fuer Mathematik, 72, (1968), 45-60.

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