

Chapter 2

Gibbs-Butzer Calculus and Pseudo-differential Operators on Local Fields

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2.1 Introduction

The main aims of this chapter are to generalize the Gibbs-Butzer calculus to that on the local fields, and to establish harmonic analysis frames on local fields, including frames of Fourier analysis on both function case and distribution case; then to establish space theory, as well as to establish fractal analysis and partial differential equations on fractals in the Gibbs-Butzer calculus sense.

After recall some definitions of local fields, we concentrate our mind to generalize Gibbs- Butzer calculus (simply, G-B calculus) on local fields. Firstly, we define G-B type derivative and integrals of functions defined on local fields by means of the pseudo-differential operator T_σ ; then generalize G-B calculus to the Schwartz distributions on local fields. The properties of G-B type calculus on local fields are studied, and the convolution kernel K_α of T_σ is determined, this kernel will play important role in the fractal PDD which is a quite new research topic.

Secondly, we deal with function space theory on local fields, such as, define Hölder type spaces by G-B type calculus; B-type space and F-type space of Triebel sense, Lipschitz classes, etc. Then an essential property of Hölder spaces is shown: It is the space in which the G-B type differentiable functions and distributions live. This property is also shown the essence of G-B calculus.

Some comparisons between structures of Euclidian spaces and local fields, as two underlying spaces, are given, such as, operation structures, topological structures, geometric structures, and analytic structures, as well as function constructions on R and K , so that one may recognize the necessary to develop G-B calculus, and may know some essential and important properties of this new calculus. Based upon the

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comparisons we suggest four principles for establishing a new calculus on other topological fields and groups.

Finally, some application examples of G-B type calculus are given, including apply to the approximation theory, partial differential equations, fractals, as well as to the clinical medical sciences and theoretical medical sciences.

2.2 Notations

Let $p \geq 2$ be a prime number. A local field, denoted by $K \equiv (K_p, \oplus, \otimes, |\square|)$ with the addition $\oplus$, multiplication $\otimes$, and non-archimedean norm $|\square|$, is a locally compact, non-trivial, totally disconnected, zero dimensional and complete topological field; the non-archimedean norm $|\cdot| : K \rightarrow R$ satisfying:

- (i) $|x| \geq 0$; $|x| = 0, \leftrightarrow x = 0$;
- (ii) $|x \otimes y| = |x| |y|$;
- (iii) $|x \oplus y| = \max\{|x|, |y|\}$ for $x, y \in K$.

It is also an ultra-metric space with an ultra-distance $d(x, y) = |x - y|$ on K .

Thus $K = \bigcup_{j=-\infty}^{\infty} B^j$ and $\{0\} = \bigcap_{j=-\infty}^{\infty} B^j$, where $B^j = \{x \in K : |x| \leq p^{-j}\}$, $j \in Z$, and $\{B^j\}_{j=-\infty}^{\infty}$ is a strictly decreasing, open and closed, and compact set sequence.

Moreover, there exists a prime element $\beta \in K$ with $|\beta| = p^{-1}$, such that each $x \in K$ has a unique expression as

$$x = x_{-s}\beta^{-s} + x_{-s+1}\beta^{-s+1} + \cdots + x_{-1}\beta^{-1} + x_0\beta^0 + x_1\beta^1 + x_1\beta^{-2} + \cdots,$$

with $s \in P = \{0, 1, 2, \dots\}$, $x_j \in \{0, 1, \dots, p-1\}$, $j = -s, -s+1, \dots$, and $x \in B^{-s}$ with $|x| = p^s$ if $x_{-s} \neq 0$. We know that the addition operator $\oplus$ has two cases:

$$\begin{aligned} x \oplus y &= (x_j \oplus y_j), & x_j \oplus y_j &= x_j + y_j \pmod{p, \text{no carry}}, \\ x \oplus y &= (x_j \oplus y_j), & x_j \oplus y_j &= x_j + y_j \pmod{p, \text{carry from left to right}}, \end{aligned}$$

Then K is called a *p-series field* for the first case, and a *p-adic number field* for the second.

We may consider the algebraic extensions of K , that is $K_q \equiv (K_q, \oplus, \otimes, |\square|)$ with number $q = p^c$, $c \in N$. However, we only concentrate self-mind to the study on $K \equiv (K_p, \oplus, \otimes, |\square|)$, and mention that there are many interesting open problems on algebraic extensions K_q [15], [16].

Let μ be the Haar measure on the addition group $K^+ \equiv (K_p, \oplus)$ with $\mu(B^0) = 1$, and $\mu(B^j) = p^{-j}$, $j \in Z$. Let Γ be the character group of the additive group K^+ of K ; that is, $\xi : K^+ \rightarrow C$ is a complex function on K for $\xi \in \Gamma$, such that the action of $\xi \in \Gamma$ on $x \in K$, denoted by $\langle \xi, x \rangle$, satisfies $\langle \xi, x \rangle \in \{z \in C : |z| = 1\}$. The annihilator of B^j , denoted by $\Gamma_j = \{\xi \in \Gamma : \langle \xi, x \rangle = 1 \text{ for all } x \in B^j\}$, then we have $\Gamma = \bigcup_{j=-\infty}^{\infty} \Gamma_j$, and $\bigcap_{j=-\infty}^{\infty} \Gamma_j$ is the identity character $I \in \Gamma_j$, $\forall j \in Z$, that is,

$\{I\} = \bigcap_{j=-\infty}^{\infty} \Gamma_j$. In the sequel, we denote $\langle \xi, x \rangle$ by $\chi_\xi(x)$, $\xi \in \Gamma$, $x \in K$, thus, there exists one-one correspondence $\chi_\xi \leftrightarrow \xi$, and $\chi_\xi(x) \in \{z \in C : |z| = 1\}$.

The set sequence $\{\Gamma_j\}_{j=-\infty}^{\infty}$ is a strictly increasing, open and closed, compact set sequence in Γ . For the Haar measure μ of K^+ , let λ be the Haar measure on Γ , chosen such that $\lambda(\Gamma_0) = \mu(B^0) = 1$, thus $\lambda(\Gamma_j) = (\mu(B^j))^{-1} = p^j$, and $\Gamma_j = \{\xi \in \Gamma : |\xi| \leq p^j\}$, $j \in \mathbb{Z}$.

When we take $p = 2$, then the dyadic (2-aidc) series field $(K_2, \oplus, \otimes, |\square|)$ is familiar for us, and the character group Γ_2 of K_2 is the Walsh system $\Gamma_2 = \{wal_k(x) : x \in [0, 1]\}$. So that local fields and the character groups are generalized from the dyadic group and the Walsh system.

For some properties of K and Γ , B^j and Γ_j , we refer to [15], [16].

2.3 G-B type derivative and G-B type integrals

C.W. Onneweer defined fractional differentiations on local fields in earlier 80's of the last century [3], [4]. W.X. Zheng defined derivatives on local fields in 1985 by a series form [19], [20]; their excellent jobs are a part of the G-B calculus.

In this section, we define the G-B type derivative (sometimes called p -type derivative) and G-B type integrals by the so called pseudo-differential operators. These definitions are generalizations of G-B calculus for functions and Schwartz distributions, and also have lots of advantages to study and develop G-B calculus.

Firstly, we define test function class $S(K)$ and the symbol class $S_{\rho, \delta}^m(K \times \Gamma)$.

Definition 2.1 *The test function class $S(K)$ is defined as: Each function $\varphi \in S(K)$ has the form*

$$\varphi(x) = \sum_{j=1}^n c_j \Phi_{k_j}(x - h_j), \quad h_j \in K, k_j \in \mathbb{Z}, c_j \in C, n \in \mathbb{N}, \quad (2.1)$$

where Φ_{k_j} is the characteristic function of B^{k_j} , $\Phi_{k_j}(x) = \begin{cases} 1, & x \in B^{k_j}, \\ 0, & x \notin B^{k_j}. \end{cases}$

$S(K)$ is a topological linear space with certain topology and is dense in $L^r(K)$, $r \geq 1$.

The symbol class $S_{\rho, \delta}^m(K \times \Gamma)$ is defined as: Let $m \in \mathbb{R}$, $\rho, \delta \geq 0$. Each $\sigma(x, \xi) \in S_{\rho, \delta}^m(K \times \Gamma)$ satisfies

(i) There exists a constant $c > 0$ such that

$$|\sigma(x, \xi)| \leq c \langle \xi \rangle^m, \quad \xi \in \Gamma, \quad (2.2)$$

(ii) For any $(\mu, \nu) \in P \times P$, there exists a constant $c_{\mu, \nu} > 0$ such that

$$|\Delta_h^x \Delta_k^\xi \sigma(x, \xi)| \leq c_{\mu, \nu} |h|^\mu |k|^\nu \langle \xi \rangle^{m + \delta\mu - \rho\nu}, \quad (2.3)$$

where $h \in K, k \in \Gamma$, $\Delta_h^x \Delta_k^\xi \sigma(x, \xi)$ is the second order difference. Moreover,

$$|\Delta_h^x \sigma(x, \xi)| \leq c_\mu |h|^\mu \langle \xi \rangle^{m+\delta\mu}, \quad (2.4)$$

and

$$|\Delta_k^\xi \sigma(x, \xi)| \leq c_\nu |k|^\nu \langle \xi \rangle^{m-\rho\nu}, \quad |k| < \langle \xi \rangle, \quad (2.5)$$

Δ_h^x and Δ_k^ξ are the first order differences.

With certain semi-norm, $S_{\rho, \delta}^m(K \times \Gamma)$ is a Frechet space. An element $\sigma \in S_{\rho, \delta}^m(K \times \Gamma)$ is called a symbol.

Now we turn to define pseudo-differential operator for $f \in S(K)$.

Definition 2.2 The pseudo-differential operator T_σ with the symbol $\sigma \in S_{\rho, \delta}^m(K \times \Gamma)$ is defined by

$$T_\sigma f(x) = \int_\Gamma \left\{ \int_K \sigma(x, \xi) f(t) \overline{\chi}_\xi(t-x) dt \right\} dx \quad (2.6)$$

for $f \in S(K)$, where $\chi_\xi : K \rightarrow \mathbb{C}$ is chosen such that it is trivial on B^0 , but non-trivial on the other part of K [20].

The following lemma is obvious.

Lemma 2.1 For $m \in \mathbb{R}$ and $\rho, \delta \geq 0$, we have $\langle \xi \rangle^m \in S_{\rho, \delta}^m(K \times \Gamma)$.

By Lemma 2.1, p -type derivatives can be defined as follows.

Definition 2.3 Let $\sigma(x, \xi) = \langle \xi \rangle^m$, $m \geq 0$. If for a Haar measurable function f on K , the integral

$$T_{\langle \square \rangle^m} f(x) = \int_\Gamma \left\{ \int_K \langle \xi \rangle^m f(t) \overline{\chi}_\xi(t-x) dt \right\} d\xi, \quad (2.7)$$

exists at $x \in K$, then it is called an m -order point-wise G - B -type derivative of $f(x)$ at x , denoted by $f^{(m)} \equiv T_{\langle \square \rangle^m} f(x)$. Let

$$f_n(x) = \begin{cases} f(x), & |x| \leq p^n, \\ 0, & |x| > p^n, \end{cases} \quad n \in \mathbb{Z}. \quad (2.8)$$

If there exists $g \in L^r(K)$, $1 \leq r < +\infty$, such that

$$\|g(\square) - T_{\langle \square \rangle^m} f(\square)\|_{L^r(K)} \rightarrow 0, \quad n \rightarrow +\infty, \quad (2.9)$$

then g is called an m -order L^r -strong G - B -type derivative of f , denoted by $D^{(m)} f$.

If f has any order point-wise G-B type derivatives or any order L^r -strong G-B type derivatives, then it is called infinitely point-wise G-B type differentiable, or infinitely strong G-B type differentiable, respectively. Clearly, these definitions include the cases of integer order and fractional order derivatives for any $m \in [0, +\infty)$.

If we use the following form instead of that in (2.7),

$$T_{(\square)^{-m}}f(x) = \int_{\Gamma} \left\{ \int_K \langle \xi \rangle^{-m} f(t) \overline{\chi}_{\xi}(t-x) dt \right\} d\xi, \quad (2.10)$$

where $m \geq 0$, then it is called an m -order point-wise G-B type integral of f at x , denoted by $f_{(\cdot)^m}(x) \equiv T_{(\cdot)^{-m}}f(x)$. An m -order L^r -strong G-B type integral of f , denoted by $I_{(m)}f$, can be defined similarly.

Lemma 2.2 *If $\gamma > 0$, then there exists a constant $c > 0$ such that for any $x \in K$ with $|x| > p^{-n}$, we have $\int_{|\xi| \leq p^n} \frac{|\chi_x(\xi) - 1|}{|\xi|^{\gamma+1}} d\xi \geq c|x|^{\gamma}$.*

Lemma 2.3 *For any $\alpha \in P$, we have*

$$|\chi_k(x) - 1| \leq |x|^{\alpha} |k|^{\alpha}, \quad x \in K, \quad k \in \Gamma,$$

where $c > 0$ is a constant.

For the proofs of above two Lemmas, we refer to [10]. By these two lemmas, we get:

Theorem 2.1 *The Fourier transform $f^{\wedge}(\xi) = \int_K f(x) \overline{\chi}_{\xi}(x) dx$, $\xi \in \Gamma$, is a homeomorphism from $S(K)$ onto $S(\Gamma)$.*

Theorem 2.2 *The operator $T_{(\square)^m}$, $m \in R$, satisfies the following*

- (i) $|T_{(\square)^{-m}}\varphi(x)| \leq c_{\alpha}|x|^{-\alpha} \leq c_{\alpha}\langle x \rangle^{-\alpha}$, $\alpha \in P$, $x \in K$,
- (ii) $|\Delta_h^{\alpha} T_{(\square)^{-m}}\varphi(x)| \leq c_{\alpha\gamma}|h|^{\alpha}|x|^{-\gamma} = c_{\alpha\gamma}|h|^{\alpha}\langle x \rangle^{-\gamma}$, $\alpha, \gamma \in P$, $x, h \in K$.

The following theorem is a key one.

Theorem 2.3 *The operator $T_{(\square)^m}$, $m \in R$, is a homeomorphism from $S(K)$ onto $S(\Gamma)$.*

Note that, by the equivalence expression of (2.7)

$$T_{(\square)^m}f(x) = (\langle \square \rangle^m f^{\wedge}(\square))^{\vee}(x), \quad x \in K, \quad (2.11)$$

and by Theorem 2.1 for $f \in S(K)$, we can get its proof.

Theorem 2.4 *Every function $\varphi \in S(K)$ is infinitely G-B type differentiable both in point-wise and L^r -sense for all $1 \leq r < +\infty$; also infinitely G-B type integrable both in pointwise and L^r -sense. Moreover, it holds*

$$\varphi^{(m)}(x) = D^{(m)}\varphi(x), \quad \varphi_{(m)}(x) = I_{(m)}\varphi(x), \quad m \geq 0. \quad (2.12)$$

Proof. Without loss of generality, we only need to prove the theorem for $\varphi(x) = \tau_h \Phi_{B^s}(x)$, $x \in K$, $s \in \mathbb{Z}$, where $\Phi_{B^s}(x)$ is the characteristic function of B^s . By the formula

$$[\tau_h \Phi_{B^s}(\square)]^\wedge(\xi) = p^{-s} \overline{\chi}_h(\xi) \Phi_{\Gamma_s}(\xi), \quad (2.13)$$

where $\Phi_{\Gamma_s}(\xi)$ is the characteristic function of Γ_s , we have

$$T_{\langle \square \rangle^m}[\tau_h \Phi_{B^s}(\square)](x) = p^{-s} \int_{\Gamma} \langle \xi \rangle^m \Phi_{\Gamma_s}(\xi) \chi_{x-h}(\xi) d\xi.$$

By the formula

$$\int_{|\xi|=p^j} \chi_x(\xi) d\xi = \begin{cases} p^j(1-p^{-1}), & |\xi| \leq p^{-j}, \\ -p^{j-1}, & |\xi| = p^{-j+1}, \\ 0, & |\xi| > p^{-j+1}, \end{cases}$$

we have for $s \leq 0$ that

$$T_{\langle \square \rangle^m}[\tau_h \Phi_{B^s}(\square)](x) = \tau_h \Phi_{B^s}(x). \quad (2.14)$$

And for $s > 0$, by a delicate computing we get

$$T_{\langle \square \rangle^m}[\tau_h \Phi_{B^s}(\square)](x) = I_1 + I_2,$$

where

$$I_1 = \begin{cases} p^{-k}, & |x-h| \leq 1, \\ 0, & |x-h| > 1, \end{cases}$$

and

$$I_2 = \begin{cases} p^{-s} \cdot p^{m+1} (1-p^{-1}) \frac{p^{s(m+1)}-1}{p^{m+1}-1}, & x \in h+B^s, \\ p^{-s} \cdot p^{m+1} (1-p^{-1}) \frac{p^{(s-1)(m+1)}-1}{p^{m+1}-1} - p^{sm-1}, & x \in (h+B^{s-1}) \setminus (h+B^k), \\ \vdots & \vdots \\ p^{-s} \cdot p^{m+1} (1-p^{-1}) \frac{p^{2(m+1)}-1}{p^{m+1}-1} - p^{3m-(s-2)}, & x \in (h+B^2) \setminus (h+B^3), \\ p^{-s} \cdot p^{m+1} (1-p^{-1}) \frac{p^{(m+1)}-1}{p^{m+1}-1} - p^{2m-(s-1)}, & x \in (h+B^1) \setminus (h+B^2), \\ p^{m-s}, & x \in (h+B^0) \setminus (h+B^1), \\ 0, & x \notin (h+B^0). \end{cases}$$

Hence $T_{\langle \square \rangle^m}[\tau_h \Phi_{B^s}(\square)](x)$ takes value of a constant on each co-set of $h+B^s$, its support is $h+B^0$, so it belongs to $S(K)$. Moreover, since $S(K) \subset L^r(K)$, $1 \leq r < +\infty$, we conclude that each $\psi \in S(K)$ has m -order point-wise p -type and L^r -strong p -type derivatives and integrals for any $m \geq 0$. And it holds (2.12)

$$\Psi^{(m)}(x) = D^{(m)}\varphi(x), \quad \Psi_{\langle m \rangle}(x) = I_{\langle m \rangle}\varphi(x), \quad m \geq 0.$$

The proof is complete.

By Theorems 2.1 and 2.3, and the uniqueness theorem of Fourier transform, we may prove the following theorem.

Theorem 2.5 *If $f \in S(K)$, then for $m \geq 0$ holds*

$$(f^{(m)}(\square))^{\wedge}(\xi) = \langle \xi \rangle^m f^{\wedge}(\xi), \quad \xi \in \Gamma, \quad (2.15)$$

$$(f_{(m)}(\square))^{\wedge}(\xi) = \langle \xi \rangle^{-m} f^{\wedge}(\xi), \quad \xi \in \Gamma. \quad (2.16)$$

Similarly, the formulae holds for $D^{(m)}f(x)$ and $I_{(m)}f(x)$.

Theorem 2.4 has an important meaning: The operator $T_{(\square)^m}$, regarded as an operation, is closed in the space $S(K)$, so that we may generalize the concept of G-B type derivatives and integrals to that of the distribution space $S^*(K)$ of $S(K)$. With the $\mathcal{W}^*$ -topology, $S^*(K)$ is a topological space.

For $f \in S^*(K)$, we define the Fourier transform of f .

Definition 2.4 *For $f \in S^*(K)$, if there exists a distribution $g \in S^*(K)$, satisfying*

$$\langle g, \varphi^{\wedge} \rangle = \langle f, \varphi \rangle, \quad \forall \varphi \in S(K), \quad (2.17)$$

then g is called the Fourier transform of the distribution $f \in S^*(K)$, denoted by $g = f^{\varphi}$.

That is $\langle f^{\wedge}, \varphi^{\wedge} \rangle = \langle f, \varphi \rangle$ for each $\varphi \in S(K)$.

Theorem 2.6 *The Fourier transform is a homeomorphism from $S^*(K)$ onto $S^*(\Gamma)$.*

We give the following definition of the derivative of $f \in S^*(K)$.

Definition 2.5 *For $f \in S^*(K)$, $m \geq 0$, if there exists a distribution $g \in S^*(K)$ satisfying*

$$\langle g, \varphi \rangle = \langle f, \varphi^{(m)} \rangle, \quad \forall \varphi \in S(K), \quad (2.18)$$

then g is called the m -order G-B type derivative of the distribution $f \in S^*(K)$, denoted by $f^{(m)}$. That is $\langle f_{(m)}, \varphi \rangle = \langle f, \varphi_{(m)} \rangle$ for each $\varphi \in S(K)$.

Similarly, for $f \in S^*(K)$, $m \geq 0$, the m -order G-B type integral of the distribution f is defined as a distribution, denoted by $f_{(m)}$, satisfying

$$\langle f_{(m)}, \varphi \rangle = \langle f, \varphi_{(m)} \rangle, \quad \forall \varphi \in S(K). \quad (2.19)$$

That is $\langle f_{(m)}, \varphi \rangle = \langle f, \varphi_{(m)} \rangle$ for each $\varphi \in S(K)$.

The following theorem is a generalization of Theorem 2.5.

Theorem 2.7 *If $f \in S^*(K)$ then for $m \geq 0$ holds*

$$(f^{(m)})^{\wedge} = \langle \xi \rangle^m f^{\wedge}, \quad (f_{(m)})^{\wedge} = \langle \xi \rangle^{-m} f^{\wedge}, \quad (2.20)$$

in the distribution sense.

By Theorems 2.1 and 2.3, and the uniqueness theorem of Fourier transform, we may prove the following theorem.

Theorem 2.5 *If $f \in S(K)$, then for $m \geq 0$ holds*

$$(f^{(m)}(\square))^{\wedge}(\xi) = \langle \xi \rangle^m f^{\wedge}(\xi), \quad \xi \in \Gamma, \quad (2.15)$$

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Similarly, for $f \in S^*(K)$, $m \geq 0$, the m -order G-B type integral of the distribution f is defined as a distribution, denoted by $f_{(m)}$, satisfying

$$\langle f_{(m)}, \varphi \rangle = \langle f, \varphi_{(m)} \rangle, \quad \forall \varphi \in S(K). \quad (2.19)$$

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$$(f^{(m)})^{\wedge} = \langle \xi \rangle^m f^{\wedge}, \quad (f_{(m)})^{\wedge} = \langle \xi \rangle^{-m} f^{\wedge}, \quad (2.20)$$

in the distribution sense.

Proof Taking $\forall \varphi \in S(K)$, by Definition 2.5, one has $\langle f^{(m)}, \varphi \rangle = \langle f, \varphi^{(m)} \rangle$. Then, by (2.17), (2.18), it follows that

$$\langle (f^{(m)})^\wedge, \varphi^\wedge \rangle = \langle f^{(m)}, \varphi \rangle = \langle f \varphi^{(m)} \rangle = \langle f, T_{(\cdot)^m} \varphi \rangle. \quad (2.21)$$

Since

$$\begin{aligned} T_{(\square)^m} \varphi(x) &= \int_{\Gamma} \langle \xi \rangle^m \int_K \varphi(t) \overline{\chi}_\xi(t) \chi_x(\xi) dt d\xi = \int_{\Gamma} \langle \xi \rangle^m \varphi^\wedge(\xi) \chi_x(\xi) d\xi \\ &= \int_{\Gamma} \langle \xi \rangle^m \varphi^\wedge(\xi) \chi_x(\xi) d\xi = (\langle \square \rangle^m \varphi^\wedge(\square))^\vee(x). \end{aligned}$$

Thus, (2.21) becomes

$$\begin{aligned} \langle (f^{(m)})^\wedge, \varphi^\wedge \rangle &= \langle f, T_{(\square)^m} \varphi \rangle = \langle f, (\langle \square \rangle^m \varphi^\wedge(\square))^\vee \rangle \\ &= \langle f^\wedge, \langle \xi \rangle^m \varphi^\wedge \rangle = \langle \langle \xi \rangle^m f^\wedge, \varphi^\wedge \rangle, \end{aligned}$$

which implies that $(f^{(m)})^\wedge = \langle \xi \rangle^m f^\wedge$ in the distribution sense. The proof is complete.

Example 2.1 Evaluate m -order G-B type derivative of a character $\chi_\lambda(x)$, $m \geq 0$. Since $\chi_\lambda \in L_{loc}(K) \subset S^*(K)$, we have for any $\varphi \in S(K)$ that

$$\begin{aligned} \langle (\chi_\lambda)^\wedge, \varphi^\wedge \rangle &= \langle \chi_\lambda, \varphi \rangle = \int_K \chi_\lambda \varphi(x) dx = \int_K \varphi(x) \overline{\chi}_\lambda(-x) dx \\ &= \int_K \varphi(-x) \overline{\chi}_\lambda(-x) dx = \varphi^\wedge(\lambda) = \langle \delta_\lambda, \varphi^\wedge \rangle. \end{aligned}$$

This implies that $(\chi_\lambda)^\wedge = \delta_\lambda^\wedge$. Then by Theorem 2.7,

$$(\chi_\lambda(\square)^{(m)})^\wedge = \langle \xi \rangle^m \chi_\lambda^\wedge = \langle \xi \rangle^m \delta_\lambda.$$

Hence,

$$\chi_\lambda^{(m)} = (\langle \xi \rangle^m \chi_\lambda^\wedge)^\vee = \langle \lambda^m \rangle \delta_\lambda = \langle \lambda \rangle^m \chi_\lambda,$$

by the formula $(\langle \xi \rangle^m \delta_\lambda)^\vee = \langle \lambda \rangle^m \delta_\lambda^\vee$. Then the important formula

$$\chi_\lambda^{(m)} = \langle \lambda \rangle^m \chi_\lambda, \quad (2.22)$$

has been obtained.

The meaning of the formula (2.22) is the following. It is the eigenequation of the G-B type derivative when $m = 1$, since in this case $\chi_\lambda^{(1)} = \langle \lambda \rangle \chi_\lambda$, and the character χ_λ is the eigenfunction, while λ is the eigenvalue.

The following theorem shows that the two operations, G-B type derivative and G-B type integral, are mutually inverse.

Theorem 2.8 If $\varphi \in S(K)$, then for $m \geq 0$, holds

$$(\varphi^{(m)}(\square))_{(m)}(x) = \varphi(x), \quad (\varphi_{(m)}(\square))^{(m)}(x) = \varphi(x). \quad (2.23)$$

For the distribution $f \in S^*(K)$, it holds for $m \geq 0$

$$(f^{(m)})_{(m)} = f, \quad (f_{(m)})^{(m)} = f, \quad (2.24)$$

in the distribution sense.

It is very significant and very useful to determine the kernel k_α of the G-B type derivative operator T_α , $\alpha \in R$. For this, we can use the following theorem from [5].

Theorem 2.9 The convolution kernel of G-B type derivative operator T_α , $\alpha \in R$ is a distribution

$$k_\alpha = \begin{cases} \left(\frac{1-p^\alpha}{1-p^{-\alpha-1}} \pi_{-\alpha} + \frac{1-p^\alpha}{1-p^{\alpha+1}} \right) \Phi_{B^0}, & \alpha \neq 0, -1, \\ \left(1 - \frac{1}{p} \right) (1 - \log_p |x|) \Phi_{B^0}, & \alpha = -1, \\ \delta, & \alpha = 0, \end{cases} \quad (2.25)$$

with

$$\langle \pi_{-\alpha}, \varphi \rangle = \int_{K_p} |x|^{\alpha-1} (\varphi(x) - \varphi(0)) dx, \quad \forall \varphi \in S(K_p), \quad (2.26)$$

and δ is the Dirac distribution.

Proof First, we rewrite the G-B type derivative operator $T_{(\square)}^m$, $m \in R$, defined in (2.7) and (2.10), as

$$T_\alpha \varphi(x) = (\langle \square \rangle^\alpha \varphi^\wedge(\square))^\vee(x), \quad \varphi \in S(K). \quad (2.27)$$

And for $f \in S^*(K)$,

$$\langle T_\alpha f, \varphi \rangle = \langle f, T_\alpha \varphi \rangle, \quad \varphi \in S(K). \quad (2.28)$$

Then, we give the sketch of the proof, and refer to [5] for the details.

(a) π_α defined in (2.25) has the following form for $\forall \varphi \in S(K_p)$

$$\begin{aligned} \langle \pi_\alpha, \varphi \rangle &= \int_{B^0} |x|^{\alpha-1} (\varphi(x) - \varphi(0)) dx \\ &\quad + \int_{K_p \setminus B^0} |x|^{\alpha-1} \varphi(x) dx + \frac{1-p^{-1}}{1-p^\alpha} \varphi(0). \end{aligned} \quad (2.29)$$

It is analytic on $\alpha \in C$ except $\alpha_k = \frac{2k\pi i}{\ln p}$, $k \in Z$, and for any $\alpha \in R$, $\alpha \neq 0$, it is well defined.

(b) Let $\alpha \in R$, then for $\alpha \neq 0$, it follows by evaluating the integral

$$\int_{B^0} |x|^{-\alpha-1} (\chi(-\xi x) - 1) dx = \left(\frac{p^{-\alpha} - p^{-\alpha-1}}{1 - p^{-\alpha}} + \frac{1 - p^{-\alpha-1}}{1 - p^{\alpha}} |\xi|^{\alpha} \right) (1 - \Delta_0),$$

$$\text{with } \Delta_0 = \begin{cases} 1, & x \in B^0, \\ 0, & x \notin B^0, \end{cases}, B^0 = \{x \in K : |x| \leq 1\}.$$

(c) For $\alpha \neq 0, -1$, by (2.29) and the Fubini theorem, it follows that

$$\left\langle \left(\frac{1 - p^{\alpha}}{1 - p^{-\alpha-1}} \pi_{-\alpha} \Delta_0 \right)^{\wedge}, \varphi \right\rangle = \langle |\xi|^{\alpha} (1 - \Delta_0) + \frac{1 - p^{-1}}{1 - p^{-\alpha-1}} \Delta_0, \varphi \rangle.$$

Thus,

$$(k_{\alpha})^{\wedge} = |\xi|^{\alpha} (1 - \Delta_0) + \frac{1 - p^{-1}}{1 - p^{-\alpha-1}} \Delta_0 + \frac{1 - p^{\alpha}}{1 - p^{\alpha+1}} \Delta_0 = |\xi|^{\alpha} (1 - \Delta_0) + \Delta_0 = \langle \xi \rangle^{\alpha}.$$

(d) For $\alpha = -1$,

$$\begin{aligned} (k_{-1})^{\wedge} &= \left(1 - \frac{1}{p} \right) \int_{B^0} (1 - \log_p |x|) \chi(-\xi x) dx \\ &= \left(1 - \frac{1}{p} \right) \left(\Delta_0 - \int_{B^0} \log_p |x| \chi(-\xi x) dx \right). \end{aligned}$$

Computing $\int_{B^0} \log_p |x| \chi(-\xi x) dx$, we get

$$(k_{-1})^{\wedge}(\xi) = \left(1 - \frac{1}{p} \right) \left(1 - \frac{1}{1-p} \right) = 1, \quad |\xi| \leq 1.$$

and

$$(k_{-1})^{\wedge}(\xi) = \left(1 - \frac{1}{p} \right) \left(-\frac{|\xi|^{-1}}{p^{-1}-1} \right) = |\xi|^{-1}, \quad |\xi| > 1.$$

Thus,

$$(k_{-1})^{\wedge}(\xi) = \langle \xi \rangle^{-1}.$$

(e) For $\alpha = 0$, we get

$$(k_{\alpha})^{\wedge}(\xi) = \delta^{\wedge} = 1 = \langle \xi \rangle^0.$$

The proof is complete.

The kernel k_{α} and the operator T_{α} , $\alpha \in \mathbb{R}$, has important and interesting properties:

1. $\text{supp}(k_\alpha) \subset B^0$,
2. $k_\alpha^\wedge = \langle \xi \rangle^\alpha$,
3. $k_\alpha * k_\beta = k_{\alpha+\beta}$,
4. $T^\alpha f = k_\alpha * f$, $\forall f \in S^*(K)$,
5. $\lim_{\beta \rightarrow \alpha} T^\beta f = T^\alpha f$, $\forall f \in S^*(K)$,
6. $T^\alpha \delta = k_\alpha$ in $S^*(K)$,
7. $T^\alpha 1 = 1$ in $S^*(K)$,
8. The eigenvalues of T^α are $\{\lambda_n\}_{n=0}^{+\infty} = \begin{cases} \{1, p^\alpha, p^{2\alpha}, \dots\}, & \alpha > 0, \\ \{1\}, & \alpha = 0, \\ \{\dots, p^{2\alpha}, p^\alpha, 1\}, & \alpha < 0. \end{cases}$
9. The eigenfunctions of T^α are $\{\Psi_{n,j,I} : n \in \mathbb{Z}, j = 1, \dots, p-1, I = z_I + B^0\}$, with $\Psi_{n,j,I}(x) = p^{\frac{-n}{2}} \chi_j(p^{n-1}x) \Phi_{B^0}(p^{n-1}x - z_I)$, and

$$T^\alpha \Psi_{1-n,j,I}(x) = \begin{cases} p^{n\alpha} \Psi_{1-n,j,I}(x) & n > 0, \\ \Psi_{1-n,j,I}(x) & n \leq 0, \end{cases}$$

The above properties reveal certain essential properties of G-B calculus.

2.4 Function Spaces on Local Fields

To define some function spaces on a local field K , we use the non-homogeneous unit decompositions on K and Γ , respectively,

$$1 = \Phi_{B^0}(x) + \sum_{j=1}^{+\infty} \Phi_{B^{-j} \setminus B^{-j+1}}(x), \quad x \in K. \quad (2.30)$$

$$1 = \Phi_{\Gamma_0}(x) + \sum_{j=1}^{+\infty} \Phi_{\Gamma_{-j} \setminus \Gamma_{j-1}}(\xi), \quad \xi \in \Gamma. \quad (2.31)$$

where Φ_A is the characteristic function of the set A .

By using the unit decompositions, we define the Littlewood-Paley decomposition of a distribution.

Definition 2.6 A decomposition $f = \sum_{j=1}^{+\infty} u_j$ of $f \in S^*(K)$ is called the Littlewood-Paley decomposition, if u_j satisfies

$$\text{supp } u_0^\wedge \subset \Gamma^0, \quad \text{supp } u_j^\wedge \subset \Gamma_j \setminus \Gamma_{j-1}, \quad j \in \mathbb{N}, \quad (2.32)$$

where $u_j^\wedge$ is the Fourier transform of u_j in the distribution sense.

We also define the B -type spaces and F -type spaces in Triebel sense [17], [18] over a local field K .

Take a function sequence $\varphi = \{\varphi_j\}_{j=0}^{+\infty} \subset S(\Gamma)$ satisfying

- (i) $\text{supp } \varphi_0 \subset \{\xi \in \Gamma : |\xi| < p^{0+1}\}$, $\text{supp } \varphi_j \subset \{\xi \in \Gamma : p^{j-1} < |\xi| < p^{j+1}\}$, $j \in N$,
- (ii) $1 = \sum_{j=1}^{+\infty} \varphi_j(\xi)$, $\xi \in \Gamma$,
- (iii) $|(\varphi_j^\vee)^{(s)}(x)| \leq c_s p^{-j+js}$, $s \geq 0$, $j \in P$, $x \in K$.

where $(\varphi_j^\vee)^{(s)}(x)$ is the s -order G-B type derivative of $\varphi_j^\vee(x)$. Denote by

$$A(\Gamma) = \left\{ \varphi = \{\varphi_j\}_{j=0}^{+\infty} \subset S(\Gamma) : \varphi \text{ with (i)-(iii)} \right\}.$$

Specially, for a local field K , we take $\{\varphi_j\}_{j=0}^{+\infty} \subset A(\Gamma)$ as follows

$$\varphi_0 = \Phi_{\Gamma_0}(\xi), \quad \varphi_j(\xi) = \Phi_{\Gamma_j \setminus \Gamma_{j-1}}(\xi), \quad j \in N. \quad (2.33)$$

Then $\{\varphi_j\}_{j=0}^{+\infty} = \{\Phi_{\Gamma_0}, \Phi_{\Gamma_j \setminus \Gamma_{j-1}}\}_{j=1}^{+\infty} \subset A(\Gamma)$ satisfies

- (i) $\text{supp } \Phi_{\Gamma_0}(\xi) \subset \Gamma_0$, $\text{supp } \Phi_{\Gamma_j \setminus \Gamma_{j-1}}(\xi) \subset \Gamma_j \setminus \Gamma_{j-1}$, $j \in N$,
- (ii) $1 = \sum_{j=1}^{+\infty} \Phi_{\Gamma_j \setminus \Gamma_{j-1}}(\xi)$, $\xi \in \Gamma$,
- (iii) $|(\Phi_{\Gamma_0}^\vee)^{(s)}(x)| \leq c_s p^{-0+0 \cdot s}$, $|(\Phi_{\Gamma_j \setminus \Gamma_{j-1}}^\vee)^{(s)}(x)| \leq c_s p^{-j+js}$, $s \leq 0$, $j \in P$, $x \in K$.

Definition 2.7 The B -type space of Triebel sense on K is defined as

$$B_{rt}^s(K) = \{f \in S^*(K) : \|f\|_{B_{rt}^s(K)} < +\infty\}, \quad s \in R, \quad 0 < r, t \leq +\infty,$$

with $\{\varphi_j\}_{j=0}^{+\infty}$ in (2.33), and

$$\|f\|_{B_{rt}^s(K)} = \|p^{sj}[\varphi_j(\xi)f^\wedge(\xi)]^\vee(x)\|_{l_t(L^r(K))} = \left\{ \sum_{j=0}^{+\infty} \|p^{sj}[\varphi_j(\xi)f^\wedge(\xi)]^\vee(x)\|_{L^r(K)}^t \right\}^{\frac{1}{t}}.$$

The F -type spaces of Triebel sense on K is defined as

$$F_{rt}^s(K) = \{f \in S^*(K) : \|f\|_{F_{rt}^s(K)} < +\infty\}, \quad s \in R, \quad 0 < r, t < +\infty,$$

with $\{\varphi_j\}_{j=0}^{+\infty}$ in (2.33), and

$$\|f\|_{F_{rt}^s(K)} = \|p^{sj}[\varphi_j(\xi)f^\wedge(\xi)]^\vee(x)\|_{L^r(l_t(K))} = \left\{ \sum_{j=0}^{+\infty} |p^{sj}[\varphi_j(\xi)f^\wedge(\xi)]^\vee(x)|^t \right\}^{\frac{1}{t}} \|_{L^r(K)}.$$

The B -type and F -type spaces have more higher summarization, they contain many familiar spaces in usual as special cases, and are very useful in various scientific research. More important spaces in harmonic analysis, fractal analysis, partial differential equations are introduced in the following by means of B -type spaces.

Definition 2.8 The Hölder type spaces $C^\sigma(K)$, $\sigma \in R$, are defined as follows

- (i) For $\sigma = 0$, we define $C^0(K) = C(K)$ as the continuous function space on K ,
(ii) For $\sigma \in (0, +\infty)$, we define $C^\sigma(K)$ as the set of all distributions $f \in S^*(K)$ with the Littlewood-Paley decomposition $f = \sum_{j=0}^{+\infty} f_j$ satisfying
1. $\text{supp } f_0^\wedge \subset \Gamma^0$, $\text{supp } f_j^\wedge \subset \Gamma_j \setminus \Gamma_{j-1}$, $j \in N$, 2. $\|f_j\|_{L^\infty(K)} \leq cp^{-j\sigma}$, $j \in P$,
and define the norm of $C^\sigma(K)$ as $\|f\|_{C^\sigma(K)} = \text{supp}_j \{p^{j\sigma} \|f_j\|_{L^\infty(K)}\}$ for $\sigma \in (0, +\infty)$ such that $C^\sigma(K)$ is a Banach space;
(iii) For $\sigma \in (-\infty, 0)$, we define $C^\sigma(K) = B_{\infty, \infty}^\sigma(K)$, where $B_{\infty, \infty}^\sigma(K)$ is the B-type space $B_{rt}^s(K)$ with $s = \sigma$, $r = t = \infty$.

Theorem 2.10 *The Hölder type space $C^\sigma(K)$ for $\sigma \in [0, +\infty)$ has the following properties:*

1. If $f \in C^\sigma(K)$, then f has G-B type derivatives $f^{(\lambda)}(x) = T_{(\cdot)}^\lambda f(x)$, $x \in K$, for each $0 \leq \lambda \leq \sigma$, and $f^{(\lambda)} \in C^{\sigma-\lambda}(K)$,
2. If $f^{(\sigma)}(x) = T_{(\square)}^\sigma f \in C^0(K)$, then f has G-B type derivative $f^{(\sigma-\lambda)}(x) = T_{(\square)}^{\sigma-\lambda} f(x)$, $x \in K$, for each $0 \leq \lambda \leq \sigma$, and $f^{(\sigma-\lambda)} \in C^\lambda(K)$.

Proof We present a sketch of the proof, for more details see [12].

1. $f \in C^\sigma(K)$, $\sigma \in [0, +\infty)$, $\rightarrow f = \sum_{j=0}^{+\infty} f_j$ by definition of $C^\sigma(K)$
 $\rightarrow$ evaluate $T_{(\square)}^\lambda f(x)$ for $f = \sum_{j=0}^{+\infty} f_j$, $0 \leq \lambda \leq \sigma \rightarrow T_{(\square)}^\lambda f(x) = \sum_{j=0}^{+\infty} p^{j\lambda} f_j(x)$ exists,
 $\rightarrow \|T_{(\square)}^\sigma f(x)\|_{L^\infty(K)} = \text{supp}_j \{p^{j\sigma} \|f_j\|_{L^\infty(K)}\} \leq c$, by $f \in C^\sigma(K) \rightarrow T_{(\square)}^\sigma f(x)$ exists,
 $\rightarrow \forall \varepsilon > 0$, estimate $J = T_{(\square)}^\lambda f(x+h) - T_{(\square)}^\lambda f(x)$ for $|h| < \delta$, $0 \leq \lambda \leq \sigma$, get $|J| < \varepsilon$,
 $\rightarrow T_{(\square)}^\sigma f \in C^0(K)$ and $f^{(\lambda)} = T_{(\square)}^\lambda f \in C^{\sigma-\lambda}(K)$ for $0 \leq \lambda \leq \sigma$.
2. $T_{(\square)}^\sigma f \in C^0(K)$, $\sigma \in [0, +\infty) \rightarrow$ by the L-P decomposition of $f \in S^*(K)$

$$f(x) = f(x)\Phi_{B^0}(x) + \sum_{j=1}^{+\infty} f(x)(\Phi_{B^{-j}}(x) - \Phi_{B^{-j+1}}(x)),$$

$$\rightarrow T_{(\square)}^{\sigma-\lambda} f(x) = T_{(\square)}^{\sigma-\lambda} \left\{ \Phi_{B^0}(x) + \sum_{j=1}^{+\infty} f(x)(\Phi_{B^{-j}}(x) - \Phi_{B^{-j+1}}(x)) \right\} = \sum_{j=0}^{+\infty} p^{j(\sigma-\lambda)} f_j(x),$$

converges at $x \in K$ for $0 \leq \lambda \leq \sigma$ by $T_{(\square)}^\sigma f \in C^0(K) \rightarrow p^{j(\sigma-\lambda)} f_j(x) = o(1)$,

$$\rightarrow \|f_j(\square)\|_{L^\infty(K)} \leq cp^{-j(\sigma-\lambda)} \rightarrow f \in C^{\sigma-\lambda}(K) \text{ for } 0 \leq \lambda \leq \sigma \rightarrow f \in C^\sigma(K),$$

$$\rightarrow f^{(\sigma-\lambda)} = T_{(\square)}^{\sigma-\lambda} f \in f \in C^\lambda(K) \text{ by (1)}.$$

The meaning of Theorem 2.10 is the following. The Hölder spaces $C^\sigma(K)$, $\sigma \in [0, +\infty)$, are the spaces in which the G-B type differentiable functions live.

Theorem 2.11 *The following holds*

1. $0 \leq \sigma_1 < \sigma_2 \rightarrow C^{\sigma_2}(K) \subset C^{\sigma_1}(K)$,
2. $\sigma \in \mathbb{R} \rightarrow B_{\infty, \infty}^\sigma = C^\sigma(K)$.

A distribution $f \in S^*(K)$ has Littlewood-Paley decomposition $f = f_0 + \sum_{j=1}^{+\infty} f_j$, and for $\sigma \in (0, +\infty)$, taking $\{\varphi_j\}_{j=0}^{+\infty} \subset A(\Gamma)$, we have $p^{\sigma j}[\varphi_j f^\wedge(\xi)]^\vee(x) = p^{\sigma j} f_j(x)$, thus

$$\begin{aligned} \|f\|_{B_{\infty, \infty}^\sigma(K)} &= \|p^{\sigma j}[\varphi_j f^\wedge(\xi)]^\vee(x)\|_{l_\infty(L_\infty(K))} \\ &= \sup_j \left\{ \sup_{x \in K} |p^{\sigma j}[\varphi_j(\xi) f^\wedge(\xi)]^\vee(x)| \right\}, \\ &= \sup_j \left\{ p^{\sigma j} \|f_j\|_{L_\infty(K)} \right\} = \|f\|_{C^\sigma(K)}, \end{aligned}$$

then we conclude that $B_{\infty, \infty}^\sigma(K) = C^\sigma(K)$, $\sigma \in (0, +\infty)$.

The relation (1) in Theorem 2.11 shows that the spaces with higher differentiability are contained in that with lower differentiability.

The other important function space is the Lipschitz class.

Denote $X \equiv X(K) = \begin{cases} C(K), \\ L^r(K), \end{cases} 1 \leq r < \infty$, where K is a local field.

Definition 2.9 *For $\alpha > 0$, we define the function class*

$$Lip(\alpha, X) = \{f \in X : \|f(x+h) - f(x)\|_{X(K)} = O(|h|^\alpha)\},$$

called the Lipschitz class on a local field K , simply, Lip class.

Theorem 2.12 *For a local field K , we have*

$$Lip(\alpha, C(K)) = C^\alpha(K), \quad \alpha \in (0, +\infty).$$

The meaning of Theorem 2.12: In the case of local fields, the parameter α in the Lip class can be taken in the interval $(0, +\infty)$. This is an interesting fact: $Lip(\alpha, R)$ is a constant class for $\alpha > 0$. However, we have $Lip(\alpha, X(K)) \neq \text{constant class}$ for $\alpha > 0$.

We also define other two function spaces on local fields which will play important roles in fractal PDE :

Lebesgue type space (Bessel potential space) $L_r^s(K)$, $s \in \mathbb{R}$, $1 \leq r \leq +\infty$

$$L_r^s(K) = \left\{ f \in S^*(K) : \|f\|_{L_r^s(K)} = \|(\langle \square \rangle^s f^\wedge(\square))^\vee\|_{L_r^s(K)} < +\infty \right\},$$

where $\langle x \rangle = \max\{1, |x|\}$.

Sobolev type space $W^s(K)$, $s \in (0, +\infty)$

$$W^s(K) = \left\{ f \in S^*(K) : \|f\|_{L_2^s(K)} = \|(\langle \square \rangle^s f^\wedge(\square))^\vee\|_{L^2(K)} < +\infty \right\} \equiv L_2^s(K).$$

2.5 A Comparison of Newton calculus with G-B calculus

Let $C^m \equiv C_{2\pi}^m$ be the function space of all 2π periodic m -times, $m \in \mathbb{Z}^+$, continuous differentiable functions on $\mathbb{R}$; $Lip\alpha \equiv Lip(\alpha, C_{2\pi})$, $0 < \alpha \leq 1$, the Lipschitz class on $C_{2\pi}$.

For the smoothness of functions, we have the following chain:

$$\dots \subsetneq C^m \subsetneq C^{m-1} \subsetneq \dots \subsetneq C^1 \subsetneq Lip1 \subsetneq Lip\alpha \subsetneq Lip\beta \subsetneq C, \quad (2.34)$$

for $m \in \mathbb{N} = \{1, 2, 3, \dots\}$, $1 > \alpha > \beta > 0$.

The above chain holds also for $C^r \equiv C_R^r$. In more details, the chain between C^1 and C is:

$$\begin{array}{ccccccc} C^1 & \subsetneq & Lip1 & \subsetneq & Lip^*1 & \subsetneq & Lip^*\alpha = Lip\alpha Lip\beta = Lip^*\beta \subsetneq C \\ \text{unequal} & & 1 > \alpha > 0 & \text{equal} & 1 > \alpha > \beta > 0 & \text{equal} & \end{array} \quad (2.35)$$

We note that there is a "gap" between $Lip\alpha$ and $Lip^*\alpha$ at the case $\alpha = 1$, that is

$$0 < \alpha < 1 \rightarrow Lip\alpha = Lip^*\alpha, \quad (2.36)$$

but $Lip1 \subsetneq Lip^*1$.

This "gap" also appears in the equivalent theorems in approximation theory:

$$\begin{array}{l} 0 < \alpha < 1 : f \in Lip\alpha \leftrightarrow E_n(C_{2\pi}, f) = O(n^{-\alpha}), \\ \alpha = 1 : f \in Lip1 \rightarrow E_n(C_{2\pi}, f) = O(n^{-1}), \\ \alpha = 1 : f \in Lip^*1 \leftrightarrow E_n(C_{2\pi}, f) = O(n^{-1}). \end{array}$$

where $E_n(C_{2\pi}, f)$ is the best approximation of $f \in C_{2\pi}$ by trigonometric polynomials of degree n .

Moreover, the "gap" appears in the Holder space $C^\alpha = C^\alpha(\mathbb{R})$, $\alpha \in (0, +\infty) \setminus \mathbb{N}$, and the Zygmund class $Z = \{f \in C_{2\pi} : |f(x+h) - 2f(x) + f(x-h)| = O(|h|)\}$, that is

$$\begin{array}{lll} 0 < \alpha < 1 & C^\alpha & \leftrightarrow Lip\alpha \\ \alpha = 1 & \text{no Holder space} & Lip1 \subset Z(\text{Zygmund class}) \\ 1 < \alpha < 2 & C^\alpha & \text{no Lip class} \end{array}$$

where the Zygmund class Z defined by A. Zygmund in 1945, that is the Lip^*1 .

However, the equivalent theorem in approximation theory for local field K is: for $\alpha > 0$

$$f \in Lip(\alpha, C(K)) \leftrightarrow E_{p^k}(C(K), f) = O((p^k)^{-\alpha}), k \rightarrow +\infty, \quad (2.37)$$

where E_{p^n} is the p^n -order best approximation of continuous function $f \in C(K)$ over local field K , and $Lip(\alpha, C(K))$ is the Lipschitz class on K , $\alpha > 0$. Furthermore, for the Holder type space and Lipschitz class, there is no "gap" also:

$$C^\alpha(K) \leftrightarrow Lip(\alpha, C(K)), \quad \alpha \in (0, +\infty). \quad (2.38)$$

The above interesting results motivate us to compare the algebraic, geometric and analytic structures of Euclidean spaces and local fields.

Comparison structures of R^n (take $n = 1$) with K :

1) **Algebraic structure:** The addition and multiplication operations of R and K are quite different. The former one is the usual addition and multiplication on the real line, and the later one is the addition of sequences coordinate-wise in mod p , multiplication of Cauchy product in mod p (including "no carrying" , and "carrying form lift to right").

2) **Topological structures:** The topological structures of R and K are quite different. The former one is the usual Euclidean ε -topology, and the later one is endowed with the non- archimedean norm. Thus, they are both locally compact, but R is connected and K is totally disconnected.

3) **Geometric structures:** The geometric structures of R and K are quite different. For example, two balls in R have three positions: no intersection, having intersection; and one is contained in the other one. However, two balls in K only have two positions: no intersection, and one is contained in the other one. This is an essential difference, really.

4) **Analytic structures:** The structures of character groups of R and K are different. By the dual theory, $\Gamma_R \leftrightarrow R$, $\Gamma_K \leftrightarrow K$, thus Γ_R and Γ_K are connected and totally disconnected, respectively.

Spaces	Groups	Character groups, functions, values
Euclidean R	Compact $x \in [-1, 1]$	$\{\exp(2\pi i k x) : k \in \mathbb{Z}\}, \mu = ik$
	Locally compact $x \in R$	$\{\exp(2\pi i y x) : y \in R\}, \mu = iy$
local field K	Compact $x \in B^0$	$\Gamma_0 = \{\exp(\frac{2\pi i}{p} k \otimes x) : k \in P\}, \mu = k$
	Locally compact $x \in K$	$\Gamma\{\exp(\frac{2\pi i}{p} \xi \otimes x) : \xi \in K\}, \mu = \xi$

Moreover, the character equations (eigen-equations), character functions (eigen-functions), character values (eigen-values) of two cases are completely different. For example, the character equations are $y' = \mu y$ with traditional derivative $'$, and $y^{(1)} = \mu y$ with G-B type derivative $y^{(1)}$, for R and K , respectively.

5) **Function structures:** From the point of view of construction theory of functions, the best approximation theory are quite different for R and K . For example, the

equivalent theorems are essentially different: for Euclidean spaces, we have (2.34)-(2.37); and the equivalent theorems not only need Lip class, but also the Lip^* class. However, for local fields, we have (2.37)-(2.38), and the equivalent theorems only need Lip class not Lip^* .

We list the equivalent theorem on the compact group $B^0 = \{x \in K : |x| \leq 1\}$ in K .

Theorem 2.13 *The following statements are equivalent. For $\alpha > 0$, $r \in P$,*

1. $f^{(r)} \in Lip(\alpha, L(B^0))$,
2. $\omega(f^{(r)}, p^{-n}, L(B^0)) = O(p^{-n\alpha})$,
3. $E_{p^n}(f, L(B^0)) = O(p^{-n\alpha - nr})$, $n \rightarrow +\infty$,
4. $\|f(\cdot) - S_{p^n}(f, \cdot)\|_{L(B^0)} = O(p^{-n\alpha - nr})$, $n \rightarrow +\infty$.

We see that the approximation properties of functions defined on local fields are determined completely by one order continuity modulus, so by Lip class.

The following is the equivalent theorem on K .

Theorem 2.14 *For $s \geq 0$, the following statements are equivalent*

1. $f^{(s)} \in Lip(\alpha, L^1(K))$,
2. $\omega(f^{(s)}, p^{-n}, L^1(K)) = O(p^{-n\alpha})$, $n \rightarrow +\infty$,
3. $E_{p^n}(f, L^1(K)) = O(p^{-n\alpha - nr})$, $n \rightarrow +\infty$.

The above comparison motivates us to suggest four principles to establish a new calculus:

Principle 1 as operations, **differential operator must have the inverse operator, the integral operator**, and they are opposite each to other

Classical calculus on R	$\int_a^b \frac{df}{dx} = \frac{d}{dx} \int_a^b f dx$ with $f(a) = 0$,
Gibbs-Butzer calculus on K	$(f^{(m)})_{(m)} = (f_{(m)})^{(m)}$, $m \geq 0$.

Principle 2 as spectra, **the Fourier transform of a derivative has certain spectrum transformation formula**

Classical calculus on R	$\left(\frac{d^n f}{dx^n}\right)(\xi) = (i\xi)^n f^\wedge(\xi)$, $n \in N$,
Gibbs-Butzer calculus on K	$(f^{(m)}(\cdot))(\xi) = \langle \xi \rangle^m f^\wedge(\xi)$, $m \in P$.

Principle 3 in the point of view of construction theory of function, **derivatives should satisfy the direct and inverse approximation theorems** (Jackson and Bernstein theorems), that is, the more a function is smoother, the more the best approximation tends to zero faster; and vice versa

Classical calculus on R	see (2.37),
Gibbs-Butzer calculus on K	see Theorems 2.13 and 2.14.

Principle 4 in the point of view of motion in physics, **eigen-equations, eigen-functions, eigen-values** describe the eigen-vibrations. Correspondingly, derivatives should satisfy eigen-equations

Classical calculus on R	$\frac{dy(x)}{dx} = \mu y(x), x \in R,$
Gibbs-Butzer calculus on K	$y^{(1)}(x) = \langle \mu \rangle y(x), x \in K.$

and characters in the character groups should be the eigen-functions

Classical calculus on R	$y_\lambda(x) = \exp 2\pi\lambda x, \lambda \in \Gamma_R,$
Gibbs-Butzer calculus on K	$\chi_\xi(x) = \exp \frac{2\pi}{p} \xi \otimes x, \xi \in \Gamma_K,$

with character values $\lambda \in \Gamma_R(\leftrightarrow R)$ and $\xi \in \Gamma_K(\leftrightarrow K)$.

2.6 Applications of Gibbs-Butzer-type Calculus

In this section, we give some examples to deal with some mathematical problems and certain medical science problems by using G-B type derivatives.

2.6.1 Application to Approximation Theory on Local Fields

We define the Poisson kernel $R_y(x)$ on a local field K [10]

$$R_y(x) = |y|^{-1} \Phi_{B^s}(xy^{-1}), \quad x, y \in K, \quad s \in \mathbb{Z}, \quad (2.39)$$

by evaluating G-B type derivative, it follows that for $m > 0$

$$\begin{aligned} T_{(\square)}^m R_y(x) &= \int_{\Gamma} \langle \xi \rangle^m \left\{ \int_K |y|^{-1} \Phi_{B^s}(xy^{-1}) \overline{\chi}_\xi(t) dt \right\} \chi_x(\xi) d\xi \\ &= |y|^{-1} \int_{\Gamma} \langle \xi \rangle^m \left\{ \int_K \Phi_{B^s}(xy^{-1}) \overline{\chi}_\xi(t) dt \right\} \chi_x(\xi) d\xi \\ &= \int_{|\xi| \leq |y|^{-1}} \chi_x(\xi) d\xi = \begin{cases} |y|^{-1} \Phi_{B^s}(xy^{-1}), & |y|^{-1} \geq 1, \\ \Phi_{B^s}(xy^{-1}) + J_y(x), & |y|^{-1} < 1. \end{cases} \end{aligned}$$

where $J_y(x) \in S(G)$ and $\|R_y^{(m)}(\square)\|_{L^1(K)} = O(|y|^{-m}), y \rightarrow 0$. So that we may prove the approximation theorem.

Theorem 2.15 *Let $1 \leq r < +\infty$, for any $m \geq 0$ and $\alpha > 0$. Then,*

1. $f^{(m)} \in Lip(\alpha, L^r(K)) \rightarrow \|R_y^* f(\square) - f(\square)\|_{L^r(K)} = O(|y|^{m+\alpha}), \quad |y| \rightarrow 0,$
2. *If $f \in L^r(K)$ satisfies $\|R_y^* f(\square) - f(\square)\|_{L^r(K)} = O(|y|^{m+\alpha}), \quad |y| \rightarrow 0$, then $f^{(m)} \in L^r(K)$, and holds $\omega(f^{(m)}, \delta, L^r(K)) = \begin{cases} O(\delta^\alpha), & 0 < \delta < 1, \\ O(\delta |\ln \delta|), & \delta = 1, \end{cases}$*

It is clear that the G-B type derivatives defined by pseudo-differential operators is suitable to describe the approximation properties of functions defined on local

fields. That is, the more a function is smoother, the more the best approximation tends to zero faster ; and conversely, the smoothness of a function is determined by the best approximation.

Other approximation identity kernels, such as the Abel-Poisson type kernel, the typical means kernel, the de la Vellee-Poussin type kernel, the radial approximation identity kernel, and so on, are constructed on local fields [15]. The G-B type calculus plays nice and important roles in study of properties of these kernels, see [15] for details.

2.6.2 Application to determine function space for which a function belongs to it

For example, let $p = 2$, and $x \in B^0$, $x = 0.x_1x_2x_3 \cdots$, $x_j \in \{0, 1\}$, $j \in N$ (dyadic expression). We define a function $f: B^0 \rightarrow R$,

$$f(x) = \begin{cases} 0, & 0 = x, \\ \cdots & \cdots \\ 3^{-3}, & 0.001 < x \leq 0.01, \\ 3^{-2}, & 0.01 < x \leq 0.1, \\ 3^{-1}, & 0.1 < x \leq 1. \end{cases} \quad (2.40)$$

Take $h = 0.\underbrace{0 \cdots 01}_{k-1} = \frac{1}{2^k}$, $k = 1, 2, \dots$, then at $x = 0$, it follows that

$$|f(0 \oplus x) - f(0)| = |3^{-k} - 0| = 3^{-k} = (2^{-k})^{\log_2(3)} = O(|h|^{\log_2(3)}).$$

Moreover, for $x = 0$, and $0.x_1x_2x_3 \cdots$, $x_j \in \{0, 1\}$, $j \in N$, and $h = 0.\underbrace{0 \cdots 01}_{k-1} = \frac{1}{2^k}$,

we have

$$\begin{aligned} |f(0 \oplus x) - f(0)| &= |f(0.x_1 \cdots x_{k-1}(x_k + 1)x_{k+1} \cdots) - f(0.x_1 \cdots x_{k-1}x_kx_{k+1} \cdots)| \\ &= O(|h|^{\log_2(3)}). \end{aligned}$$

This shows that $f(x)$ is not a constant, and $f \in Lip(K, \log_2(3))$.

Moreover, we evaluate the G-B type derivative. Rewrite $f(x)$ as

$$f(x) = \begin{cases} \sum_{k=1}^{+\infty} \frac{1}{3^k} \Phi_{\beta^{k-1} + B_k}(x), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

where $\Phi_{\beta^{k-1} + B_k}(x)$ is the characteristic function of the ball $\beta^{k-1} + B_k \subset K$, $|\beta| = 2^{-1}$.

By the definition of G-B type derivative, we evaluate

$$f^{(m)}(x) = \int_{\Gamma} \langle \xi \rangle^m \int_K f(t) \overline{\chi}_{\xi}(t-x) dt d\xi,$$

and by some special technique for local fields, we get that

$$f^{(m)}(x) = \sum_{k=1}^{+\infty} \left\{ \frac{1-2^m}{6^k} \Phi_{B_0}(x) + \sum_{j=1}^{k-2} \frac{2^{jm+1}(1-2^m)}{6^k} \Phi_{B_j}(x) \right. \\ \left. + \frac{2^{mk}(2^{m-1}-2^{-1})}{3^k} \Phi_{B_{k-1}}(x) - \frac{2^{km}}{3^k} \Phi_{B_k}(x) \right\}$$

exists for every $m \leq \log_2(3)$, the above series is convergent for $x \in B^0$, a.e., so that the m -th order G-B type derivative $f^{(m)}(x)$ of $f(x)$ exists for every $m \leq \log_2(3)$. Then, this shows that $f \in C^{\alpha}(K)$ with $\alpha = \log_2(3)$.

2.6.3 Application to partial differential equations with fractal boundaries

We deal with the G-B type differential equation problem with fractal boundary

$$\begin{cases} \frac{\partial^{(2)} u(t,x,y)}{\partial t^{(2)}} = \frac{\partial^{(2)} u(t,x,y)}{\partial x^{(2)}} + \frac{\partial^{(2)} u(t,x,y)}{\partial t^{(2)}}, & t > 0, (x,y) \in \Omega, \\ u(t,x,y)|_{t=0} = \phi(x,y), \\ \frac{\partial^{(1)} u(t,x,y)}{\partial t^{(1)}}|_{t=0} = \psi(x,y), & (x,y) \in \Omega, \\ u(t,x,y)|_{\gamma} = 0, & t > 0, \end{cases}$$

where the domain $\Omega \subset K \times K$ has boundary γ , the von Koch type curve; the approximations of γ are curves $\gamma_1, \gamma_2, \dots$, and corresponding domains are $\Omega_1, \Omega_2, \dots$, respectively.

Figure 2.1 shows graphs of $\gamma_1, \gamma_2, \gamma_4$ for $p = 5$.

Suppose that the initial conditions $\phi(x,y) = \lim_{k \rightarrow +\infty} \phi_k(x,y)$, $\psi(x,y) = \lim_{k \rightarrow +\infty} \psi_k(x,y)$ are convergent in Ω , and the domains of $\phi_k(x,y)$ and $\psi_k(x,y)$ are Ω_k , $k = 1, 2, \dots$

The k -th approximation problem of (2.41) is

$$\begin{cases} \frac{\partial^{(2)} u_k(t,x,y)}{\partial t^{(2)}} = \frac{\partial^{(2)} u_k(t,x,y)}{\partial x^{(2)}} + \frac{\partial^{(2)} u_k(t,x,y)}{\partial y^{(2)}}, & t > 0, (x,y) \in \Omega_k, \\ u_k(t,x,y)|_{t=0} = \phi_k(x,y), & (x,y) \in \Omega_k \\ \frac{\partial^{(1)} u_k(t,x,y)}{\partial t^{(1)}}|_{t=0} = \psi_k(x,y), & (x,y) \in \Omega_k \\ u_k(t,x,y)|_{\gamma_k} = 0, & t > 0, \end{cases} \quad (2.41)$$

where $\gamma_1, \gamma_2, \dots, \gamma_k, \dots$, are the 1-st, 2-nd, ..., k -th ... von Koch type approximation curves, and the initial functions $\phi(x,y) = \lim_{k \rightarrow +\infty} \phi_k(x,y)$ and $\psi(x,y) = \lim_{k \rightarrow +\infty} \psi_k(x,y)$ with domains Ω_k , $k = 1, 2, \dots$, converging in Ω .

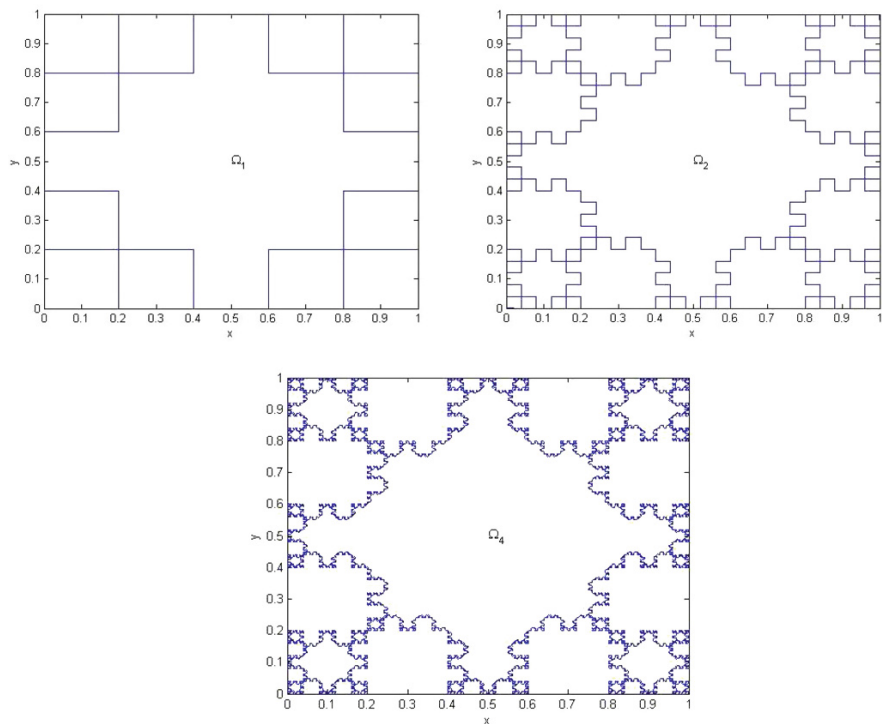


Fig. 2.1: Graphs of $\gamma_1, \gamma_2, \gamma_4$ for $p = 5$.

Let the solution of (2.41) has the form $u_k(t, x, y) = T_k(t)v_k(x, y)$ for $k = 1, 2, \dots$, then

$$\begin{aligned}\frac{\partial^{(2)} u_k(t, x, y)}{\partial t^{(2)}} &= \frac{d^{(2)} T_k(t)}{dt^{(2)}} v_k(x, y), \\ \frac{\partial^{(2)} u_k(t, x, y)}{\partial x^{(2)}} &= T_k(t) \frac{\partial^{(2)} v_k(x, y)}{\partial x^{(2)}}, \\ \frac{\partial^{(2)} u_k(t, x, y)}{\partial y^{(2)}} &= T_k(t) \frac{\partial^{(2)} v_k(x, y)}{\partial y^{(2)}}.\end{aligned}$$

Substituting in the equation $\frac{\partial^{(2)} u_k(t, x, y)}{\partial t^{(2)}} = \frac{\partial^{(2)} u_k(t, x, y)}{\partial x^{(2)}} + \frac{\partial^{(2)} u_k(t, x, y)}{\partial y^{(2)}}$, we get

$$\frac{d^{(2)} T_k(t)}{dt^{(2)}} v_k(x, y) = T_k(t) \left(\frac{\partial^{(2)} v_k(x, y)}{\partial x^{(2)}} + \frac{\partial^{(2)} v_k(x, y)}{\partial y^{(2)}} \right).$$

If $T_k(t) \neq 0$, $v_k(x, y) \neq 0$, then

$$\frac{1}{T_k(t)} \frac{dt^{(2)} T_k(t)}{dt^{(2)}} = \frac{1}{v_k(x, y)} \left(\frac{\partial^{(2)} v_k(x, y)}{\partial x^{(2)}} + \frac{\partial^{(2)} v_k(x, y)}{\partial y^{(2)}} \right).$$

And let $\lambda^2 + \mu^2$, $\lambda > 0$, $\mu > 0$, so that

$$\frac{1}{T_k(t)} \frac{dt^{(2)} T_k(t)}{dt^{(2)}} = \frac{1}{v_k(x, y)} \left(\frac{\partial^{(2)} v_k(x, y)}{\partial x^{(2)}} + \frac{\partial^{(2)} v_k(x, y)}{\partial y^{(2)}} \right) = \lambda^2 + \mu^2, \quad \lambda > 0, \mu > 0.$$

Hence,

$$\frac{\partial^{(2)} v_k}{\partial x^{(2)}} + \frac{\partial^{(2)} v_k}{\partial y^{(2)}} = (\lambda^2 + \mu^2) v_k, \quad (2.42)$$

$$\frac{d^{(2)} T_k}{dt^{(2)}} = (\lambda^2 + \mu^2) T_k. \quad (2.43)$$

It is easy to verify that

$$v_{k, \lambda, \mu}(x, y) = \sin \left(\frac{2\pi}{p} (\lambda \otimes x) \right) \sin \left(\frac{2\pi}{p} (\mu \otimes y) \right)$$

is a solution of (2.42). To determine λ and μ , by the conditions we have $v_{k, \lambda, \mu}|_{\gamma_k} = 0$ that

$$\lambda_{k, m, p} \equiv \lambda(k, m, p), \quad \mu_{k, m, p} \equiv \mu(k, m, p), \quad m, n = 1, 2, \dots$$

Then, the solution of (2.42) is

$$v_{k, \lambda, \mu}(x, y) \equiv \sin \left(\frac{2\pi}{p} (\lambda_{k, m, p} \otimes x) \right) \sin \left(\frac{2\pi}{p} (\mu_{k, m, p} \otimes y) \right), \quad m, n = 1, 2, \dots \quad (2.44)$$

Certainly, the solution of (2.43) is

$$T_{k, m, n}(t) = A_{k, m, n} \cos \left(\frac{2\pi}{p} (\lambda_{k, m, p} \otimes t) \right) + B_{k, m, n} \sin \frac{2\pi}{p} ((\mu_{k, m, p} \otimes t)). \quad (2.45)$$

Combining the above results, we get

$$\begin{aligned} u_{k, m, n}(t, x, y) &= T_{k, m, n}(t) v_{k, m, n}(x, y) \\ &= \left(A_{k, m, n} \cos \left(\frac{2\pi}{p} (\lambda_{k, m, p} \otimes t) \right) + B_{k, m, n} \sin \left(\frac{2\pi}{p} (\mu_{k, m, p} \otimes t) \right) \right) \\ &\quad \sin \left(\frac{2\pi}{p} (\lambda_{k, m, p} \otimes x) \right) \sin \left(\frac{2\pi}{p} (\mu_{k, m, p} \otimes y) \right), \quad m, n = 1, 2, \dots, \end{aligned} \quad (2.46)$$

with

$$A_{k,m,n} = \frac{\int_{\Omega_k} \phi_k(x,y) \cdot \sin\left(\frac{2\pi}{p} \lambda_{k,m,p} \otimes x\right) \cdot \sin\left(\frac{2\pi}{p} \mu_{k,n,p} \otimes y\right) dx dy}{\int_{\Omega_k} \left\{ \sin\left(\frac{2\pi}{p} \lambda_{k,m,p} \otimes x\right) \right\}^2 \cdot \left\{ \sin\left(\frac{2\pi}{p} \mu_{k,n,p} \otimes y\right) \right\}^2 dx dy}, \quad (2.47)$$

$$B_{k,m,n} = \frac{\int_{\Omega_k} \psi_k(x,y) \cdot \sin\left(\frac{2\pi}{p} \lambda_{k,m,p} \otimes x\right) \cdot \sin\left(\frac{2\pi}{p} \mu_{k,m,p} \otimes y\right) dx dy}{q \int_{\Omega_k} \left\{ \sin\left(\frac{2\pi}{p} \lambda_{k,m,p} \otimes x\right) \right\}^2 \cdot \left\{ \sin\left(\frac{2\pi}{p} \mu_{k,m,p} \otimes y\right) \right\}^2 dx dy}, \quad (2.48)$$

where $q = p^k \sqrt{m^2 + n^2}$.

Thus, $u_k(t,x,y) = \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} T_{k,m,n}(t) v_{k,m,n}(x,y)$, and the problem (2.41) has a formal solution

$$u(t,x,y) = \lim_{k \rightarrow +\infty} u_k(t,x,y) = \lim_{k \rightarrow +\infty} \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} u_{k,m,n}(t,x,y). \quad (2.49)$$

A numerical example

By taking

$$\phi(x,y) = 0, \quad \psi_k(x,y) = \begin{cases} \frac{1}{k^2}, & (x,y) \in \left[\frac{2}{5}, \frac{3}{5}, \frac{2}{5}, \frac{3}{5}\right], \text{ for } p = 5 \\ 0, & \text{otherwise,} \end{cases}$$

for $p = 5$, we have the solution by the formula (2.49) and the approximation graphs of (2.49) as in Figure 2.2.

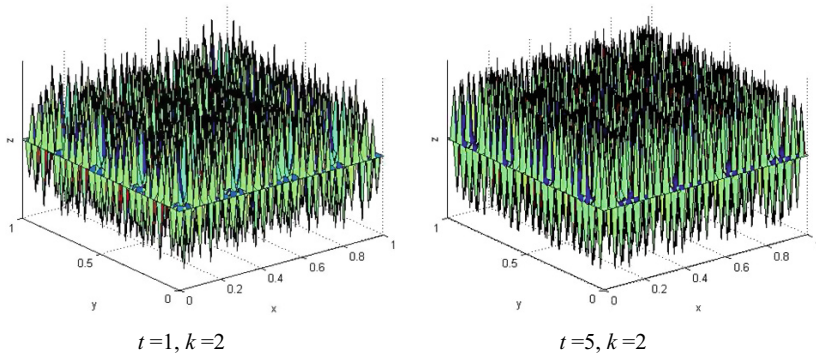


Fig. 2.2: The approximation graphs of (2.49).

Under certain conditions, we may have fractal solutions of the problem (2.41). About fractal PDE, there are lots of open problems, we refer to [15].

Application to study of fractals

We define the tri-adic Cantor type set $C_3 = D \setminus \bigcup_{j=1}^{+\infty} V_j$ on a local field K for $p = 3$:

$$\begin{aligned} D &\equiv B^0 = \{x \in K_3 : |x| \leq 1\} = (0 \cdot \beta^0 + B^1) \cup (1 \cdot \beta^0 + B^1) \cup (2 \cdot \beta^0 + B^1), \\ V_1 &= B^1 = \{x \in K_3 : |x| \leq 3^{-1}\} = (0 \cdot \beta^0 + B^2) \cup (1 \cdot \beta^0 + B^2) \cup (2 \cdot \beta^0 + B^2), \\ V_2 &= (1 \cdot \beta^0 + B^2) \cup (2 \cdot \beta^0 + B^2), \\ V_3 &= (1 \cdot \beta^0 + 1 \cdot \beta^1 + B^3) \cup (1 \cdot \beta^0 + 2 \cdot \beta^1 + B^3) \cup (2 \cdot \beta^0 + 1 \cdot \beta^1 + B^3) \\ &\quad \cup (2 \cdot \beta^0 + 2 \cdot \beta^1 + B^3), \\ \dots &= \dots \end{aligned}$$

Correspondingly, the tri-adic Cantor type function (devil's ladder) is defined as $\vartheta(x)$, $x \in K_3$

1. $\text{supp } \vartheta(x) = D$,
2. $\forall x \in D$, let $x = \sum_{j=0}^{k-2} x_j \beta^j$, we define
If $\exists k \geq 1$, such that $x_j \neq 0$ for $0 \leq j \leq k-2$, and $x_{k-1} = 0$, then

$$\vartheta(x) = \sum_{j=0}^{k-2} (x_j - 1) \left(\frac{1}{2}\right)^{j-1} + \left(\frac{1}{2}\right)^k,$$

If $x_j \neq 0$ for $0 \leq j < +\infty$, then

$$\vartheta(x) = \sum_{j=0}^{+\infty} (x_j - 1) \left(\frac{1}{2}\right)^{j-1},$$

If $x \notin D$, then $\vartheta(x) = 0$.

Then, we have

Theorem 2.16 *The tri-adic Cantor type function $\vartheta(x)$ is infinitely G-B type integrable; and it is m-order G-B-type differentiable with $m \leq \frac{\ln 2}{\ln 3}$; moreover, for $x \in D$, we have*

$$\begin{aligned} \vartheta^{(m)}(x) &= \frac{1}{2} + \sum_{l=1}^{+\infty} \frac{3^{lm}}{6^l} \chi(\beta^{-l}x) \left(\frac{1}{2} + \omega + \frac{1}{2} \chi(\beta^{-l}x) + \overline{\omega} \chi(\beta^{-l}x) \right) \\ &\quad \times \prod_{j=1}^{l-1} (2 - \chi(\beta^{-j}x) - \chi^2(\beta^{-j}x)). \end{aligned}$$

And for $x \notin D$, $\vartheta^{(m)}(x)$, where $\omega = e^{\frac{2\pi i}{3}}$, and $\chi(\beta^{-j}) = \begin{cases} \omega, & j = 1, \\ 1, & \text{otherwise.} \end{cases}$

Figure 2.3 shows the graph of C_3 and $\vartheta(x)$.

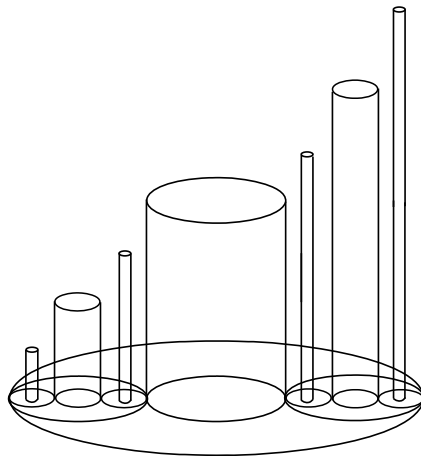


Fig. 2.3: The graph of C_3 and $\vartheta(x)$.

This is a very interesting example: the G-B type differentiability of the tri-adic Cantor type function $\vartheta(x)$ is connected with the Hausdorff dimension $\dim_H C_3 = \frac{\ln 2}{\ln 3}$ of the tri-adic Cantor type set C_3 . Combining the result of Example 2), $f \in C^\alpha(K)$ with $\alpha = \log_2 3$ and $f^{(m)} \in C(K)$ exists for $m \leq \log_2 3$, we would ask what deep essential properties of G-B calculus are ?

For more examples of applications to fractal analysis, we refer to [15].

2.6.4 Application to medical science

1. To determine the malignant cases of liver's cancer Doctors ask for to determine the malignant of a cancer by math accurately, such that they can make a treatment plans for a patient, for example, chemotherapy, actinotherapy, or medical- therapy.

By estimating the Hausdorff dimension or box dimension of a Liver's cancer block, then determine the malignant cases. We have do almost 200 cases of patients with liver' cancers, and may get a mathematical model to describe the relationship between the malignant cases and fractal dimensions.

2. auxiliary partial orthotopic liver transplantation (APOLT) Use some mathematical models to evaluate blood flow, blood volume, mean transit time and time to peak, then protract a graph to describe the cases of hepatic arterial perfusion, portal

with liver' cancers, and may get a mathematical model to describe the relationship between the malignant cases and fractal dimensions.

2. auxiliary partial orthotopic liver transplantation (APOLT) Use some mathematical models to evaluate blood flow, blood volume, mean transit time and time to peak, then protract a graph to describe the cases of hepatic arterial perfusion, portal venous perfusion and total liver perfusion, and hepatic perfusion index of a patient. How much volume to transplant?

3. Determine what genes control the liver' cancers Use track records in CMOS chips to analyze the effects of each gene to liver's cancers, and then may know the effects of each gene.

For the details of applications to medical science, we refer to [15].

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