

## Chapter 2

# Application to Markov Processes

### 2.1 Problem

Let  $X_t$  be a standard Markov process with the state space  $S$ . The time interval  $[0, \infty)$  is denoted by  $T$ . Let  $a$  be a fixed state and  $\sigma_a$  the hitting time for  $a$ . We impose the following four assumptions.

- (A-1)  $P_b(\sigma_a < \infty) = 1$ ;
- (A-2)  $E_b(\sigma_a \wedge 1) \rightarrow 0$  as  $b \rightarrow a$ ;
- (A-3)  $\inf_{b \in U^c} E_b(\sigma_a \wedge 1) > 0$  for every neighborhood  $U$  of  $a$ ;
- (A-4)  $a$  is a *discontinuous exit state*.

We will explain the meaning of this condition.

$s$  is called an *exit time* from  $a$  for the path  $(X_t(\omega))$  if, for every  $\varepsilon > 0$ ,

$$\{t : X_t(\omega) = a\} \cap (s - \varepsilon, s) \neq \emptyset$$

and if, for some  $\varepsilon > 0$ ,

$$\{t : X_t(\omega) = a\} \cap (s, s + \varepsilon) = \emptyset.$$

All exit times from  $a$  for the path  $(X_t(\omega))$  form a countable set depending on  $\omega$ .

An exit time  $s$  from  $a$  for the path  $(X_t(\omega))$  is called a *continuous* or *discontinuous* exit time according as

$$X_s(\omega) = a \quad \text{or} \quad X_s(\omega) \neq a.$$

$a$  is called a *discontinuous exit state* if all exit times from  $a$  for the path  $(X_t(\omega))$  are discontinuous a.s. with respect to  $P_a$ .

Let  $X_t^0 = X_{t \wedge \sigma_a}$ . Since the hitting time  $\sigma_a^0$  of the path  $(X_t^0(\omega))$  is the same as  $\sigma_a$ , the conditions (A-1), (A-2) and (A-3) are equivalent to

$$(A^0-1) \quad P_b(\sigma_a^0 < \infty) = 1;$$

$$(A^0-2) \quad E_b(\sigma_a^0 \wedge 1) \rightarrow 0, \text{ as } b \rightarrow a;$$

$$(A^0-3) \quad \inf_{b \in U^c} E_b(\sigma_a^0 \wedge 1) > 0 \text{ for every neighborhood } U \text{ of } a.$$

By the strong Markov property of  $(X_t)$ , the probability laws of the path  $(X_t)$  is determined by the probability laws of the path  $(X_t^0)$  and the probability law of the path  $(X_t)$  starting at  $a$ . Symbolically we have

$$\text{p.l. of } (X_t) = \text{p.l. of } (X_t^0) + \text{p.l. of } (X_t) \text{ starting at } a. \quad (2.1)$$

Since the path  $(X_t)$  starting at  $a$  behaves outside of  $a$  in the same way as the path  $(X_t^0)$ , the union relation in (2.1) is no disjoint union. We want to extract some information  $I$  from the probability law of the path  $(X_t)$  starting at  $a$  to obtain a symbolic information relation

$$\text{p.l. of } (X_t) = \text{p.l. of } (X_t^0) + I \quad (\text{disjoint union}). \quad (2.1')$$

In the subsequent sections we will prove that  $I$  consists of two elements: *jumping-in measure*  $k(db)$  and *stagnancy rate*  $m$ .

## 2.2 The Poisson Point Process Attached to a Markov Process at a State $a$

We use the same notations as in Sect. 2.1 and impose the conditions (A-1), (A-2), (A-3) and (A-4).

Let  $A(t)$  be a local time of  $(X_t)$  at  $a$ . By our assumptions (A-1) and (A-2),  $A(t)$  is determined up to a multiplicative constant and we have

$$P_b(A(t) < \infty \text{ for every } t) = 1$$

and

$$P_b(A(t) \rightarrow \infty \text{ as } t \rightarrow \infty) = 1.$$

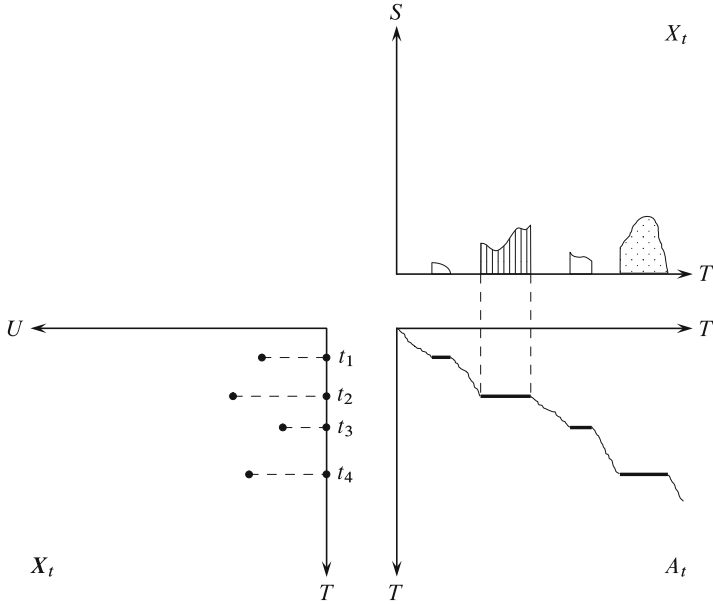
We refer the reader to Blumenthal and Gettoor [1] for the definition and the properties of local times.

Let  $U$  be the space of all right continuous functions:  $T \rightarrow S$  with left limits. The sample path of  $(X_t)$  belongs to  $U$  a.s. for every starting point.

From now on we will refer to  $P_a$  for the probability law of  $X_t(\omega)$  unless the contrary is explicitly stated. Let us define a point process  $X : T \rightarrow U$  by

$$D_X = \{A(s) : s \text{ moves over all exit times from } a \text{ for the path}\},$$

$$X_\omega(t) = (X \circ \theta_{A^{-1}(t-)})^0 \quad \text{for } t \in D_X,$$



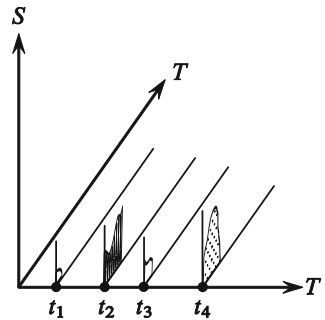
**Fig. 2.1**  $X_t$ ,  $A_t$  and  $X_t$

where  $\theta_t$  is a shift operator and  $0$  is a stopping operator. Note that  $D_X$  consists of all values of  $A(t)$  corresponding to the flat  $t$ -intervals of  $A(t)$  and that  $X_\omega(t)$  is a function  $: T \rightarrow S$  belonging to  $U$  for  $\omega$  and  $t$  fixed and (Fig. 2.1)

$$X_\omega(t)(s) = X_{s \wedge \sigma_a(\theta_\tau \omega)}(\theta_\tau \omega) \quad \text{for } s \in T, \quad \tau = A^{-1}(t-).$$

Figure 2.2 is an intuitive picture of  $X_t$ .

**Fig. 2.2** An intuitive picture of  $X_t$



**Theorem and Definition 2.2.1** *The point process  $X$  defined above is a Poisson point process:  $T \rightarrow U$ ; it is called the Poisson point process attached to the Markov process  $(X_t)$ .*

*Proof* Let  $\{\mathcal{B}_t\}$  be a family of sub- $\sigma$ -algebras of  $\mathcal{B}$  in the definition of the Markov process  $(X_t)$ . Since  $A^{-1}(t+)$  and  $A^{-1}(t-) = \sup_n A^{-1}(t - \frac{1}{n}+)$  are both stopping times,  $\mathcal{B}_{A^{-1}(t-)}$  and  $\mathcal{B}_{A^{-1}(t+)}$  are well-defined.

Let  $\mathcal{B}_t(X)$  be the  $\sigma$ -algebra generated by  $X|_d[0, t)$ . Then

$$\mathcal{B}_t(X) \subset \mathcal{B}_{A^{-1}(t-)} \subset \mathcal{B}_{A^{-1}(t+)}.$$

For  $t$  fixed, we have

$$P(X(A^{-1}(t+)) = a) = 1.$$

Thus, for  $B \in \mathcal{B}_t(X)$  and  $M \in \mathcal{P}$ , we have

$$P_a(B \cap ((\theta_t X) \in M)) = P_a(B)P_a(X \in M)$$

because of the additivity of  $A(t)$ , the strong Markov property of  $(X_t)$  and the definition of  $X$ . Thus  $X$  has renewal property. This implies that  $X$  is differential and stationary.

To prove the  $\sigma$ -discreteness of  $X$  we will introduce a map  $h : U \rightarrow T$  by

$$h(u) = \inf\{t : u(t) = a\}.$$

If  $u$  is the path of  $(X_t(\omega))$ , then  $h(u)$  will be  $\sigma_a(\omega)$ .

Let  $U_n$  be the set of all  $u$  such that

$$h(u) > \frac{1}{n}.$$

By (A-4) we have

$$X = \bigvee_n X|_r U_n \quad \text{a.s.}$$

Since

$$N(X, [0, t) \times U_n) \leq n \cdot A^{-1}(t-) < \infty \quad \text{a.s.,}$$

$X_\omega|_r U_n$  is discrete a.s. for each  $n$ .  $X$  is therefore  $\sigma$ -discrete. □

## 2.3 The Jumping-In Measure and the Stagnancy Rate

Let us consider a map  $e : U \rightarrow S$  by

$$e(u) = u(0), \quad u \in U.$$

Since the path of  $X_t$  has no discontinuities of the second kind and since  $A^{-1}(t) < \infty$  for  $t < \infty$ , the distance  $\rho(X_t, a)$  between  $X_t$  and  $a$  can be larger than  $\varepsilon(>0)$  a finite number of times during  $[0, A^{-1}(t))$  a.s. for  $t < \infty$ . This implies that  $e \cdot X$  is a  $\sigma$ -discrete point process. By Theorem 1.6.1, we see that  $e \cdot X$  is a Poisson point process.

**Definition 2.3.1** The characteristic measure  $k$  of  $e \cdot X$  is called the *jumping-in measure* of the Markov process  $X_t$  from  $a$ .

It is obvious that  $k = n_X e^{-1}$ . Since  $n_X$  is concentrated on the paths starting at points in  $S - \{a\}$  by (A-4),  $k$  is concentrated on  $S - \{a\}$ . It is obvious that the total measure of  $k$  is the same as that of  $n_X$ . Since  $X$  is  $\sigma$ -discrete, the total measure of  $k$  is  $\sigma$ -finite.

Since  $A^{-1}(t)$  is known to be an increasing homogeneous Lévy process (=a subordinator), it can be written as

$$A^{-1}(t) = m \cdot t + J(t), \quad m > 0,$$

when  $J(t)$  is a pure jump process.

**Definition 2.3.2** The coefficient  $m$  is called the *stagnancy rate* of the Markov process  $(X_t)$ .

The following theorem shows that the characteristic measure  $n_X$  is determined by the measure  $k$  and the probability law of the path of  $(X_t^0)$ .

**Theorem 2.3.3**

$$n_X(V) = \int_S k(db) P_b(X^0 \in V)$$

where  $X^0$  denotes the path  $(X_t^0(\omega), t \in T)$ .

*Proof* Let  $S_i$  denote the set  $\{b \in S : \rho(a, b) > 1/i\}$  for  $i = 1, 2, \dots$ . Then  $\cup S_i = S - \{a\}$ . Let  $U_i = \{u \in U : u(0) \in S_i\} = e^{-1}(S_i)$ . Then  $U_i$  increases with  $i$  and the limit  $U_\infty$  is the space of all paths in  $U$  starting from points in  $S - \{a\}$ . We have

$$X = X|_r U_\infty = \bigvee_i X_i, \quad X_i = X|_r U_i$$

by (A-4). The set  $A^{-1}(D_{X_i}) \cap [0, A^{-1}(t+)]$  is included in the set of the time points  $s \in [0, A^{-1}(t+)]$  for which  $\rho(X_s, X_{s-}) > 1/i$ . Since the sample path of  $(X_t)$  has no

discontinuity points of the second kind, the latter set is finite and so is  $A^{-1}(D_{X_i}) \cap [0, A^{-1}(t+)]$ . This implies  $D_{X_i} \cap [0, t]$  is finite.  $X_i$  is therefore a discrete Poisson point process.

By Theorem 1.4.1, we have

$$n_{X_i}(V_i) = \lambda_i P_a(X_i(\tau_i) \in V_i), \quad V_i \in \mathcal{U}_i \equiv U_i \cap \mathcal{U},$$

where  $\lambda_i = n_{X_i}(U_i)$  and  $\tau_i$  is the smallest element in  $D_{X_i}$ . By the definition we have

$$X_i(\tau_i) = X(\tau_i) = (X \circ \theta_{\sigma_i})^0, \quad \sigma_i = A^{-1}(\tau_i-).$$

Since  $n_{X_i} = n_X|_{U_i}$  and since  $\sigma_i$  is a stopping time with respect to  $\{\mathcal{B}_t\}$ , we have, for  $V \in \mathcal{U}$ ,

$$\begin{aligned} n_X(U_i \cap V) &= \lambda_i P_a((X \circ \theta_{\sigma_i})^0 \in V \cap U_i) \\ &= \lambda_i \int_{S_i} P_a(X_{\sigma_i} \in db) P_b(X^0 \in V \cap U_i) \end{aligned}$$

Set  $V = e^{-1}(B_i)$ ,  $B_i \in \mathcal{S}_i \equiv S_i \cap \mathcal{S}$ . Then  $V \subset e^{-1}(S_i) = U_i$  and so

$$k(B_i) = \lambda_i P_a(X_{\sigma_i} \in B_i).$$

Thus we have

$$n_X(U_i \cap V) = \int_{S_i} k(db) P_b(X^0 \in V \cap U_i).$$

Letting  $i \uparrow \infty$ , we have

$$n_X(V) = \int_{S-\{a\}} k(db) P_b(X^0 \in V) = \int_S k(db) P_b(X^0 \in V),$$

which completes the proof.  $\square$

The jumping-in measure  $k$  is not arbitrary. We have:

**Theorem 2.3.4**  *$k$  is concentrated on  $S - \{a\}$  and*

$$\int_S E_b(\sigma_a^0 \wedge 1) k(db) < \infty.$$

*Proof*  $h \cdot X$  is also a Poisson point process whose integrated process is the discontinuous part of the increasing homogeneous Lévy process  $A^{-1}(t)$ . Therefore  $h \cdot X$  is summable and so

$$\int_0^\infty (1 \wedge t) n_{h \cdot X}(dt) < \infty$$

by virtue of Theorem 1.7.3. Since  $n_{h \cdot X} = n_X h^{-1}$ , this can be written

$$\int_0^\infty (1 \wedge t) \int_S k(db) P_b(\sigma_a^0 \in dt) < \infty$$

by the previous theorem, namely

$$\int_S k(db) E_b(\sigma_a^0 \wedge 1) < \infty.$$

□

*Remark* By this theorem and the condition (A-3), we have

$$k(U^c) < \infty$$

for every neighborhood  $U$  of  $a$ .

If  $k(S) < \infty$ , then  $X$  is discrete. Then the set  $\{t : X_t(\omega) = a\}$  is a sequence of disjoint intervals ordered linearly and  $A(t, \omega)$  is the sojourn time at a singleton  $\{a\}$  up to a multiplicative constant. Thus we have  $m > 0$  in the decomposition:

$$A^{-1}(t) = mt + J(t),$$

$J(t)$  being the discontinuous part of  $A^{-1}(t)$ . Therefore we obtain:

**Theorem 2.3.5**  $m \geq 0$  in general, and  $m > 0$  in case  $k(S) < \infty$ .

$A(t)$  is determined up to a multiplicative constant and  $m$  and  $k$  depend on which version of  $A(t)$  we take. Let  $A_i(t)$ ,  $i = 1, 2$  be two versions of  $A(t)$  and write the corresponding  $m$  and  $k$  as  $m_i$  and  $k_i$ ,  $i = 1, 2$ . Then we have a constant  $c > 0$  such that

$$A_2(t) = cA_1(t).$$

Consider the decompositions

$$A_i^{-1}(s) = m_i s + J_i(s), \quad i = 1, 2.$$

Then

$$\begin{aligned} A_2^{-1}(cs) &= A_1^{-1}(s), \\ m_2 cs + J_2(cs) &= m_1 s + J_1(s) \end{aligned}$$

and so

$$m_2 = \frac{1}{c} m_1.$$

Writing  $\#A$  for the number of points in  $A$ , we have

$$\begin{aligned}
 \varepsilon \cdot k_2(B) &= E_a[\#\{s : 0 \leq s \leq \varepsilon, e(X_s) \in B\}] \\
 &= E_a[\#\{s : 0 \leq s \leq \varepsilon, X(A_2^{-1}(s-)) \in B\}] \\
 &= E_a[\#\{t : 0 \leq A_2(t) \leq \varepsilon, X(t) \in B\}] \\
 &= E_a[\#\{t : 0 \leq cA_1(t) \leq \varepsilon, X(t) \in B\}] \\
 &= E_a\left[\#\left\{t : 0 \leq A_1(t) \leq \frac{1}{c}\varepsilon, X(t) \in B\right\}\right] \\
 &= \frac{1}{c}\varepsilon k_1(B)
 \end{aligned}$$

and so

$$k_2 = \frac{1}{c}k_1.$$

Thus we have:

**Theorem 2.3.6** *If  $A_2(t) = cA_1(t)$ , then  $m_2 = \frac{1}{c}m_1$  and  $k_2 = \frac{1}{c}k_1$ .*

Therefore  $m$  and  $k$  are determined up to a common multiplicative constant.

To have  $m$  and  $k$  determined uniquely, we have to take a standard version of the local time  $A(t)$ .

**Definition 2.3.7**  $A(t)$  is called *standard* if

$$E_a\left(\int_0^\infty e^{-t} dA(t)\right) = 1,$$

in which case

$$E_b\left(\int_0^\infty e^{-t} dA(t)\right) = E_b(e^{-\sigma_a^0}) \quad \text{for every } b.$$

The  $m$  and  $k$  that correspond to the standard  $A(t)$  are called the *standard stagnancy rate* and the *standard jumping-in measure*.

**Theorem 2.3.8** *The standard stagnancy rate  $m$  and the stagnancy jumping-in measure  $k$  satisfy the following conditions.*

- (a)  $m \geq 0$  in general and  $m > 0$  in case  $k(S) < \infty$ .
- (b)  $k$  is concentrated on  $S - \{a\}$  and

- (i)  $\int_S k(db) E_b(\sigma_a^0 \wedge 1) < \infty$ ;
- (ii)  $m + \int_S k(db) E_b(1 - e^{-\sigma_a^0}) = 1$ .

*Proof* By Theorems 2.3.4 and 2.3.5 it is enough to prove (b)-(ii). Since  $m$  and  $k$  are standard, the corresponding  $A(t)$  satisfies



$$E_a \left( \int_0^\infty e^{-t} dA(t) \right) = 1.$$

But the left side is

$$\begin{aligned} E_a \left( \int_0^\infty e^{-A^{-1}(t)} dt \right) &= \int_0^\infty E_a(e^{-A^{-1}(t)}) dt \\ &= \int_0^\infty e^{-mt - t \cdot \int_0^\infty (1-e^{-s}) \int_S k(db) P_b(\sigma_a^0 \in ds)} dt \end{aligned}$$

(see the proof of Theorem 2.3.4 and use the Lévy–Khinchin formula)

$$\begin{aligned} &= \left( m + \int_0^\infty (1 - e^{-s}) \int_S k(db) P_b(\sigma_a^0 \in ds) \right)^{-1} \\ &= \left( m + \int_S k(db) E_b(1 - e^{-\sigma_a^0}) \right)^{-1}. \end{aligned}$$

This proves (ii). □

## 2.4 The Existence and Uniqueness Theorem

Suppose that  $X_t$  is a standard Markov process with the state space  $S$  and that  $a$  is a fixed state. We assume (A-1), (A-2), (A-3) and (A-4) in Sect. 2.1.

Let  $X_t^0 = X_{t \wedge \sigma_a}$ ,  $m$  the standard stagnancy rate and  $k$  the jumping-in measure for  $X_t$ . Then we have proved:

- (i)  $X_t^0$  is a standard Markov process which satisfies (A<sup>0</sup>-1), (A<sup>0</sup>-2), (A<sup>0</sup>-3).
- (ii)  $m$  and  $k$  satisfy (a) and (b) in Theorem 2.3.8.

Now we want to construct  $X_t$  for  $X_t^0$ ,  $m$  and  $k$  given.

**Theorem 2.4.1** *Suppose that  $X_t^0$ ,  $m$  and  $k$  satisfy (i) and (ii). Then there exists a standard Markov process  $X_t$  satisfying (A-1), (A-2) and (A-3) such that  $X_{t \wedge \sigma_a}$  is equivalent to  $X_t^0$  and that the standard stagnancy rate and the standard jumping-in measure are respectively equal to  $m$  and  $k$ . Such  $X_t$  is unique up to equivalence.*

*Proof of existence* First we will construct the Poisson point process  $X$  attached to the Markov process  $X_t$  that is to be constructed.

Let  $U$  be the space of all right continuous functions:  $T \rightarrow S$  with left limits. Define a  $\sigma$ -finite measure  $n$  on  $U$  by

$$n(V) = \int_S k(db) P_b(X^0 \in V)$$

and construct a Poisson point process  $X : T \rightarrow U$  with  $n_X = n$  by Theorem 1.3.5, or by Theorem 1.5.4.

Set

$$\tilde{A}(s) = ms + \sum_{\substack{\alpha \leq s \\ \alpha \in D_X}} h(X(\alpha))$$

where  $h(u) = \inf\{\alpha \in T : u(\alpha) = a\}$ .

Define  $Y(t)$  as follows.

$$Y(t) = \begin{cases} X(s)(t - \tilde{A}(s-)) & \text{if } \tilde{A}(s-) \leq t < \tilde{A}(s) \\ a & \text{if } \tilde{A}(s-) = t = \tilde{A}(s). \end{cases}$$

Now define the probability law  $P_a$  of the path of  $X_t$  starting at  $a$  by

$$P_a(X_\bullet \in V) = P(Y(\cdot) \in V)$$

and the probability law  $P_b$  of the path of  $X_t$  starting at a general state  $b$  by

$$P_b(X_{\bullet \wedge \sigma_a} \in V_1, X \circ \theta_{\sigma_a} \in V_2) = P_b(X_\bullet^0 \in V_1)P_a(X_\bullet \in V_2).$$

It is needless to say that the definition of  $P_a$  is suggested by Fig. 2.1 and that the definition of  $P_b$  is suggested by the strong Markov property.

First we will prove that

$$P(\tilde{A}(s) < \infty \text{ for every } s \text{ and } \tilde{A}(\infty) = \infty) = 1, \quad (2.2)$$

so that  $Y(t)$  is well-defined for every  $t$ . If  $k(S) = 0$ , then  $m > 0$  and

$$\tilde{A}(s) = ms < \infty \quad \text{and} \quad A(\infty) = \infty.$$

If  $k(S) > 0$ , then  $h \cdot X$  is a Poisson point process with

$$n_{h \cdot X} = nh^{-1} = \int_S k(db) P_b(\sigma_a^0 \in \cdot).$$

Since

$$\int_0^\infty (t \wedge 1) n_{h \cdot X}(dt) = \int_S k(db) E_b(\sigma_a^0 \wedge 1) < \infty,$$

$h \cdot X$  is summable and so

$$J(s) = \sum_{\substack{\alpha \leq s \\ \alpha \in D_X}} h(X(\alpha)) < \infty$$

for every  $s < \infty$  and  $J(s)$  is a homogeneous Lévy process with increasing paths. Since  $n_{h \cdot X}([0, \infty)) = k(S) > 0$ , we have

$$P(J(\infty) = \infty) = 1.$$

This proves (2.2).

Now we will prove that the process  $X_t$  defined above is a standard Markov process with (A-1), (A-2), (A-3) and (A-4).

**Case 1.**  $k(S) < \infty$ . In this case we have  $m > 0$ .

Since

$$n_X(U) = \int_S k(db) P_b(X_\bullet^0 \in U) = k(S),$$

$X$  is discrete.

Set

$$D_X = \{\tau_1 < \tau_1 + \tau_2 < \tau_1 + \tau_2 + \tau_3 < \dots\}$$

and set

$$\xi_i = X(\tau_1 + \tau_2 + \dots + \tau_i), \quad i = 1, 2, \dots$$

Then  $\tau_1, \tau_2, \dots, \xi_1, \xi_2, \dots$  are independent and

$$\begin{aligned} P(\tau_i > t) &= e^{-tk(S)}, \\ P(\xi_i \in V) &= \frac{1}{k(S)} \int_S k(db) P_b(X^0(\cdot) \in V), \quad V \in \mathcal{U}. \end{aligned}$$

In other words the probability law of  $\xi_i$  is the probability law of the path of  $X^0$  with the initial distribution  $k(db)/k(S)$ .

By the definition of  $Y(t)$  we have  $Y(t) = a$  for

$$\begin{aligned} &m\tau_1 + h(\xi_1) + \dots + m\tau_{i-1} + h(\xi_{i-1}) \\ &\leq t < m\tau_1 + h(\xi_1) + \dots + m\tau_{i-1} + h(\xi_{i-1}) + m\tau_i \end{aligned}$$

and

$$Y(t) = \xi_i(t - m\tau_1 - h(\xi_1) - \dots - m\tau_{i-1} - h(\xi_{i-1}) - m\tau_i)$$

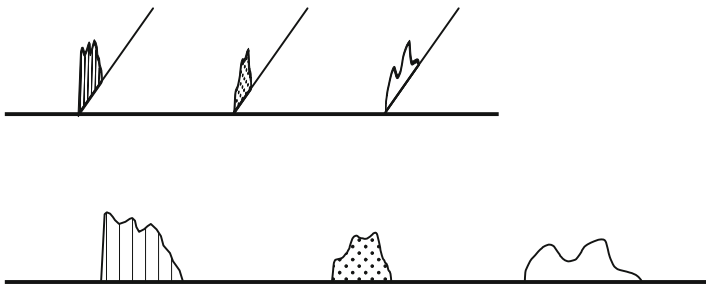
for

$$\begin{aligned} &m\tau_1 + h(\xi_1) + \dots + m\tau_{i-1} + h(\xi_{i-1}) + m\tau_i \\ &\leq t < m\tau_1 + h(\xi_1) + \dots + m\tau_i + h(\xi_i). \end{aligned}$$

Since

$$P(m\tau_i > t) = P(\tau_i > t/m) = e^{-tk(S)/m},$$

$X(t)$  can be described as follows. If it starts at  $a$ , it stays at  $a$  for an exponential holding time with the parameter  $= k(S)/m$ , then jumps into  $db$  with probability



**Fig. 2.3** Markov process from a point process

$k(db)/k(S)$  and moves in the same way as  $X_t^0$  does until it hits  $a$ , it will repeat the same motion afterwards independently of its past history. If it starts at  $b \neq a$ , it performs the same motion as  $X_t^0$  until it hits  $a$  and then it will act as above. We can verify the strong Markov property of this motion by routine. It is easy to check the other properties of  $X_t$  stated above (see Fig. 2.3).

**Case 2.**  $k(S) = \infty$ . Everything can be verified by routine except the fact that the sample path of  $Y(t)$  belongs to  $U$  a.s. Since it is obvious that  $Y(t)$  is right continuous and has left limits as far as it is in  $S - \{a\}$ , the only fact that needs proof is that the set of  $s$  such that

$$\sigma_\varepsilon(X(s)) < \infty, \quad \sigma_\varepsilon(u) = \inf\{t : \rho(a, u(t)) \geq \varepsilon\}$$

forms a discrete set a.s. for every  $\varepsilon > 0$ . Since  $X(s)(t) = a$  for  $t \geq h(X(s))$  a.s.,  $\sigma_\varepsilon(X(s)) < \infty$  is equivalent to

$$\sigma_\varepsilon(X(s)) < h(X(s))$$

a.s. It is therefore enough to prove that

$$X|_r V_\varepsilon, \quad V_\varepsilon = \{u : \sigma_\varepsilon(u) < h(u)\}$$

is discrete a.s., namely that

$$n_X(V_\varepsilon) < \infty.$$

Set

$$\delta = \inf\{E_b(\sigma_a^0 \wedge 1) : \rho(b, a) \geq \varepsilon\}.$$

Then  $\delta > 0$  by (A<sup>0</sup>-3).

Observe that

$$\begin{aligned}
 \int_U h(u) \wedge 1 n_X(du) &\geq \int_{V_\varepsilon} h(u) \wedge 1 n_X(du) \\
 &\geq \int_{V_\varepsilon} (h(u) - \sigma_\varepsilon(u)) \wedge 1 n_X(du) \\
 &= \int_V (h(u) - \sigma_\varepsilon(u)) \wedge 1 \int_S k(db) P_b(X^0 \in du) \\
 &= \int_S k(db) E_b[(\sigma_a^0 - \sigma_\varepsilon(X^0)) \wedge 1, \sigma_a^0 > \sigma_\varepsilon(X^0)] \\
 &= \int_S k(db) E_b[E_{X(\sigma_\varepsilon(X^0))}(\sigma_a^0 \wedge 1), \sigma_a^0 > \sigma_\varepsilon(X^0)] \\
 &\geq \delta \int_S k(db) P_b(\sigma_a^0 > \sigma_\varepsilon(X^0)) \\
 &= \delta \int_S k(db) P_b(X^0 \in V_\varepsilon) \\
 &= \delta n_X(V_\varepsilon)
 \end{aligned}$$

and that

$$\begin{aligned}
 \int_U h(u) \wedge 1 n_X(du) &= \int_U h(u) \wedge 1 \int_S k(db) P_b(X^0 \in du) \\
 &= \int_S k(db) E_b(h(X^0) \wedge 1) \\
 &= \int_S k(db) E_b(\sigma_a^0 \wedge 1).
 \end{aligned}$$

Thus we have  $n_X(V_\varepsilon) < \infty$ .

The *proof of uniqueness* is easy, because the probability law of the path of  $(X_t^0)$  and  $k$  determine  $n_X$  and so the probability law of  $X$ , which, combined with  $m$  determines the probability law of the path of  $X_t$ .  $\square$

## 2.5 The Resolvent Operator and the Generator of the Markov Process Constructed in Sect. 2.4

The generator of a Markov process is defined in many ways which are not always equivalent to each other. We will adopt the following definition due to E.B. Dynkin.

Let  $X_t$  be a Markov process with right continuous paths. The *transition probability*  $p(t, b, E)$  is defined by

$$p(t, b, E) = P_b(X_t \in E),$$

and the *transition operator*  $p_t$  is defined by

$$p_t f(b) = \int_S p(t, b, dc) f(c) = E_b(f(X_t)).$$

$p_t$  carries the space  $\mathbb{B}(S)$  of all bounded real Borel measurable functions into itself. It has the *semi-group property*:

$$p_{t+s} = p_t p_s, \quad p_0 = I \text{ (=identity operator)}.$$

The *resolvent operator* (potential operator of order  $\alpha$ )  $R_\alpha$  ( $\alpha > 0$ ) is defined by

$$R_\alpha = \int_0^\infty e^{-\alpha t} p_t dt$$

i.e.,

$$R_\alpha f(b) = \int_0^\infty e^{-\alpha t} p_t f(b) dt = E_b \left( \int_0^\infty e^{-\alpha t} f(X_t) dt \right).$$

It satisfies the *resolvent equations*:

$$R_\alpha - R_\beta + (\alpha - \beta) R_\alpha R_\beta = 0.$$

The *Dynkin subspace*  $\mathbb{L}$  of  $\mathbb{B}(S)$  is defined by

$$\mathbb{L} = \{f \in \mathbb{B}(S) : \lim_{t \downarrow 0} p_t f(b) = f(b) \text{ for every } b\}.$$

$\mathbb{L}$  is a linear subspace of  $\mathbb{B}(S)$ .

Because of the right continuity of the path of  $X_t$  we have

$$\mathbb{C}(S) \subset \mathbb{L} \subset \mathbb{B}(S),$$

$\mathbb{C}(S)$  being the space of all bounded continuous real functions on  $S$ .

It is easy to see that

$$p_t \mathbb{L} \subset \mathbb{L}, \quad R_\alpha \mathbb{L} \subset \mathbb{L}.$$

In view of this fact we will regard  $p_t$  and  $R_\alpha$  as operators  $\mathbb{L} \rightarrow \mathbb{L}$ , unless the contrary is stated explicitly.

By virtue of the resolvent equation  $\mathcal{R} = R_\alpha \mathbb{L}$  is independent of  $\alpha$ .  $R_\alpha : \mathbb{L} \rightarrow \mathcal{R}$  is 1-1 and so  $R_\alpha^{-1}$  is well-defined.

**Definition 2.5.1** The generator  $\mathcal{G}$  of  $(X_t)$  is defined by

$$\mathcal{D}(\mathcal{G}) = \left\{ f \in \mathbb{L} : \frac{1}{t}(p_t f - f) \text{ converges boundedly as } t \rightarrow 0 \text{ to a function } \in \mathbb{L} \right\}$$

and

$$\mathcal{G}f(b) = \lim_{t \downarrow 0} \frac{1}{t}(p_t f(b) - f(b)), \quad f \in \mathcal{D}(\mathcal{G}).$$

**Theorem 2.5.2**  $\mathcal{D}(\mathcal{G}) = \mathcal{R} = R_\alpha \mathbb{L}$ ,  $\mathcal{G}f = \alpha f - R_\alpha^{-1}f$ ,  $f \in \mathcal{D}(\mathcal{G})$ .

Let  $X_t$  be a standard Markov process and  $a$  be a fixed state. We assume that (A-1), (A-2) and (A-3) are satisfied. Let  $X_t^0 = X_{t \wedge \sigma_a}$ . Then  $X_t^0$  is also a standard Markov process with (A<sup>0</sup>-1), (A<sup>0</sup>-2) and (A<sup>0</sup>-3). We will denote the transition operator, the resolvent operator and the generator of  $X_t$  respectively by  $p_t$ ,  $R_\alpha$  and  $\mathcal{G}$  and the corresponding operators for  $X_t^0$  are denoted by  $p_t^0$ ,  $R_\alpha^0$  and  $\mathcal{G}^0$ .

**Theorem 2.5.3**  $\mathcal{D}(\mathcal{G}) \subset \mathcal{D}(\mathcal{G}^0)$  and

$$\begin{aligned} \mathcal{G}f(b) &= \mathcal{G}^0 f(b), \quad b \neq a, \\ \mathcal{G}^0 f(a) &= 0. \end{aligned}$$

*Proof* If  $f \in \mathcal{D}(\mathcal{G})$ , then

$$f = R_\alpha g, \quad g \in \mathbb{L}.$$

By Dynkin's formula we have

$$\begin{aligned} f(b) &= E_b \left( \int_0^\infty e^{-\alpha t} g(X_t) dt \right) \\ &= E_b \left( \int_0^{\sigma_a} e^{-\alpha t} g(X_t) dt \right) + E_b(e^{-\alpha \sigma_a} f(X_{\sigma_a})) \\ &= E_b \left( \int_0^{\sigma_a} e^{-\alpha t} g(X_t) dt \right) + E_b(e^{-\alpha \sigma_a}) f(a). \end{aligned}$$

Set

$$g^0(b) = \begin{cases} g(b) & b \neq a, \\ \alpha R_\alpha g(a) = \alpha f(a) & b = a. \end{cases}$$

Then

$$\begin{aligned} R_\alpha^0 g^0(b) &= E_b \left( \int_0^\infty e^{-\alpha t} g^0(X_t^0) dt \right) \\ &= E_b \left( \int_0^{\sigma_a} e^{-\alpha t} g(X_t) dt \right) + E_b(e^{-\alpha \sigma_a}) R_\alpha^0 g^0(a). \end{aligned}$$

Since

$$R_\alpha^0 g^0(a) = \int_0^\infty e^{-\alpha t} \alpha f(a) dt = f(a),$$

we have

$$f(b) = R_\alpha^0 g^0(b).$$

To complete the proof of  $\mathcal{D}(\mathcal{G}) \subset \mathcal{D}(\mathcal{G}^0)$ , we need only prove that  $g^0$  belongs to the Dynkin space  $\mathbb{L}^0$  of  $X_t^0$ . Since  $X_t^0 = a$  for  $t \geq \sigma_a$ , we have

$$p_t^0 g^0(a) = g^0(a) \longrightarrow g^0(a) \quad \text{as } t \downarrow 0.$$

Suppose  $b \neq a$ . Then  $P_b(\sigma_a > 0) = 1$  and so

$$\lim_{t \downarrow 0} P_b(\sigma_a \leq t) = 0.$$

Therefore

$$\begin{aligned} & |p_t^0 g^0(b) - g^0(b)| \\ &= |E_b(g^0(X_t^0) - g^0(b))| \\ &= |E_b(g(X_t), t < \sigma_a) + E_b(g^0(a), t \geq \sigma_a) - g(b)| \\ &= |E_b(g(X_t)) - E_b(g(X_t), t \geq \sigma_a) + E_b(g^0(a), t \geq \sigma_a) - g(b)| \\ &= |E_b(g(X_t)) - g(b)| + (\|g\| + |g^0(a)|)P_b(t \geq \sigma_a) \longrightarrow 0, \end{aligned}$$

where  $\|g\| = \sup_{c \in S} |g(c)|$ . Since

$$f = R_\alpha g = R_\alpha^0 g^0,$$

we have

$$\mathcal{G}f = \alpha f - g, \quad \mathcal{G}^0 f = \alpha f - g^0$$

and so

$$\mathcal{G}^0 f(b) = \mathcal{G}f(b) \quad \text{for } b \neq a$$

and

$$\mathcal{G}^0 f(a) = \alpha f(a) - g^0(a) = 0. \quad \square$$

Let  $X_t$  be the Markov process constructed from  $X_t^0$ ,  $m$  and  $k$  in Sect. 2.4. The resolvent and the generator for  $X_t$  are denoted respectively by  $R_\alpha$  and  $\mathcal{G}$  and the corresponding operators for  $X_t^0$  are denoted respectively by  $R_\alpha^0$  and  $\mathcal{G}^0$ .



We will discuss the relation between  $(R_\alpha, \mathcal{G})$  and  $(R_\alpha^0, \mathcal{G}^0)$ . Let us make three cases.

**Case 1.**  $k(S) = 0$ . In this trivial case  $a$  is a trap for  $X_t$  and  $(X_t)$  is equivalent to  $(X_t^0)$ , so that

$$R_\alpha = R_\alpha^0 \quad \text{and} \quad \mathcal{G} = \mathcal{G}^0.$$

**Case 2.**  $0 < k(S) < \infty$ . ( $m > 0$  in this case.)  $a$  is an exponential holding state with the rate  $k(S)/m$ .

**Theorem 2.5.4** *If  $0 < k(S) < \infty$ , then*

$$R_\alpha g(b) = R_\alpha^0 g(b) + E_b(e^{-\alpha\sigma_a^0}) R_\alpha g(a) \quad \text{for } b \neq a; \quad (2.3)$$

$$R_\alpha g(a) = \frac{mg(a) + \int_S k(db) R_\alpha^0 g(b)}{\alpha m + \int_S k(db) E_b(1 - e^{-\alpha\sigma_a^0})}; \quad (2.4)$$

$$\mathcal{G}f(b) = \mathcal{G}^0 f(b) \quad \text{for } b \neq a; \quad (2.5)$$

$$m\mathcal{G}f(a) = \int_S k(db)(f(b) - f(a)). \quad (2.6)$$

*Proof* Equation (2.3) is obvious by Dynkin's formula.

To prove (2.4), set

$$f(a) = R_\alpha g(a) \quad \text{and} \quad f^0(a) = R_\alpha^0 g(a).$$

The Poisson point process  $X$  attached to  $(X_t)$  is discrete. Let  $\sigma$  be the first point in  $D_X$  and  $\tau$  be the first exit time from  $a$  for  $(X_t)$ . Let  $Y_t$  be the process derived from  $X$  in Sect. 2.4. By Dynkin's formula, we have

$$\begin{aligned} f(a) &= E_a \left( \int_0^\infty e^{-\alpha t} g(X_t) dt \right) \\ &= E_a \left( \int_0^\tau e^{-\alpha t} g(X_t) dt \right) + E_a(e^{-\alpha\tau} f(X_\tau)) \\ &= E \left( \int_0^{m\sigma} e^{-\alpha t} g(a) dt \right) + E[e^{-\alpha m\sigma} f(X_\sigma(0))] \\ &= g(a) E \left[ \frac{1 - e^{-\alpha m\sigma}}{\alpha} \right] + E[e^{-\alpha m\sigma}] E[f(X_\sigma(0))]. \end{aligned}$$

Observe

$$\begin{aligned} E(e^{-\alpha m \sigma}) &= \int_0^\infty e^{-\alpha m t} e^{-tk(S)} k(S) dt \\ &= \frac{k(S)}{\alpha m + k(S)} \end{aligned}$$

and

$$E[f(X_\sigma(0))] = \frac{1}{k(S)} \int_S k(db) f(b).$$

Therefore we have

$$f(a) = \frac{mg(a) + \int_S k(db) f(b)}{\alpha m + k(S)}, \quad (2.7)$$

which, combined with (2.3), implies

$$f(a) = \frac{mg(a) + \int_S k(db) f^0(b) + \int_S k(db) E_b(e^{-\alpha \sigma_a^0}) f(a)}{\alpha m + k(S)}.$$

Solving this for  $f(a)$ , we have

$$f(a) = \frac{mg(a) + \int_S k(db) f^0(b)}{\alpha m + \int_S k(db) E_b(1 - e^{-\alpha \sigma_a^0})},$$

which proves (2.4). Equation (2.5) is obvious by Theorem 2.5.3.

It follows from (2.7) that

$$m(\alpha f(a) - g(a)) = \int_S k(db)(f(b) - f(a)),$$

which proves (2.6). □

**Case 3.**  $k(S) = \infty$ .  $a$  is an instantaneous state for  $(X_t)$ .

**Theorem 2.5.5** *Theorem 2.5.4 holds also in case  $k(S) = \infty$ :*

$$\left( \int_S k(db)(f(b) - f(a)) \right) = \lim_{\varepsilon \downarrow 0} \int_{\rho(b,a) > \varepsilon} k(db)(f(b) - f(a))$$

with the following proviso. If  $m > 0$ , (2.4) holds for  $g$  with

$$\lim_{b \rightarrow a} g(b) = g(a) \quad (2.8)$$

and (2.6) holds for  $f = R_\alpha g$  with  $g$  satisfying the same condition.

*Proof* (2.3) and (2.5) are obvious. Let  $\varepsilon > 0$  and set

$$\begin{aligned} S^{1,\varepsilon} &= \{b \in S : \rho(b, a) \geq \varepsilon\}, \\ S^{2,\varepsilon} &= S - S^{1,\varepsilon}, \\ U^{i,\varepsilon} &= \{u \in U : u(0) \in S^{i,\varepsilon}\}, \quad i = 1, 2, \\ X^{i,\varepsilon} &= X|_r U^{i,\varepsilon}, \quad i = 1, 2. \end{aligned}$$

Let  $Y^{2,\varepsilon}(t)$  be the process derived from  $X^{2,\varepsilon}$  in the same way as  $Y_t$  was derived from  $X$  in Sect. 2.4. Since we fix  $\varepsilon$  for the moment, we omit  $\varepsilon$  in  $S^{i,\varepsilon}$ ,  $U^{i,\varepsilon}$  etc.

Let

$$J(t, X) = \sum_{\substack{s \leq t \\ s \in D_X}} h(X_s).$$

Similarly for  $J(t, X^i)$ .  $X^1$  is discrete. Let  $\sigma$  be the first element in  $D_{X^1}$ .

Noticing that

$$s \in D_X, \quad s < \sigma \implies s \in D_{X^2},$$

we have

$$J(\sigma-, X) = J(\sigma-, X^2), \quad X_\sigma = X_\sigma^1$$

and

$$Y_t = Y_t^2 \quad \text{for } t < m\sigma + J(\sigma-, X) = m\sigma + J(\sigma-, X^2).$$

$$\begin{aligned} f(a) &\equiv R_\alpha g(a) \\ &= E \left( \int_0^\infty e^{-\alpha t} g(Y_t) dt \right) \\ &= E \left( \int_0^{m\sigma + J(\sigma-, X)} e^{-\alpha t} g(Y_t) dt \right) \\ &\quad + E \left[ e^{-\alpha m\sigma - \alpha J(\sigma-, X)} \int_0^{h(X_\sigma)} e^{-\alpha t} g(X_\sigma(t)) dt \right] \\ &\quad + E \left[ e^{-\alpha m\sigma - \alpha J(\sigma-, X) - \alpha h(X_\sigma)} \int_0^\infty e^{-\alpha t} g(Y(t, \theta_\sigma X|_d(0, \infty))) dt \right] \\ &= E \left( \int_0^{m\sigma + J(\sigma-, X^2)} e^{-\alpha t} g(Y_t^2) dt \right) \\ &\quad + E \left[ e^{-\alpha m\sigma - \alpha J(\sigma-, X^2)} \int_0^{h(X_\sigma^1)} e^{-\alpha t} g(X_\sigma^1(t)) dt \right] \\ &\quad + E \left[ e^{-\alpha m\sigma - \alpha J(\sigma-, X^2) - \alpha h(X_\sigma^1)} \int_0^\infty e^{-\alpha t} g(Y(t, \theta_\sigma X|_d(0, \infty))) dt \right] \\ &= I_1 + I_2 + I_3. \end{aligned}$$

$X^1$  and  $X^2$  are independent.  $\sigma$  and  $X_\sigma^1$  are  $\mathcal{B}(X^1)$  measurable and independent of each other.  $X^2$ ,  $\sigma$  and  $X_\sigma^1$  are therefore independent of each other. Thus we have

$$\begin{aligned} I_2 &= E[e^{-\alpha m \sigma - \alpha J(\sigma-, X^2)}] E\left[\int_0^{h(X_\sigma^1)} e^{-\alpha t} g(X_\sigma^1(t)) dt\right] \\ &= \int_0^\infty P(\sigma \in dt) e^{-\alpha m t} E[-\alpha J(t-, X^2)] \int_{S^1} \frac{k(db)}{k(S^1)} E_b\left(\int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0) dt\right). \end{aligned}$$

Since  $J(t, X^2)$  is a Lévy process increasing with jumps whose Lévy measure is equal to

$$n_{h, X^2}(dt) = \int_{S^2} k(db) P_b(\sigma_a^0 \in dt),$$

we have

$$\begin{aligned} E[e^{-\alpha J(t-, X^2)}] &= e^{-t \int_0^\infty (1 - e^{-\alpha s}) n_{h, X^2}(ds)} \\ &= e^{-t \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0})}. \end{aligned}$$

It is obvious that

$$P(\sigma \in dt) = e^{-k(S^1)t} k(S^1) dt.$$

Therefore

$$I_2 = \frac{\int_{S^1} k(db) E_b\left(\int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0) dt\right)}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0})}.$$

By the strong renewal property of  $X$  we have

$$\begin{aligned} I_3 &= E[e^{-\alpha m \sigma - \alpha J(\sigma-, X^2) - \alpha h(X_\sigma^1)}] E\left[\int_0^\infty e^{-\alpha t} g(Y_t) dt\right] \\ &= E[e^{-\alpha m \sigma - \alpha J(\sigma-, X^2)}] E[e^{-\alpha h(X_\sigma^1)}] f(a) \\ &= \frac{f(a) \int_{S^1} k(db) E_b(e^{-\alpha \sigma_a^0})}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0})}. \end{aligned}$$

Thus we have

$$f(a) = I_1 + \frac{\int_{S^1} k(db) \left[ E_b\left(\int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0) dt\right) + E_b(e^{-\alpha \sigma_a^0}) f(a) \right]}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0})}. \quad (2.9)$$

To evaluate  $I_1$ , consider

$$\begin{aligned}
 f^2(a) &\equiv E\left(\int_0^\infty e^{-\alpha t} g(Y_t^2) dt\right) \\
 &= I_1 + E\left(e^{-\alpha m\sigma - \alpha J(\sigma-, X^2)} \int_0^\infty e^{-\alpha t} g(Y(t, \theta_\sigma X^2|_d(0, \infty))) dt\right) \\
 &= I_1 + \int_0^\infty P(\sigma \in ds) \\
 &\quad \times E\left(e^{-\alpha ms - \alpha J(s-, X^2)} \int_0^\infty e^{-\alpha t} g(Y(t, \theta_s X^2|_d(0, \infty))) dt\right) \\
 &= I_1 + \int_0^\infty P(\sigma \in ds) \\
 &\quad \times E(e^{-\alpha ms - \alpha J(s-, X^2)}) E\left(\int_0^\infty e^{-\alpha t} g(Y(t, X^2)) dt\right) \\
 &\quad \text{(by the renewal property of } X^2) \\
 &= I_1 + \int_0^\infty P(\sigma \in ds) E(e^{-\alpha ms - \alpha J(s-, X^2)}) f^2(a) \\
 &= I_1 + E(e^{-\alpha m\sigma - \alpha J(\sigma-, X^2)}) f^2(a).
 \end{aligned}$$

This implies

$$\begin{aligned}
 I_1 &= f^2(a)[1 - E(e^{-\alpha m\sigma - \alpha J(\sigma-, X^2)})] \\
 &= f^2(a)\left[1 - \frac{k(S^1)}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}\right] \\
 &= f^2(a) \frac{\alpha m + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}.
 \end{aligned}$$

From (2.9) we have

$$\begin{aligned}
 f(a) &= f^2(a) \frac{\alpha m + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})} \\
 &\quad + \frac{\int_{S^1} k(db) \left[ E_b\left(\int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0) dt\right) + E_b(e^{-\alpha\sigma_a^0}) f(a) \right]}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha\sigma_a^0})}.
 \end{aligned} \tag{2.10}$$

Solving this for  $f$  we have

$$f(a) = \frac{f^2(a) \left( \alpha m + \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0}) \right) + \int_{S^1} k(db) E_b \left( \int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0) dt \right)}{\alpha m + \int_S k(db) E_b(1 - e^{-\alpha \sigma_a^0})}. \quad (2.11)$$

Let  $\varepsilon \downarrow 0$ , then

$$\int_{S^1} k(db) E_b \left( \int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0) dt \right) \longrightarrow \int_S k(db) E_b \left( \int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0) dt \right);$$

notice that

$$\begin{aligned} & \left| \int_S k(db) E_b \left( \int_0^{\sigma_a^0} e^{-\alpha t} g(X_t^0) dt \right) \right| \\ & \leq \|g\| \int_S k(db) E_b(1 - e^{-\alpha \sigma_a^0}) \\ & < \infty, \quad \| \cdot \| = \text{sup. norm} \end{aligned}$$

by virtue of  $\int_S k(db) E_b(\sigma_a^0 \wedge 1) < \infty$ . It is obvious that

$$\int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0}) \longrightarrow 0 \quad \text{as } \varepsilon \downarrow 0.$$

It follows from (2.10) and (2.3) that

$$\begin{aligned} f(a) &= f^2(a) \frac{\alpha m + \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0})}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0})} \\ &+ \frac{\int_{S^1} k(db) f(b)}{\alpha m + k(S^1) + \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0})} \end{aligned}$$

so that

$$\begin{aligned} & m(\alpha f(a) - \alpha f^2(a)) + f(a) \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0}) \\ &= f^2(a) \int_{S^2} k(db) E_b(1 - e^{-\alpha \sigma_a^0}) + \int_{S^1} k(db) (f(b) - f(a)). \end{aligned} \quad (2.12)$$

If  $m = 0$ , we can derive (2.4) and (2.6) from (2.11) and (2.12), letting  $\varepsilon \downarrow 0$  and noticing that

$$|f^2(a)| \equiv |f^{2,\varepsilon}(a)| \leq \|g\|/\alpha.$$

If  $m > 0$ , we need only prove that

$$\lim_{\varepsilon \downarrow 0} f^{2,\varepsilon}(a) = \frac{g(a)}{\alpha}, \quad (2.13)$$

in order to derive (2.4) and (2.6) from (2.11) and (2.12).

Let  $\eta > 0$  and set

$$\begin{aligned} V^1 &= V^{1,\eta} = \{u \in U : \sup_t \rho(u(t), a) \geq \eta\}, \\ V^2 &= V^{2,\eta} = U - V^1, \\ Y^i &= Y^{i,\varepsilon,\eta} = X^{2,\varepsilon}|_r V^{i,\eta}, \quad i = 1, 2. \end{aligned}$$

By the argument in the last step of the existence proof of Theorem 2.4.1, we have

$$\begin{aligned} \lambda &\equiv \lambda_{\varepsilon,\eta} \equiv n_{Y^1}(V^1) = n_{X^{2,\varepsilon}}(V^{1,\eta}) \\ &\leq \left( \inf_{\rho(b,a) > \eta} E_b(\sigma_a^0 \wedge 1) \right)^{-1} \int_{S^{2,\varepsilon}} k(db) E_b(\sigma_a^0 \wedge 1) \\ &\longrightarrow 0, \quad \varepsilon \downarrow 0 \quad \text{for } \eta \text{ fixed.} \end{aligned}$$

$Y^1$  is a discrete Poisson point process. Let  $\tau = \tau_{\varepsilon,\eta}$  be the first element in  $D_{Y^1}$ . Then  $\tau$  is exponentially distributed with rate  $= \lambda_{\varepsilon,\eta}$ . Using the same argument as in deriving (2.9), we obtain

$$\begin{aligned} &\left| f^{2,\varepsilon}(a) - \frac{g(a)}{\alpha} \right| \\ &\leq E \left( \int_0^\infty e^{-\alpha t} g_0(Y_t^{2,\varepsilon}) dt \right), \quad g_0(b) = |g(b) - g(a)| \\ &= E \left( \int_0^{m\tau + J(\tau-, Y^2)} e^{-\alpha t} g_0(Y(t, Y^2)) dt \right) \\ &\quad + E(e^{-\alpha m\tau - \alpha J(\tau-, Y^2)}) E \left( \int_0^{h(Y_\tau^1)} e^{-\alpha t} g_0(Y_\tau^1(t)) dt \right) \\ &\quad + E(e^{-\alpha m\tau - \alpha J(\tau-, Y^2) - \alpha h(Y_\tau^1)}) E \left( \int_0^\infty e^{-\alpha t} g_0(Y_t^{2,\varepsilon}) dt \right). \end{aligned}$$

Since  $\rho(Y(t, Y^2), a) < \eta$  for  $0 < t < m\tau + J(\tau-, Y^2)$ , we have

$$\left| f^{2,\varepsilon}(a) - \frac{g(a)}{\alpha} \right| \leq \delta(\eta) \frac{1}{\alpha} + E(e^{-\alpha m\tau}) \frac{\|g_0\|}{\alpha} + E(e^{-\alpha m\tau}) \frac{\|g_0\|}{\alpha}$$

where  $\delta(\eta) = \sup\{g_0(b), \rho(b, a) < \eta\} \rightarrow 0$  ( $\eta \downarrow 0$ ) by (2.8). Since  $\tau$  is exponentially distributed with rate  $\lambda_{\varepsilon, \eta}$ , we have

$$\begin{aligned} E(e^{-\alpha m \tau}) &= \int_0^\infty e^{-\alpha m t} e^{-\lambda_{\varepsilon, \eta} t} \lambda_{\varepsilon, \eta} dt = \frac{\lambda_{\varepsilon, \eta}}{\alpha m + \lambda_{\varepsilon, \eta}} \\ &\longrightarrow 0, \quad \varepsilon \downarrow 0 \end{aligned}$$

by  $m > 0$ . Thus we have

$$\limsup_{\varepsilon \downarrow 0} \left| f^{2, \varepsilon}(a) - \frac{g(a)}{\alpha} \right| \leq \delta(\eta) \cdot \frac{1}{\alpha} \longrightarrow 0, \quad \eta \downarrow 0.$$

This completes the proof. □

## 2.6 Examples

*Example 1* Let  $S = [0, \infty)$  and  $X^0$  be a diffusion in  $S$  stopped at 0 such that the generator of  $X^0$  is

$$\mathcal{G}^0 = \frac{d}{dm} \frac{d}{dx}.$$

Let 0 be an *exit or regular* (i.e., *exit* in Feller's new terminology) *boundary*, i.e.,

$$\int_0^1 m(\xi, 1) d\xi < \infty.$$

Then  $X^0$  satisfies (A<sup>0</sup>-1), (A<sup>0</sup>-2) and (A<sup>0</sup>-3) in Sect. 2.1; notice that

$$\inf_{\rho(b, 0) > \varepsilon} E_b(\sigma_0^0 \wedge 1) = E_\varepsilon(\sigma_0^0 \wedge 1) > 0.$$

We will investigate the condition (i) in Theorem 2.3.8:

$$\int_S k(db) E_b(\sigma_0^0 \wedge 1) < \infty. \tag{2.14}$$

This is equivalent to

$$\int_S k(db) E_b(1 - e^{-\sigma_0^0}) < \infty.$$



Since  $u(b) = E_b(e^{-\sigma_0^0})$  is a decreasing positive solution of

$$\begin{aligned} \frac{d}{dm} \frac{d}{dx} u &= u, \quad u(0) = 1, \\ u'(1) - u'(\xi) &= \int_{\xi}^1 u(\xi) m(d\xi) \approx m(\xi, 1) \quad (\xi \downarrow 0), \\ u(0) - u(b) &\approx \int_0^b (m(\xi, 1) - u'(1)) d\xi \end{aligned}$$

$(\alpha(\xi) \approx \beta(\xi) \quad (\xi \downarrow 0))$  means that we have  $c_1, c_2 > 0$  independent of  $\xi$  such that  $c_1\beta(\xi) < \alpha(\xi) < c_2\beta(\xi)$  near  $\xi = 0$ .

**Case 1. (regular case)** If 0 is a *regular* (i.e., *exit and entrance* in Feller's new terminology) *boundary*, i.e.,  $m(0, 1) < \infty$ , then

$$E_b(1 - e^{-\sigma_0^0}) = u(0) - u(b) \approx b \quad (b \downarrow 0).$$

Since  $E_b(1 - e^{-\sigma_0^0}) \rightarrow 1$  as  $b \rightarrow \infty$ ,  $E_b(1 - e^{-\sigma_0^0}) \approx b \wedge 1$  in  $0 < b < \infty$ . Therefore our condition (2.14) turns out to be

$$\int_0^\infty k(db)(b \wedge 1) < \infty.$$

**Case 2. (exit case)** If 0 is an *exit* (i.e., *exit and non-entrance* in Feller's new terminology) *boundary*, i.e.,  $m(0, 1) = \infty$ , then (2.14) turns out to be

$$\int_0^\infty k(db) \left[ \int_0^b m(\xi, 1) d\xi \wedge 1 \right] < \infty.$$

*Example 2* Let  $S = [0, \infty)$  and  $X^0$  be a deterministic motion with constant speed “−1”. Then  $P_b(\sigma_0^0 = b) = 1$  and so (2.14) is written as

$$\int_0^\infty k(db)(b \wedge 1) < \infty.$$

## Reference

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