

Chapter 2

Discrete-Time Sliding Mode Control

Abstract In this study, two different approaches to the sliding surface design are presented. First, a discrete-time integral sliding mode (ISM) control scheme for sampled-data systems is discussed. The control scheme is characterized by a discrete-time integral switching surface which inherits the desired properties of the continuous-time integral switching surface, such as full order sliding manifold with eigenvalue assignment, and elimination of the reaching phase. In particular, comparing with existing discrete-time sliding mode control, the scheme is able to achieve more precise tracking performance. It will be shown that, the control scheme achieves $O(T^2)$ steady-state error for state regulation and reference tracking with the widely adopted delay-based disturbance estimation. Another desirable feature is that the Discrete-time ISM control prevents the generation of overlarge control actions which are usually inevitable due to the deadbeat poles of a reduced order sliding manifold designed for sampled-data systems. Second, a terminal sliding mode control scheme is discussed. Terminal Sliding Mode (TSM) control is known for its high gain property nearby the vicinity of the equilibrium while retaining reasonably low gain elsewhere. This is desirable in digital implementation where the limited sampling frequency may incur chattering if the controller gain is overly high. The overall sliding surface integrates a linear switching surface with a terminal switching surface. The switching surface can be designed according to the precision requirement. The design is implemented on a specific SISO system example, but, the approach can be used in exactly the same way for any other system as long as it is SISO. The analysis and experimental investigation show that the TSM controller design outperforms the linear SM control.

2.1 Introduction

Research in discrete-time control has been intensified in recent years. A primary reason is that most control strategies nowadays are implemented in discrete-time. This also necessitated a rework in the sliding mode (SM) control strategy for sampled-data systems, [150, 154]. In such systems, the switching frequency in control variables is limited by T^{-1} ; where T is the sampling period. This has led researchers to approach

discrete-time sliding mode control from two directions. The first is the emulation that focuses on how to map continuous-time sliding mode control to discrete-time, and the switching term can be preserved, [62, 67]. The second is based on the equivalent control design and disturbance observer, [150, 154]. In the former, although high-frequency switching is theoretically desirable from the robustness point of view, it is usually hard to achieve in practice because of physical constraints, such as processor computational speed, A/D and D/A conversion delays, actuator bandwidth, etc. The use of a discontinuous control law in a sampled-data system will bring about chattering phenomenon in the vicinity of the sliding manifold, hence lead to a boundary layer with thickness $O(T)$, [150].

The effort to eliminate the chattering has been paid over 30 years. In continuous-time SM control, smoothing schemes such as boundary layer (saturation) are widely used, which in fact results in a continuous nonlinear feedback instead of switching control. Nevertheless, it is widely accepted by the community that this class of controllers can still be regarded as SM control. Similarly, in discrete-time SM control, by introducing a continuous control law, chattering can be eliminated. In such circumstance, the central issue is to guarantee the precision bound or the smallness of the error.

In [154] a discrete-time equivalent control was proposed. This approach results in the motion in $O(T^2)$ vicinity of the sliding manifold. The main difficulty in the implementation of this control law is that we need to know the disturbances for calculating the equivalent control. Lack of such information leads to an $O(T)$ error boundary.

The control proposed in [150] drives the sliding mode to $O(T^2)$ in one-step owing to the incorporation of deadbeat poles in the closed-loop system. State regulation was not considered in [150]. In fact, as far as the state regulation is concerned, the same SM controller design will produce an accuracy in $O(T)$ instead of $O(T^2)$ boundary. Moreover, the SM control with deadbeat poles requires large control efforts that might be undesirable in practice. Introducing saturation in the control input endangers the global stability or accuracy of the closed-loop system.

In this chapter we begin by discussing a discrete-time integral sliding manifold (ISM). With the full control of the system closed-loop poles and the elimination of the reaching phase, like the continuous-time integral sliding mode control, [40, 60, 155], the closed-loop system can achieve the desired control performance while avoiding the generation of overly large control inputs. It is worth highlighting that the discrete-time ISM control does not only drive the sliding mode into the $O(T^2)$ boundary, but also achieve the $O(T^2)$ boundary for state regulation.

After focusing on state feedback based ISM regulation, we consider the situation where output tracking and output feedback is required. Based on output feedback two approaches arose—design based on observers to construct the missing states, [55, 177], or design based on the output measurement only, [54, 146]. Recently integral sliding-mode control has been developed to improve controller design and consequently the control performance, [40, 60, 155], which use full state information. The first objective of this work is to extend ISM control to output-tracking problems.

We present three ISM control design approaches associated with state feedback, output feedback, and output feedback with state estimation, respectively.

After the discussion on ISM is concluded a second approach to discrete-time Sliding Mode control is presented. The approach is called terminal sliding mode control and has been developed in [150] to achieve finite time convergence of the system dynamics in the terminal sliding mode. In [3, 43, 60], the first-order terminal sliding mode control technique is developed for the control of a simple second-order nonlinear system and an 4th order nonlinear rigid robotic manipulator system with the result that the output tracking error can converge to zero in finite time.

Most of the Terminal Sliding Mode (TSM) approaches have been developed from the continuous-time point of view, [40, 150, 154, 155], however less work exists in the design from the discrete-time point of view. In [67] a continuous-time terminal sliding mode controller is first discretized and then applied to a sampled-data system. While it is possible to achieve acceptable result via this approach it makes more sense if the design was tackled entirely from the discrete-time point of view. This would allow us to gain more insight on the performance and stability issues and, thereby, achieve the best possible performance. In this paper, a revised terminal sliding mode control law is developed from the discrete-time point of view. It is shown that the new method can achieve better performance than with the linear SM control owing to the high gain property of the Terminal Sliding Mode in the vicinity of the origin.

In each of the above approaches, robustness is enhanced by the use of disturbance observers or estimators. Different disturbance observer approaches will be presented depending on the control problem. When the system states are accessible, the disturbance can be directly estimated using state and control signals delayed by one sampling period. The resulting control can perform arbitrary trajectory tracking or state regulation with $O(T^2)$ accuracy. When only outputs are accessible, the delayed disturbance estimation cannot be performed. In that case a dynamic disturbance observer design based on an unconventional Sliding Mode approach will be used. It will be shown that the SM observer can quickly and effectively estimate the disturbance and avoid the undesirable deadbeat response inherent in conventional SM based designs for sampled-data systems; in the sequel, avoid the generation of overly large estimation signals in the controller.

2.2 Problem Formulation

Consider the following continuous-time system with a nominal linear time invariant model and matched disturbance

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{B}\mathbf{f}(\mathbf{x}, t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}\tag{2.1}$$

where the state $\mathbf{x}(t) \in \Re^n$, the output $\mathbf{y}(t) \in \Re^m$, the control $\mathbf{u}(t) \in \Re^m$, and the disturbance $\mathbf{f}(\mathbf{x}, t) \in \Re^m$. The state matrix is $A \in \Re^{n \times n}$ and the control matrix is $B \in \Re^{n \times m}$ and the output matrix is $C \in \Re^{m \times n}$. For the system (2.1) the following assumptions are made:

Assumption 2.1 The pair (A, B) is Controllable and the pair (A, C) is Observable.

Assumption 2.2 Controllability and Observability is not lost upon sampling.

Assumption 2.3 The disturbance $\mathbf{f}(\mathbf{x}, t) \equiv \mathbf{f}(t)$ is smooth and uniformly bounded.

Proceeding further, the discretized counterpart of (2.1) can be given by

$$\begin{aligned} \mathbf{x}_{k+1} &= \Phi \mathbf{x}_k + \Gamma \mathbf{u}_k + \mathbf{d}_k, \quad \mathbf{x}_0 = \mathbf{x}(0) \\ \mathbf{y}_k &= C \mathbf{x}_k \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} \Phi &= e^{AT}, \quad \Gamma = \int_0^T e^{A\tau} d\tau B \\ \mathbf{d}_k &= \int_0^T e^{A\tau} B \mathbf{f}((k+1)T - \tau) d\tau \end{aligned}$$

and T is the sampling period. Here the disturbance $\mathbf{d}_k$ represents the influence accumulated from kT to $(k+1)T$, in the sequel it shall directly link to $\mathbf{x}_{k+1} = \mathbf{x}((k+1)T)$. From the definition of Γ it can be shown that

$$\Gamma = BT + \frac{1}{2!}ABT^2 + \dots = BT + MT^2 + O(T^3) \Rightarrow BT = \Gamma - MT^2 + O(T^3) \quad (2.3)$$

where M is a constant matrix because T is fixed.

The control objective is to design a discrete-time sliding manifold and a discrete-time SM control law for the sampled-data system (2.2), hence achieve as precisely as possible state regulation or output tracking.

Remark 2.1 The smoothness assumption made on the disturbance is to ensure that the disturbance bandwidth is sufficiently lower than the controller bandwidth, or the ignorance of high frequency components does not significantly affect the control performance. Indeed, if a disturbance has frequencies nearby or higher than the Nyquist frequency, for instance a non-smooth disturbance, a discrete-time SM control will not be able to handle it.

In order to proceed further, the following definition is necessary:

Definition 2.1 The magnitude of a variable v is said to be $O(T^r)$ if, and only if, there is a $C > 0$ such that for any sufficiently small T the following inequality holds

$$|v| \leq CT^r$$

where r is an integer. Denote $O(T^0) = O(1)$.

Remark 2.2 Note that $O(T^r)$ can be a scalar function or a vector valued function.

Associated with the above definition if there exists two variables v_1 and v_2 such that $v_1 \in O(T^r)$ and $v_2 \in O(T^{r+1})$ then $v_1 \gg v_2$ and, therefore, the following relations hold

$$O(T^{r+1}) + O(T^r) = O(T^r) \quad \forall r \in \mathbb{Z}$$

$$O(T^r) \cdot O(1) = O(T^r) \quad \forall r \in \mathbb{Z}$$

$$O(T^r) \cdot O(T^{-s}) = O(T^{r-s}) \quad \forall r, s \in \mathbb{Z}$$

where $\mathbb{Z}$ is the set of integers.

Based on (2.3) and the Definition, the magnitude of Γ is $O(T)$. Note that, as a consequence of sampling, the disturbance originally matched in continuous-time will contain mismatched components in the sampled-data system. This is summarized in the following lemma:

Lemma 2.1 If the disturbance $\mathbf{f}(t)$ in (2.1) is bounded and smooth, then

$$\mathbf{d}_k = \int_0^T e^{A\tau} B \mathbf{f}((k+1)T - \tau) d\tau = \Gamma \mathbf{f}_k + \frac{1}{2} \Gamma \mathbf{v}_k T + O(T^3) \quad (2.4)$$

where $\mathbf{v}_k = \mathbf{v}(kT)$, $\mathbf{v}(t) = \frac{d}{dt} \mathbf{f}(t)$, $\mathbf{d}_k - \mathbf{d}_{k-1} \in O(T^2)$, and $\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2} \in O(T^3)$.

Proof Consider the Taylor's series expansion of $\mathbf{f}((k+1)T - \tau)$

$$\mathbf{f}(kT + T - \tau) = \mathbf{f}_k + \mathbf{v}_k(T - \tau) + \frac{1}{2!} \mathbf{w}_k(T - \tau)^2 + \dots = \mathbf{f}_k + \mathbf{v}_k(T - \tau) + \xi(T - \tau)^2 \quad (2.5)$$

where $\mathbf{v}(t) = \frac{d}{dt} \mathbf{f}(t)$, $\mathbf{w}(t) = \frac{d^2}{dt^2} \mathbf{f}(t)$ and $\xi = \frac{1}{2!} \mathbf{w}(\mu)$ and μ is a time value between kT and $(k+1)T$, [43]. Substituting (2.5) into the expression of $\mathbf{d}_k$

$$\mathbf{d}_k = \int_0^T e^{A\tau} B \mathbf{f}_k d\tau + \int_0^T e^{A\tau} B \mathbf{v}_k(T - \tau) d\tau + \int_0^T e^{A\tau} B \xi(T - \tau)^2 d\tau. \quad (2.6)$$

For clarity, each integral will be analyzed separately. Since $\mathbf{f}_k$ is independent of τ it can be taken out of the first integral

$$\int_0^T e^{A\tau} B \mathbf{f}_k d\tau = \int_0^T e^{A\tau} B d\tau \mathbf{f}_k = \Gamma \mathbf{f}_k. \quad (2.7)$$

In order to solve the second integral term, it is necessary to expand $e^{A\tau}$ into series form. Thus,

$$\int_0^T e^{A\tau} B \mathbf{v}_k(T - \tau) d\tau = \int_0^T \left[e^{A\tau} B - \left(B + AB\tau + \frac{1}{2!} A^2 B \tau^2 + \dots \right) \tau \right] d\tau \mathbf{v}_k. \quad (2.8)$$

Solving the integral leads to

$$\int_0^T e^{A\tau} B \mathbf{v}_k(T - \tau) d\tau = \left[\Gamma - \left(\frac{1}{2!} BT + \frac{1}{3!} ABT^2 + \dots \right) \right] T \mathbf{v}_k. \quad (2.9)$$

Simplifying the result with the aid of (2.3)

$$\int_0^T e^{A\tau} B \mathbf{v}_k(T - \tau) d\tau = \left[\Gamma - \frac{1}{2} \Gamma + \frac{1}{2} MT^2 - \left(\frac{1}{3!} ABT^2 + \frac{1}{4!} A^2 BT^2 + \dots \right) \right] T \mathbf{v}_k. \quad (2.10)$$

Simplifying the above expression further

$$\int_0^T e^{A\tau} B \mathbf{v}_k(T - \tau) d\tau = \frac{1}{2} \Gamma \mathbf{v}_k T + \hat{M} \mathbf{v}_k T^3 \quad (2.11)$$

where $\hat{M}$ is a constant matrix. Finally, note that in (2.6) the third integral is $O(T^3)$, since, the term inside the integral is already $O(T^2)$, therefore

$$\int_0^T e^{A\tau} B \xi(T - \tau)^2 d\tau = O(T^3). \quad (2.12)$$

Thus, combining (2.7), (2.10) and (2.12) leads to

$$\mathbf{d}_k = \Gamma \mathbf{f}_k + \frac{1}{2} \Gamma \mathbf{v}_k T + \hat{M} T^3 \mathbf{v}_k + O(T^3) = \Gamma \mathbf{f}_k + \frac{1}{2} \Gamma \mathbf{v}_k T + O(T^3). \quad (2.13)$$

Now evaluate

$$\mathbf{d}_k - \mathbf{d}_{k-1} = \Gamma(\mathbf{f}_k - \mathbf{f}_{k-1}) + \frac{1}{2} \Gamma(\mathbf{v}_k - \mathbf{v}_{k-1})T + O(T^3). \quad (2.14)$$

From (2.5) and letting $\tau = 0$, $\mathbf{f}_k - \mathbf{f}_{k-1} \in O(T)$. From (2.3), $\Gamma \in O(T)$. In the sequel $\mathbf{d}_k - \mathbf{d}_{k-1} \in O(T^2)$, if the assumptions on the boundedness and smoothness of $\mathbf{f}(t)$

hold. Finally, we notice that (2.14) is the difference of the first order approximation, whereas

$$\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2} = \Gamma(\mathbf{f}_k - 2\mathbf{f}_{k-1} + \mathbf{f}_{k-2}) + \frac{1}{2}\Gamma(\mathbf{v}_k - 2\mathbf{v}_{k-1} + \mathbf{v}_{k-2}) + O(T^3) \quad (2.15)$$

is the difference of the second order approximation. Accordingly the magnitude of $\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}$ is $O(T^3)$, [43].

Note that the magnitude of the mismatched part in the disturbance $\mathbf{d}_k$ is of the order $O(T^3)$.

2.3 Classical Discrete-Time Sliding Mode Control Revisited

2.3.1 State Regulation

Consider the well established discrete-time sliding surface, [150, 154], shown below

$$\sigma_k = D\mathbf{x}_k \quad (2.16)$$

where $\sigma_k \in \mathbb{R}^m$ and D is a constant matrix of rank m . The objective is to steer the states towards and force them to stay on the sliding manifold $\sigma_k = 0$ at every sampling instant. The control accuracy of this class of sampled-data SM control is given by the following lemma:

Lemma 2.2 *With $\sigma_k = D\mathbf{x}_k$ and equivalent control based on a disturbance estimate*

$$\hat{\mathbf{d}}_k = \mathbf{x}_k - \Phi\mathbf{x}_{k-1} - \Gamma\mathbf{u}_{k-1}$$

then there exists a matrix D such that the control accuracy of the closed-loop system is

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k\| \leq O(T).$$

Proof Discrete-time equivalent control is defined by solving $\sigma_{k+1} = 0$, [150]. This leads to

$$\mathbf{u}_k^{\text{eq}} = -(D\Gamma)^{-1} D(\Phi\mathbf{x}_k + \mathbf{d}_k) \quad (2.17)$$

with D selected such that the closed-loop system achieves desired performance and $D\Gamma$ is invertible, [3]. Under practical considerations, the control cannot be implemented in the same form as in (2.33) because of the lack of prior knowledge regarding the discretized disturbance $\mathbf{d}_k$. However, with some continuity assumptions on the

disturbance, $\mathbf{d}_k$ can be estimated by its previous value $\mathbf{d}_{k-1}$, [150]. The substitution of $\mathbf{d}_k$ by $\mathbf{d}_{k-1}$ will at most result in an error of $O(T^2)$. With reasonably small sampling period as in motion control or mechatronics, such a substitution will be effective. Let

$$\hat{\mathbf{d}}_k = \mathbf{d}_{k-1} = \mathbf{x}_k - \Phi \mathbf{x}_{k-1} - \Gamma \mathbf{u}_{k-1} \quad (2.18)$$

where $\hat{\mathbf{d}}_k$ is the estimate of $\mathbf{d}_k$. Thus, analogous to the equivalent control law (2.33), the practical control law is

$$\mathbf{u}_k = -(D\Gamma)^{-1} D (\Phi \mathbf{x}_k + \mathbf{d}_{k-1}). \quad (2.19)$$

Substituting the sampled-data dynamics (2.2), applying the above control law, and using the conclusions in Lemma 2.1, yield

$$\sigma_{k+1} = D (\Phi \mathbf{x}_k + \Gamma \mathbf{u}_k + \mathbf{d}_k) = D (\mathbf{d}_k - \mathbf{d}_{k-1}) = O(T^2) \quad (2.20)$$

which is the result shown in [150]. The closed-loop dynamics is

$$\mathbf{x}_{k+1} = \left(\Phi - \Gamma (D\Gamma)^{-1} D \Phi \right) \mathbf{x}_k + \left(I - \Gamma (D\Gamma)^{-1} D \right) \mathbf{d}_{k-1} + \mathbf{d}_k - \mathbf{d}_{k-1} \quad (2.21)$$

where the matrix $(\Phi - \Gamma (D\Gamma)^{-1} D \Phi)$ has m zero eigenvalues and $n - m$ eigenvalues to be assigned inside the unit circle in the complex z -plane. It is possible to simplify (2.21) further to

$$\mathbf{x}_{k+1} = \left(\Phi - \Gamma (D\Gamma)^{-1} D \Phi \right) \mathbf{x}_k + \delta_k \quad (2.22)$$

where $\delta_k = (I - \Gamma (D\Gamma)^{-1} D) \mathbf{d}_{k-1} + \mathbf{d}_k - \mathbf{d}_{k-1}$. From Lemma 2.1,

$$\begin{aligned} \delta_k &= \mathbf{d}_k - \mathbf{d}_{k-1} + \left(I - \Gamma (D\Gamma)^{-1} D \right) \left(\Gamma \mathbf{f}_{k-1} + \frac{1}{2} \Gamma \mathbf{v}_{k-1} T + O(T^3) \right) \\ &= O(T^2) + \left(I - \Gamma (D\Gamma)^{-1} D \right) O(T^3) = O(T^2). \end{aligned} \quad (2.23)$$

In the preceding derivation, we use the relations $(I - \Gamma (D\Gamma)^{-1} D) \Gamma = 0$, $\|I - \Gamma (D\Gamma)^{-1} D\| \leq 1$ and $O(1) \cdot O(T^3) = O(T^3)$. Note that since m eigenvalues of the matrix $(\Phi - \Gamma (D\Gamma)^{-1} D \Phi)$ are deadbeat, it can be written as

$$(\Phi - \Gamma (D\Gamma)^{-1} D \Phi) = PJP^{-1} \quad (2.24)$$

where P is a transformation matrix and J is the Jordan matrix of the eigenvalues of $(\Phi - \Gamma (D\Gamma)^{-1} D\Phi)$. The matrix J can be written as

$$J = \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} \quad (2.25)$$

where $J_1 \in \mathfrak{N}^{m \times m}$ and $J_2 \in \mathfrak{N}^{(n-m) \times (n-m)}$ and are given by

$$J_1 = \begin{bmatrix} \mathbf{0} & I_{m-1} \\ 0 & \mathbf{0} \end{bmatrix} \quad \& \quad J_2 = \begin{bmatrix} \lambda_{m+1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

where λ_j are the eigenvalues of $(\Phi - \Gamma (D\Gamma)^{-1} D\Phi)$. For simplicity it is assumed that the non-zero eigenvalues are designed to be distinct and that their continuous time counterparts are of order $O(1)$. Then the solution of (2.22) is

$$\mathbf{x}_k = P J^k P^{-1} \mathbf{x}_0 + P \left(\sum_{i=0}^{k-1} J^i P^{-1} \delta_{k-i-1} \right). \quad (2.26)$$

Rewriting (2.26) as

$$\mathbf{x}_k = P J^k P^{-1} \mathbf{x}_0 + P \left(\sum_{i=0}^{k-1} \begin{bmatrix} J_1^i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^{-1} \delta_{k-i-1} \right) + P \left(\sum_{i=0}^{k-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2^i \end{bmatrix} P^{-1} \delta_{k-i-1} \right) \quad (2.27)$$

it is easy to verify that $J_1^i = \mathbf{0}$ for $i \geq m$. Thus, (2.27) becomes (for $k \geq m$)

$$\mathbf{x}_k = P J^k P^{-1} \mathbf{x}_0 + P \left(\sum_{i=0}^m \begin{bmatrix} J_1^i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} P^{-1} \delta_{k-i-1} \right) + P \left(\sum_{i=0}^{k-1} \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2^i \end{bmatrix} P^{-1} \delta_{k-i-1} \right). \quad (2.28)$$

Notice $\|J_1\| = 1$ and $\|J_2\| = \lambda_{\max} = \max\{\lambda_{m+1}, \dots, \lambda_n\}$. Hence, from (2.28)

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k\| \leq \|P\| \left(\sum_{i=0}^m \left\| \begin{bmatrix} J_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \right\|^i \|P^{-1}\| \cdot \|\delta_{k-i-1}\| + \sum_{i=0}^{k-1} \left\| \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & J_2 \end{bmatrix} \right\|^i \|P^{-1}\| \cdot \|\delta_{k-i-1}\| \right). \quad (2.29)$$

Since $\lambda_{\max} < 1$ for a stable system,

$$\sum_{i=0}^{\infty} \|J_2\|^i = \frac{1}{1 - \lambda_{\max}} \quad \& \quad \sum_{i=0}^m \|J_1\|^i = m.$$

Using Tustin's approximation

$$\lambda_{\max} = \frac{2 + Tp}{2 - Tp} \Rightarrow \frac{1}{1 - \lambda_{\max}} = \frac{1}{1 - \frac{2+Tp}{2-Tp}} = \frac{2 - Tp}{-2Tp} \leq O(T^{-1}) \quad (2.30)$$

where $p \geq O(1)$ is the corresponding pole in continuous-time. Assuming $m \in O(1)$, and using the fact $\|P^{-1}\| = \|P\|^{-1}$, it can be derived from (2.29) that

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k\| \leq O(1) \cdot O(T^2) + O(T^{-1}) \cdot O(T^2) = O(T). \quad (2.31)$$

Remark 2.3 Under practical considerations, it is generally advisable to select the pole p large enough such that the system has a fast enough response. With the selection of a small sampling period T , a pole of order $O(T)$ would lead to an undesirably slow response. Thus, it makes sense to select a pole of order $O(1)$ or larger.

Remark 2.4 The SM control in [150] guarantees that the sliding variable σ_k is of order $O(T^2)$, but cannot guarantee the same order of magnitude of steady-state errors for the system state variables. In the next section, we show that an integral sliding mode design can achieve a more precise state regulation.

2.3.2 Output Tracking

Consider the discrete-time sliding manifold given below, [150, 154],

$$\sigma_k = D_o (\mathbf{r}_k - \mathbf{y}_k) \quad (2.32)$$

where D_o is a constant matrix of rank m and $\mathbf{r}_k \in \mathfrak{R}^m$. The objective is to force the output $\mathbf{y}_k$ to track the reference $\mathbf{r}_k$. The property for this class of sampled-data SM control is given by the following lemma:

Lemma 2.3 For $\sigma_k = D_o (\mathbf{r}_k - \mathbf{y}_k)$ and control based on a disturbance estimate

$$\hat{\mathbf{d}}_k = \mathbf{x}_k - \Phi \mathbf{x}_{k-1} - \Gamma \mathbf{u}_{k-1}$$

the closed-loop system has the following properties

$$\sigma_{k+1} \in O\left(T^2\right)$$

$$\mathbf{r}_{k+1} - \mathbf{y}_{k+1} \in O\left(T^2\right)$$

Proof Similar to the regulation problem the discrete-time equivalent control is defined by solving $\sigma_{k+1} = 0$, [150]. This leads to

$$\mathbf{u}_k^{\text{eq}} = (D_o C \Gamma)^{-1} D_o (\mathbf{r}_{k+1} - C \Phi \mathbf{x}_k - C \mathbf{d}_k) \quad (2.33)$$

with D_o selected such that $D_o C \Gamma$ is invertible. As in the regulation case, the control cannot be implemented in the same form as in (2.33) because of the lack of knowledge of $\mathbf{d}_k$ which requires a priori knowledge of the disturbance $\mathbf{f}(t)$. Thus, the delayed disturbance $\mathbf{d}_{k-1}$ will be used

$$\hat{\mathbf{d}}_k = \mathbf{d}_{k-1} = \mathbf{x}_k - \Phi \mathbf{x}_{k-1} - \Gamma \mathbf{u}_{k-1} \quad (2.34)$$

Thus, the control becomes

$$\mathbf{u}_k = (D_o C \Gamma)^{-1} D_o (\mathbf{r}_{k+1} - C \Phi \mathbf{x}_k - C \mathbf{d}_{k-1}) \quad (2.35)$$

The closed-loop system under the control given by (2.35) is

$$\mathbf{x}_{k+1} = [\Phi - \Gamma (D_o C \Gamma)^{-1} D_o C \Phi] \mathbf{x}_k + \Gamma (D_o C \Gamma)^{-1} D_o \mathbf{r}_{k+1} + \mathbf{d}_k - \Gamma (D_o C \Gamma)^{-1} D_o C \mathbf{d}_{k-1}. \quad (2.36)$$

Note that $(D_o C \Gamma)^{-1} D_o = (C \Gamma)^{-1}$. Simplifying (2.36) further gives

$$\mathbf{x}_{k+1} = [\Phi - \Gamma (C \Gamma)^{-1} C \Phi] \mathbf{x}_k + \Gamma (C \Gamma)^{-1} \mathbf{r}_{k+1} + \mathbf{d}_k - \Gamma (C \Gamma)^{-1} C \mathbf{d}_{k-1}. \quad (2.37)$$

where the eigenvalues of the matrix $[\Phi - \Gamma (C \Gamma)^{-1} C \Phi]$ are the transmission zeros of the system, [167]. Postmultiplication of (2.37) with C results in,

$$\mathbf{y}_{k+1} = C \mathbf{x}_{k+1} = \mathbf{r}_{k+1} + C (\mathbf{d}_k - \mathbf{d}_{k-1}) = \mathbf{r}_{k+1} + O\left(T^2\right). \quad (2.38)$$

Substitution of (2.38) into the forward expression of (2.32) results in

$$\sigma_{k+1} = D_o (\mathbf{r}_{k+1} - \mathbf{y}_{k+1}) \in O\left(T^2\right). \quad (2.39)$$

The above result shows that with the control given by (2.35) the output is stable and that the tracking error converges to a bound of order $O(T^2)$. However, the stability of the whole system is guaranteed only if the transmission zeros are stable. Looking back at (2.37), it is very simple to show that $[\Phi - \Gamma (C\Gamma)^{-1} C\Phi]$ has m eigenvalues in the origin. Note, that those m deadbeat eigenvalues correspond to the output deadbeat response. If the matrices are partitioned as shown

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \quad \& \quad C = [C_1 \mid C_2]$$

where $(\Phi_{11}, C_1, \Gamma_1) \in \mathbb{R}^{m \times m}$, $(\Phi_{12}, C_2) \in \mathbb{R}^{m \times n-m}$, $(\Phi_{21}, \Gamma_2) \in \mathbb{R}^{n-m \times m}$ and $\Phi_{22} \in \mathbb{R}^{n-m \times n-m}$. The eigenvalues of $[\Phi - \Gamma (C\Gamma)^{-1} C\Phi]$ can be found from

$$\det[\lambda I_n - (\Phi - \Gamma (C\Gamma)^{-1} C\Phi)] = \det \begin{bmatrix} \lambda I_m - \Phi_{11} + \Gamma_1 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & -\Phi_{12} + \Gamma_1 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \\ -\Phi_{21} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & \lambda I_{n-m} - \Phi_{22} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \end{bmatrix} = 0$$

If the top row is premultiplied with C_1 and the bottom row premultiplied by C_2 and the results summed and used as the new top row, the following is obtained

$$\det \begin{bmatrix} \lambda C_1 & \lambda C_2 \\ -\Phi_{21} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & \lambda I_{n-m} - \Phi_{22} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \end{bmatrix} = 0 \quad (2.40)$$

This can be further simplified to

$$\lambda^m \det \begin{bmatrix} C_1 & C_2 \\ -\Phi_{21} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & \lambda I_{n-m} - \Phi_{22} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \end{bmatrix} = 0 \quad (2.41)$$

The above result shows that there are m eigenvalues at the origin.

Remark 2.5 The conventional method guarantees that the sliding surface is of order $O(T^2)$ and deadbeat tracking error of order $O(T^2)$. However, deadbeat response is not practical as it requires large control effort and the addition of input saturation would sacrifice global stability.

2.4 Discrete-Time Integral Sliding Mode Control

2.4.1 State Regulation with ISM

Consider the new discrete-time integral sliding manifold defined below

$$\begin{aligned}\sigma_k &= D\mathbf{x}_k - D\mathbf{x}_0 + \boldsymbol{\varepsilon}_k \\ \boldsymbol{\varepsilon}_k &= \boldsymbol{\varepsilon}_{k-1} + E\mathbf{x}_{k-1}\end{aligned}\tag{2.42}$$

where $\sigma_k \in \mathbb{R}^m$, $\boldsymbol{\varepsilon}_k \in \mathbb{R}^m$, and matrices D and E are constant and of rank m . The term $D\mathbf{x}_0$ is used to eliminate the reaching phase. (2.42) is the discrete-time counterpart of the following sliding manifold, [40]

$$\sigma(t) = D\mathbf{x}(t) - D\mathbf{x}(0) + \int_0^t E\mathbf{x}(\tau)d\tau = 0.\tag{2.43}$$

Theorem 2.1 *Based on assumption that the pair (Φ, Γ) in (2.2) is controllable, there exists a matrix K such that the eigenvalues of $\Phi - \Gamma K$ are distinct and within the unit circle. Choose the control law*

$$\mathbf{u}_k = (D\Gamma)^{-1} D\mathbf{x}_0 - (D\Gamma)^{-1} \left((D\Phi + E) \mathbf{x}_k + D\hat{\mathbf{d}}_k + \boldsymbol{\varepsilon}_k \right)\tag{2.44}$$

where $D\Gamma$ is invertible,

$$E = -D(\Phi - I - \Gamma K)\tag{2.45}$$

and $\hat{\mathbf{d}}_k$ is the disturbance compensation (2.67). Then the closed-loop dynamics is

$$\mathbf{x}_{k+1} = (\Phi - \Gamma K) \mathbf{x}_k + \boldsymbol{\zeta}_k\tag{2.46}$$

with $\boldsymbol{\zeta}_k \in \mathbb{R}^n$ is $O(T^3)$, and

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k\| \leq O(T^2).$$

Proof Consider a forward expression of (2.42)

$$\begin{aligned}\sigma_{k+1} &= D\mathbf{x}_{k+1} - D\mathbf{x}_0 + \boldsymbol{\varepsilon}_{k+1} \\ \boldsymbol{\varepsilon}_{k+1} &= \boldsymbol{\varepsilon}_k + E\mathbf{x}_k\end{aligned}\tag{2.47}$$

Substituting $\boldsymbol{\varepsilon}_{k+1}$ and (2.2) into the expression of the sliding manifold in (2.47) leads to

$$\sigma_{k+1} = (D\Phi + E) \mathbf{x}_k + D(\Gamma \mathbf{u}_k + \mathbf{d}_k) + \varepsilon_k - D\mathbf{x}_0. \quad (2.48)$$

The equivalent control is found by solving for $\sigma_{k+1} = 0$

$$\mathbf{u}_k^{\text{eq}} = (D\Gamma)^{-1} D\mathbf{x}_0 - (D\Gamma)^{-1} ((D\Phi + E) \mathbf{x}_k + D\mathbf{d}_k + \varepsilon_k). \quad (2.49)$$

Similar to the classical case with control given by (2.33), implementation of (2.49) would require a priori knowledge of the disturbance $\mathbf{d}_k$. By replacing the disturbance in (2.49) with its estimate $\hat{\mathbf{d}}_k$, which is defined in (2.67), the practical control law is

$$\mathbf{u}_k = (D\Gamma)^{-1} D\mathbf{x}_0 - (D\Gamma)^{-1} ((D\Phi + E) \mathbf{x}_k + D\hat{\mathbf{d}}_k + \varepsilon_k) \quad (2.50)$$

Substitution of $\mathbf{u}_k$ defined by (2.50) into (2.2) leads to the closed-loop equation in the sliding mode

$$\begin{aligned} \mathbf{x}_{k+1} = & \left[\Phi - \Gamma (D\Gamma)^{-1} (D\Phi + E) \right] \mathbf{x}_k - \Gamma (D\Gamma)^{-1} \varepsilon_k + \Gamma (D\Gamma)^{-1} D\mathbf{x}_0 \\ & + \mathbf{d}_k - \Gamma (D\Gamma)^{-1} D\hat{\mathbf{d}}_k. \end{aligned} \quad (2.51)$$

Let us derive the sliding dynamics. Rewriting (2.47)

$$\sigma_{k+1} = D\mathbf{x}_{k+1} + E\mathbf{x}_k - D\mathbf{x}_0 + \varepsilon_k. \quad (2.52)$$

Substitution of (2.51) into (2.52) leads to

$$\sigma_{k+1} = D\mathbf{d}_k - D\hat{\mathbf{d}}_k = D\mathbf{d}_k - D\mathbf{d}_{k-1} \in O(T^2), \quad (2.53)$$

that is, the introduction of ISM control leads to the same sliding dynamics as in [150]. Next, solving ε_k in (2.42) in terms of $\mathbf{x}_k$ and σ_k

$$\varepsilon_k = \sigma_k - D\mathbf{x}_k + D\mathbf{x}_0, \quad (2.54)$$

and substituting it into (2.51), the closed-loop dynamics becomes

$$\mathbf{x}_{k+1} = \left[\Phi - \Gamma (D\Gamma)^{-1} (D(\Phi - I) + E) \right] \mathbf{x}_k - \Gamma (D\Gamma)^{-1} \sigma_k + \mathbf{d}_k - \Gamma (D\Gamma)^{-1} D\hat{\mathbf{d}}_k. \quad (2.55)$$

In (2.55), σ_k can be substituted by $\sigma_k = D\mathbf{d}_{k-1} - D\mathbf{d}_{k-2}$ as can be inferred from (2.53). Also, under the condition (2.45), $D(\Phi - I) + E = D\Gamma K$. Therefore, $\Phi - \Gamma (D\Gamma)^{-1} (D(\Phi - I) + E) = \Phi - \Gamma K$. Since it is assumed that the pair (Φ, Γ) is controllable, there exists a matrix K such that eigenvalues of $\Phi - \Gamma K$ can be placed anywhere inside the unit circle. Note that, the selection of matrix D is arbitrary as long as it guarantees the invertibility of $D\Gamma$ while matrix E , computed using (2.45),

guarantees the desired closed-loop performance. Thus, we have

$$\mathbf{x}_{k+1} = (\Phi - \Gamma K) \mathbf{x}_k + \mathbf{d}_k - \Gamma (D\Gamma)^{-1} D \mathbf{d}_{k-1} - \Gamma (D\Gamma)^{-1} D (\mathbf{d}_{k-1} - \mathbf{d}_{k-2}). \quad (2.56)$$

Note that in (2.56), the disturbance estimate $\hat{\mathbf{d}}_k$ has been replaced by $\mathbf{d}_{k-1}$. Further simplification of (2.56) leads to

$$\mathbf{x}_{k+1} = (\Phi - \Gamma K) \mathbf{x}_k + \boldsymbol{\zeta}_k \quad (2.57)$$

where

$$\boldsymbol{\zeta}_k = \mathbf{d}_k - 2\Gamma (D\Gamma)^{-1} D \mathbf{d}_{k-1} + \Gamma (D\Gamma)^{-1} D \mathbf{d}_{k-2}. \quad (2.58)$$

The magnitude of $\boldsymbol{\zeta}_k$ can be evaluated as below. Adding and subtracting $2\mathbf{d}_{k-1}$ and $\mathbf{d}_{k-2}$ from the right hand side of (2.58) yield

$$\boldsymbol{\zeta}_k = (\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}) + \left(I - \Gamma (D\Gamma)^{-1} D \right) (2\mathbf{d}_{k-1} - \mathbf{d}_{k-2}). \quad (2.59)$$

In Lemma 2.1, it has been shown that $(\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}) \in O(T^3)$. On the other hand, from (2.4) we have

$$\begin{aligned} & \left(I - \Gamma (D\Gamma)^{-1} D \right) (2\mathbf{d}_{k-1} - \mathbf{d}_{k-2}) \\ &= \left(I - \Gamma (D\Gamma)^{-1} D \right) \left(\Gamma (2\mathbf{f}_{k-1} - \mathbf{f}_{k-2}) + \frac{T}{2} \Gamma (2\mathbf{v}_{k-1} - \mathbf{v}_{k-2}) + O(T^3) \right) \end{aligned}$$

Note that $(I - \Gamma (D\Gamma)^{-1} D) \Gamma = 0$, thus

$$\left(I - \Gamma (D\Gamma)^{-1} D \right) \left(\Gamma (2\mathbf{f}_{k-1} - \mathbf{f}_{k-2}) + \frac{T}{2} \Gamma (2\mathbf{v}_{k-1} - \mathbf{v}_{k-2}) \right) = 0.$$

Furthermore, $\| (I - \Gamma (D\Gamma)^{-1} D) \| \leq 1$, thus $(I - \Gamma (D\Gamma)^{-1} D) O(T^3)$ remains $O(T^3)$. This concludes that

$$\boldsymbol{\zeta}_k \in O(T^3).$$

Comparing (2.57) with (2.22), the difference is that $\boldsymbol{\delta}_k \in O(T^2)$ whereas $\boldsymbol{\zeta}_k \in O(T^3)$. Further, by doing a similarity decomposition for dynamics of (2.57), only the J_2 matrix of dimension n exists. Thus the derivation procedure shown in (2.22)–(2.31) holds for (2.57), and the solution is

$$\mathbf{x}_k = (\Phi - \Gamma K)^k \mathbf{x}_0 + \sum_{i=0}^{k-1} (\Phi - \Gamma K)^i \boldsymbol{\zeta}_{k-i-1}. \quad (2.60)$$

Assuming distinct eigenvalues of $\Phi - \Gamma K$ and following the procedure that resulted in (2.31), it can be shown that

$$\lim_{k \rightarrow \infty} \left\| \sum_{i=0}^{k-1} (\Phi - \Gamma K)^i \boldsymbol{\zeta}_{k-i-1} \right\| \in O(T^2). \quad (2.61)$$

Finally, it is concluded that

$$\lim_{k \rightarrow \infty} \|\mathbf{x}_k\| \leq O(T^2). \quad (2.62)$$

Remark 2.6 From the foregoing derivations, it can be seen that the state errors are always one order higher than the disturbance term $\boldsymbol{\zeta}_k$ in the worst case due to convolution as shown by (2.61). After incorporating the integral sliding manifold, the off-set from the disturbance can be better compensated, in the sequel leading to a smaller steady state error boundary.

Remark 2.7 It is evident from the above analysis that, for the class of systems considered in this work and in [150, 154], the equivalent control based SM control with disturbance observer guarantees the motion of the states within an $O(T^2)$ bound, which is smaller than $O(T)$ for T sufficiently small, and is lower than what can be achieved by SM control using switching control, [40, 155]. In such circumstance, without the loss of precision we can relax the necessity of incorporating a switching term, in the sequel avoid exciting chattering.

2.4.2 Output-Tracking ISM Control: State Feedback Approach

In this section we discuss the state-feedback-based output ISM control. We first present the controller design using an appropriate integral sliding surface and a delay based disturbance estimation. Next the stability condition of the closed-loop system and the error dynamics under output ISM control are derived. The ultimate tracking error bound is analyzed.

2.4.2.1 Controller Design

Consider the discrete-time integral sliding surface defined below,

$$\begin{aligned} \boldsymbol{\sigma}_k &= \mathbf{e}_k - \mathbf{e}_0 + \boldsymbol{\varepsilon}_k \\ \boldsymbol{\varepsilon}_k &= \boldsymbol{\varepsilon}_{k-1} + E\mathbf{e}_{k-1} \end{aligned} \quad (2.63)$$

where $\mathbf{e}_k = \mathbf{r}_k - \mathbf{y}_k$ is the tracking error, $\sigma_k, \boldsymbol{\varepsilon}_k \in \mathbb{R}^m$ are the sliding function and integral vectors, and $E \in \mathbb{R}^{m \times m}$ is an integral gain matrix.

By virtue of the concept of equivalent control, a SM control law can be derived by letting $\sigma_{k+1} = 0$. From (2.63), $-\mathbf{e}_0 + \boldsymbol{\varepsilon}_k = \sigma_k - \mathbf{e}_k$, we have

$$\begin{aligned}\sigma_{k+1} &= \mathbf{e}_{k+1} - \mathbf{e}_0 + \boldsymbol{\varepsilon}_{k+1} = \mathbf{e}_{k+1} - \mathbf{e}_0 + \boldsymbol{\varepsilon}_k + E\mathbf{e}_k \\ &= \mathbf{e}_{k+1} - (I_m - E)\mathbf{e}_k + \sigma_k.\end{aligned}\quad (2.64)$$

From the system dynamics (2.2), the output error $\mathbf{e}_{k+1}$ is

$$\mathbf{e}_{k+1} = \mathbf{r}_{k+1} - [C\Phi\mathbf{x}_k + C\Gamma\mathbf{u}_k + C\mathbf{d}_k]$$

and

$$\begin{aligned}\sigma_{k+1} &= \mathbf{r}_{k+1} - [C\Phi\mathbf{x}_k + C\Gamma\mathbf{u}_k + C\mathbf{d}_k] - (I_m - E)\mathbf{e}_k + \sigma_k \\ &= \mathbf{a}_k - C\Gamma\mathbf{u}_k - C\mathbf{d}_k.\end{aligned}\quad (2.65)$$

where $\mathbf{a}_k = \mathbf{r}_{k+1} - \Lambda\mathbf{e}_k - C\Phi\mathbf{x}_k + \sigma_k$, and $\Lambda = I_m - E$. Assuming $\sigma_{k+1} = 0$, we can derive the equivalent control

$$\mathbf{u}_k^{\text{eq}} = (C\Gamma)^{-1}(\mathbf{a}_k - C\mathbf{d}_k). \quad (2.66)$$

Note that the control (2.66) is based on the current value of the disturbance $\mathbf{d}_k$ which is unknown and therefore cannot be implemented in the current form. To overcome this, the disturbance estimate will be used. When the system states are accessible, a delay based disturbance estimate can be easily derived from the system (2.2)

$$\hat{\mathbf{d}}_k = \mathbf{d}_{k-1} = \mathbf{x}_k - \Phi\mathbf{x}_{k-1} - \Gamma\mathbf{u}_{k-1}. \quad (2.67)$$

Note that $\mathbf{d}_{k-1}$ is the exogenous disturbance and bounded, therefore $\hat{\mathbf{d}}_k$ is bounded for all k . Using the disturbance estimation (2.67), the actual ISM control law is given by

$$\mathbf{u}_k = (C\Gamma)^{-1}(\mathbf{a}_k - C\hat{\mathbf{d}}_k). \quad (2.68)$$

2.4.2.2 Stability Analysis

Since the integral switching surface (2.63) consists of outputs only, it is necessary to examine the closed-loop stability in state space when the ISM control (2.68) and disturbance estimation (2.67) are used.

Expressing $\mathbf{e}_k = \mathbf{r}_k - C\mathbf{x}_k$, the ISM control law (2.68) can be rewritten as

$$\begin{aligned}\mathbf{u}_k &= (C\Gamma)^{-1} \left(\mathbf{r}_{k+1} - \Lambda \mathbf{e}_k - C\Phi \mathbf{x}_k + \sigma_k - C\hat{\mathbf{d}}_k \right) \\ &= -(C\Gamma)^{-1} (C\Phi - \Lambda C) \mathbf{x}_k - (C\Gamma)^{-1} C\hat{\mathbf{d}}_k \\ &\quad + (C\Gamma)^{-1} (\mathbf{r}_{k+1} - \Lambda \mathbf{r}_k) + (C\Gamma)^{-1} \sigma_k.\end{aligned}\quad (2.69)$$

Substituting the above control law (2.69) into the system (2.2) yields the closed-loop state dynamics

$$\begin{aligned}\mathbf{x}_{k+1} &= \left[\Phi - \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda C) \right] \mathbf{x}_k + \mathbf{d}_k - \Gamma (C\Gamma)^{-1} C\hat{\mathbf{d}}_k \\ &\quad + \Gamma (C\Gamma)^{-1} (\mathbf{r}_{k+1} - \Lambda \mathbf{r}_k) + \Gamma (C\Gamma)^{-1} \sigma_k.\end{aligned}\quad (2.70)$$

It can be seen from the dynamics (2.70) that the stability of $\mathbf{x}_k$ is determined by the matrix $[\Phi - \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda C)]$ and the boundedness of σ_k .

Lemma 2.4 *The eigenvalues of $[\Phi - \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda C)]$ are the eigenvalues of Λ and the non-zero eigenvalues of $[\Phi - \Gamma (C\Gamma)^{-1} C\Phi]$.*

Proof Consider the matrices Φ , Γ and C which can be partitioned as shown

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{bmatrix}, \quad \Gamma = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} \quad \& \quad C = [C_1 \mid C_2]$$

where $(\Phi_{11}, C_1, \Gamma_1) \in \Re^{m \times m}$, $(\Phi_{12}, C_2) \in \Re^{m \times n-m}$, $(\Phi_{21}, \Gamma_2) \in \Re^{n-m \times m}$ and $\Phi_{22} \in \Re^{n-m \times n-m}$. The eigenvalues of $[\Phi - \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda C)]$ are found from

$$\det \left[\lambda I_n - \Phi + \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda C) \right] = 0 \quad (2.71)$$

$$\det \begin{bmatrix} \lambda I - \Phi_{11} + \Gamma_1 (C\Gamma)^{-1} \left(C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} - \Lambda C_1 \right) & -\Phi_{12} + \Gamma_1 (C\Gamma)^{-1} \left(C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} - \Lambda C_2 \right) \\ -\Phi_{21} + \Gamma_2 (C\Gamma)^{-1} \left(C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} - \Lambda C_1 \right) & \lambda I - \Phi_{22} + \Gamma_2 (C\Gamma)^{-1} \left(C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} - \Lambda C_2 \right) \end{bmatrix} \quad (2.72)$$

= 0

If the top row is premultiplied with C_1 and the bottom row is premultiplied with C_2 and the results summed and used as the new top row, using the fact that $C_1 \Gamma_1 + C_2 \Gamma_2 = C\Gamma$ the following is obtained

$$\det \begin{bmatrix} (\lambda I_m - \Lambda)C_1 & (\lambda I_m - \Lambda)C_2 \\ -\Phi_{21} + \Gamma_2(C\Gamma)^{-1} \left(C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} - \Lambda C_1 \right) & \lambda I - \Phi_{22} + \Gamma_2(C\Gamma)^{-1} \left(C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} - \Lambda C_2 \right) \end{bmatrix} = 0 \quad (2.73)$$

factoring the term $(\lambda I_m - \Lambda)$ and premultiplying the top row with $\Gamma_2(C\Gamma)^{-1}\Lambda$ and adding to the bottom row leads to

$$\det(\lambda I_m - \Lambda) \det \begin{bmatrix} C_1 & C_2 \\ -\Phi_{21} + \Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & \lambda I_{n-m} - \Phi_{22} + \Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \end{bmatrix} = 0 \quad (2.74)$$

Thus, we can conclude that m eigenvalues of $[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda C)]$ are the eigenvalues of Λ .

Now, consider

$$\det \begin{bmatrix} C_1 & C_2 \\ -\Phi_{21} + \Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & \lambda I_{n-m} - \Phi_{22} + \Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \end{bmatrix} = 0 \quad (2.75)$$

Using the following relations

$$C_2\Phi_{21} - C_2\Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} = -C_1\Phi_{11} + C_1\Gamma_1(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} \quad (2.76)$$

$$-C_2\Phi_{22} + C_2\Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} = -C_1\Phi_{12} + C_1\Gamma_1(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \quad (2.77)$$

Multiplying (2.75) with $\lambda^{-m}\lambda^m$ we get

$$\lambda^{-m} \det \begin{bmatrix} \lambda C_1 & \lambda C_2 \\ -\Phi_{21} + \Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & \lambda I_{n-m} - \Phi_{22} + \Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \end{bmatrix} = 0 \quad (2.78)$$

Premultiplying the bottom row with C_2 and subtracting from the top row and using the result as the new top row we get

$$\lambda^{-m} \det \begin{bmatrix} \lambda C_1 + C_2\Phi_{21} - C_2\Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & C_2\Phi_{22} - C_2\Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \\ -\Phi_{21} + \Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & \lambda I_{n-m} - \Phi_{22} + \Gamma_2(C\Gamma)^{-1}C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \end{bmatrix} = 0 \quad (2.79)$$

using relations (2.76) and (2.77) we finally get

$$\lambda^{-m} \det \begin{bmatrix} \lambda C_1 - C_1 \Phi_{11} + C_1 \Gamma_1 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & -C_1 \Phi_{12} + C_1 \Gamma_1 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \\ -\Phi_{21} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & \lambda I_{n-m} - \Phi_{22} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \end{bmatrix} = 0 \quad (2.80)$$

We can factor out the matrix C_1 from the top row to get

$$\lambda^{-m} \det(C_1) \det \begin{bmatrix} \lambda I_m - \Phi_{11} + \Gamma_1 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & -\Phi_{12} + \Gamma_1 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \\ -\Phi_{21} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{11} \\ \Phi_{21} \end{bmatrix} & \lambda I_{n-m} - \Phi_{22} + \Gamma_2 (C\Gamma)^{-1} C \begin{bmatrix} \Phi_{12} \\ \Phi_{22} \end{bmatrix} \end{bmatrix} = 0 \quad (2.81)$$

which finally simplifies to

$$\lambda^{-m} \det(C_1) \det[\Phi - \Gamma(C\Gamma)C\Phi] = 0 \quad (2.82)$$

It is well known that $[\Phi - \Gamma(C\Gamma)C\Phi]$ has atleast m zero eigenvalues which would be cancelled out by λ^{-m} and, thus, we finally conclude that the eigenvalues of $[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda C)]$ are the eigenvalues of Λ and the non-zero eigenvalues of $[\Phi - \Gamma(C\Gamma)C\Phi]$.

According to Lemma 2.1, the matrix $[\Phi - \Gamma(C\Gamma)^{-1}(C\Phi - \Lambda C)]$ has m poles to be placed at desired locations while the remaining $n - m$ poles are the open-loop zeros of the system (Φ, Γ, C) . Since, the system (2.2) is assumed to be minimum-phase, the $n - m$ poles are stable. Therefore, stability of the closed-loop state dynamics is guaranteed. Note that if Λ is a zero matrix then m poles are zero and the performance will be the same as the conventional deadbeat sliding-mode controller design.

Since we use the estimate of the disturbance, $\sigma_k \neq 0$. To show the boundedness and facilitate later analysis on the tracking performance, we derive the relationship between the switching surface and the disturbance estimate, as well as the relationship between the output tracking error and the disturbance estimate.

Theorem 2.2 Assume that the system (2.2) is minimum-phase and the eigenvalues of the matrix Λ are within the unit circle. Then by the control law (2.68) we have

$$\sigma_{k+1} = C(\hat{\mathbf{d}}_k - \mathbf{d}_k) \quad (2.83)$$

and the error dynamics

$$\mathbf{e}_{k+1} = \Lambda \mathbf{e}_k + \delta_k \quad (2.84)$$

where $\delta_k = C(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1})$.

Proof In order to verify the first part of *Theorem 2.2*, rewrite (2.65) as

$$\begin{aligned}\sigma_{k+1} &= \mathbf{a}_k - C\Gamma\mathbf{u}_k - C\mathbf{d}_k = \mathbf{a}_k - C\Gamma\mathbf{u}_k^{\text{eq}} - C\mathbf{d}_k + C\Gamma(\mathbf{u}_k^{\text{eq}} - \mathbf{u}_k) \\ &= C\Gamma(\mathbf{u}_k^{\text{eq}} - \mathbf{u}_k),\end{aligned}$$

where we use the property of equivalent control $\sigma_{k+1} = \mathbf{a}_k - C\Gamma\mathbf{u}_k^{\text{eq}} - C\mathbf{d}_k = 0$. Comparing two control laws (2.66) and (2.68), we obtain

$$\sigma_{k+1} = C(\hat{\mathbf{d}}_k - \mathbf{d}_k).$$

Note that the switching surface σ_{k+1} is no longer zero as desired but a function of the difference $\mathbf{d}_k - \hat{\mathbf{d}}_k$. This, however, is acceptable since the difference is $\mathbf{d}_k - \hat{\mathbf{d}}_k = \mathbf{d}_k - \mathbf{d}_{k-1}$ by the delay based disturbance estimation; thus, according to *Lemma 2.1* the difference is $O(T^2)$ which is quite small in practical applications.

To derive the second part of *Theorem 2.2* regarding the error dynamics, rewriting (2.64) as

$$\mathbf{e}_{k+1} = \Lambda\mathbf{e}_k + \sigma_{k+1} - \sigma_k$$

and substituting the relationship (2.83), lead to

$$\begin{aligned}\mathbf{e}_{k+1} &= \Lambda\mathbf{e}_k + C(\hat{\mathbf{d}}_k - \mathbf{d}_k) - C(\hat{\mathbf{d}}_{k-1} - \mathbf{d}_{k-1}) \\ &= \Lambda\mathbf{e}_k + C(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1}) = \Lambda\mathbf{e}_k + \delta_k.\end{aligned}\quad (2.85)$$

Since $\hat{\mathbf{d}}_k = \mathbf{d}_{k-1}$, δ_k is bounded, from *Lemma 2.1* we can conclude the boundedness of $\mathbf{e}_k$.

2.4.2.3 Tracking Error Bound

The tracking performance of the ISM control can be evaluated in terms of the error dynamics (2.84).

Theorem 2.3 *Using the delay based disturbance estimation (2.67), the ultimate tracking error bound with ISM control is given by*

$$\|\mathbf{e}_k\| = O(T^2)$$

where $\|\cdot\|$ represents the Euclidean norm.

Proof In order to calculate the tracking error bound we must find the bound of δ_k . From 2.2

$$\delta_k = C \left(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1} \right).$$

Substituting $\hat{\mathbf{d}}_k = \mathbf{d}_{k-1}$ from (2.67)

$$\delta_k = C (\mathbf{d}_{k-1} - \mathbf{d}_k + \mathbf{d}_{k-1} - \mathbf{d}_{k-2})$$

which simplifies to

$$\delta_k = -C (\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2}). \quad (2.86)$$

According to Lemma 2.1, $\mathbf{d}_k - 2\mathbf{d}_{k-1} + \mathbf{d}_{k-2} = O(T^3)$, therefore $\delta_k = O(T^3)$. Also from Lemma 2.2 the ultimate error bound on $\|\mathbf{e}_k\|$ in the expression $\mathbf{e}_{k+1} = \Lambda \mathbf{e}_k + \delta_k$ will be one order higher than the bound on δ_k due to convolution. Since the bound on δ_k is $O(T^3)$, the ultimate bound on $\|\mathbf{e}_k\|$ is $O(T^2)$.

Remark 2.8 In practical control a disturbance could be piece-wise smooth. The delay based estimation (2.67) can quickly capture the varying disturbance after one sampling period. Assume that the disturbance $\mathbf{f}(t)$ undergoes an abrupt change at the time interval $[(k-1)T, kT]$, then Lemma 2.1 does not hold for the time instant k because $\mathbf{v}(t)$ becomes extremely big. Nevertheless, Lemma 2.1 will be satisfied immediately after the time instant k if the disturbance becomes smooth again. From this point of view, the delay based estimation has a very small time-delay or equivalently a large bandwidth.

Remark 2.9 Although the state-feedback approach may seem to be not very practical for a number of output tracking tasks, this section serves as a precursor to the output-feedback-based and state-observer-based approaches to be explored in subsequent sections.

2.4.3 Output Tracking ISM: Output Feedback Approach

In this section we derive ISM control that only uses the output tracking error. The new design will require a reference model and a dynamic disturbance observer due to the lack of the state information. The reference model will be constructed such that its output is the reference trajectory $\mathbf{r}_k$.

2.4.3.1 Controller Design

In order to proceed we will first define a reference model

$$\begin{aligned}\mathbf{x}_{m,k+1} &= (\Phi - K_1) \mathbf{x}_{m,k} + K_2 \mathbf{r}_{k+1} \\ \mathbf{y}_{m,k} &= C \mathbf{x}_{m,k}\end{aligned}\quad (2.87)$$

where $\mathbf{x}_{m,k} \in \mathbb{R}^n$ is the state vector, $\mathbf{y}_{m,k} \in \mathbb{R}^m$ is the output vector, and $\mathbf{r}_k \in \mathbb{R}^m$ is a bounded reference trajectory. K_1 is selected such that $(\Phi - K_1)$ is stable. The selection criteria for the matrices K_1 and K_2 will be discussed in detail in Sects. 2.4.3.2 and 2.4.3.5.

Now consider a new sliding surface

$$\begin{aligned}\sigma_k &= D (\mathbf{x}_{m,k} - \mathbf{x}_k) + \boldsymbol{\varepsilon}_k \\ \boldsymbol{\varepsilon}_k &= \boldsymbol{\varepsilon}_{k-1} + E D (\mathbf{x}_{m,k-1} - \mathbf{x}_{k-1})\end{aligned}\quad (2.88)$$

where $D = C\Phi^{-1}$, $\sigma_k, \boldsymbol{\varepsilon}_k \in \mathbb{R}^m$ are the switching function and integral vectors, $E \in \mathbb{R}^{m \times m}$ is an integral gain matrix. Note that $D\mathbf{x}_k = C\Phi^{-1}(\Phi\mathbf{x}_{k-1} + \Gamma\mathbf{u}_{k-1} + \mathbf{d}_{k-1}) = \mathbf{y}_{k-1} + D(\Gamma\mathbf{u}_{k-1} + \mathbf{d}_{k-1})$ is independent of the states, such a simplification was first proposed in [101].

The equivalent control law can be derived from $\sigma_{k+1} = 0$. From (2.88) $\boldsymbol{\varepsilon}_k = \sigma_k - D(\mathbf{x}_{m,k} - \mathbf{x}_k)$, we have

$$\begin{aligned}\sigma_{k+1} &= D(\mathbf{x}_{m,k+1} - \mathbf{x}_{k+1}) + \boldsymbol{\varepsilon}_{k+1} \\ &= D(\mathbf{x}_{m,k+1} - \mathbf{x}_{k+1}) + \boldsymbol{\varepsilon}_k + E D(\mathbf{x}_{m,k} - \mathbf{x}_k) \\ &= D(\mathbf{x}_{m,k+1} - \mathbf{x}_{k+1}) + \sigma_k - D(\mathbf{x}_{m,k} - \mathbf{x}_k) + E D(\mathbf{x}_{m,k} - \mathbf{x}_k) \\ &= D\mathbf{x}_{m,k+1} - D\mathbf{x}_{k+1} + \sigma_k - \Lambda D(\mathbf{x}_{m,k} - \mathbf{x}_k)\end{aligned}\quad (2.89)$$

where $\Lambda = I - E$. Substituting the system dynamics (2.2) into (2.89) yields

$$\begin{aligned}\sigma_{k+1} &= D\mathbf{x}_{m,k+1} - D(\Phi\mathbf{x}_k + \Gamma\mathbf{u}_k + \mathbf{d}_k) + \sigma_k - \Lambda D(\mathbf{x}_{m,k} - \mathbf{x}_k) \\ &= \mathbf{a}_k - D\Gamma\mathbf{u}_k - D\mathbf{d}_k\end{aligned}\quad (2.90)$$

where $\mathbf{a}_k = -(D\Phi - \Lambda D)\mathbf{x}_k + (D\mathbf{x}_{m,k+1} - \Lambda D\mathbf{x}_{m,k}) + \sigma_k$.

Letting $\sigma_{k+1} = 0$, solving for the equivalent control $\mathbf{u}_k^{\text{eq}}$, we have

$$\begin{aligned}\mathbf{u}_k^{\text{eq}} &= (D\Gamma)^{-1}(\mathbf{a}_k - D\mathbf{d}_k) \\ &= -(D\Gamma)^{-1}(D\Phi - \Lambda D)\mathbf{x}_k + (D\Gamma)^{-1}(D\mathbf{x}_{m,k+1} - \Lambda D\mathbf{x}_{m,k}) - (D\Gamma)^{-1}D\mathbf{d}_k.\end{aligned}\quad (2.91)$$

Control law (2.91) is not implementable as it requires a priori knowledge of the disturbance. Thus, the estimation of the disturbance should be used

$$\mathbf{u}_k = (D\Gamma)^{-1} \left(\mathbf{a}_k - D\hat{\mathbf{d}}_k \right) \quad (2.92)$$

where $\hat{\mathbf{d}}_k$ is the disturbance estimation.

However, note that the disturbance estimate used in the state feedback controller designed in Sect. 2.4.2 requires full state information which is not available in this case. Therefore, an observer that is based on output feedback is proposed and will be detailed in Sect. 2.4.3.2.

2.4.3.2 Disturbance Observer Design

Note that according to Lemma 2.1, the disturbance can be written as

$$\mathbf{d}_k = \Gamma \mathbf{f}_k + \frac{1}{2} \Gamma \mathbf{v}_k T + O(T^3) = \Gamma \boldsymbol{\eta}_k + O(T^3) \quad (2.93)$$

where $\boldsymbol{\eta}_k = \mathbf{f}_k + \frac{1}{2} \mathbf{v}_k T$. If $\boldsymbol{\eta}_k$ can be estimated, then the estimation error of $\mathbf{d}_k$ would be $O(T^3)$ which is acceptable in practical applications.

Define the observer

$$\begin{aligned} \mathbf{x}_{d,k} &= \Phi \mathbf{x}_{d,k-1} + \Gamma \mathbf{u}_{k-1} + \Gamma \hat{\boldsymbol{\eta}}_{k-1} \\ \mathbf{y}_{d,k-1} &= C \mathbf{x}_{d,k-1} \end{aligned} \quad (2.94)$$

where $\mathbf{x}_{d,k-1} \in \mathfrak{N}^n$ is the observer state vector, $\mathbf{y}_{d,k-1} \in \mathfrak{N}^m$ is the observer output vector, $\hat{\boldsymbol{\eta}}_{k-1} \in \mathfrak{N}^m$ is the disturbance estimate and will act as the ‘control input’ to the observer, therefore we can write $\hat{\mathbf{d}}_{k-1} = \Gamma \hat{\boldsymbol{\eta}}_{k-1}$. Since the disturbance estimate will be used in the final control signal, it must not be overly large. Therefore, it is wise to avoid a deadbeat design. For this reason we design the disturbance observer based on an integral sliding surface

$$\begin{aligned} \boldsymbol{\sigma}_{d,k} &= \mathbf{e}_{d,k} - \mathbf{e}_{d,0} + \boldsymbol{\varepsilon}_{d,k} \\ \boldsymbol{\varepsilon}_{d,k} &= \boldsymbol{\varepsilon}_{d,k-1} + E_d \mathbf{e}_{d,k-1} \end{aligned} \quad (2.95)$$

where $\mathbf{e}_{d,k} = \mathbf{y}_k - \mathbf{y}_{d,k}$ is the output estimation error, $\boldsymbol{\sigma}_{d,k}, \boldsymbol{\varepsilon}_{d,k} \in \mathfrak{N}^m$ are the sliding function and integral vectors, and E_d is an integral gain matrix.

Note that the sliding surface (2.95) is analogous to (2.63), that is, the set $(\mathbf{y}_k, \mathbf{x}_{d,k}, \mathbf{u}_k + \hat{\boldsymbol{\eta}}_k, \mathbf{y}_{d,k}, \boldsymbol{\sigma}_{d,k})$ has duality with the set $(\mathbf{r}_k, \mathbf{x}_k, \mathbf{u}_k, \mathbf{y}_k, \boldsymbol{\sigma}_k)$, except for an one-step delay in the observer dynamics (2.94). Therefore, let $\boldsymbol{\sigma}_{d,k} = 0$ we can derive the virtual equivalent control $\mathbf{u}_{k-1} + \hat{\boldsymbol{\eta}}_{k-1}$, thus, analogous to (2.69),

$$\hat{\eta}_{k-1} = (C\Gamma)^{-1} [\mathbf{y}_k - \Lambda_d \mathbf{e}_{d,k-1} - C\Phi \mathbf{x}_{d,k-1} + \sigma_{d,k-1}] - \mathbf{u}_{k-1} \quad (2.96)$$

where $\Lambda_d = I_m - E_d$.

In practice, the quantity $\mathbf{y}_{k+1}$ is not available at the time instant k when computing $\hat{\eta}_k$. Therefore we can only compute $\hat{\eta}_{k-1}$, and in the control law (2.92) we use the delayed estimate $\hat{\mathbf{d}}_k = \Gamma \hat{\eta}_{k-1}$.

The stability and convergence properties of the observer (2.94) and the disturbance estimation (2.96) are analyzed in the following theorem:

Theorem 2.4 *The observer output $\mathbf{y}_{d,k}$ converges asymptotically to the true outputs $\mathbf{y}_k$, and the disturbance estimate $\hat{\mathbf{d}}_k$ converges to the actual disturbance $\mathbf{d}_{k-1}$ with the precision order $O(T^2)$.*

Proof Substituting (2.96) into (2.94), and using $\mathbf{e}_{d,k-1} = C(\mathbf{y}_{k-1} - \mathbf{y}_{d,k-1})$, it is obtained that

$$\begin{aligned} \mathbf{x}_{d,k} = & \left[\Phi - \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda_d C) \right] \mathbf{x}_{d,k-1} + \Gamma (C\Gamma)^{-1} [\mathbf{y}_k - \Lambda_d \mathbf{y}_{k-1}] \\ & + \Gamma (C\Gamma)^{-1} \sigma_{d,k-1}. \end{aligned} \quad (2.97)$$

Since the control and estimate $\mathbf{u}_{k-1} + \hat{\eta}_{k-1}$ are chosen such that $\sigma_{d,k} = 0$ for any $k > 0$, (2.97) renders to

$$\mathbf{x}_{d,k} = \left[\Phi - \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda_d C) \right] \mathbf{x}_{d,k-1} + \Gamma (C\Gamma)^{-1} [\mathbf{y}_k - \Lambda_d \mathbf{y}_{k-1}]. \quad (2.98)$$

The second term on the right hand side of (2.98) can be expressed as

$$\begin{aligned} \Gamma (C\Gamma)^{-1} [\mathbf{y}_k - \Lambda_d \mathbf{y}_{k-1}] = & \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda_d C) \mathbf{x}_{k-1} + \Gamma \mathbf{u}_{k-1} \\ & + \Gamma (C\Gamma)^{-1} C \mathbf{d}_{k-1} \end{aligned}$$

by using the relations $\mathbf{y}_k = C\Phi \mathbf{x}_{k-1} + C\Gamma \mathbf{u}_{k-1} + C\mathbf{d}_{k-1}$ and $\mathbf{y}_{k-1} = C\mathbf{x}_{k-1}$. Therefore (2.98) can be rewritten as

$$\mathbf{x}_{d,k} = \Phi \mathbf{x}_{d,k-1} + \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda_d C) \Delta \mathbf{x}_{d,k-1} + \Gamma \mathbf{u}_k + \Gamma (C\Gamma)^{-1} C \mathbf{d}_{k-1} \quad (2.99)$$

where $\Delta \mathbf{x}_{d,k-1} = \mathbf{x}_{k-1} - \mathbf{x}_{d,k-1}$.

Further subtracting (2.99) from the system (2.2) we obtain

$$\Delta \mathbf{x}_{d,k} = \left[\Phi - \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda_d C) \right] \Delta \mathbf{x}_{d,k-1} + \left[I - \Gamma (C\Gamma)^{-1} C \right] \mathbf{d}_{k-1} \quad (2.100)$$

where $[I - \Gamma (C\Gamma)^{-1} C] \mathbf{d}_{k-1}$ is $O(T^3)$ because

$$[I - \Gamma (C\Gamma)^{-1} C] [\Gamma \boldsymbol{\eta}_{k-1} + O(T^3)] = [I - \Gamma (C\Gamma)^{-1} C] O(T^3) = O(T^3).$$

Applying *Lemma 2.1*, $\Delta \mathbf{x}_{d,k-1} = O(T^2)$.

From (2.100) we can see that the stability of the disturbance observer depends only on the matrix $[\Phi - \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda_d C)]$ and is guaranteed by the selection of the matrix Λ_d and the fact that system (Φ, Γ, C) is minimum phase. It should also be noted that the residue term $[I - \Gamma (C\Gamma)^{-1} C] \mathbf{d}_{k-1}$ in the state space is orthogonal to the output space, as $C [I - \Gamma (C\Gamma)^{-1} C] \mathbf{d}_{k-1} = 0$. Therefore premultiplication of (2.100) with C yields the output tracking error dynamics

$$\mathbf{e}_{d,k} = \Lambda_d \mathbf{e}_{d,k-1} \quad (2.101)$$

which is asymptotically stable through choosing a stable matrix Λ_d .

Finally we discuss the convergence property of the estimate $\hat{\mathbf{d}}_{k-1}$. Subtracting (2.94) from (2.2) with one-step delay, we obtain

$$\Delta \mathbf{x}_{d,k} = \Phi \Delta \mathbf{x}_{d,k-1} + \Gamma (\boldsymbol{\eta}_{k-1} - \hat{\boldsymbol{\eta}}_{k-1}) + O(T^3). \quad (2.102)$$

Premultiplying (2.102) with C , and substituting (2.101) that describes $C \Delta \mathbf{x}_{d,k}$, yield

$$\hat{\boldsymbol{\eta}}_{k-1} = \boldsymbol{\eta}_{k-1} + (C\Gamma)^{-1} (C\Phi - \Lambda_d C) \Delta \mathbf{x}_{d,k-1} + (C\Gamma)^{-1} O(T^3). \quad (2.103)$$

The second term on the right hand side of (2.103) is $O(T)$ because $\Delta \mathbf{x}_{d,k-1} = O(T^2)$ but $(C\Gamma)^{-1} = O(T^{-1})$. As a result, from (2.103) we can conclude that $\hat{\boldsymbol{\eta}}_{k-1}$ approaches $\boldsymbol{\eta}_{k-1}$ with the precision $O(T)$. In terms of the relationship

$$\mathbf{d}_{k-1} - \hat{\mathbf{d}}_k = \Gamma (\boldsymbol{\eta}_{k-1} - \hat{\boldsymbol{\eta}}_{k-1}) + O(T^3)$$

and $\Gamma = O(T)$, we conclude $\hat{\mathbf{d}}_k$ converges to $\mathbf{d}_{k-1}$ with the precision of $O(T^2)$.

Remark 2.10 At the time k , we can guarantee the convergence of $\hat{\boldsymbol{\eta}}_{k-1}$ to $\boldsymbol{\eta}_{k-1}$ with the precision $O(T)$. In other words, we can guarantee the convergence of the disturbance estimate at the time k , $\hat{\mathbf{d}}_k$, to the actual disturbance at time $k-1$, $\mathbf{d}_{k-1}$, with the precision $O(T^2)$. This result is consistent with the state-based estimation presented in Sect. 2.3.1 in which $\hat{\mathbf{d}}_k$ is made equal to $\mathbf{d}_{k-1}$. Comparing differences between the state-based and output-based disturbance estimation, the former has only one-step delay with perfect precision, where as the latter is asymptotic with $O(T^2)$ precision.

2.4.3.3 Stability Analysis

To analyze the stability of the closed-loop system, substitute $\mathbf{u}_k$ in (2.92) into the system (2.2) leading to the closed-loop equation in the sliding mode

$$\begin{aligned} \mathbf{x}_{k+1} = & \left[\Phi - \Gamma (D\Gamma)^{-1} (D\Phi - \Lambda_d D) \right] \mathbf{x}_k + \mathbf{d}_k - \Gamma (D\Gamma)^{-1} D \hat{\mathbf{d}}_k \\ & + \Gamma (D\Gamma)^{-1} \left[D\mathbf{x}_{m,k+1} - \Lambda_d D\mathbf{x}_{m,k} + \sigma_k \right]. \end{aligned} \quad (2.104)$$

The stability of the above sliding equation is summarized in the following theorem:

Theorem 2.5 *Using the control law (2.92) the sliding mode is*

$$\sigma_{k+1} = D \left(\hat{\mathbf{d}}_k - \mathbf{d}_k \right).$$

Further, the state tracking error $\Delta \mathbf{x}_k = \mathbf{x}_{m,k} - \mathbf{x}_k$ is bounded if system (Φ, Γ, D) is minimum-phase and the eigenvalues of the matrix Λ_d are within the unit circle.

Proof In order to verify the first part of Theorem 2.5, rewrite the dynamics of the sliding mode (2.90)

$$\begin{aligned} \sigma_{k+1} &= \mathbf{a}_k - D\Gamma \mathbf{u}_k - D\mathbf{d}_k = \mathbf{a}_k - D\Gamma \mathbf{u}_k^{\text{eq}} - D\mathbf{d}_k + D\Gamma (\mathbf{u}_k^{\text{eq}} - \mathbf{u}_k) \\ &= D\Gamma (\mathbf{u}_k^{\text{eq}} - \mathbf{u}_k) \end{aligned}$$

where we use the property of equivalent control $\sigma_{k+1} = \mathbf{a}_k - D\Gamma \mathbf{u}_k^{\text{eq}} - D\mathbf{d}_k = 0$. Comparing two control laws (2.91) and (2.92), we obtain

$$\sigma_{k+1} = D \left(\hat{\mathbf{d}}_k - \mathbf{d}_k \right).$$

Note that if there is no disturbance or we have perfect estimation of the disturbance, then $\sigma_{k+1} = 0$ as desired. From the results of Theorem 2.3 and Lemma 2.1

$$\hat{\mathbf{d}}_k - \mathbf{d}_k = \hat{\mathbf{d}}_k - \mathbf{d}_{k-1} - (\mathbf{d}_k - \mathbf{d}_{k-1}) = O(T^2)$$

as $k \rightarrow \infty$. Thus $\sigma_{k+1} \rightarrow O(T^2)$ which is acceptable in practice.

To prove the boundedness of the state tracking error $\Delta \mathbf{x}_k$, first derive the state error dynamics. Subtracting both sides of (2.104) from the reference model (2.87), and substituting $\sigma_k = D \left(\hat{\mathbf{d}}_{k-1} - \mathbf{d}_{k-1} \right)$, yields

$$\begin{aligned} \Delta \mathbf{x}_{k+1} = & \left[\Phi - \Gamma (D\Gamma)^{-1} (D\Phi - \Lambda_d D) \right] \Delta \mathbf{x}_k \\ & + \left[I - \Gamma (D\Gamma)^{-1} D \right] (K_2 \mathbf{r}_{k+1} - K_1 \mathbf{x}_{m,k}) - \zeta_k \end{aligned} \quad (2.105)$$

where

$$\boldsymbol{\zeta}_k = \mathbf{d}_k - \Gamma (D\Gamma)^{-1} D \left(\hat{\mathbf{d}}_k - \hat{\mathbf{d}}_{k-1} + \mathbf{d}_{k-1} \right). \quad (2.106)$$

The stability of (2.105) is dependent on $[\Phi - \Gamma (D\Gamma)^{-1} (D\Phi - \Lambda_d D)]$. From Lemma 2.1 the closed-loop poles of (2.105) are the eigenvalues of Λ_d and the open-loop zeros of the system (Φ, Γ, D) . Thus, m poles of the closed-loop system can be selected by the proper choice of the matrix Λ_d while the remaining poles are stable only if the system (Φ, Γ, D) is minimum-phase. Note that both $\mathbf{r}_{k+1}$ and $\mathbf{x}_{m,k}$ are reference signals and are bounded. Therefore we need only to show the boundedness of $\boldsymbol{\zeta}_k$ which is

$$\boldsymbol{\zeta}_k = (I - \Gamma (D\Gamma)^{-1} D) \mathbf{d}_k + \Gamma (D\Gamma)^{-1} D \left(\mathbf{d}_k - \hat{\mathbf{d}}_k \right) - \Gamma (D\Gamma)^{-1} D \left(\mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1} \right). \quad (2.107)$$

From Theorem 2.3, the second and third terms on the right hand side of (2.107) are $O(T^2)$. From Lemma 2.1, the first term on the right hand side of (2.107) can be written as

$$(I - \Gamma (D\Gamma)^{-1} D) \mathbf{d}_k = (I - \Gamma (D\Gamma)^{-1} D) \left(\Gamma \boldsymbol{\eta}_k + O(T^3) \right) = O(T^3).$$

Therefore $\boldsymbol{\zeta}_k = O(T^2)$ which is bounded.

We have established the stability condition for the closed-loop system, but, have yet to establish the ultimate tracking error bound. From (2.105) it can be seen that the tracking error bound is dependent on the disturbance estimate $\hat{\mathbf{d}}_k$ as well as the selection of K_1 and K_2 . Up to this point, not much was discussed in terms of the selection of the reference model (2.87). As it can be seen from (2.105) the selection of the reference model can effect the overall tracking error bound. Since we consider an arbitrary reference $\mathbf{r}_k$, the reference model must be selected such that its output is the reference signal $\mathbf{r}_k$. To achieve this requirement, we explore two possible selections of the reference model.

2.4.3.4 Error Bound for a Minimum-Phase (Φ, Γ, C)

For this case the reference model requires that the matrices $K_1 = \Gamma (C\Gamma)^{-1} C\Phi$ and $K_2 = \Gamma (C\Gamma)^{-1}$, therefore, the reference model (2.87) can be written as

$$\begin{aligned} \mathbf{x}_{m,k+1} &= \left[\Phi - \Gamma (C\Gamma)^{-1} C\Phi \right] \mathbf{x}_{m,k} + \Gamma (C\Gamma)^{-1} \mathbf{r}_{k+1} \\ \mathbf{y}_{m,k} &= C\mathbf{x}_{m,k} = \mathbf{r}_k. \end{aligned} \quad (2.108)$$

It can be seen from (2.108) that it is stable if the matrix $[\Phi - \Gamma (C\Gamma)^{-1} C\Phi]$ is stable, i.e., the system (Φ, Γ, C) is minimum-phase. Substituting the selected matrices K_1 and K_2 into (2.105) and using the fact that $[I - \Gamma (D\Gamma)^{-1} D] \Gamma = 0$, we obtain

$$\Delta \mathbf{x}_{k+1} = [\Phi - \Gamma (D\Gamma)^{-1} (D\Phi - \Lambda_d D)] \Delta \mathbf{x}_k - \boldsymbol{\zeta}_k. \quad (2.109)$$

where $\boldsymbol{\zeta}_k = O(T^2)$ according to *Theorem 2.5*. As for the output tracking error bound, from *Lemma 2.1*, the ultimate error bound on $\|\Delta \mathbf{x}_k\|$ will be one order higher than the bound on $\boldsymbol{\zeta}_k$. Thus, the ultimate bound on the output tracking error is

$$\|\mathbf{e}_k\| \leq \|C\| \cdot \|\Delta \mathbf{x}_k\| = O(T). \quad (2.110)$$

2.4.3.5 Error Bound for a Minimum-Phase (Φ, Γ, D)

In the case that it is only possible to satisfy (Φ, Γ, D) to be minimum-phase, a different reference model needs to be selected. For this new reference model, select the matrices $K_1 = \Gamma (D\Gamma)^{-1} D\Phi$ and $K_2 = \Gamma (D\Gamma)^{-1}$. Then the reference model (2.87) can be written as

$$\begin{aligned} \mathbf{x}_{m,k+1} &= [\Phi - \Gamma (D\Gamma)^{-1} D\Phi] \mathbf{x}_{m,k} + \Gamma (D\Gamma)^{-1} \mathbf{r}_{k+1} \\ \mathbf{y}_{m,k} &= D\mathbf{x}_{m,k} = \mathbf{r}_k. \end{aligned} \quad (2.111)$$

The matrix $[\Phi - \Gamma (D\Gamma)^{-1} D\Phi]$ is stable only if (Φ, Γ, D) is minimum-phase. Substituting the selected matrices K_1 and K_2 into (2.105), and using the property $[I - \Gamma (D\Gamma)^{-1} D] \Gamma = 0$, we have

$$\Delta \mathbf{x}_{k+1} = [\Phi - \Gamma (D\Gamma)^{-1} (D\Phi - \Lambda_d D)] \Delta \mathbf{x}_k - \boldsymbol{\zeta}_k. \quad (2.112)$$

We can see from (2.112) that the tracking error bound is only dependent on the disturbance estimation $\boldsymbol{\zeta}_k$.

On the other hand, the disturbance observer requires (Φ, Γ, C) to be minimum-phase, hence is not implementable in this case. Without the disturbance estimator, using *Lemma 2.1*, (2.106) becomes

$$\begin{aligned} \boldsymbol{\zeta}_k &= \mathbf{d}_k - \Gamma (D\Gamma)^{-1} \mathbf{d}_{k-1} \\ &= \mathbf{d}_k - \mathbf{d}_{k-1} + [I - \Gamma (D\Gamma)^{-1} D] (\Gamma \boldsymbol{\eta}_{k-1} + O(T^3)) \\ &= O(T^2) + O(T^3) = O(T^2). \end{aligned} \quad (2.113)$$

As a result, the closed-loop system is

$$\Delta \mathbf{x}_{k+1} = \left[\Phi - \Gamma (D\Gamma)^{-1} (D\Phi - \Lambda_d D) \right] \Delta \mathbf{x}_k + O(T^2). \quad (2.114)$$

By *Lemma 2.1*, the ultimate bound on $\|\Delta \mathbf{x}_k\| = O(T)$, and therefore, the ultimate bound on the tracking error is

$$\|\mathbf{e}_k\| \leq \|D\| \cdot \|\Delta \mathbf{x}_k\| = O(T). \quad (2.115)$$

While this approach gives a similar precision in output tracking performance, it only requires (Φ, Γ, D) to be minimum-phase and can be used in the cases (Φ, Γ, C) is not minimum-phase.

2.4.4 Output Tracking ISM: State Observer Approach

In this section we explore the observer-based approach for the unknown states when only output measurement is available. By virtue of state estimation, it is required that (Φ, Γ, C) to be minimum-phase, thus, the observer based disturbance estimation approach in Sect. 2.4.3.2 is applicable. From the derivation procedure in Sect. 2.4.3.2, we can see that the error dynamics (2.101) of the disturbance observer (2.94) is independent of the control inputs $\mathbf{u}_k$. Therefore the same disturbance observer (2.94)–(2.96) can be incorporated in the state-observer approach directly without any modification. In this section we focus on the design and analysis of the control law and state observer.

2.4.4.1 Controller Structure and Closed-Loop System

With the state estimation, the ISM control can be constructed according to the preceding state-feedback based ISM control design (2.69) by substituting $C\Phi \mathbf{x}_k$ with $C\Phi \mathbf{x}_k - C\Phi \tilde{\mathbf{x}}_k$ where $\tilde{\mathbf{x}}_k = \mathbf{x}_k - \hat{\mathbf{x}}_k$ is the state estimation error,

$$\begin{aligned} \mathbf{u}_k = & -(C\Gamma)^{-1} (C\Phi - \Lambda_d C) \mathbf{x}_k - (C\Gamma)^{-1} C\hat{\mathbf{d}}_k + (C\Gamma)^{-1} (\mathbf{r}_{k+1} - \Lambda_d \mathbf{r}_k) \\ & + (C\Gamma)^{-1} \sigma_k + (C\Gamma)^{-1} C\Phi \tilde{\mathbf{x}}_k. \end{aligned} \quad (2.116)$$

In comparison to the control law (2.69), the control law (2.116) has an additional term $(C\Gamma)^{-1} C\Phi \tilde{\mathbf{x}}_k$ due to the state estimation error. Substituting $\mathbf{u}_k$ in (2.116) into (2.2) yields the closed-loop dynamics

$$\begin{aligned} \mathbf{x}_{k+1} = & \left[\Phi - \Gamma (C\Gamma)^{-1} (C\Phi - \Lambda_d C) \right] \mathbf{x}_k + \mathbf{d}_k - \Gamma (C\Gamma)^{-1} C\hat{\mathbf{d}}_k \\ & + \Gamma (C\Gamma)^{-1} (\mathbf{r}_{k+1} - \Lambda_d \mathbf{r}_k) + \Gamma (C\Gamma)^{-1} \sigma_k \\ & + \Gamma (C\Gamma)^{-1} C\Phi \tilde{\mathbf{x}}_k \end{aligned} \quad (2.117)$$

which, comparing with the state-feedback (2.70), is almost the same except for an additional term $\Gamma (C\Gamma)^{-1} C\Phi \tilde{\mathbf{x}}_k$. Hence, following the proof of *Theorem 2.2*, the properties of the closed-loop system (2.117) can be derived.

Theorem 2.6 *Assume that the system (2.2) is minimum-phase and the eigenvalues of the matrix Λ_d are within the unit circle. Then by the control law (2.116) we have*

$$\sigma_{k+1} = C \left(\hat{\mathbf{d}}_k - \mathbf{d}_k \right) - C\Phi \tilde{\mathbf{x}}_k \quad (2.118)$$

and the error dynamics

$$\mathbf{e}_{k+1} = \Lambda_d \mathbf{e}_k + \delta_k \quad (2.119)$$

$$\text{where } \delta_k = C \left(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1} \right) - C\Phi (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1}).$$

Proof In order to prove (2.118), notice that (2.64) can be written as

$$\sigma_{k+1} = \mathbf{r}_{k+1} - C\mathbf{x}_{k+1} - \Lambda_d \mathbf{r}_k + \Lambda_d C\mathbf{x}_k + \sigma_k. \quad (2.120)$$

Substituting the closed-loop dynamics (2.117) into (2.120) and simplifying we obtain

$$\sigma_{k+1} = C \left(\hat{\mathbf{d}}_k - \mathbf{d}_k \right) - C\Phi \tilde{\mathbf{x}}_k$$

which proves the first part of the theorem.

In order to prove the second part, premultiply (2.117) with C and simplify to obtain the following result

$$\mathbf{y}_{k+1} = \Lambda_d \mathbf{y}_k + C\mathbf{d}_k - C\hat{\mathbf{d}}_k + \mathbf{r}_{k+1} - \Lambda_d \mathbf{r}_k + C\Phi \tilde{\mathbf{x}}_k + \sigma_k \quad (2.121)$$

From the result (2.118) it can be obtained that $\sigma_k = C \left(\hat{\mathbf{d}}_{k-1} - \mathbf{d}_{k-1} \right) - C\Phi \tilde{\mathbf{x}}_{k-1}$. Substituting in (2.121) and using the fact that $\mathbf{e}_k = \mathbf{r}_k - \mathbf{y}_k$ we obtain

$$\begin{aligned} \mathbf{e}_{k+1} &= \Lambda_d \mathbf{e}_k + C \left(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1} \right) - C\Phi (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1}) \\ &= \Lambda_d \mathbf{e}_k + \delta_k \end{aligned} \quad (2.122)$$

$$\text{where } \delta_k = C \left(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1} \right) - C\Phi (\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1}).$$

As in the state-feedback approach, the output tracking error depends on the proper selection of the eigenvalues of Λ_d , as well as the disturbance estimation and state estimation precision. The influence of the disturbance estimation has been discussed in *Theorem 2.4*. The effect of $\tilde{\mathbf{x}}_k$ on the tracking error bound will be evaluated, we will discuss the state observer in the next section.

2.4.4.2 State Observer

State estimation will be accomplished with the following state-observer

$$\hat{\mathbf{x}}_{k+1} = \Phi \hat{\mathbf{x}}_k + \Gamma \mathbf{u}_k + L (\mathbf{y}_k - \hat{\mathbf{y}}_k) + \hat{\mathbf{d}}_k \quad (2.123)$$

where $\hat{\mathbf{x}}_k$, $\hat{\mathbf{y}}_k$ are the state and output estimates and L is a design matrix. Observer (2.123) is well-known, however, the term $\hat{\mathbf{d}}_k$ has been added to compensate for the disturbance. It is necessary to investigate the effect of the disturbance estimation on the state estimation. Subtracting (2.123) from (2.2) we get

$$\tilde{\mathbf{x}}_{k+1} = [\Phi - LC] \tilde{\mathbf{x}}_k + \mathbf{d}_k - \hat{\mathbf{d}}_k. \quad (2.124)$$

It can be seen that the state estimation is independent of the control inputs. Under the assumption that (Φ, Γ, C) is controllable and observable, we can choose L such that $\Phi - LC$ is asymptotically stable. From *Theorem 2.4*, $\mathbf{d}_k - \hat{\mathbf{d}}_k = O(T^2)$, thus, from *Lemma 2.1* the ultimate bound of $\tilde{\mathbf{x}}_k$ is $O(T)$. Later we will show that, for systems of relative degree greater than 1, by virtue of the integral action in the ISM control, the state estimation error will be reduced to $O(T^2)$ in the closed-loop system.

2.4.4.3 Tracking Error Bound

In order to calculate the tracking error bound we need first calculate the bound of $\boldsymbol{\zeta}_k$ in *Theorem 2.5*. From the error dynamics of the state estimation (2.124), the solution trajectory is

$$\tilde{\mathbf{x}}_k = [\Phi - LC]^k \tilde{\mathbf{x}}_0 + \sum_{i=0}^{k-1} \left([\Phi - LC]^{k-1-i} (\mathbf{d}_i - \hat{\mathbf{d}}_i) \right). \quad (2.125)$$

The difference $\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1}$ can be calculated as

$$\begin{aligned} \tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1} &= [(\Phi - LC) - I_n] (\Phi - LC)^{k-1} \tilde{\mathbf{x}}_0 + \sum_{i=0}^{k-1} \left([\Phi - LC]^{k-1-i} (\mathbf{d}_i - \hat{\mathbf{d}}_i) \right) \\ &\quad - \sum_{i=0}^{k-2} \left([\Phi - LC]^{k-1-i} (\mathbf{d}_i - \hat{\mathbf{d}}_i) \right) \end{aligned} \quad (2.126)$$

which can be simplified to

$$\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1} = [(\Phi - LC) - I_n] (\Phi - LC)^{k-1} \tilde{\mathbf{x}}_0 + (\mathbf{d}_k - \hat{\mathbf{d}}_k). \quad (2.127)$$

Since $(\Phi - LC)$ is asymptotically stable, the ultimate bound is

$$\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1} = \mathbf{d}_k - \hat{\mathbf{d}}_k, \quad (2.128)$$

and δ_k can be expressed ultimately as

$$\begin{aligned} \delta_k &= C \left(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1} \right) - C\Phi \left(\tilde{\mathbf{x}}_k - \tilde{\mathbf{x}}_{k-1} \right) \\ &= C \left(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1} \right) - C\Phi \left(\mathbf{d}_k - \hat{\mathbf{d}}_k \right). \end{aligned} \quad (2.129)$$

From *Theorem 2.4*, the disturbance estimation error $\mathbf{d}_k - \hat{\mathbf{d}}_k$ is $O(T^2)$. Therefore we have

$$\delta_k = C \cdot \left(O(T^2) + O(T^2) \right) - C\Phi \cdot O(T^2) = O(T^2).$$

The ultimate bound on σ_k is $O(T^2)$ according to (2.118), and, from (2.122) the ultimate bound on $\|\mathbf{e}_k\|$ is $O(T)$.

Remark 2.11 Note that the guaranteed tracking precision is $O(T)$ because the control problem becomes highly challenging in the presence of state estimation and disturbance estimation errors, and meanwhile aiming at arbitrary reference tracking.

In many motion control tasks the system relative degree is 2, for instance from the torque or force to position tracking in motion control. Now we derive an interesting property by the following corollary.

Corollary 2.1 *For a continuous system of relative degree greater than 1, the ultimate bound of $\|\mathbf{e}_k\|$ is $O(T^2)$.*

Proof From (2.84) in *Theorem 2.5* and *Lemma 2.1*, $\|\mathbf{e}_k\|$ is $O(T^2)$ if $\delta_k = O(T^3)$. When the system relative degree is 2, $CB = 0$, and

$$C\Gamma = C \left(BT + \frac{1}{2!}ABT^2 + \frac{1}{3!}A^2BT^3 + \dots \right) = \frac{1}{2!}CABT^2 + \frac{1}{3!}CA^2BT^3 + \dots = O(T^2). \quad (2.130)$$

Similarly

$$C\Phi\Gamma = C \left(I + AT + \frac{1}{2!}A^2T^2 + \dots \right) \Gamma = C(I + O(T))\Gamma = C\Gamma + O(T^2) = O(T^2). \quad (2.131)$$

Now rewrite

$$\begin{aligned}\delta_k &= C \left(\hat{\mathbf{d}}_k - \mathbf{d}_k + \mathbf{d}_{k-1} - \hat{\mathbf{d}}_{k-1} \right) - C\Phi \left(\mathbf{d}_k - \hat{\mathbf{d}}_k \right) \\ &= C\Gamma \left(\hat{\boldsymbol{\eta}}_k - \boldsymbol{\eta}_k + \boldsymbol{\eta}_{k-1} - \hat{\boldsymbol{\eta}}_{k-1} \right) - C\Phi\Gamma \left(\boldsymbol{\eta}_k - \hat{\boldsymbol{\eta}}_k \right) + O \left(T^3 \right). \quad (2.132)\end{aligned}$$

Note that the ultimate bound of $\boldsymbol{\eta}_k - \hat{\boldsymbol{\eta}}_k$, derived in *Theorem 2.3*, is $O(T)$. Thus we conclude from (2.132)

$$\delta_k = O \left(T^2 \right) \cdot (O(T) + O(T)) - O \left(T^2 \right) \cdot O(T) + O \left(T^3 \right) = O \left(T^3 \right)$$

and consequently $\|\mathbf{e}_k\|$ is ultimately $O(T^2)$.

Remark 2.12 From the result we see that even though the state estimation error is $O(T)$ we can still obtain $O(T^2)$ output tracking by virtue of the integral action in the controller design for relative degree greater than 1.

2.4.5 Systems with a Piece-Wise Smooth Disturbance

In practice, the disturbance of a motion system, $\mathbf{f}(\mathbf{x}, t)$, may become discontinuous at certain circumstances. For example, due to the static friction force, a discontinuity occurs when the motion speed drops to zero. It is thus vital to examine the system performance around the time the discontinuity occurs.

Suppose the discontinuity of $\mathbf{f}(\mathbf{x}, t)$ occurs at the j th sampling instant. The immediate consequence of the discontinuity in $\mathbf{f}(\mathbf{x}, t)$ is the loss of the property $\mathbf{d}_j - \mathbf{d}_{j-1} = O(T)$ instead of $O(T^2)$, and the loss of the property $\mathbf{d}_j - 2\mathbf{d}_{j-1} + \mathbf{d}_{j-2} = O(T)$ instead of $O(T^3)$.

Since we focus on tracking tasks in this work, it can be reasonably assumed that the discontinuity occurs only occasionally. As such, the property $\mathbf{d}_j - \mathbf{d}_{j-1} = O(T^2)$ will be restored one step after the occurrence of the discontinuity, and the property: $\mathbf{d}_j - 2\mathbf{d}_{j-1} + \mathbf{d}_{j-2} = O(T^3)$ will be restored two step after the occurrence of the discontinuity. The discontinuity presents an impulse-like impact to the system behavior at the instant $k = j$. It is worth to investigate the applicability of *Lemma 2.1* for this case. Write

$$\mathbf{e}_k = \Lambda_d^k \mathbf{e}_0 + \sum_{i=0}^{k-1} \Lambda_d^i \delta_{k-i-1}.$$

For simplicity assume $\Lambda_d^k \mathbf{e}_0$ can be ignored and $\mathbf{e}_k = O(T^2)$ at $k = j$. Then δ_j will give $\mathbf{e}_k$ an offset with the magnitude $O(T)$, which can be viewed as a new initial error at the time instant j and will disappear exponentially with the rate Λ_d^k .

Note that Λ_d is a design matrix, thus can be chosen to be sufficiently small such that the impact from δ_j , which is $O(T)$, can be quickly attenuated. As a result, the analysis of the preceding sections still holds for discontinuous disturbances.

In the worst case the non-smooth disturbance presents for a long interval, we can consider a nonlinear switching control action. Denote $\mathbf{u}_k$ the ISM control designed in preceding sections, a new nonlinear control law is

$$\mathbf{u}_k^n = \mathbf{u}_k + \mu \text{sat}(\sigma_k) \quad (2.133)$$

where $\text{sat}(\sigma_k)$ is a vector with each of the elements given by

$$\text{sat}(x) = \begin{cases} 1 & \forall x \in (\varepsilon, \infty) \\ \frac{x}{\varepsilon} & \forall x \in [-\varepsilon, \varepsilon] \\ -1 & \forall x \in (-\infty, -\varepsilon) \end{cases}$$

and ε is the required bound on the sliding function σ_k .

Remark 2.13 Let $\varepsilon \rightarrow 0$, it is known that $\text{sat}(\cdot)$ renders to a signum function and improves the control system bandwidth, hence suppress the discontinuous disturbance. In digital implementation, however, the actual bandwidth, being limited by the sampling frequency, is $\pi/2T$ where T is the sampling period.

2.4.6 Illustrative Example

2.4.6.1 State Regulation

Consider the system (2.1) with the following parameters

$$A = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -3 & 4 \\ 5 & 6 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 2 \\ 4 & -1 & 2 \end{bmatrix} \quad \& \quad \mathbf{f}(t) = \begin{bmatrix} 0.3 \sin(4\pi t) \\ 0.3 \cos(4\pi t) \end{bmatrix}$$

The initial states are $\mathbf{x}_0 = [1 \quad 1 \quad -1]^T$. The system will be simulated for a sample interval of 1 ms. For the classical SM control, the D matrix is chosen such that the non-zero pole of the sliding dynamics is $p = -5$ in continuous-time, or $z = 0.9950$ in discrete-time. Hence, the poles of the system with SM are $[0 \quad 0.9950 \quad 0]^T$ respectively. Accordingly the D matrix is

$$D = \begin{bmatrix} 0.2621 & -0.3108 & -0.0385 \\ 3.4268 & 2.4432 & 1.1787 \end{bmatrix}.$$

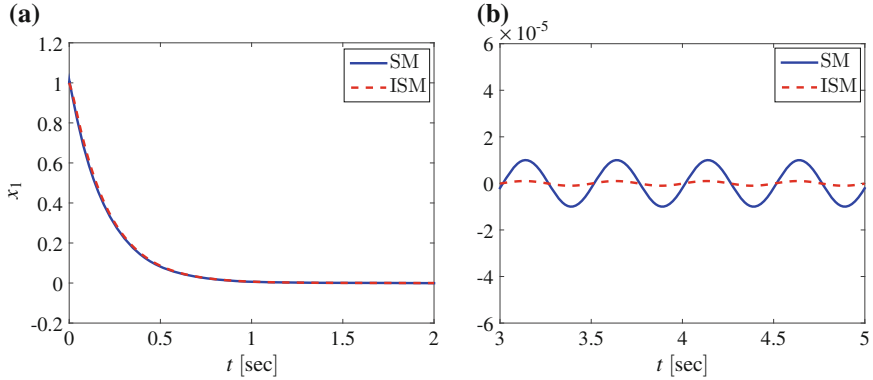


Fig. 2.1 System state x_1 **a** Transient performance **b** Steady-State performance

Using the same D matrix given above, the system with ISM is designed such that the dominant (non-zero) pole remains the same, but, the remaining poles are not deadbeat. The poles are selected as $z = [0.9048 \ 0.9950 \ 0.8958]^T$ respectively.

Using the pole placement command of Matlab, the gain matrices can be obtained

$$K = \begin{bmatrix} 66.6705 & 9.4041 & 15.8872 \\ 18.2422 & 21.3569 & 8.5793 \end{bmatrix}.$$

According to (2.45)

$$E = \begin{bmatrix} 0.0297 & -0.0313 & -0.0034 \\ 0.3147 & 0.2366 & 0.1115 \end{bmatrix}.$$

The delayed disturbance compensation is used. Figure 2.1a shows that the system state $x_1(t)$ is asymptotically stable for both discrete-time SM control and ISM control, which show almost the same behavior globally. On the other hand, the difference in the steady-state response between discrete-time SM control and ISM control can be seen from Fig. 2.1b. The control inputs are shown in Fig. 2.2. Note that the control inputs of the SM control has much larger values at the initial phase in comparison with ISM control, due to the existence of deadbeat poles. Another reason for the lower value of the control inputs in the ISM control is the elimination of the reaching phase by compensating for the non-zero initial condition in (2.50).

To demonstrate the quality of both designs, the open-loop transfer function matrices, $G_{i,j}^{OL}$, for the systems with SM and ISM are computed and Bode plots of some elements are shown in Fig. 2.3. In addition, the sensitivity function of state x_1 with respect to the disturbance components $f_1(t)$ and $f_2(t)$ is plotted in Fig. 2.4. It can be seen from Figs. 2.3 and 2.4 that ISM control greatly reduces the effect of the disturbance as compared to SM control. Moreover ISM presents a larger open-loop gain at the lower frequency band by virtue of the integral action in the sliding manifold, which ensures a more accurate closed-loop response to possible reference inputs.

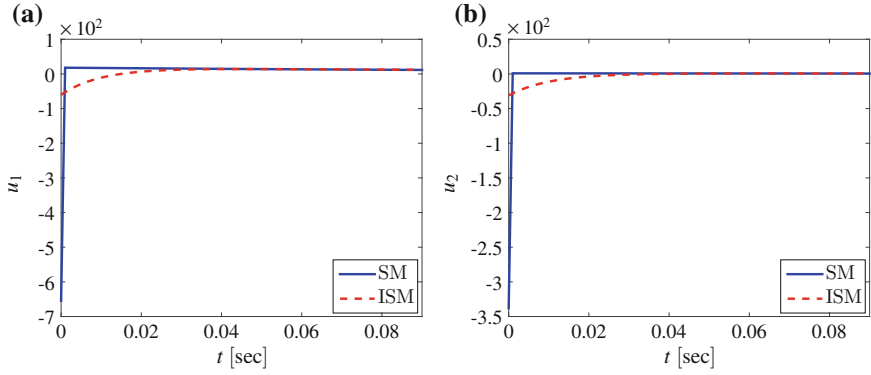


Fig. 2.2 a System control input u_1 b System control input u_2

Fig. 2.3 Bode plot of some of the elements of the open-loop transfer matrix

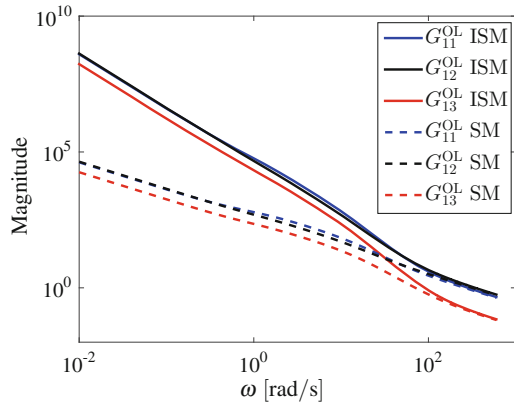


Fig. 2.4 Sensitivity function of x_1 with respect to f_1 and f_2

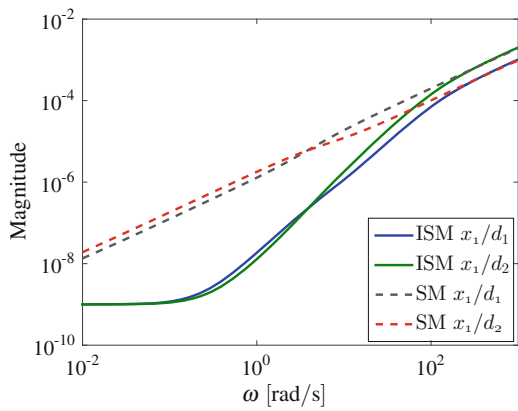
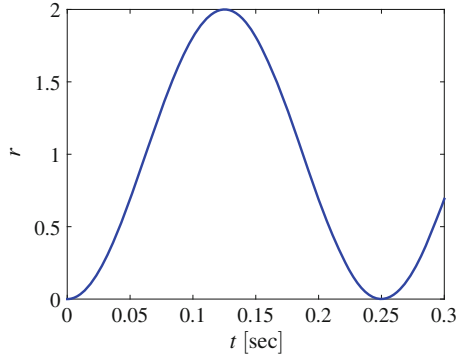


Fig. 2.5 The output reference trajectory



2.4.6.2 State Feedback Approach

Consider the following second order system

$$A = \begin{bmatrix} 10 & 1 \\ -10 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 4.2 \end{bmatrix} \quad \& \quad C = [1 \ 0].$$

The sampled-data system obtained with a sampling period $T = 1$ ms is

$$\Phi = \begin{bmatrix} 1.01 & -0.001 \\ -0.01 & 0.99 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.0040 \\ 0.0042 \end{bmatrix} \quad \& \quad C = [1 \ 0].$$

The zero of (Φ, Γ, C) is $z = 0.989$ and, therefore, the system is minimum-phase. Let the desired pole, the remaining pole of the closed-loop system to be designed, be $z = 0.75$, then the design parameter is given by $E = 0.25$. The system is simulated with an output reference $r_k = 1 + \sin(8\pi kT - \frac{\pi}{2})$, shown in Fig. 2.5. The disturbance acting on the system will be non-smooth when speed crosses zero and given by

$$f(\mathbf{x}, t) = \begin{cases} 10 & \forall x_2 \in (-\infty, -\varepsilon_f) \\ 0 & \forall x_2 \in [-\varepsilon_f, \varepsilon_f] \\ -10 & \forall x_2 \in (\varepsilon_f, \infty) \end{cases} \quad (2.134)$$

where the coefficient ε_f is sufficiently small. The system is simulated using control law (2.68). The controller performance is compared with that of a PI controller having a proportional gain of $k_p = 240$ and integral gain of $k_i = 8$. In Fig. 2.6 the tracking error is 4×10^{-6} which corresponds to $O(T^2)$ and is almost invisible as compared with the PI controller performance. Note worthy is the fact that the control signal for the PI control is much larger initially as compared to the ISM control. A smaller control signal is more desirable in practice as it would not create a heavy burden on actuators (Fig. 2.7).

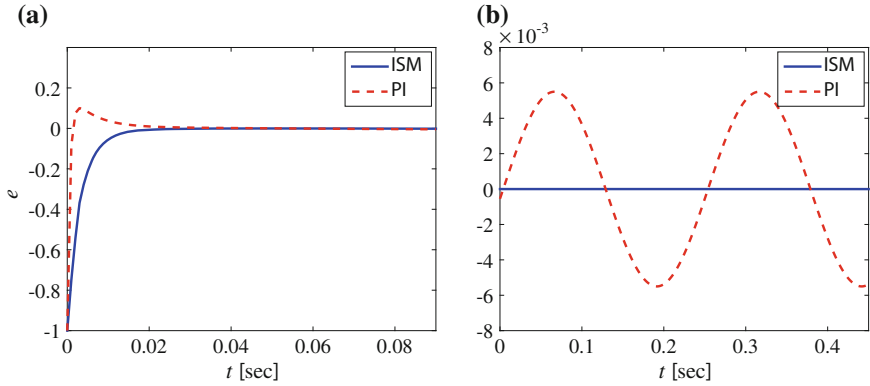
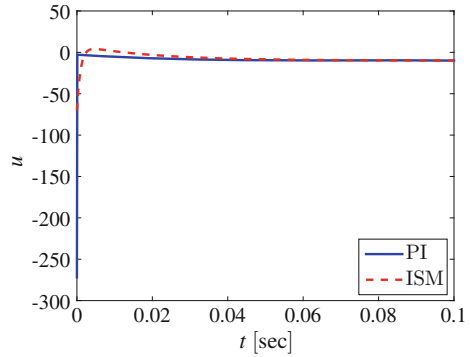


Fig. 2.6 Tracking error of ISM control and PI controllers **a** Transient performance **b** Steady-State performance

Fig. 2.7 Control inputs of ISM control with state feedback and PI



2.4.6.3 Output Feedback Approach

Consider the system

$$A = \begin{bmatrix} -60 & -10 \\ 10 & -10 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 4.2 \end{bmatrix} \quad \& \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

After sampling the system at $T = 1$ ms, the discretized system is

$$\Phi = \begin{bmatrix} 0.9417 & -0.0097 \\ 0.0097 & 0.9900 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0.0039 \\ 0.0042 \end{bmatrix} \quad \& \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

For this system the zero of (Φ, Γ, C) is $z = 1.001$ whereas, using $D = C\Phi^{-1}$, the zero of (Φ, Γ, D) is $z = 0.998$. Therefore, the output-feedback approach with the reference model in Sect. 2.4.3.5 is the only option. Using the same disturbance $f(t)$ and reference trajectory r_k , the system is simulated. The controller performance is

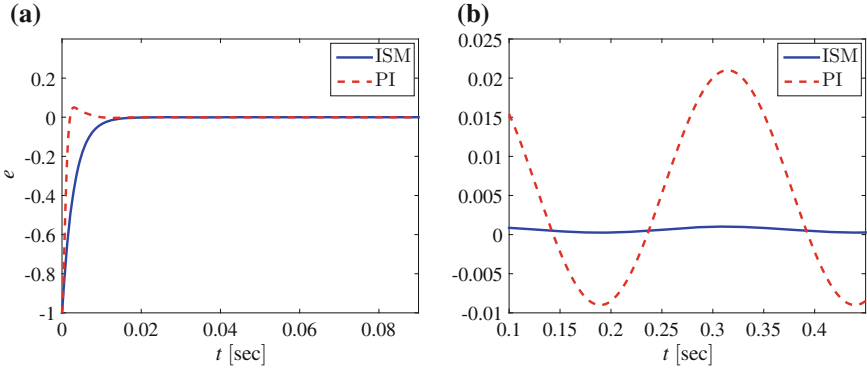
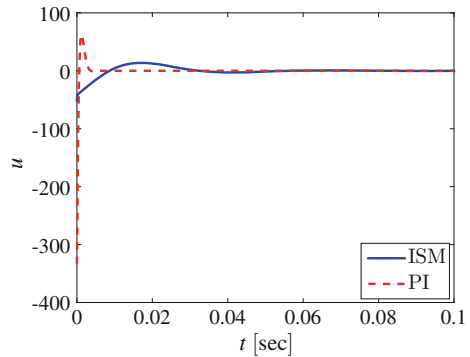


Fig. 2.8 Tracking error of ISM control and PI controllers **a** Transient performance **b** Steady-State performance

Fig. 2.9 Control inputs of ISM control and PI output feedback



compared with that of a PI controller having a proportional gain of $k_p = 200$ and integral gain of $k_i = 30$. As it can be seen from Fig. 2.8, the performance is quite good and better than that of a PI controller. The tracking error for the ISM control is about 17×10^{-6} which corresponds to $O(T^2)$ at steady state. Note that even though the worst case scenario of $O(T)$ was predicted for this approach it was possible to achieve $O(T^2)$ at steady state. Also, similar to the state feedback approach the control signal of the ISM control controller is much smaller than that of the PI controller at the onset of motion (Fig. 2.9).

2.4.6.4 State Observer Approach

For this approach we will go back to the system (2.135), and estimate x_2 using the observer (2.123). The observer has a gain of $L = [1.19 \ 342.23]$ and is designed such that two poles are at $z = 0.4$ allowing a fast enough convergence. From Fig. 2.10 the estimation of error $\tilde{x}_{2,k}$ is plotted. As we can see the estimation is quite good

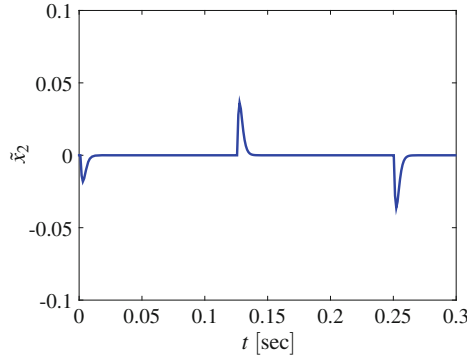


Fig. 2.10 Observer state estimation error $\tilde{x}_2$

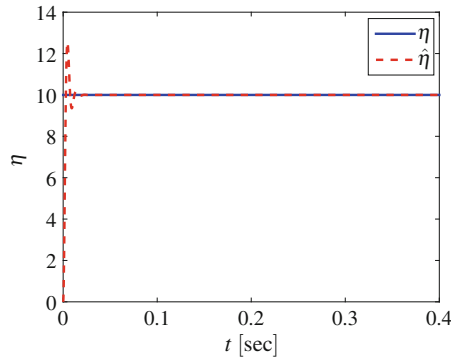


Fig. 2.11 Disturbance η and estimate $\hat{\eta}$

and deviating only when the discontinuities occur but attenuates very quickly. The disturbance estimation is seen in Fig. 2.11 and the estimation converges quickly to the actual disturbance. From Fig. 2.12 we can see the tracking error performance. The tracking error is about 6×10^{-6} which matches the theoretical results of $O(T^2)$ bound. Again like the previous two approaches, the control signal of the ISM control is smaller than that of the PI controller at the initial phase (Fig. 2.13).

Finally we need to show the effects of a more frequently occurring discontinuous disturbance and how adding a nonlinear term, (2.133), would improve the performance. The disturbance is shown in Fig. 2.15. As it can be seen from Fig. 2.14 the rapid disturbance degrades the performance of the ISM control and PI controllers, however, addition of a switching term with $\mu = 10$ and $\varepsilon = 0.01$ improves the tracking performance. Note also from Fig. 2.15 that the disturbance estimate $\hat{\eta}_k$ converges quickly to the disturbance.

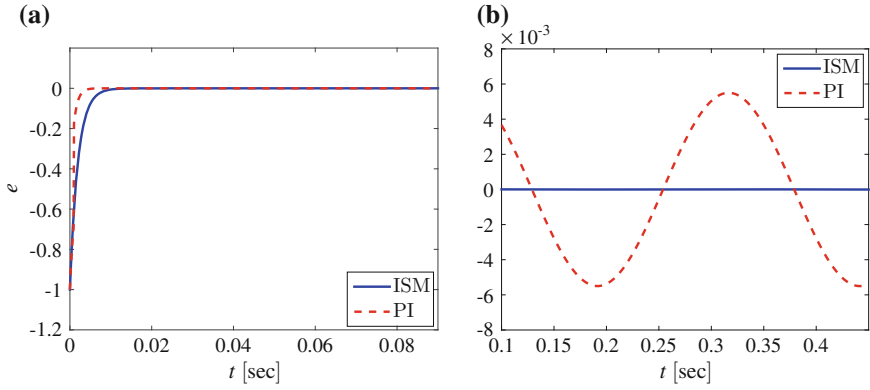


Fig. 2.12 Tracking error of ISM control and PI controllers **a** Transient performance **b** Steady-State performance

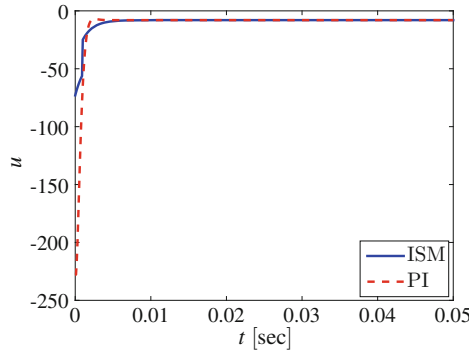


Fig. 2.13 Control inputs of ISM control with state observer and PI

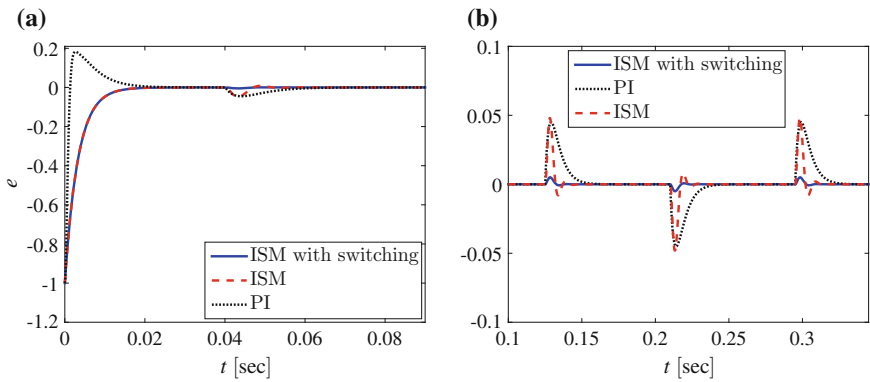


Fig. 2.14 Tracking errors of ISM control, PI and ISM control with switching under a more frequent discontinuous disturbance **a** Transient performance **b** Steady-State performance

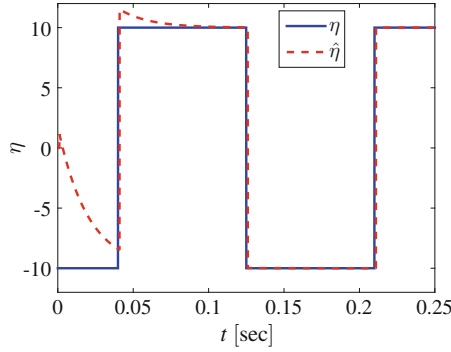


Fig. 2.15 Disturbance η and estimate $\hat{\eta}$

2.5 Discrete-Time Terminal Sliding Mode Control

In this section we will discuss the design of the tracking controller for the system. The controller will be designed based on an appropriate sliding surface. Further, the stability conditions of the closed-loop system will be analyzed. The relation between TSM control properties and the closed-loop eigenvalue will be explored. Note that the subsequent analysis and derivations will be based on a 2nd order system example, however, it is possible to apply the same principles for a system of higher order as long as it is SISO.

2.5.1 Controller Design and Stability Analysis

Consider a 2nd order SISO version of (2.2) given by

$$\begin{aligned}\mathbf{x}_{k+1} &= \Phi \mathbf{x}_k + \gamma u_k + \mathbf{d}_k \\ y_k &= \mathbf{c} \mathbf{x}_k = x_{1,k}, \quad y_0 = y(0)\end{aligned}\tag{2.135}$$

where $\mathbf{x}_k \in \mathbb{R}^2$, $u_k \in \mathbb{R}$, $\mathbf{d}_k \in \mathbb{R}^2$, $y_k \in \mathbb{R}$, $\Phi \in \mathbb{R}^{2 \times 2}$, $\gamma \in \mathbb{R}^2$ and $\mathbf{c} \in \mathbb{R}^2$ respectively. Further, consider the discrete-time sliding surface below,

$$\sigma_k = \mathbf{s} \mathbf{e}_k + \beta e_{k-1}^p\tag{2.136}$$

where $\mathbf{e}_k^\top = [e_{1,k} \mid e_{2,k}]$, $e_{1,k} = r_{1,k} - x_{1,k}$, $e_{2,k} = r_{2,k} - x_{2,k}$, $r_{1,k}$, $r_{2,k}$ are arbitrary time-varying references, σ_k is the sliding function, and $\mathbf{s}$, β , p are positive design constants. The tracking problem is to force $x_{1,k} \rightarrow r_{1,k}$. The selection of $p < 1$ guarantees a steeper slope of the sliding surface as the states approach the origin which is desirable as seen in Fig. 2.16. Also note that p should be selected as

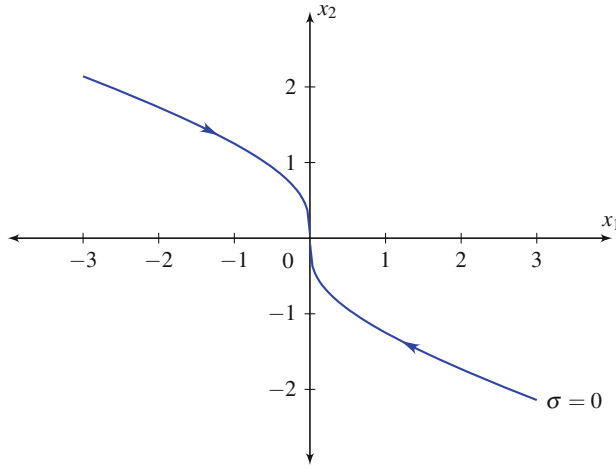


Fig. 2.16 Phase portrait of the sliding surface

a rational number with odd numerator and denominator to guarantee that the sign of the error remains intact. Let us first derive the discrete-time TSM control law by using the concept of equivalent control and discuss the TSM control properties associated with stability.

Theorem 2.7 *The new TSM control law proposed is*

$$u_k = (\mathbf{s}\boldsymbol{\gamma})^{-1} (\mathbf{s}\mathbf{r}_{k+1} - \mathbf{s}\Phi\mathbf{r}_k + \mathbf{s}\Phi\mathbf{e}_k + \beta e_k^p) - (\mathbf{s}\boldsymbol{\gamma})^{-1} \hat{\mathbf{s}}\mathbf{d}_k \quad (2.137)$$

where $\mathbf{r}_k^\top = [r_{1,k} \ r_{2,k}]$ and $\hat{\mathbf{d}}_k$ is the estimate of the disturbance. The controller (2.137) drives the sliding variable to

$$\sigma_{k+1} = \mathbf{s} (\hat{\mathbf{d}}_k - \mathbf{d}_k)$$

and results in the closed-loop error dynamics

$$\mathbf{e}_{k+1} = [\Phi - \boldsymbol{\gamma} (\mathbf{s}\boldsymbol{\gamma})^{-1} \mathbf{s}\Phi] \mathbf{e}_k - \beta (\mathbf{s}\boldsymbol{\gamma})^{-1} e_k^p + \delta_k$$

where $\delta_k = \boldsymbol{\gamma} (\mathbf{s}\boldsymbol{\gamma})^{-1} \hat{\mathbf{s}}\mathbf{d}_k - \mathbf{d}_k$ is due to the disturbance estimation error and is of $O(T^2)$. Further, the stable range of closed-loop system is nonlinearly dependent on the tracking error e_k .

Proof The control law (2.137) can be derived using the design method based on equivalent control. To proceed, consider a forward expression of (2.136)

$$\sigma_{k+1} = \mathbf{s}\mathbf{e}_{k+1} + \beta e_k^p. \quad (2.138)$$

The objective of a sliding mode controller is to achieve $\sigma_{k+1} = 0$, therefore, we need to derive an explicit expression in terms of the error dynamics. For this we rewrite the system dynamics (2.2) in terms of the error dynamics. It can be shown that the error dynamics is of the form

$$\mathbf{e}_{k+1} = \Phi \mathbf{e}_k - \gamma u_k - \mathbf{d}_k + \mathbf{r}_{k+1} - \Phi \mathbf{r}_k. \quad (2.139)$$

Substitution of (2.139) into (2.138) and equating the resulting expression of σ_{k+1} to zero we obtain the expression for the equivalent control u_k^{eq} ,

$$u_k^{\text{eq}} = (\mathbf{s}\gamma)^{-1} (\mathbf{s}\mathbf{r}_{k+1} - \mathbf{s}\Phi \mathbf{r}_k + \mathbf{s}\Phi \mathbf{e}_k + \beta e_k^p) - (\mathbf{s}\gamma)^{-1} \mathbf{s}\mathbf{d}_k. \quad (2.140)$$

Note that the control (2.140) is based the current value of the disturbance $d_{1,k}$ which is unknown and therefore cannot be implemented in this current form. To overcome this, the disturbance will be estimated with the so called delay estimate as follows,

$$\hat{\mathbf{d}}_k = \mathbf{d}_{k-1} = \mathbf{x}_k - \Phi \mathbf{x}_{k-1} - \gamma u_{k-1} \quad (2.141)$$

therefore, the final controller structure is given by

$$u_k = (\mathbf{s}\gamma)^{-1} (\mathbf{s}\mathbf{r}_{k+1} - \mathbf{s}\Phi \mathbf{r}_k + \mathbf{s}\Phi \mathbf{e}_k + \beta e_k^p) - (\mathbf{s}\gamma)^{-1} \mathbf{s}\hat{\mathbf{d}}_k. \quad (2.142)$$

In order to verify the second part of *Theorem 2.7* with regard to the closed-loop stability, first derive the closed-loop error dynamics. Substitute u_k in (2.137) into (2.139), we obtain

$$\begin{aligned} \mathbf{e}_{k+1} = & \left[\Phi - \gamma (\mathbf{s}\gamma)^{-1} \mathbf{s}\Phi \right] \mathbf{e}_k - \beta (\mathbf{s}\gamma)^{-1} e_k^p - \mathbf{d}_k \\ & + \gamma (\mathbf{s}\gamma)^{-1} \mathbf{s}\hat{\mathbf{d}}_k + \left(I - \gamma (\mathbf{s}\gamma)^{-1} \mathbf{s} \right) (\mathbf{r}_{k+1} - \Phi \mathbf{r}_k). \end{aligned} \quad (2.143)$$

Next, in order to eliminate the term $(I - \gamma (\mathbf{s}\gamma)^{-1} \mathbf{s}) (\mathbf{r}_{k+1} - \Phi \mathbf{r}_k)$ from the closed-loop dynamics (2.143), we note that since the objective is to have $\mathbf{x}_k \rightarrow \mathbf{r}_k$ then there must exist a control input $u_{m,k}$ such that $\mathbf{r}_{k+1} = \Phi \mathbf{r}_k + \gamma u_{m,k}$. Thus,

$$\left(I - \gamma (\mathbf{s}\gamma)^{-1} \mathbf{s} \right) (\mathbf{r}_{k+1} - \Phi \mathbf{r}_k) = \left(I - \gamma (\mathbf{s}\gamma)^{-1} \mathbf{s} \right) \gamma u_{m,k} = 0. \quad (2.144)$$

and the final closed-loop error dynamics is

$$\mathbf{e}_{k+1} = \left[\Phi - \gamma (\mathbf{s}\gamma)^{-1} \mathbf{s}\Phi \right] \mathbf{e}_k - \beta (\mathbf{s}\gamma)^{-1} e_k^p + \delta_k \quad (2.145)$$

where $\delta_k = \gamma (\mathbf{s}\gamma)^{-1} \mathbf{s}\hat{\mathbf{d}}_k - \mathbf{d}_k$ and is of $O(T^2)$, [1]. The sliding surface dynamics is obtained by substituting (2.145) into (2.138) to get

$$\sigma_{k+1} = \mathbf{s} \left(\hat{\mathbf{d}}_k - \mathbf{d}_k \right) = \mathbf{s} \left(\mathbf{d}_{k-1} - \mathbf{d}_k \right) = O \left(T^2 \right). \quad (2.146)$$

To evaluate the stable range of (2.145), rewrite (2.145) in the form

$$\mathbf{e}_{k+1} = \left[\Phi - \boldsymbol{\gamma} (\mathbf{s}\boldsymbol{\gamma})^{-1} \left(\mathbf{s}\Phi + \beta e_k^{p-1} C \right) \right] \mathbf{e}_k + \boldsymbol{\delta}_k \quad (2.147)$$

where $C = \text{diag}(1, 0)$. Denote $\mathbf{l}_k = [l_{1,k} \ ; \ l_{2,k}] = (\mathbf{s}\boldsymbol{\gamma})^{-1} \left(\mathbf{s}\Phi + \beta e_k^{p-1} C \right)$ the control gain vector, where $l_{1,k}$ is error dependent. The error dynamics (2.148) can be rewritten as

$$\mathbf{e}_{k+1} = [\Phi - \boldsymbol{\gamma}\mathbf{l}_k] \mathbf{e}_k + \boldsymbol{\delta}_k. \quad (2.148)$$

From (2.148) we see that there must exist certain range for the first element of the gain vector, $l_{1,k}$, such that the closed-loop system is stable. Let $l_{1,\min} \leq l_{1,k} \leq l_{1,\max}$ where $l_{1,\min}$ and $l_{1,\max}$ denote the minimum and maximum allowable values for $l_{1,k}$.

From the definition of $\mathbf{l}_k$ we can obtain

$$\beta e_k^{p-1} + s_1 \phi_{1,1} + s_2 \phi_{2,1} = \boldsymbol{\gamma} \mathbf{l}_{1,k} \quad (2.149)$$

and

$$s_1 \phi_{1,2} + s_2 \phi_{2,2} = \boldsymbol{\gamma} \mathbf{l}_{2,k} \quad (2.150)$$

where $\phi_{i,j}$ are elements of the matrix Φ . From (2.149) we can derive the following inequality

$$\boldsymbol{\gamma} \mathbf{l}_{1,\min} < \beta e_k^{p-1} + s_1 \phi_{1,1} + s_2 \phi_{2,1} < \boldsymbol{\gamma} \mathbf{l}_{1,\max} \quad (2.151)$$

from which we can obtain, when $p < 1$,

$$\begin{aligned} |e_k| &> \left(\frac{\beta}{\boldsymbol{\gamma} \mathbf{l}_{1,\max} - s_1 \phi_{1,1} - s_2 \phi_{2,1}} \right)^{\frac{1}{1-p}} \\ |e_k| &< \left(\frac{\beta}{\boldsymbol{\gamma} \mathbf{l}_{1,\min} - s_1 \phi_{1,1} - s_2 \phi_{2,1}} \right)^{\frac{1}{1-p}} \end{aligned} \quad (2.152)$$

The first relation gives the minimum-bound of the error and the second relation gives the stable operation range. Note, that by selecting a proper s_1 and s_2 such that the denominator in the second relation is zero for a non-zero β then it is possible to guarantee global stability outside of the minimum-error bound.

2.5.2 TSM Control Tracking Properties

First derive the ultimate tracking error bound of (2.1) when the proposed discrete-time TSM control is applied. Using the discrete-time TSM control law (2.137), in the following we show that the ultimate bound of the tracking error is $O(T^2)$. From the previous section we obtained a minimum error bound based on the selection of β which if selected small enough would result in a small error bound.

However, due to the existence of a disturbance term δ_k the ultimate error bound may be large. Note that the solution of the closed-loop system (2.148) is

$$\mathbf{e}_k = \left(\Pi_{i=0}^k [\Phi - \gamma \mathbf{l}_i] \right) \mathbf{e}_0 + \sum_{i=0}^{k-1} \left(\Pi_{j=0}^i [\Phi - \gamma \mathbf{l}_j] \right) \delta_{k-i-1}. \quad (2.153)$$

According to our earlier discussion the ultimate error bound would be an order higher than δ_k , which means that the error bound will be $O(T)$. This property holds if the gain $\mathbf{l}_k$ constant and the term $\left(\Pi_{j=0}^i [\Phi - \gamma \mathbf{l}_j] \right) \delta_{k-i-1}$ is an infinite series.

According to [1], the series will be of the order $O\left(\frac{1}{1-\lambda_{\max}}\right) \cdot O(\delta)$ where $\lambda_{\max}$ is the dominant eigenvalue. Therefore, if the dominant eigenvalue is designed close to the edge of the unit circle, then $O\left(\frac{1}{1-\lambda_{\max}}\right) = O(T^{-1})$. This implies a rather bad rejection of the exogenous disturbance. To enhance disturbance rejection, it is desirable to choose the dominant eigenvalue closer to the origin during steady state motion, then $O\left(\frac{1}{1-\lambda_{\max}}\right) = O(1)$ and

$$O\left(\frac{1}{1-\lambda_{\max}}\right) \cdot O(\delta) = O(\delta) = O(T^2). \quad (2.154)$$

However, in practical consideration of sampled-data processes during transient motion, an eigenvalue closer to the origin will result in large initial control effort of the order $O\left(\frac{1}{1-\lambda_{\max}}\right) = O(T^{-1})$.

A very useful property that is acquired by using the terminal switching surface is that the system gain $l_{1,k}$ will increase as the error approaches zero because of the nonlinear term e_k^{1-p} in $\mathbf{l}_k$. This means that it is possible to move the dominant eigenvale of the closed-loop system from an initial position nearby the unit circle towards the origin, thus avoid the large initial control effort during the transient period, obtain very stable operation at steady state, and quickly attenuate exogenous disturbances. We will explore more about this property in the next section.

Finally, we look at what happens to the tracking error when there is a discontinuity in the disturbance. Consider the disturbance term associated with the closed-loop system (2.148). It can be reasonably assumed that the discontinuity occurs rarely, therefore, if we assume that the discontinuity occurs at the k th sampling point, then $\delta_k = O(T)$ rather than $O(T^2)$ as the difference $\mathbf{d}_{k-1} - \mathbf{d}_k$ will no longer be $O(T^2)$

but of the order of $\mathbf{d}_k$ which is $O(T)$. If the discontinuity occurs at a time instant k' , then $\delta_k = O(T)$ at $k = k', k' + 1, k' + 2$, and return to $\delta_k = O(T^2)$ for subsequent sampling instants. Therefore the solution of (2.148) would lead to the worst case error bound

$$|e_k| = O(T) \quad (2.155)$$

for certain time interval but $O(T^2)$ ultimately.

2.5.3 Determination of Controller Parameters

In order to discuss the controller design, consider the case when the system parameters are

$$\Phi = \begin{bmatrix} 1 & 9.313 \times 10^{-4} \\ 0 & 0.8659 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 2.861 \times 10^{-6} \\ 5.6 \times 10^{-3} \end{bmatrix} \quad \& \quad \mathbf{c} = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

The control law requires the design of three parameters: the vector $\mathbf{s}$, the parameter β , and the power p . As was discussed earlier the parameters p and β determine the dynamics of the eigenvalue and, using (2.148), this can be seen from Figs. 2.17, 2.18, 2.19 and 2.20. We can see from these figures, since e_1 leads to high gain feedback, both closed-loop eigenvalues of the discrete-time TSM control will eventually exceed unity and become unstable when e_1 is sufficiently small. This is consistent with the discussion made in previous section that there exists a minimum-bound of tracking error specified by (2.152). From the eigenvalue figures it is clear that the minimum error bound is determined by a critical value of e_1 where at least one eigenvalue becomes marginal stable. The minimum error bound can be reduced by shifting the curves of eigenvalues leftwards. As shown in Figs. 2.17, 2.18, 2.19 and 2.20, this can be achieved by either reducing β or increasing p . However, a smaller β implies a

Fig. 2.17 System eigenvalue λ_1 w.r.t e_1 for different choices of β and $p = \frac{5}{9}$

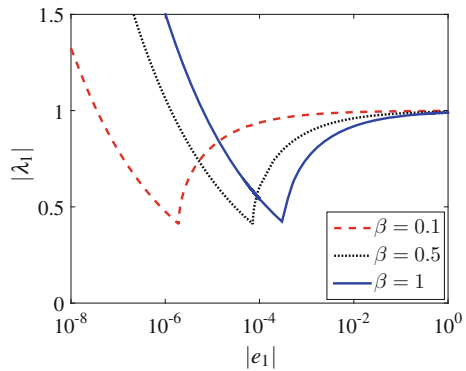


Fig. 2.18 System eigenvalue λ_2 w.r.t e_1 for different choices of β and $p = \frac{5}{9}$

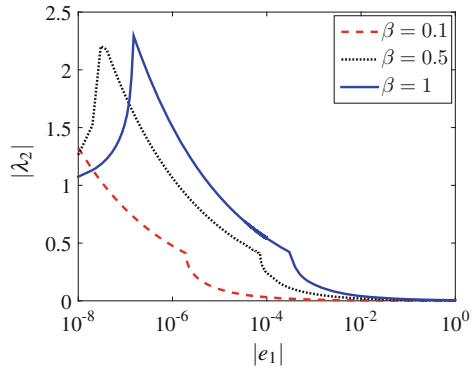


Fig. 2.19 System eigenvalue λ_1 w.r.t e_1 for different choices of p and $\beta = 0.5$

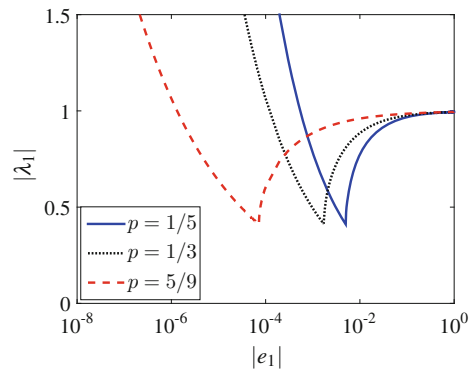
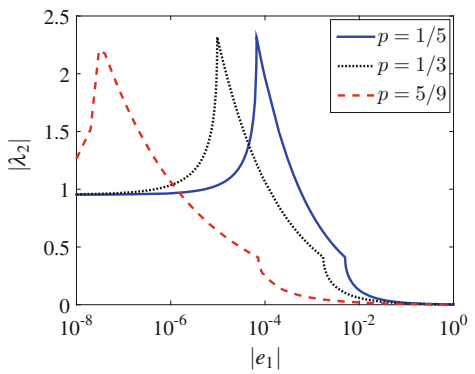


Fig. 2.20 System eigenvalue λ_2 w.r.t e_1 for different choices of p and $\beta = 0.5$



smaller range of stability as shown in (2.152). Therefore in TSM control design, a relatively larger p is preferred.

From Figs. 2.17 and 2.19, the first eigenvalue, λ_1 , is always near unity when the error e_1 is large. Hence this eigenvalue does not generate large initial control efforts, but may generate large steady state error in the presence of disturbance if the eigenvalue does not decrease with respect to error. By incorporating TSM, λ_1 will first drop when e_1 decreases, then rise when e_1 further decreases. A smaller β and a larger p will speed up this variation pattern as e_1 decreases.

The variation pattern of the second eigenvalue, λ_2 , is opposite to the first eigenvalue as can be seen from Figs. 2.18 and 2.20.

For the system $\mathbf{e}_{k+1} = [\Phi - \boldsymbol{\gamma}\mathbf{l}_k] \mathbf{e}_k$ the solution is $\mathbf{e}_k = V \cdot \text{diag}(\lambda_1^k, 0) V^{-1} \mathbf{e}_0$ where the matrix V consists of the eigenvectors of $[\Phi - \boldsymbol{\gamma}\mathbf{l}_k]$. This leads us to conclude that the control law (2.142) is proportional to V^{-1} which is a function of λ_1 and will take large values as λ_1 moves towards 0. Thus, it is desirable to have λ_1 closer to the edge of the unit circle so that V^{-1} does not take large values. This is evident from Figs. 2.21 and 2.22 where the controller gains, at the initial time step of $k = 1$, $(\mathbf{s}\boldsymbol{\gamma})^{-1} \mathbf{s}\Phi V \cdot \text{diag}(\lambda_1, 0) V^{-1}$ was plotted w.r.t λ_1 . Note, that the nonlinear term βe_k^p has been disregarded as its contribution at the initial time step is negligible w.r.t the linear term.

Based on the above discussions the design guideline for discrete TSM control is determined. The controller gains $\mathbf{l}_k$ can be determined according to the selection of closed-loop eigenvalues. Note that eigenvalue λ_1 should take a larger value initially and drop when approaching steady state. Thus we can choose λ_1 varying from the initial value 0.995 to the final 0. The other eigenvalue is $\lambda_2 = 0$ when the closed-loop system is in sliding mode. As functions of eigenvalues, the range of the feedback gain vectors can be calculated as $[894 \ 155] \leq \mathbf{l}_k \leq [1.79 \times 10^5 \ 240]$.

Next, from relations (2.150) and (2.152) we can obtain $\mathbf{s}$, β , and p . The selected $\mathbf{s}$ should ensure that the denominator of the second expression in (2.152) is close to zero so that the upper limit of e_k is maximized, and at the same time (2.150) should be satisfied for the given range of l_2 . It is not necessary to select $\mathbf{s}$ to make the upper bound on e_1 infinite, as the real system has a maximum displacement limitation of 60×10^{-3} m. Thus, we select the denominator to take a value of 0.01 and select $l_2 = 200$ from the specified range of l_2 . Solving the two simultaneous equations (2.150) and the denominator of the second expression in (2.152) being zero for s_1 and s_2 gives $\mathbf{s} = [0.49 \ 0.100]$.

To determine the parameters β and p , first look into the relations between these two parameters and the closed-loop eigenvalues. The behavior of the eigenvalues under these parameters are shown in Figs. 2.23 and 2.24. It is possible to divide the plots into three regions in which the system has different behaviors. The transient response region is the region in which the error is large enough and the dominant eigenvalue, λ_1 , remains close to the edge of the unit circle. As a result, the control effort can be kept at appropriate level despite any large initial error. At this region, disturbance rejection is not a main concern as an $O(T)$ or $O(T^2)$ disturbance would be much smaller than the state errors.

Fig. 2.21 First element of the system gain $(s\gamma)^{-1} s\Phi V \cdot \text{diag}(\lambda_1, 0) V^{-1}$ w.r.t λ_1

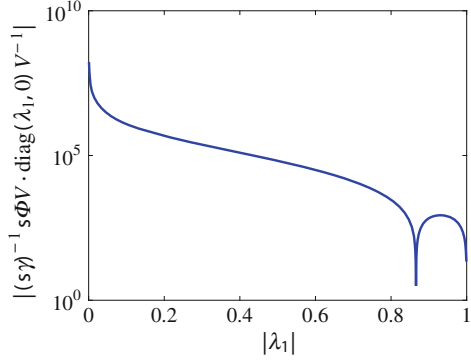


Fig. 2.22 Second element of the system gain $(s\gamma)^{-1} s\Phi V \cdot \text{diag}(\lambda_1, 0) V^{-1}$ w.r.t λ_1

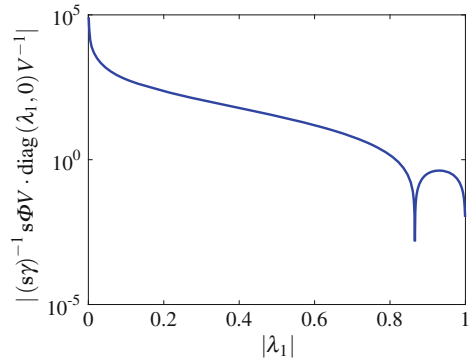
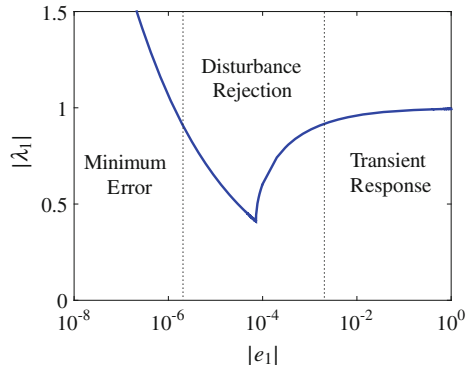
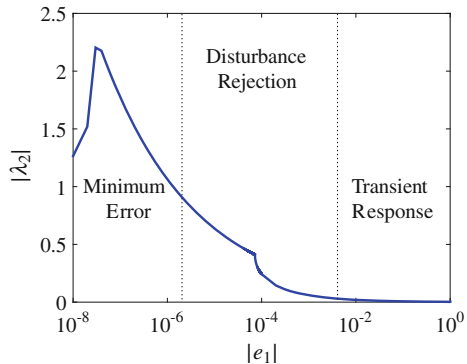


Fig. 2.23 System eigenvalue λ_1 w.r.t e_1



The disturbance rejection region is when the state errors become smaller, reaching $O(T)$ or $O(T^2)$ level. Now, since the eigenvalue also becomes smaller as the controller gain increases, the robustness and disturbance rejection property of the system are enhanced. Finally, the minimum error region is the region in which the eigenvalue goes beyond the unit circle. Therefore, the error will stay around the

Fig. 2.24 System eigenvalue λ_2 w.r.t e_1



boundary between the disturbance rejection region and the minimum error region, and the boundary determines the minimum error bound.

Based on the above discussions, from Figs. 2.17 and 2.19 it can be seen that a larger $\beta = 1$ would lead to a faster response because the dominant eigenvalue, λ_1 , drops quickly in the transient response region. However, it may also lead to a larger steady state error because λ_1 rises quickly and produces a rather large minimum error region. When smaller $\beta = 0.1$ is used, we can achieve a much smaller minimum error region, but λ_1 drops slowly in the transient response region. In the real-time implementation, we choose a mid value $\beta = 0.5$ as a tradeoff.

Looking into Figs. 2.18 and 2.20, we can observe that the variation of p will produce the similar trend as β . Namely, when p is close to 0, the transient response region is improved but the minimum error region is larger. When p is approaching 1, on the other hand, the transient response is slower as the dominant eigenvalue λ_1 drops slowly, but the minimum error region is getting smaller. In the real-time implementation, we choose a mid value $p = \frac{5}{9}$ as a tradeoff.

It should also be noted that, in the real-time implementation, the presence of the disturbance will limit the best possible tracking error e_1 to $O(T^2)$ or $O(10^{-6})$ for a sampling period of 1 ms. Therefore, any selection of larger p and smaller β which result in the minimum tracking region below $O(T^2)$ would lead to little or no improvements.

It is well known that sliding mode control requires a fast switching frequency in order to maintain the sliding motion. The proposed TSM control however does not produce much chattering though with the limited sampling frequency.

Finally, it is also interesting to check the tracking error bound according to the theoretical analysis and experimental result. Through analysis we have shown the tracking error bound is proportional to the size of the sampling period T . Therefore, we can expect a smooth control response and low tracking error when the sampling period is 1 ms. In Chap. 7 it will be shown that the magnitude of the tracking error obtained in experiment is consistent with the theoretical error bound of $O(T^2) = O(0.001^2) = O(10^{-6})$ when $T = 1$ ms.

2.6 Conclusion

This chapter discussed a discrete-time integral sliding (ISM) control design and a discrete-time terminal sliding mode (TSM) control design for sampled-data systems.

It was shown that with the discrete-time integral type sliding manifold, the ISM design retains the deadbeat structure of the discrete-time sliding mode, and at the same time allocates the closed-loop eigenvalues for the full-order multi-input system. The discrete-time ISM control achieves accurate control performance for the sliding mode, state regulation and output tracking, meanwhile eliminates the reaching phase and avoids overlarge control efforts.

In addition it was shown through theoretical investigation that the revised discrete-time TSM controller can achieve very good performance and along with discrete-time ISM control is a superior alternative to classical SM control.

Advanced Discrete-Time Control

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