Chapter 2
Limb Kinematics

Abstract The purpose of this chapter is to introduce you to the *kinematics* of limbs. Kinematics is the study of movements without regard to the forces and torques that produce them. In essence, it is the fundamental description of the articulations and motions of which a limb is capable. This chapter serves as the foundation upon which we can build a common conceptual language, and begin to discuss limb function in the context of mechanics.

2.1 What Is a Limb?

A clear understanding of the kinematics of limbs is necessary to compare and contrast the capabilities and limitations of biological and robotic limbs. Kinematics is the study of the motions and positions of rigid bodies, like limbs, *without* regard to the forces that produce them. The kinematic *degrees of freedom* (DOFs) are the articulations between the links. These articulations (i.e., anatomical joints) are the mechanical structures that allow for changes in the configuration of the links with respect to each other. In general, there are two kinds of engineered DOFs that are convenient to define and use mathematically: the linear or *prismatic DOF* like telescoping tubes that change the lengths of a link in the limb; and the revolute or *rotational DOF* like a hinge that changes the orientation of adjacent links in the limb. This allows us to use the state of each DOF to define a specific limb configuration, shape, and size. In addition, by having motors act on each DOF using linear or rotational motors, we can produce specific limb forces and accelerations.

I will focus on rotational joints because most vertebrate limbs are approximated as behaving in this way,[1] as in Fig. 2.1. Universal joints, such as those used to represent 2 DOF rotational joints like the metacarpophalangeal (MCP) joint of the index finger (which is the base knuckle of the finger), consist of two pin joints with intersecting and perpendicular rotational axes. Ball-and-socket joints, like the shoulder or hip, consist of three intersecting and perpendicular rotational joints. Other joints like the

[1] In biological systems, joint kinematics arise from the interaction of the contact of bony articulating surfaces held by ligamentous structures. A joint is, therefore, a complex system whose kinematics can be load dependent [1].

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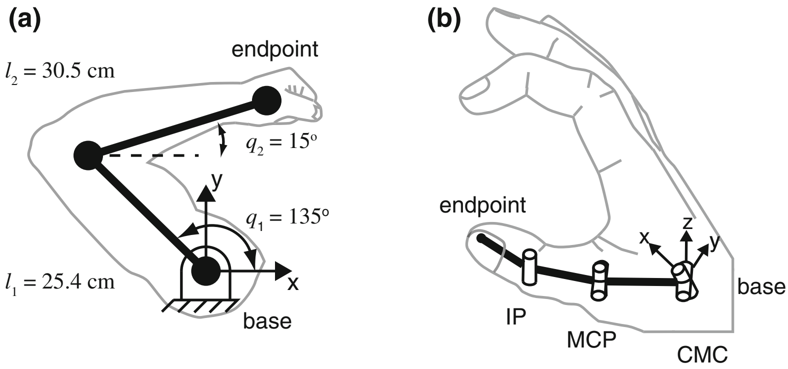

Fig. 2.1 **a** Human arm modeled as planar 2 DOF serial manipulator. The angle of each DOF is often described as the q_i variable, where i is the index of the DOF. **b** Human thumb modeled as 3D serial manipulator with 5 rotational DOFs contained in the carpometacarpal (CMC with 2 DOFs), metacarpophalangeal (MCP with 2 DOFs), and interphalangeal (IP with 1 DOF) joints. Note the axes of rotation of the different DOFs need not be parallel or perpendicular to each other [10]

knee or jaw have more complex kinematics that involve both rotation and sliding, but they are often approximated as pure rotational joints [2–5].

Note that in this book it is necessary to alternate between anatomical and robotics terminology. Robotic limbs were inspired by vertebrate limbs, and vertebrate limbs are analyzed using the mathematics developed for robotic limbs.

An important distinction between engineered and vertebrate limbs is that the former are often *torque-driven limbs*[2] where motors act on each joint to rotate them, whereas the latter are *tendon-driven limbs* where muscles pull on tendons that act on joints by spanning 1 or more DOFs. The mathematics and theory of torque-driven limbs is well developed, and serves as a starting point to discuss tendon-driven limbs. Therefore, this chapter gives a brief overview of the theory behind the basic kinematic analysis of torque-driven *robotic manipulators*.[3] That is, robotic limbs that have motors actuating on rotational joints. Subsequent chapters leverage this fundamental understanding to discuss the case of tendon-driven limbs. This is important because tendon-driven limbs can be, and often are, simplified into their mathematically equivalent torque-driven limbs—but with important conceptual consequences as mentioned in Chap. 1. This chapter is intended to give the reader a sense of the mathematical principles of these topics and some associated issues, but is by nomeans

[2]We use the term torque-driven instead of joint-driven because this is more common in the robotics literature.

[3]In robotics, the term *manipulator* is used synonymously with *robot*, *robotic arm*, *robotic limb*, or any other mechanism that is actuated and controlled.

comprehensive and further study is likely required to use this knowledge in detail. Specialized monographs [6, 7] and our prior publications [5, 8] contain a wealth of detailed information and background material.

The heart of the matter in many debates in neuromechanics is that much of the theory of robotics analysis and design takes a torque-driven approach. And rightfully so as it grows historically out of the mathematical description of serial linkage systems with pin-joints (rotational joints with 1 DOF), universal joints (with 2 DOFs), or ball-and-socket joints (with 3 DOFs), driven by idealized rotational *actuators*.[4] That is, whatever kind of electric, pneumatic, hydraulic, etc. linear or rotational actuator is used in practice (via gears, pulleys, capstans, belts, direct-drive, etc.), it can be analyzed as inducing torques at these rotational joints.break However, some subtle but important points arise when tendon-driven vertebrate limbs are represented as torque-driven systems, including:

- Actuation in vertebrate limbs is often asymmetric—whereas engineered actuators are symmetric. That is, the angular velocities, accelerations and torques that can be produced in one rotational direction (say, flexion) are not necessarily equal to those that can be produced in the other (say, extension). While most reasonable engineers would design and build systems with symmetric actuation, most biological systems lack such symmetry. For example, compare the flexor versus extensor muscles of your fingers.
- Most muscles cross multiple DOFs. Again, most reasonable engineers would design and build systems with tendons carefully routed (often threaded through the inside of the links) to selectively actuate a single joint. But most biological limbs have *musculotendons* (the combined entity of the tendon of origin, the muscle fibers, and tendon of insertion [9]) with muscles that lie outside the bones, and tendons routed along the length of the limb actuating the multiple DOFs they cross. Thus, the individual DOFs of a vertebrate limb cannot be actuated in strict isolation.
- Most DOFs are crossed by multiple muscles. Again, while most reasonable engineers would design and build systems where only two opposing tendons drive a given DOF, the vast majority of vertebrate DOFs are not actuated in this way. Thus there is no unique set of muscle forces to produce a given net joint torque.

As we shall see, these mathematically inconvenient features of vertebrate limbs require either careful application, or extension, of the analytical approaches that were developed for torque-driven robotic systems. This situation highlights the unavoidable conceptual struggle in neuromechanics between mathematical rigor and expediency versus biological realism.

[4] Actuator is the generic engineering term for a motor or some other device that produces forces or mechanical work.

2.2 Forward Kinematic Analysis of Limbs

The *forward kinematics* of a limb determine the location and orientation of its *endpoint* with respect to its *base*, given the relative configurations of each pair of adjacent links of the limb [6]. The base is usually the origin of the fixed, reference coordinate system—(x, y) or (x, y, z) in Fig. 2.1—chosen to represent the Cartesian coordinates of the workspace of the limb. The endpoint is the final, functional part of the limb. It is the point of interest, such as the hand when we speak of the arm for reach tasks, the foot when we speak of the legs for locomotion, or the fingertips when we speak of the hand for manipulation. Here I briefly present a simplified version of well-established methods to calculate forward kinematics of limbs. These simple kinematic formulations are common in neuromechanics studies, and sufficient to address important debates of motor control. In [6, 7] you can find an in-depth and generalized treatment of these topics.

This basic kinematic problem is: given a mathematical representation of the robotic or biological limb, and its joint angles and angular velocities, what is the position and velocity of its endpoint? To do so we must first understand how to create a mathematical representation of the forward kinematics of the limb.

Consider the example of a human arm, modeled as the planar 2 DOF serial manipulator shown in Fig. 2.1a. It is called a *planar model* because it is constrained to lie on a 2D plane; in contrast to a *spatial model* like the thumb model in Fig. 2.1b that allows motion in 3D space. The parameters of the arm model needed to calculate the position of the hand (i.e., the endpoint) are the lengths of the forearm and upper arm, and the angles of the shoulder and elbow joints as shown in Fig. 2.1a. Using the sample parameter values shown in the figure, it requires only basic knowledge of geometry to calculate the endpoint location by inspection as

$$(x, y) = (25.4\cos(135^\circ) + 30.5\cos(15^\circ), 25.4\sin(135^\circ) + 30.5\sin(15^\circ)) \quad (2.1)$$

$$(x, y) = (11.5, 25.9) \text{ in cm} \quad (2.2)$$

That was simple enough. However, now consider the 3D model of the thumb shown in Fig. 2.1b, in which there is a universal joint at both the carpometacarpal (CMC) and metacarpophalangeal (MCP) joints, and one hinge joint at the IP (interphalangeal) joint [10]. Say the metacarpal bone (closest to the wrist) has length 5.08 cm, the proximal phalanx (middle bone) has length 3.18 cm, and the distal

Table 2.1 Sample joint angles for the spatial thumb model in Fig. 2.1b

Joint	Angle
CMC adduction-abduction	-45°
CMC flexion-extension	20°
MCP adduction-abduction	-10°
MCP flexion-extension	-30°
IP flexion-extension	-20°

phalanx (the bone on the thumbtip) has length 2.54 cm, and the joint angles are as in Table 2.1. Where is the endpoint then?

The mathematical expression for calculating the thumb endpoint coordinates for any set of joint angles is quite complicated, and even difficult to calculate by inspection. However, we are able to calculate the endpoint position in relation to a base frame in a systematic way if we use *homogeneous transformations*. Appendix A provides a brief introduction to these tools. It is imperative that you read it before continuing if you have not worked with the fundamentals of linear algebra or robot kinematics recently.

2.3 The Forward Kinematic Model

A limb is an *open kinematic chain* because it is a serial arrangement of articulated rigid bodies, Fig. 2.2. The posture of the limb is determined by its kinematic DOFs of the system, defined in this book by variables $q_1, q_2, q_3, \ldots, q_N$, that are also called *generalized coordinates* in mechanical analysis. In the case where anatomical joints are assumed to be rotational joints, the generalized coordinates are angles; but they can also be linear displacements for prismatic joints in robotic systems or in anatomical joints that can slide. As mentioned above, using pure rotational joints (pin, universal, or ball-and-socket joints) is common in musculoskeletal models [5], but it is a critical assumption that can have important consequences to the validity and utility of the model [2–4, 11].

Based on the techniques presented in Appendix A, a forward kinematic model of a limb is created using the following steps:

1. *Create the necessary homogenous transformation matrices, one for each DOF.* Take Fig. 2.2 as an example. Recall that we describe rigid bodies by attaching a *frame of reference* to each body, and homogeneous transformations are used to relate adjacent frames of reference. From now on, we do not speak of the rigid links any more, but only treat the frames of reference. This allows you to find[5]

$$T_{base}^{endpoint} = T_0^N \tag{2.3}$$

[5] A note about typesetting conventions set forth in Appendix A. Capital letters as superscripts or subscripts (italicized or not) like M or N indicate extremes of ranges. Thus the endpoint of a limb is assigned frame N, and dimensionality of a vector or matrix are $\mathbf{v} \in \mathbb{R}^N$ or $A \in \mathbb{R}^{M \times N}$, respectively. Indices that are lowercase italicized letters like n, i, or j signify a number within a range. The letter M need not stand for muscles, or n for an intermediate frame of reference. They are simply letters to indicate dimensions and indices, and change with the context of the material. Vectors are lowercase letters typeset as $\mathbf{v}$, which can be also specified to be expressed in a given frame of reference, say frame 0, as $\mathbf{v}_0$. Or if the start and end of a vector are specified, it will be typeset as $\mathbf{p}_{0,N}$. Matrices are written as *italicized* upper case letters, such at the matrix T, which can also carry subscripts and superscripts depending on their meaning like $T_{base}^{endpoint}$. I use lowercase italics for general scalars (i.e., numbers).

If there is a single DOF between each rigid body (like pin joints in the figure), there is usually one frame of reference per link, with one homogeneous transformation per DOF—a total of $N-1$ in this case. However, this example requires N homogeneous N transformations because the last frame of reference is needed to describe the location and orientation of the endpoint with respect to frame $N-1$. But there are no DOFs between frames $N-1$ and N as both frames are fixed to the same rigid body. The addition of such extra (or 'dummy') frames of reference is sometimes necessary to define the forward kinematic model of the limb. To avoid confusion, the end of a range will always be a capital letter like N. Thus,

$$T_0^N = T_0^1 \; T_1^2 \; \dots \; T_{N-2}^{N-1} \; T_{N-1}^N \tag{2.4}$$

where

$$T_0^N = \begin{bmatrix} \;\; R_0^N & \mathbf{p}_{0,N} \\ 0 \; 0 \; 0 & 1 \end{bmatrix} \tag{2.5}$$

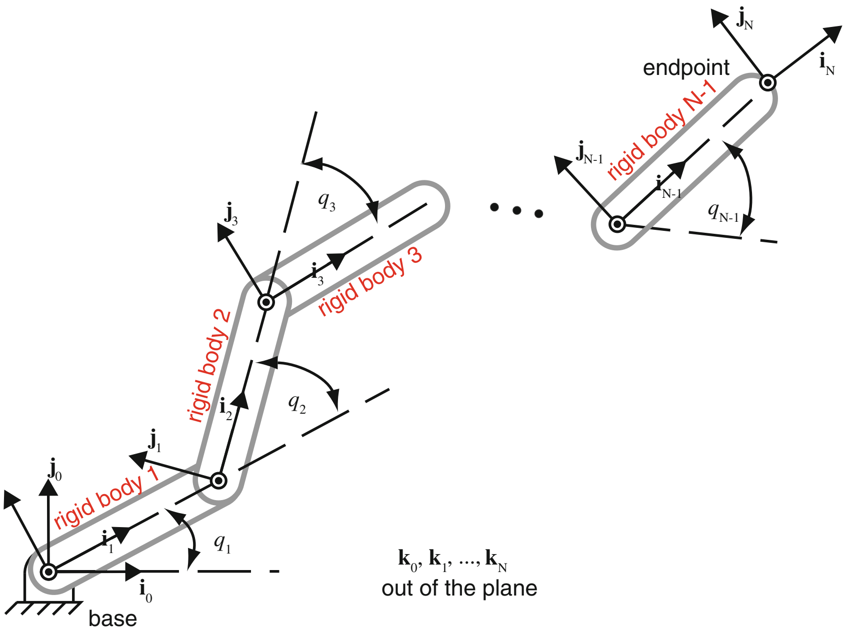

Fig. 2.2 In this book, limbs are modeled as serial linkage mechanisms (also called open kinematic chains). When defining the kinematics of such arrangements of rigid bodies, it is typical to attach a unique frame of reference to each rigid body, and thereafter perform the analysis on those frames of reference. However, this example has N frames of reference but only $N-1$ rigid bodies and DOFs. The last frame of reference is used to describe the location and orientation of the endpoint with respect to frame $N-1$

If there are 2 or more DOFs between two rigid bodies, like the CMC joint at the base of the thumb in Fig. 2.1, then intermediate frames of reference, and their respective homogeneous transformations, are needed to represent these DOFs. Appendix A and Sect. 2.4 discuss the importance of defining and allocating the DOFs of a limb in a specific order. See, for example, the kinematic models of the thumb in [10, 11].

2. *Extract the position of the endpoint from the homogeneous transformation* T_0^N. Note that in Eq. 2.5 the vector $\mathbf{p}_0^N$ is the location of the endpoint with respect to the base.
3. *Extract the orientation of the endpoint from the homogeneous transformation* T_0^N. The orientation of the last link is given by the matrix R_0^N. But notice that the limb in Fig. 2.2 is generic enough that, by having many DOFs, it raises the issue of kinematic redundancy. More on this in Sect. 7.1.

2.4 Application of the Forward Kinematic Model to a Simple Planar Limb

Consider the planar, 2 DOF, limb shown in Fig. 2.3. This limb is anchored to ground, where ground is the base frame, or frame 0. The first DOF (i.e., q_1) rotates the first link in a positive (as per the right-hand-rule) direction. The second DOF (i.e., q_2) *rotates the second link with respect to the first link* as shown in Fig. 2.2. This is an important convention in kinematic analysis in robotics: the generalized rotational coordinates q_i are relative to the prior body. Other branches of engineering and physics prefer all angles to be measured with respect to the base frame. But it will become apparent

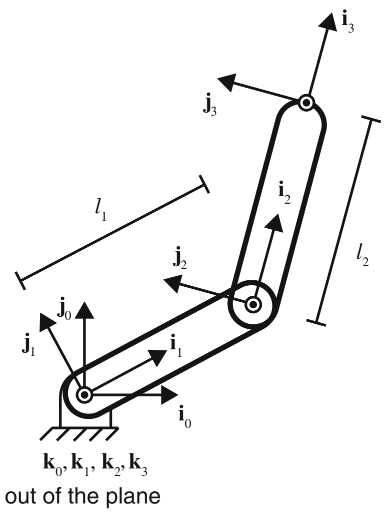

Fig. 2.3 A 2-link, 2 DOF planar serial kinematic chain with its body-fixed frames of reference. Note that, even though there is no third DOF, we add the third 'dummy' frame of reference to place the frame of reference of the endpoint at the end of the second link

why this simplifies the arithmetic of this kinematic analysis. Importantly, this also means that we need to define a reference posture of the limb for which all angles are 0, as all adjacent frames of reference are aligned with each other. For this we use the configuration where the kinematic chain is straight and aligned with the horizontal axis of the base frame, by definition making all q_i equal to zero, all $\mathbf{i}_i$ parallel and pointing to the right, all $\mathbf{j}_i$ pointing up, and all $\mathbf{k}_i$ coming out of the page.

The first question is to define T_0^1 as described in Appendix A. Well, we see that it is a rotation about the first joint, where the origin of the first joint is the same as the origin of the base frame. This transformation has a rotation about the $\mathbf{k}_0$ axis of magnitude q_1, and no translation.

Go ahead and use Eq. A.35 to obtain the following matrices, where for succinctness I use the common shorthand of c_1 for $cos(q_1)$, s_1 for $sin(q_1)$; and c_{12}, s_{12} for $cos(q_1 + q_2)$ and $sin(q_1 + q_2)$, respectively,

$$T_0^1 = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.6}$$

$$T_1^2 = \begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.7}$$

$$T_2^3 = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

It is important to point out that, even though there is no third DOF q_3, we add the third 'dummy' transformation T_2^3 to place the frame of reference of the endpoint at the end of the second link. As mentioned in Sect. 2.3, one should not confuse the number of DOFs of the system with the number of kinematic transformations used to describe the system. We then concatenate all three coordinate transformation matrices to obtain

$$T_0^3 = T_0^1 T_1^2 T_2^3 = \begin{bmatrix} c_1c_2 - s_1s_2 & -c_1s_2 - s_1c_2 & 0 & c_1(l_2c_2 + l_1) - s_1(l_2s_2) \\ s_1c_2 + c_1s_2 & -s_1s_2 + c_1c_2 & 0 & s_1(l_2c_2 + l_1) + c_1(l_2s_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.9}$$

which with the help of the trigonometric identities

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \tag{2.10}$$

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \tag{2.11}$$

simplifies to

$$T_0^3 = T_0^1 T_1^2 T_2^3 = \begin{bmatrix} c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\ s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.12}$$

It follows that

$$R_0^3 = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.13}$$

and

$$\mathbf{p}_{0,3} = \begin{pmatrix} l_1c_1 + l_2c_{12} \\ l_1s_1 + l_2s_{12} \\ 0 \end{pmatrix} \tag{2.14}$$

Look closely at Eq. 2.13 and notice that its structure says that it represents a right-handed rotation about the $\mathbf{k}_0$ axis of a magnitude equal to $q_1 + q_2$. Also, there are no other rotations, as is expected from a planar limb. Similarly, the vector $\mathbf{p}_{0,3}$ represents a displacement on the $\mathbf{i}_0$—$\mathbf{j}_0$ plane, with no component in the $\mathbf{k}_0$ direction, also as expected for a planar limb.

Therefore, in this case the forward kinematic model (also called the geometric model), $G(\mathbf{q})$, is

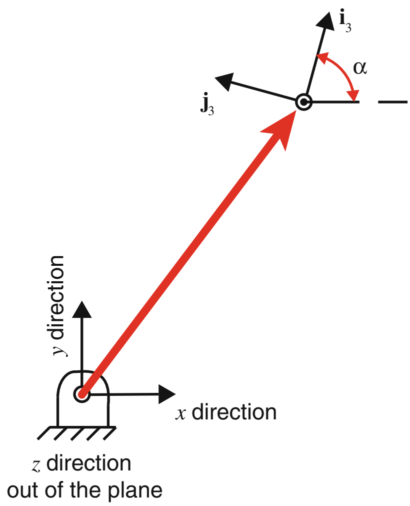

Fig. 2.4 The forward kinematic model of the planar limb in Fig. 2.3 in more familiar Cartesian coordinates

$$G(\mathbf{q}) = \begin{pmatrix} \text{displacement in } \mathbf{i}_0 \text{ direction} \\ \text{displacement in } \mathbf{j}_0 \text{ direction} \\ \text{rotation about the } \mathbf{k}_0 \text{ axis} \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ q_1 + q_2 \end{pmatrix} \tag{2.15}$$

This example raises an important concept in retrospect: How many kinematic DOFs does a rigid body have on the plane? The answer is three, which are two displacements and one rotation—as revealed by the elements of T_0^3 and written out explicitly in $G(\mathbf{q})$. We will explore the relationships between the kinematic DOFs of the endpoint and the kinematic DOFs of the limb later in this book. But for now, this simple example allows us to validate this intuition because you could have just as easily written out the forward kinematic model of this limb by inspection. Try it.

But the definition of the forward kinematic model in Eq. 2.15 holds other surprises. On one hand, creating this vector function is natural as it describes the position and orientation of the endpoint frame of reference. On the other hand, you may have probably never seen a vector with mixed units. The first two elements are distances, and the last element is an angle.

Such vectors are necessary to express forward kinematic models—they are a natural consequence of the homogeneous transformation matrices that contain a combination of rotations and displacements [6]. As you will soon see this gives rise to the very convenient mathematical formulations of twists and wrenches presented in Sect. 2.6.

For the sake of clarity, I define that forward kinematic vector as $\mathbf{x}$ using the more familiar variables shown in Fig. 2.4 as follows

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} = G(\mathbf{q}) = \begin{pmatrix} G_x(\mathbf{q}) \\ G_y(\mathbf{q}) \\ G_\alpha(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} l_1 c_1 + l_2 c_{12} \\ l_1 s_1 + l_2 s_{12} \\ q_1 + q_2 \end{pmatrix} \tag{2.16}$$

2.5 Using the Forward Kinematic Model to Obtain Endpoint Velocities

How is the velocity of the endpoint ($\dot{\mathbf{x}}$) related to the angular velocities of the joints ($\dot{\mathbf{q}}$)? Note that a dot above a variable is a shorthand that indicates the time derivative, thus $\dot{a} = \frac{da}{dt}$. This is also part of the forward kinematics problem because the joint angular velocities are inputs that produce the endpoint velocities as an output.

Consider the case of a limb with N kinematic DOFs, where we know that the forward kinematic model specifies the location and orientation of the endpoint as a function of the joint angles

$$\mathbf{x} = G(\mathbf{q}) \tag{2.17}$$

We now want to know the time derivative of the forward kinematic model such that

$$\dot{\mathbf{x}} = \frac{G(\mathbf{q})}{dt} = \frac{\partial G(\mathbf{q})}{\partial \mathbf{q}} \frac{d\mathbf{q}}{dt} = \frac{\partial G(\mathbf{q})}{\partial \mathbf{q}} \dot{\mathbf{q}} \tag{2.18}$$

For a specific example where

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} \tag{2.19}$$

$$\mathbf{q} = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix} \tag{2.20}$$

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{pmatrix} \tag{2.21}$$

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_N \end{pmatrix} \tag{2.22}$$

The definition of the partial derivatives of the vector function $G(\mathbf{q})$ is

$$\frac{\partial G(\mathbf{q})}{\partial \mathbf{q}} = J(\mathbf{q}) = \begin{bmatrix} \frac{\partial G_x(\mathbf{q})}{\partial q_1} & \frac{\partial G_x(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_x(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_y(\mathbf{q})}{\partial q_1} & \frac{\partial G_y(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_y(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_\alpha(\mathbf{q})}{\partial q_1} & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_N} \end{bmatrix} \tag{2.23}$$

where N is the number of DOFs, and $J(\mathbf{q})$ is called the *Jacobian* of the system. In this case $J \in \mathbb{R}^{3\times N}$. The instantaneous 3D endpoint velocity vector can be calculated using the following equation:

$$\dot{\mathbf{x}} = J(\mathbf{q})\,\dot{\mathbf{q}} \tag{2.24}$$

And when the Jacobian is invertible (see Sect. 3.5) we can find the instantaneous joint angular velocities associated with a given endpoint velocity vector using the following equation

$$\dot{\mathbf{q}} = J(\mathbf{q})^{-1}\,\dot{\mathbf{x}} \tag{2.25}$$

2.6 General Case of the Jacobian in the Context of Screws, Twists, and Wrenches

The Jacobian of a serial linkage system is fundamental to the calculation of the feasible motions and forces that it can produce. The general definition of a Jacobian needs to address the fact that the endpoint of the kinematic chain, as a rigid body, has 6 DOFs: three translations and three rotations. In the formal kinematics of rigid body mechanics [12], this falls within the field of screw theory [6]. Such a combined vector is called a *screw*. It consists of a pair of 3D vectors, in this case the translations and rotations a rigid body can have—which I still call $\mathbf{x}$ because it represents the forward kinematic model of the endpoint as in Eq. 2.16.

In the general case where we consider all 6 DOFs of the frame of reference fixed at the endpoint,

$$\mathbf{x} = \begin{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \end{pmatrix} \in \mathbb{R}^6 \tag{2.26}$$

containing the 3D position and orientation vectors.

The time derivative of this positional/rotational screw vector is the *twist* vector of linear and angular velocities, respectively—which I call $\dot{\mathbf{x}}$

$$\dot{\mathbf{x}} = \left(\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} \begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} \right) \tag{2.27}$$

As we will see further on, this screw concept that combines elements of different units extends to the force and torque vectors an endpoint can produce—called the endpoint *wrench* vector [13]

$$\mathbf{w} = \left(\begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} \begin{pmatrix} \tau_\alpha \\ \tau_\beta \\ \tau_\gamma \end{pmatrix} \right) \tag{2.28}$$

where f and τ are the components of force and torque along their respective dimensions. This means that in the general case the full Jacobian has 6 rows and N columns [14], $J \in \mathbb{R}^{6\times N}$

$$\frac{\partial G(\mathbf{q})}{\partial \mathbf{q}} = J(\mathbf{q}) = \begin{bmatrix} \frac{\partial G_x(\mathbf{q})}{\partial q_1} & \frac{\partial G_x(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_x(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_y(\mathbf{q})}{\partial q_1} & \frac{\partial G_y(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_y(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_z(\mathbf{q})}{\partial q_1} & \frac{\partial G_z(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_z(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_\alpha(\mathbf{q})}{\partial q_1} & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_\alpha(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_\beta(\mathbf{q})}{\partial q_1} & \frac{\partial G_\beta(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_\beta(\mathbf{q})}{\partial q_N} \\ \frac{\partial G_\gamma(\mathbf{q})}{\partial q_1} & \frac{\partial G_\gamma(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial G_\gamma(\mathbf{q})}{\partial q_N} \end{bmatrix} \tag{2.29}$$

This succinct presentation of the general case of the Jacobian matrix for a limb raises several questions, some beyond the scope of this book. For example:

- How does one find the 6D screw vector for a generic robotic or biological limb? In [6, 7] you can find examples of this, but their derivation requires a working knowledge of kinematics.
- What is the relationship between the N kinematic DOFs of the limb and the 6 DOFs the end link can have? As you shall see in later chapters, roboticists are very mindful to design limbs with 6 or fewer DOFs so that the Jacobian matrix is easier

to compute and manipulate. But there are others who design, say, snake robots, that have many more kinematic DOFs (see discussion of kinematic redundancy in Sect. 7.1). Similarly, biological limbs are often analyzed as having 6 or fewer kinematic DOFs [5].

But as mentioned above, the types of simplified limbs presented in this book are common in neuromechanics studies, and suffice to address important debates of motor control. My goal is to present simplified systems to build intuition that can be carried forward to more complex (i.e., anatomically realistic) limbs.

2.7 Using the Jacobian of a Planar System to Find Endpoint Velocities

Many of the examples and cases we investigate will use planar systems without the loss of generality. This will make the presentation of concepts easier to describe and illustrate. In the case of a planar 2-link, 2-joint system as in Fig. 2.3, the 2D forward kinematic model for the endpoint is

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ \alpha \end{pmatrix} = \begin{pmatrix} G_x(\mathbf{q}) \\ G_y(\mathbf{q}) \\ G_\alpha(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} l_1c_1 + l_2c_{12} \\ l_1s_1 + l_2s_{12} \\ q_1 + q_2 \end{pmatrix} \tag{2.30}$$

Taking the appropriate partial derivates produces

$$J(\mathbf{q}) = \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} \\ l_1c_1 + l_2c_{12} & l_2c_{12} \\ 1 & 1 \end{bmatrix} \tag{2.31}$$

where

$$\dot{\mathbf{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\alpha} \end{pmatrix} = \begin{bmatrix} -l_1s_1 - l_2s_{12} & -l_2s_{12} \\ l_1c_1 + l_2c_{12} & l_2c_{12} \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} \tag{2.32}$$

As shown in Eq. 2.32, and graphically in Fig. 2.5, each column of the Jacobian is the instantaneous endpoint velocity vector produced by one unit of the corresponding joint angular velocity (i.e., the first column of the Jacobian is the endpoint velocity vector produced by an angular velocity of 1 rad/s at the first joint if other joint angular velocities are zero, the second column is the endpoint velocity vector produced by a 1 rad/s angular velocity at the second joint if other joint angular velocities are zero, etc.). If there are simultaneous angular velocities at both joints, their *instantaneous* effects at the endpoint simply add linearly. The last row says that the angular velocity of the endpoint is simply the sum of the angular velocity at each joint.

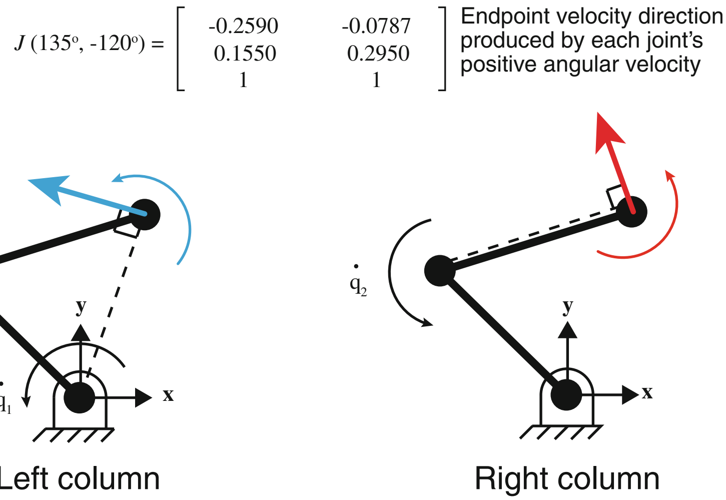

Fig. 2.5 Illustration of the Jacobian for a 2 DOF planar limb. For the posture shown, the columns of the 2×2 Jacobian show the expected *instantaneous* endpoint linear and angular velocity for isolated angular velocities of 1 rad/s at each of the joints. If both joints are actuated, then their contribution to instantaneous endpoint velocity simply add. The limb parameters are, as per the convention in Fig. 2.3, $l_i = 25.4\,\text{cm}$, $l_2 = 30.5\,\text{cm}$, $q_1 = 135°$, and $q_2 = -120°$

But let us look at the planar arm example in Fig. 2.5 and Eq. 2.32 in detail. First, we notice that (other than the last row) the values of the elements of the Jacobian matrix can be posture dependent (i.e., they change as the posture—or angles q_1 and q_2—change). Second, we are therefore forced to always speak of *instantaneous* endpoint velocities because these values only hold for that posture, and the posture is changing—by definition—given that the joints have angular velocities. And third, given that the forward kinematic model and the Jacobian involve trigonometric functions, the mapping from angular velocities to endpoint velocities changes in nonlinear ways as the motion progresses. This will naturally make the dynamical control of such systems complex because the properties of the system change in nonlinear ways as the system moves. However, for a given posture (as in the case of static force production described later), both the forward kinematic model and the Jacobian are fixed.

Further treatment of the Jacobian can be found in [6, 14]. For now, it suffices for the reader to know that the Jacobian relates joint velocities to endpoint velocities, and that it can be derived in a straightforward manner for any arbitrary serial manipulator by taking partial derivatives of the analytical expressions for the forward kinematic model. These can be derived from first principles either using homogeneous coordinate transformations, or by using other methods such as D-H parameterization, among others.

2.8 Exercises and Computer Code

Exercises and computer code for this chapter in various languages can be found at http://extras.springer.com or found by searching the World Wide Web by title and author.

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