Chapter 2
A Review of Basic Laws for a Compressible Flow

2.1 Introduction

The operation of aero engines and rockets is governed by the laws of fluid mechanics (or more specifically aerodynamics and gas dynamics) as well as thermodynamics. Understanding and analyzing the performance of aero engines and rocket motors requires a closed set of governing equations (conservation of mass and energy, linear and angular momentums, entropy) as well as several compressible flow relations that govern the isentropic flow, normal and oblique shock waves, expansion waves, and finally Fanno and Rayleigh flow. For understanding the basic physical phenomena, gas will be modeled as a *perfect gas,* and apart from the rotating elements (fans, compressors, and turbines), the flow will be assumed *one dimensional*, where its properties are assumed constant across the flow and vary only in the flow direction (axial direction). It is assumed that the students have studied a first course in both fluid mechanics and thermodynamics. A review of thermo-fluid physics and one-dimensional gas dynamics will be given in this chapter. For more details, students are asked to refer to the following set of textbooks: Shames and White [1, 2] for fluid mechanics and Shapiro, Zucrow and Hoffman, and Zucker [3–5] for gas dynamics together with Keenan, Sonntag, et al. as well as Cengel and Boles [6–8] for thermodynamics.

Macroscopic approach rather than microscopic one will be followed here. The concepts of system and control volume are followed in specifying a definite collection of material and a region in space that will be analyzed.

A.F. El-Sayed, *Fundamentals of Aircraft and Rocket Propulsion*,
DOI 10.1007/978-1-4471-6796-9_2

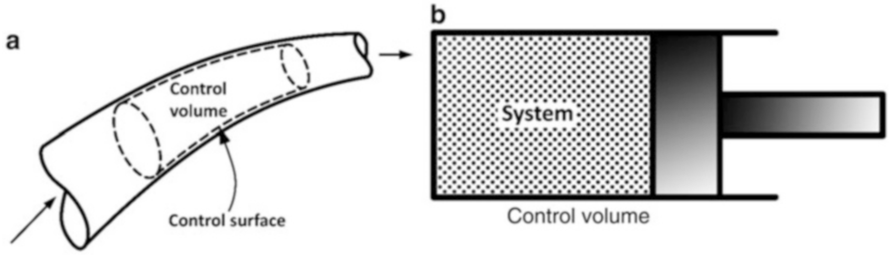


Fig. 2.1 System and control volume. (**a**) System. (**b**) Control volume

2.2 System and Control Volume

A *system* is a collection of matter of fixed identity. It may be considered enclosed by an invisible, massless, flexible surface through which may change shape, and position, but must always entail the same matter. For example, one may choose the steam in an engine cylinder (Fig. 2.1) as a system. As the piston moves, the volume of the system changes, but there is no change in the quantity and identity of mass. The terms system and control mass have identical meaning.

A *control volume* is a region of constant shape and size that is fixed in space relative to the observer. The boundary of this volume is known as the control surface. This control surface may be imagined as massless, invisible, and rigid envelope which offers no resistance to the passage of mass. The amount and identity of the matter in the control volume may change with the time, but the shape of the control volume is fixed. For instance, to study flow through a variable geometry duct, one could choose, as a control volume, the interior of the duct as shown in Fig. 2.1. We note that the control volume and the system can be infinitesimal.

2.3 Fundamental Equations

Four basic laws must be satisfied for the continuous medium (or continuum) inside aero engines and rocket motors, namely:

1. Conservation of matter (continuity equation)
2. Newton's second law (momentum and moment-of-momentum equations)
3. Conservation of energy (first law of thermodynamics)
4. Second law of thermodynamics

In addition to these general laws, there are numerous subsidiary laws, sometimes called constitutive relations, that apply to specific types of media, like the equation of state for the prefect gas and Newton's viscosity law for certain viscous fluids. Furthermore, for high-speed flows additional compressible flow features have to be

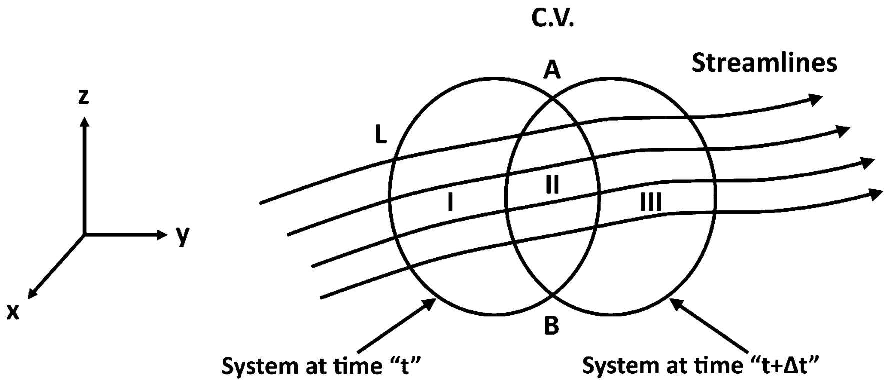


Fig. 2.2 Simplified view of moving system

governed by *isentropic, Rayleigh and Fanno flow relations together with normal and oblique shock relations* if encountered. In thermodynamics we have two kinds of properties of a substance. These whose measure depends on the amount of mass of the substance are called *extensive* properties, and those whose measure is independent of the amount of mass of the substance present are called *intensive* properties. Temperature and pressure are two famous examples for intensive properties. Examples of extensive properties are weight, momentum, volume, and energy. Each extensive variable such as enthalpy (H) and energy (E), we have $H = \iiint h\rho dv$ and $E = \iiint e\rho dv$., has its intensive properties: (h) and (e).

Consider next an arbitrary flow field $V\ (x,y,z,t)$ as seen from some frame of reference xyz wherein we observe a system of fluid of finite mass at times "t" and "$t + \Delta t$" as shown in Fig. 2.2. The streamlines correspond to those at time "t." In addition to this system, we will consider that the volume in space occupied by the system at time "t" is the control volume fixed in position and shape in xyz. Hence, at time "t" our system is identical to the fluid inside our control volume. Let us now consider some arbitrary extensive property "N" of the fluid. The distribution of "N" per unit mass will be given as "η" such that $N = \iiint \eta\rho dv$ with dv representing an element of volume.

We have divided up the overlapping systems at time "$t + \Delta t$" and at time "t" into three regions, as shown in Fig. 2.2. The region II is common to the system at both times "t" and "$t + \Delta t$." Let us compute the rate of change of N with respect to time for the system by the following limiting process:

$$\left(\frac{dN}{dt}\right)\text{system} = \frac{DN}{Dt}$$

$$\frac{DN}{Dt} = \lim_{\Delta t \to 0} \left[\frac{\left(\iiint\limits_{III} \eta\rho dv + \iiint\limits_{II} \eta\rho dv \right)_{t+\Delta t} - \left(\iiint\limits_{I} \eta\rho dv + \iiint\limits_{II} \eta\rho dv \right)_{t}}{\Delta t} \right] \tag{2.1}$$

Equation (2.1) can be rearranged to the form

$$\begin{aligned}\frac{DN}{Dt} = \lim_{\Delta t \to 0} &\left[\frac{1}{\Delta t} \left\{ \left(\iiint\limits_{II} \eta\rho dv \right)_{t+\Delta t} - \left(\iiint\limits_{II} \eta\rho dv \right)_{t} \right\} \right] \\ &+ \lim_{\Delta t \to 0} \left[\frac{1}{\Delta t} \left\{ \left(\iiint\limits_{III} \eta\rho dv \right)_{t+\Delta t} - \left(\iiint\limits_{I} \eta\rho dv \right)_{t} \right\} \right]\end{aligned} \tag{2.2}$$

After some manipulation, with the net efflux rate equal to the outlet rates efflux minus the rate influx through the control surface, we arrive at the relation

$$\frac{DN}{Dt} = \frac{\partial}{\partial t} \iiint\limits_{C.V} \eta\rho dv + \oiint\limits_{C.\,S.} \eta(\rho V \bullet dA) \tag{2.3}$$

Equation (2.3) is called *Reynolds transport equation*. This equation permits us to change from a system approach to a control–volume approach.

2.3.1 Conservation of Mass (Continuity Equation)

Now, let us apply Reynolds transport Eq. (2.3) to reach the continuity rquation. In this case:

1. The extensive property “N” is the mass of a fluid system “M.”
2. The quantity “η” is unity, since $\mathrm{M} = \iiint\limits_{C.V.} \rho dv$.

   Then Reynolds transport equation will have the form

$$\frac{DM}{Dt} = \frac{\partial}{\partial t} \iiint\limits_{C.V.} \rho\, dv + \oiint\limits_{C.S.} \left(\rho\, \overline{V} . \overline{dA}\right) = 0 \tag{2.4a}$$

Since we can choose a system of any shape at time “t,” the relation above is then valid for any control volume at time “t” as follows:

$$\oiint_{C.S.} \rho \left(\overline{V}\,.\overline{dA}\right) = -\frac{\partial}{\partial t}\iiint_{C.V.} \rho\, dv \tag{2.4b}$$

That is, the net efflux rate of mass through the control surface equals the rate of decrease of mass inside the control volume. Equation (2.4) and its simplified forms are called equation of continuity.

If the flow is steady relative to a reference fixed to the control volume, all fluid properties, including the density at any fixed position in the reference, must remain invariant with time. The right side of Eq. (2.4) can be written in the form $\iiint (\partial\rho/\partial t)dv,$, and this integral is zero. Hence, we can state that any steady flow

$$\oiint_{C.S.} \rho \left(\overline{V}\,.\overline{dA}\right) = 0 \tag{2.5a}$$

Next, consider the case of incompressible flow, in this case, ρ is constant at all positions in the domain and for all even if the velocity field is unsteady. The right side of Eq. (2.4) vanishes then, and on the left side of this equation, we can extract ρ from under the integral sign. We then arrive at the relation:

$$\oiint_{C.S.} \left(\overline{V} \bullet \overline{dA}\right) = 0 \tag{2.5b}$$

Thus, for any incompressible flow, conservation of mass reduces to conservation of volume. Let us consider the very common situation in which fluid enters some device through a pipe and leaves the device through a second pipe, as shown diagrammatically in Fig. 2.3. A dashed line indicates the chosen control surface. We assume that the flow is steady relative to the control volume and that the inlet and outlet flows are one dimensional. Applying Eq. (2.5a) for this case, we get

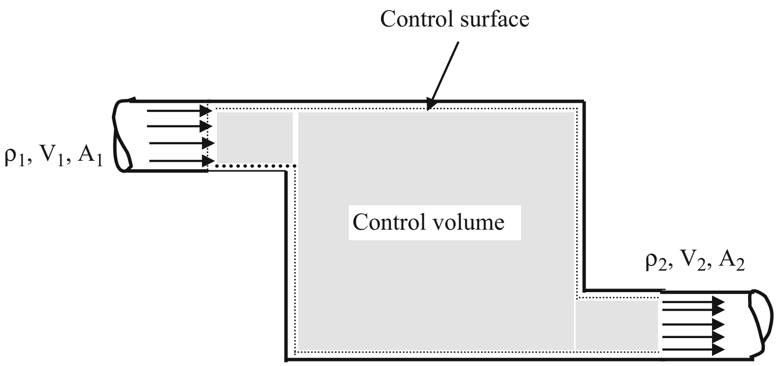


Fig. 2.3 Control volume for device with 1-D inlet and outlet

$$\oiint_{CS} \left(\rho\overline{V} \bullet \overline{dA}\right) = \iint_{A_1} \left(\rho\overline{V} \bullet \overline{dA}\right) + \iint_{A_2} \left(\rho\overline{V} \bullet \overline{dA}\right)$$

where A_1 and A_2 are, respectively, the entrance and exit areas

$$\oiint_{CS} \left(\rho\overline{V} \bullet \overline{dA}\right) = -\iint_{A_1} \rho V dA + \iint_{A_2} \rho V dA$$

With ρ and V constant at inlet and outlet sections, we obtain the following equation:

$$-\rho_1 V_1 \iint_{A_1} dA + \rho_2 V_2 \iint_{A_2} dA = 0$$

Integrating, we get

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 \tag{2.6}$$

2.3.2 Linear Momentum (Newton's Second Law)

Newton's second law states that

$$\overline{F_R} = \left(\frac{d}{dt}\right)_{\text{system}} \left[\iiint_M \overline{V} dm\right] = \left(\frac{d\overline{P}}{dt}\right)_{\text{system}} \tag{2.7}$$

where

$\overline{F_R}$ is the resultant external force and $\overline{P}$ is the linear momentum vector.

$\overline{F_R}$ is classified as the surface force and body force distributions. The surface force is denoted as $\overline{T}$ (x, y, z, t) and given as force per unit area on the boundary surfaces. The body force distribution is denoted as $\overline{B}$ (x,y,z,t) and given as force per unit mass. For example, gravity is the most common body force distribution, and thus, $\overline{B} = -g\,\overline{K}$. We can rewrite Eq. (2.7) as follows:

$$\oiint_{C.S} \overline{T} dA + \iiint_{C.V} \overline{B} \rho dv = \frac{D\overline{P}}{Dt} \tag{2.8a}$$

The linear momentum $\overline{P}$ is the extensive property to be considered in the Reynolds transport Eq. (2.3). The quantity η becomes momentum per unit mass, which is "$\overline{V}$."

Thus

$$\frac{D\overline{P}}{Dt} = \oiint_{C.S.} \overline{V}\rho\left(\overline{V}.\overline{dA}\right) + \frac{\partial}{\partial t}\iiint_{C.V.} \overline{V}\rho\, dv \tag{2.8b}$$

We then have from Eq. (2.8) the linear momentum equation expressed as

$$\oiint_{C.S.} \overline{T}dA + \iiint_{C.V.} \overline{B}\rho\, dv = \oiint_{C.S.} \overline{V}\rho\left(\overline{V}.\overline{dA}\right) + \frac{\partial}{\partial t}\iiint_{C.V.} \overline{V}\rho\, dv \tag{2.9}$$

This equation then equates the sum of these force distributions with the rate of efflux of linear momentum across the control surface plus the rate of increase of linear momentum inside the control volume. For steady flow and negligible body forces, as is often the case in *propulsion applications*, the equation above becomes

$$\oiint_{C.S.} \overline{T}dA = \oiint_{C.S} \overline{V}\rho\,\left(\overline{V}.\overline{dA}\right) \tag{2.10}$$

Since the momentum Eq. (2.9) is a vector equation, then the scalar component equations in the orthogonal x, y, and z directions may then be written as

$$\begin{aligned}\oiint_{C.S.} T_x dA + \iiint_{C.V.} B_x\rho\, dv &= \oiint_{C.S.} V_x\rho\left(\overline{V}.\overline{dA}\right) + \frac{\partial}{\partial t}\iiint_{C.V.} V_x\rho\, dv \oiint_{C.S.} T_y dA + \iiint_{C.V.} B_y\rho\, dv \\ &= \oiint_{C.S.} V_y\rho\left(\overline{V}.\overline{dA}\right) + \frac{\partial}{\partial t}\iiint_{C.V.} V_y\rho\, dv \oiint_{C.S.} T_z dA + \iiint_{C.V.} B_z\rho\, dv \\ &= \oiint_{C.S.} V_z\rho\left(\overline{V}.\overline{dA}\right) + \frac{\partial}{\partial t}\iiint_{C.V.} V_z\rho\, dv\end{aligned} \tag{2.11}$$

In using Eq. (2.11), one selects directions for the positive directions of the inertial reference axes x, y, and z. Then the positive directions of the velocities V_x, V_y, and V_z, as well as the surface and body force T_x and B_x, and so on, are established.

Example 2.1 A turbojet engine is powering an aircraft flying at a speed of (u) as shown in Fig. 2.4. Air flows into the engine at the rate of ($\dot{m}_a$) through the inlet area (A_i). Fuel is injected into the combustors at the rate of ($\dot{m}_f$). The exhaust gases are leaving the propelling nozzle at the rate of ($\dot{m}_e$) and speed of (u_e) via an exit area (A_e). The ambient and exit pressures are (P_a and P_e). Prove that the generated thrust force is expressed as

$$\tau = \dot{m}_a[(1+f)u_e - u] + (P_e - P_a)A_e$$

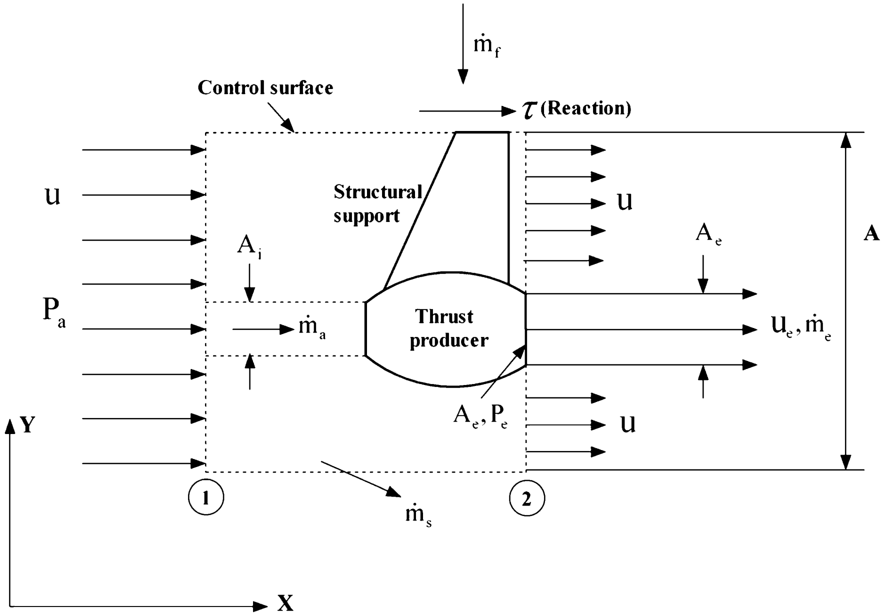


Fig. 2.4 Control volume around a turbojet engine

Figure 2.4, illustrates a turbojet engine with a part of its pod installation (which is a structural support for hanging the engine to the wing). It also defines a control volume which control surface passes through the engine outlet (exhaust) plane (2) and extends far upstream at (1). The two side faces of the control volume are parallel to the flight velocity u. The upper surface cuts the structural support, while the lower one is far below the engine. The surface area at planes (1) and (2) is equal and denoted A. The stream tube of air entering the engine has an area A_i at plane (1), while the exhaust area for gases leaving the engine is A_e. Over plane (1), the velocity and pressure are u (which is the flight speed) and P_a (ambient pressure at this altitude). The velocity and pressure over plane (2) are still u and P_a except over the exhaust area of the engine A_e which values are u_e and P_e. The x- and y-directions employed here are chosen parallel and normal to the centerline of the engine.

The following assumptions are assumed:

1. The flow is steady within the control volume; thus, all the properties within the control do not change with time.
2. The external flow is reversible; thus, the pressures and velocities are constants over the control surface except over the exhaust area P_e of the engine.

Conservation of mass across the engine gives

$$\dot{m}_a + \dot{m}_f = \dot{m}_e$$

where $\overset{\bullet}{m}_a$ and $\overset{\bullet}{m}_e$ are expressed as

$$\overset{\bullet}{m}_a = \rho u A_i, \quad \overset{\bullet}{m}_e = \rho_e u_e P_e$$

The fuel flow rate is thus expressed as

$$\overset{\bullet}{m}_f = \rho_e u_e A_e - \rho u A_i \quad \text{(A)}$$

The fuel-to-air ratio is defined here as

$$f = \frac{\overset{\bullet}{m}_f}{\overset{\bullet}{m}_a}$$

$$\overset{\bullet}{m}_e = \overset{\bullet}{m}_a(1+f) \quad \text{(B)}$$

Apply the continuity equation over the control volume

$$\frac{\partial}{\partial t}\iiint\limits_{CV} \rho dv + \oiint\limits_{CS} \rho \bar{u} \cdot d\bar{A} = 0$$

For a steady flow, $\frac{\partial}{\partial t}\iiint\limits_{CV} \rho dv = 0$, then $\oiint\limits_{CS} \rho \bar{u} \cdot d\bar{A} = 0$

$$\text{or} \quad \overset{\bullet}{m}_e + \overset{\bullet}{m}_s + \rho u(A - A_e) - \overset{\bullet}{m}_a - \overset{\bullet}{m}_f - \rho u(A - A_i) = 0$$

where ($\overset{\bullet}{m}_s$) is the side air leaving the control volume.

Rearranging and applying Eq. (A), we get the side mass flow rate as

$$\overset{\bullet}{m}_s = \rho u(A_e - A_i) \quad \text{(C)}$$

According to the momentum equation

$$\sum \vec{F} = \frac{\partial}{\partial t}\iiint\limits_{CV} \rho \vec{u} dv + \oiint\limits_{CS} \bar{u}\left(\rho \bar{u} \cdot d\bar{A}\right) = 0$$

where $\sum \bar{F}$ is the vector sum of all forces acting on the material within the control volume which are surface forces (pressure force as well as the reaction to thrust force through the structural support denoted by τ) and the body force (which is the gravitational force here).

For steady flow

$$\sum \bar{F} = \oiint\limits_{CS} \bar{u}\left(\rho \bar{u} \cdot d\bar{A}\right)$$

The x-component of the momentum equation

$$\sum F_x = (P_a - P_e)A_e + \tau = \oiint_{CS} u_x\left(\rho\bar{u}\cdot d\bar{A}\right) \quad \text{(D)}$$

If the sides of the control volume are assumed sufficient distant from the engine, then the side mass flow rate leaves the control volume nearly in the x-direction. Thus,

$$\oiint u_x\left(\rho\bar{u}\cdot d\bar{A}\right) = \dot{m}_e u_e + u[\rho u(A - A_e)] + \dot{m}_s u - \dot{m}_a u - u[\rho u(A - A_i)]$$
$$\therefore \oiint u_x\left(\rho\bar{u}\cdot d\bar{A}\right) = \dot{m}_e u_e - \dot{m}_a u - \rho u^2(A_e - A_i) + \dot{m}_s u$$

From Eq. (C)

$$\therefore \oiint u_x\left(\rho\bar{u}\cdot d\bar{A}\right) = \dot{m}_e u_e - \dot{m}_a u \quad \text{(E)}$$

From Eqs. (D) and (E) then

$$\tau - (P_e - P_a)A_e = \dot{m}_e u_e - \dot{m}_a u$$

From Eq. (B)

$$\therefore \tau = \dot{m}_a[(1+f)u_e - u] + (P_e - P_a)A_e$$

The following terminology is always used:

Net thrust = τ
Gross thrust = $\dot{m}_a[(1+f)u_e] + (P_e - P_a)A_e$
Momentum thrust = $\dot{m}_a[(1+f)u_e]$
Pressure thrust = $(P_e - P_a)A_e$
Momentum drag = $\dot{m}_a u$

Thus: Net thrust = Gross thrust – Momentum drag
Or in other words:

Net thrust = Momentum thrust + Pressure thrust – Momentum drag

Example 2.2 A fighter airplane is being refueled in flight using the hose-and-drogue system as shown in Fig. 2.5 at the rate of 300 gal/min of fuel having a specific gravity of 0.7. The inside diameter of hose is 0.12 m. The fluid pressure at the entrance of the fighter plane is 30 kPa gage. What additional thrust does the plane need to develop to maintain the constant velocity it had before the hookup?

Fig. 2.5 Aerial refueling using the hose-and-drogue system

Solution

At first, it is worthy defining *aerial refueling* (which is also identified as *air refueling, in-flight refueling (IFR), air-to-air refueling (AAR), or tanking)* as the process of transferring fuel from one aircraft (the tanker) to another (the receiver) during flight. When applied to helicopters, it is known as *HAR* for helicopter aerial refueling. A series of air refueling can give range limited only by crew fatigue and engineering factors such as engine oil consumption.

Now, back to our problem, consider a control volume starting from the probe to the fuel tank. This is an inertial control volume with the positive *x*-direction parallel to aircraft flight direction.

Thus the linear momentum equation in the *x*-direction is

$$F_x = \oiint_{C.S.} V_x \rho \left(\overline{V}.\overline{dA}\right) + \frac{\partial}{\partial t} \iiint_{C.V.} V_x \rho \, dv$$

where F_x is the force in the x-direction. Since a steady flow is assumed in refueling process, then

$$F_x = \oiint_{C.S.} V_x \rho \left(\overline{V}.\overline{dA}\right)$$

which is rewritten as: $T_x - pA = -[V_x \times (-\rho V_x A)] = \rho V_x^2 A$

where T_x is the needed additional thrust and the velocity of fuel flow into the probe is V_x. Since

$$V_x = \frac{Q}{A} = \frac{\left(300 \times 3.785 \times 10^{-3}/60\right)}{\left[\pi \times (0.12)^2/4\right]} = \frac{0.018925}{0.01131} = 1.6733 \quad \text{m/s}$$

$$\rho = 0.7 \times 1000 = 700 \quad \text{kg/m}^3$$

The additional thrust is then

$$T_x = \left(p + \rho V_x^2\right)A = \left[30 \times 10^3 + 700 \times (1.6733)^2\right] \times 0.01131 = 364\text{N}$$

Example 2.3 The idling turbojet engines of a landing airplane produce forward thrust when operating in a normal manner, but they can produce reverse thrust if the jet is properly deflected. Suppose that, while the aircraft rolls down the runway at 180 km/h, the idling engine consumes air at 40 kg/s and produces an exhaust velocity of 150 m/s.

(a) What is the forward thrust of the engine?
(b) What is the magnitude and direction (forward or reverse) if the exhaust is deflected 90° and the mass flow is kept constant?

Solution

Forward thrust has positive values and reverse thrust has negative values.

(a) The flight speed is $U = 180/3.6 = 50$ m/s.

The thrust force represents the horizontal or the x-component of the momentum equation.

$$T = \dot{m}_a(u_e - u)$$

$T = 40*(150–50) = 4000$ N

(b) Since the exhaust velocity is now vertical due to thrust reverse application, then it has a zero horizontal component; thus, the thrust equation is

$$T = \dot{m}_a(u_e - u)$$

$$T = 40*(0 - 50) = -2000 \text{ N}$$

$$T = -2000 \text{ N(reverse)}$$

2.3.3 Angular Momentum Equation (Moment of Momentum)

Consider a finite system of fluid as shown in Fig. 2.6. An element dm of the system is acted on by a force $d\overline{F}$ and has a linear momentum $(md\overline{V})$. From Newton's law, we can write

$$d\overline{F} = \frac{D}{Dt}\left(\overline{V}dm\right) \tag{2.12}$$

Now take the cross product of each side using the position vector $\overline{r}$. Thus,

$$\overline{r} \otimes d\overline{F} = \overline{r} \otimes \frac{D}{Dt}\left(\overline{V}dm\right)$$

Consider next the following operation:

$$\frac{D}{Dt}\left(\overline{r} \otimes dm\overline{V}\right) = \frac{D\overline{r}}{Dt} \otimes dm\overline{V} + \overline{r} \otimes \frac{D}{Dt}\left(dm\overline{V}\right)$$

Note that $D\overline{r}/Dt = \overline{V}$, so that the first expression on the right side is zero, since $\overline{V} \times \overline{V} = 0$.

Thus, we arrive at the relation:

$$\overline{r} \otimes d\overline{F} = \frac{D}{Dt}\left(\overline{r} \otimes dm\overline{V}\right) \tag{2.13a}$$

Next, we integrate Eq. (2.13a) over the entire system to get

$$\int \overline{r} \otimes d\overline{F} = \iiint_M \frac{D}{Dt}\left(\overline{r} \otimes \overline{V}\right)dm \tag{2.13b}$$

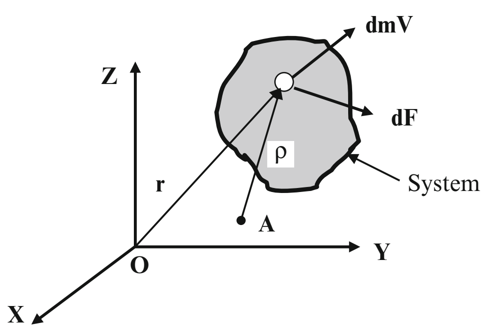


Fig. 2.6 Mass (*dm*) in a finite system

Since the mass of the system is fixed so that the limits of the integration on the right side of Eq. (2.13b) are fixed, thus we can write

$$\int \overline{r} \otimes d\overline{F} = \frac{D}{Dt}\left(\iiint\limits_{M} \overline{r} \otimes \overline{V} dm\right) = \frac{D\overline{H}}{Dt}$$

where $\overline{H}$ is the moment about a fixed point (a) in inertial space of the linear momentum of the system. The integral on the left side of the equation represents the total moment about point (a) of the external forces acting on the system and may be given as

$$\int \overline{r} \otimes d\overline{F} = \oiint\limits_{C.S} \overline{r} \otimes \overline{T} dA + \iiint\limits_{C.V} \overline{r} \otimes \overline{B} \rho \, dv \tag{2.14}$$

We may now give the moment-of-momentum equation for a finite system as follows:

$$\oiint\limits_{C.S} \overline{r} \otimes \overline{T} dA + \iiint\limits_{C.V} \overline{r} \otimes \overline{B} \rho \, dv = \frac{D\overline{H}}{Dt}$$

Next, since ($\overline{H} \equiv \iiint (\overline{r} \otimes \overline{V})\rho dv$) is the extensive property, then its intensive property (η) is ($\overline{r} \otimes \overline{V}$). Thus applying Reynolds transport equation, one gets

$$\frac{D\overline{H}}{Dt} = \oiint\limits_{CS} \left(\overline{r} \otimes \overline{V}\right)\left(\rho \overline{V} \bullet \overline{dA}\right) + \frac{\partial}{\partial t} \iiint\limits_{CV} \left(\overline{r} \otimes \overline{V}\right)(\rho dv)$$

We then have the desired moment-of-momentum equation for an inertial control volume:

$$\oiint\limits_{CS} \overline{r} \otimes \overline{T} dA + \iiint\limits_{CV} \overline{r} \otimes \overline{B} \rho dv = \oiint\limits_{CS} \left(\overline{r} \otimes \overline{V}\right)\left(\rho \overline{V} \bullet \overline{dA}\right) + \frac{\partial}{\partial t} \iiint\limits_{CV} \left(\overline{r} \otimes \overline{V}\right)(\rho dv) \tag{2.15}$$

The terms on the right side represent the efflux of moment of momentum through the control surface plus the rate of increase of moment of momentum inside the control volume where both quantities are observed from the control volume.

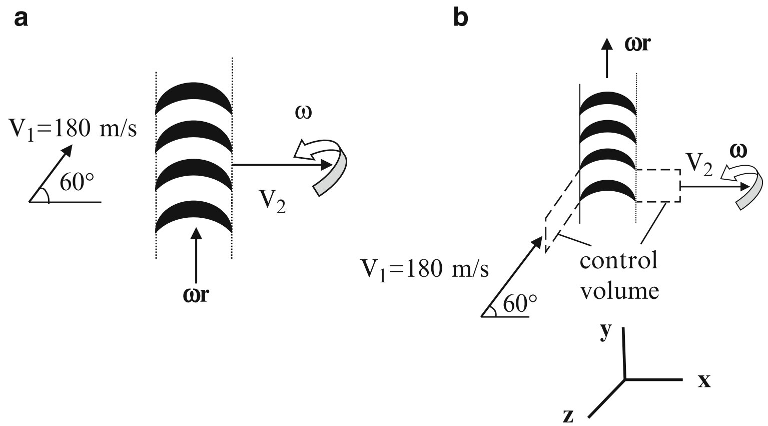


Fig. 2.7 Impulse turbine. (**a**) Layout. (**b**) Control volume

Example 2.4 An impulse turbine blade row is illustrated in Fig. 2.7a. The rotor has an average radius r of 0.6 m and rotates at a constant angular speed ω. What is the transverse torque on the turbine if the air mass flow rate is 100 kg/s?

Solution

Choosing the shown control volume described in Fig. 2.7a, and assuming the flow is steady, then Eq. (2.14) is reduced to

$$\oiint_{CS} \overline{r} \otimes \overline{T} dA = \oiint_{CS} \left(\overline{r} \otimes \overline{V}\right)\left(\rho \overline{V} \bullet \overline{dA}\right)$$

Or the torque $\overline{\tau}$ is expressed by the relation

$$\overline{\tau} = \oiint_{CS} \left(\overline{r} \otimes \overline{V}\right)\left(\rho \overline{V} \bullet \overline{dA}\right)$$

The flow is fast enough to assume a constant density; thus, the x-component of the torque which is responsible for turbine rotation is expressed by the relation:

$$\tau_x = \dot{m} \times r_z \times \left[(V_{out})_y - (V_{in})_y\right]$$

$$\tau_x = 100 \times 0.6 \times [0 - (180 \times \sin 60)] = -9,353 \text{ N.m}$$

The negative sign indicates that the turbine rotor rotates in a counterclockwise direction as shown in figure.

Another Solution

The above problem can also be solved using the linear momentum Eq. (2.11). The tangential force (T_y) is expressed by the relation:

$$\oiint\limits_{C.S.} T_y dA + \iiint\limits_{C.V.} B_y \rho \, dv = \oiint\limits_{C.S.} V_y \rho \left(\overline{V}.\overline{dA}\right) + \frac{\partial}{\partial t} \iiint\limits_{C.V.} V_y \rho \, dv$$

Again for the same assumptions of steady constant density flow, then T_y is expressed as

$$T_y = \oiint\limits_{C.S.} V_y \rho \left(\overline{V}.\overline{dA}\right) = \dot{m} \quad \times \quad \left[(V_{out})_y - (V_{in})_y\right]$$

$$T_y = 100 \times (0 - 180 \times \sin 60) = -15,588 \;\; N$$

$$\tau_x = r_z \times T_y = 0.6 \times (-15,588) = -9,353 \;\; \text{N.m} = -9.353 \;\; \text{kN.m}$$

2.3.4 Energy Equation (First Law of Thermodynamics)

The first law of thermodynamics is a statement of macroscopic experience which states that energy must at all times be conserved. It will be convenient to classify energy under two main categories: stored energy and energy in transition. The types of *stored energy* of an element of mass are:

1. Kinetic energy E_k: energy associated with the motion of the mass
2. Potential energy E_P: energy associated with the position of the mass in conservative external fields
3. Internal energy U: molecular and atomic energy associated with the internal fields of the mass

The types of *energy in transition* are heat and work. Heat is the energy in transition from one mass to another as a result of a temperature difference. On the other hand, work, as learned in mechanics, is the energy in transition to or from a system which occurs when external forces, acting on the system, move through a distance.

For an arbitrary system (shown in Fig. 2.8), the net heat added to the system and the net work done by the system on the surroundings during the time interval Δt are designated as Q and W_k, respectively.

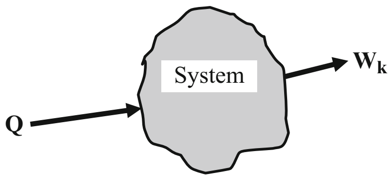


Fig. 2.8 Heat and work on system

If E represents the total stored energy of a system at any time t and its property as a point function is employed, conservation of energy demands that for a process occurring during a time interval between t_1 and t_2, then

$$Q - W_k = \Delta E = E_2 - E_1 = \left(E_k + E_p + U\right)_2 - \left(E_k + E_p + U\right)_1 \tag{2.16}$$

The differential form of Eq. (2.16) may be written in the following manner:

$$dE = dQ - dW_k$$

Accordingly, we can employ the usual derivative notation dQ/dt and dW_k/dt for time derivative. However, E is a point function and expressible in terms of spatial variables and time. Thus, we have for the time variations of stored energy and energy in transition for a system.

$$\frac{DE}{Dt} = \frac{dQ}{dt} - \frac{dW_K}{dt} \tag{2.17}$$

To develop the control–volume approach, we will consider E being the extensive property to be used in the Reynolds transport equation. The term (e) will then represent stored energy per unit mass. We can then say using the Reynolds transport equation

$$\frac{DE}{Dt} = \oiint_{C.S.} e\left(\rho\overline{V}.\overline{dA}\right) + \frac{\partial}{\partial t}\iiint_{C.V.} e\rho\, dv \tag{2.18}$$

Using Eq. (2.17) in the left side of Eq. (2.18), we get

$$\frac{dQ}{dt} - \frac{dW_k}{dt} = \oiint_{C.S.} e\left(\rho\overline{V}.\overline{dA}\right) + \frac{\partial}{\partial t}\iiint_{C.V.} e\rho\, dv \tag{2.19}$$

Equation (2.19) then states that the net rate of energy transferred into the control volume by heat and work equals the rate of efflux of stored energy from control volume plus the rate of increase of stored energy inside the control volume.

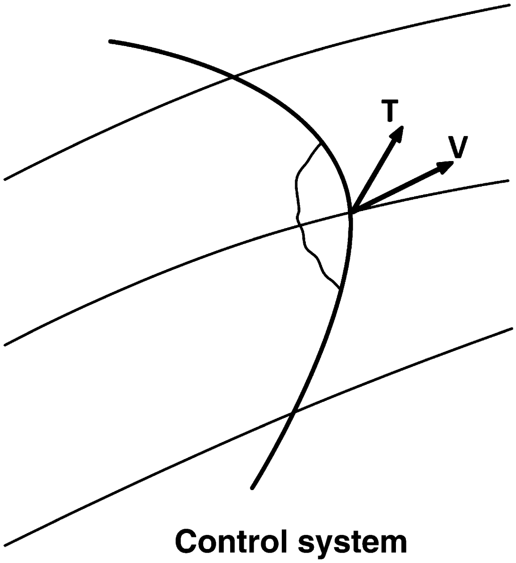
Where (e) is expressed as

$$e = \frac{V^2}{2} + gz + u \tag{2.20}$$

Next let us discuss the term dW_k/dt in Eq. (2.19) which is classified into three groups:

1. Net work done on the surroundings as a result of traction force $\overline{T}$.

Fig. 2.9 Flow work and control surface



2. Any other work transferred by direct contact between inside and outside non-fluid elements, like shafts or by electric currents. We call this work shaft work and denote it as W_S.
3. Work transferred by body forces. Since the effects of gravity have already been taken into account as the potential energy (in Eq. 2.20), so the body force $\overline{B}$ must not include gravity; it may include, for instance, contributions from magnetic and electric force distributions.

Referring to Fig. 2.9, the time rate of the work leaving the control volume—the total rate of flow work—is given as

$$\text{Total rate of flow work} = -\oiint_{CS} \overline{T}.\overline{V} dA$$

Also, the total rate of body force work leaving the control volume is given by:

$$\text{Total rate of body force work} = -\iiint_{CV} \overline{B} \cdot \overline{V} \rho dv$$

A general form of the first law can now be given as

$$\begin{aligned} &\frac{dQ}{dt} - \frac{dW_s}{dt} + \oiint_{CS} \overline{T} \cdot \overline{V} dA + \iiint_{CV} \overline{B}.\overline{V}\rho dv \\ &= \oiint_{CS} \left(\frac{V^2}{2} + gz + u\right) \left(\rho\overline{V} \cdot \overline{dA}\right) + \frac{\partial}{\partial t}\iiint_{CV} \left(\frac{V^2}{2} + gz + u\right)\rho dv \end{aligned} \tag{2.21}$$

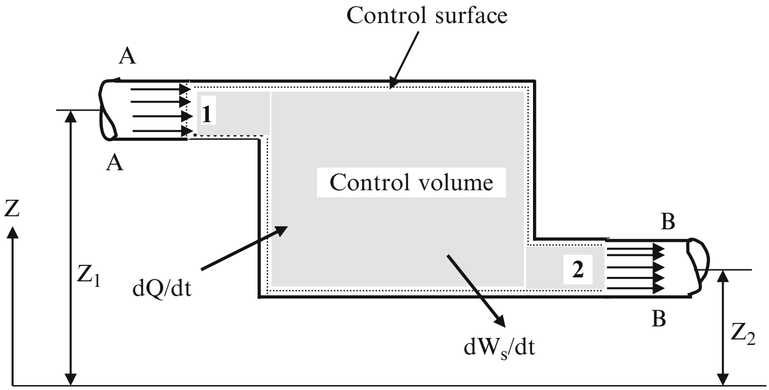


Fig. 2.10 Control volume for idealized machine

Figure 2.10 *illustrates a simple* example for a steady flow device having one-dimensional inlet and outlet flows. This may represent, for instance, a *gas turbine having inlet and outlet at* sections AA and BB. The traction force power occurs at sections AA and BB and is given as $+p_1V_1A_1$ and $-p_2V_2A_2$, respectively. Furthermore, $\rho\overline{V} \bullet d\overline{A}$ at these sections becomes $-\rho_1V_1A_1$ and $+\rho_2V_2A_2$,, respectively. The equation becomes

$$\frac{dQ}{dt} - \frac{dW_s}{dt} + p_1V_1A_1 - p_2V_2A_2 = -\left(\frac{V_1^2}{2} + gz_1 + u_1\right)\rho_1V_1A_1 + \left(\frac{V_2^2}{2} + gz_2 + u_2\right)\rho_2V_2A_2 \tag{2.22}$$

Since the products $\rho_1 v_1$ and $\rho_2 v_2$ (where v_1 and v_2 are the specific volumes) equal unity, the following form of the first law:

$$\frac{dQ}{dt} + \left(\frac{V_1^2}{2} + gz_1 + u_1 + p_1v_1\right)\rho_1V_1A_1 = \frac{dW_s}{dt} + \left(\frac{V_2^2}{2} + gz_2 + u_2 + p_2v_2\right)\rho_2V_2A_2 \tag{2.23a}$$

Since the enthalpy h is defined as $h = u + pv$, and $\rho_1V_1A_1 = \rho_2V_2A_2 = dm/dt$, then Eq. (2.23a) can be written as:

$$\frac{dQ}{dt} + \left(\frac{V_1^2}{2} + gz_1 + h_1\right)\frac{dm}{dt} = \frac{dW_s}{dt} + \left(\frac{V_2^2}{2} + gz_2 + h_2\right)\frac{dm}{dt} \tag{2.23b}$$

If the following assumptions are satisfied:

1. The flow is steady.
2. Air is an ideal gas with constant specific heats.
3. Potential energy changes are negligible ($gz_1 = gz_2 = 0$).
4. There are no work interactions ($\frac{dW_s}{dt} = 0$).
5. The diffuser is adiabatic ($\frac{dQ}{dt} = 0$).

then Eq. (2.23b) is reduced to

$$\frac{V_1^2}{2}_1 + h_1 = \frac{V_2^2}{2} + h_2 \tag{2.24}$$

Finally, if $V_1 = 0$, then the total or stagnation enthalpy (h_1) is defined as

$$h_1 = \frac{V_2^2}{2} + h_2 \tag{2.25}$$

Example 2.5 Air is decelerated in an adiabatic diffuser. The inlet conditions are pressure = 100 kPa, temperature = 50 °C, and velocity =500 m/s. The outlet conditions are pressure = 150 kPa and temperature = 50 °C. The specific heat at constant pressure is 1.007 kJ/kg. K. Calculate the velocity at outlet to diffuser.

Solution

Since the above-listed assumptions hold (steady flow with negligible changes in height, no work or heat exchanges, and the fluid is an ideal gas with constant specific heats), then Eq. (2.25) may be applied. Thus,

$$V_2 = \sqrt{2(h_1 - h_2) + V_1^2} = \sqrt{2C_p(T_1 - T_2) + V_1^2}$$

$$V_2 = \sqrt{2 \times 1007 \times (20 - 50) + (500)^2} = 435.4 \quad \text{m/s}$$

2.3.5 *The Second Law of Thermodynamics and the Entropy Equation*

The second law of thermodynamics states that: *it is impossible for a system to perform a cyclic process that produces work (say raising of a weight) and exchanges heat with a single reservoir of uniform temperature*. The second law permits the definition of the property *entropy* (s). For a system,

$$ds = \left(\frac{dQ}{T}\right)_{\text{reversible}} \tag{2.26}$$

where ds is the change of entropy during a reversible heat exchange. Irreversible processes are processes which involve one of these features: friction, heat transfer with finite temperature gradient, mass transfer with finite concentration gradient, or unrestrained expansion. For any process,

$$ds \geq \frac{dQ}{T} \tag{2.27}$$

where equality holds only for reversible process. If the process is reversible and adiabatic ($dQ = 0$), it must be isentropic ($ds = 0$).

For a small system composed of pure substance in the absence of gravity motion, then if the properties are uniform throughout the system, then the first law for incremental changes is

$$dq = du + dw$$

where q and w are the heat and work per unit mass. If the system experiences a reversible process for which the incremental work $dw = pdv$, then from Eq. (2.28), we can write

$$Tds = du + pdv \tag{2.28}$$

2.3.6 Equation of State

In compressible gases, it is necessary to define the thermodynamic state of the gas with state variables, e. g., the static pressure p, the static density ρ, and the static temperature T. Their interdependence is described by the thermal equation of state. If the law given by Boyle, Mariotte, and Gay-Lussac is used, then

$$p = \rho RT \tag{2.29}$$

The gas is called thermally perfect. For thermally non-perfect gases, other relations must be used, as, for example, the Van der Waals law. The specific gas constant R depends on the molecular weight of the gas. For air it is $R = 287$ J/kg K. The gas constant is related to the *universal* gas constant (R_u) and the molecular weight of gas (M) by the relation

$$R = \frac{R_u}{M}$$

The value of universal gas constant is $R_u = 8.31434 \quad \mathrm{kJ/(kmol.K)}$

Internal energy is a state variable, which is defined by two thermodynamic quantities, namely, the temperature T and the specific volume ($v = 1/\rho$):

$$u = u(v, T) \tag{2.30}$$

This relation is known as the caloric equation of state. The total derivative is

$$du = \left(\frac{\partial u}{\partial v}\right)_T dv + \left(\frac{\partial u}{\partial T}\right)_v dT \tag{2.31}$$

The internal energy of thermally perfect gases depends on the temperature only. It then follows that

$$du = \left(\frac{du}{dT}\right)_v dT$$

where

$$C_v = \left(\frac{du}{dT}\right)_v \tag{2.32}$$

The quantity $\left(\frac{du}{dT}\right)_v$ is called specific heat at constant volume (C_v). If C_v is constant, the gas is called calorically perfect, and the internal energy is given by

$$u = C_v T + u_r \tag{2.33}$$

The quantity u_r is a reference value.

The enthalpy h was defined earlier and repeated here is defined as

$$h = u + p\,v \tag{2.34}$$

Similar to the internal energy, the enthalpy of thermally perfect gases depends on the temperature only, or

$$dh = C_p\,dT \tag{2.35a}$$

The quantity Cp is the specific heat at constant pressure, or

$$C_P = \left(\frac{dh}{dT}\right)_p \tag{2.36a}$$

It follows from the relation for the specific heats C_v and C_P

$$C_p = C_v + R \tag{2.36b}$$

for calorically perfect gases, that C_p is constant. Hence,

$$h = C_pT + h_r \tag{2.35b}$$

where h_r is again a reference value.

The ratio of the specific heats $C_p/C_v = \gamma$, where γ, according to the gas kinetic theory, is given by the number n of degrees of freedom

$$\gamma = \frac{n + 2}{n}$$

For monatomic gases ($n = 3$) $\gamma = 1.667$, and for diatomic gases ($n = 5$) $\gamma = 1.4$. At high temperatures additional degrees of freedom are excited, and the ratio C_p/C_v decreases. For air at a temperature of 300 K, then $\gamma = 1.4$, while at temperature 3000 K, then $\gamma = 1.292$.

From Eqs. (2.28) and (2.33) and since $\frac{P}{T} = \frac{R}{v}$
then

$$ds = C_v \frac{dT}{T} + \mathrm{R} \frac{dv}{v} \tag{2.37}$$

Similarly, from Eqs. (2.28) and (2.34)

$$Tds = dh - vdP$$

From Eq. (2.34) and ideal gas relation, $\frac{v}{T} = \frac{R}{P}$, then

$$ds = C_p \frac{dT}{T} \text{-} R \frac{dP}{P} \tag{2.38}$$

Example 2.6 The constant volume-specific heat of an ideal gas varies according to the equation $C_v = aT^2$, where $a = 2.32 \times 10^{-5}$kJ/kg.K^3. If the gas is heated from 50 to 80 °C at constant volume, find the change in entropy.

Solution

From Eq. (2.40), the change in entropy is expressed as

$$\Delta s = \int_{T_1}^{T_2} C_v \frac{dT}{T} = a \int_{T_1}^{T_2} T^2 \frac{dT}{T} = a \int_{T_1}^{T_2} TdT = a \left. \frac{[T^2]}{2} \right|_{T_1}^{T_2}$$

$$T_1 = 50 + 273 = 323 \ \text{K}$$

$$T_2 = 80 + 273 = 353 \ \text{K}$$

$$\Delta s = 2.32 \times 10^{-5} \left[353^2 - 323^2\right]/2 = 0.235 \quad \text{kJ/kg.K}$$

2.4 Steady One-Dimensional Compressible Flow

One-dimensional flow refers to flow involving uniform distributions of fluid properties over any flow cross section area. It provides accurately the stream-wise variation of average fluid properties. The flow in diffusers, combustors, and nozzles exhibits the major characteristics of one-dimensional flow. Though one-dimensional analysis for the flow in rotating elements (fans, compressors, and turbines) provides also the mean flow features, it is more appropriate to extend the analysis of flow within them to either two dimensional (2-D) or three dimensional (3-D). This is attributed to the large variations normal to streamlines, which are no longer limited to the thin layer adjacent to the surface and known as boundary layer.

2.4.1 Isentropic Relations

It follows from the conservation equations for one-dimensional, steady, compressible flow that the sum of the kinetic energy ($u^2/2$) and the static enthalpy (h) remains constant. The value of this constant is given by the stagnation (or total) enthalpy, and Eq. (2.25) may be rewritten as

$$h_0 = h + u^2/2 \tag{2.25}$$

Generally, the stagnation state is a theoretical state in which the flow is brought into a complete motionless condition in isentropic process without other forces (e.g., gravity force).

Several properties can be represented by this theoretical process which includes temperature, pressure, density, etc. and denoted by the subscript "0."

For calorically perfect gases, the enthalpy can be replaced by the product of static temperature and the specific heat at constant pressure (C_pT), thus,

$$C_pT_0 = C_pT + u^2/2$$

or

$$T_0 = T + \frac{u^2}{2C_p} \tag{2.39}$$

Introducing the thermal equation of state there in (2.39) results

$$\frac{\gamma}{\gamma - 1}\frac{p_0}{\rho_0} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho} + \frac{u^2}{2} \tag{2.40}$$

and with the definition of the speed of sound (a) as

Table 2.1 Sonic speeds at different temperatures for air and helium

	Temperature (K)	200	300	1000
Air	Sonic speed [m/s]	284	347	634
Helium	Sonic speed [m/s]	832	1019	1861

$$a^2 = \frac{\gamma p}{\rho} = \gamma RT$$

Equation (2.40) will be reduced to

$$\frac{a_0^2}{\gamma - 1} = \frac{a^2}{\gamma - 1} + \frac{u^2}{2} \tag{2.41}$$

The speed of sound depends on the gas constant (R) and temperature (T); thus, the sonic speed for air and helium ($R_{\text{air}} = 287\ \text{J/kg.K}, R_{\text{Helium}} = 2077\ \text{J/kg.K}$) at different temperatures are given in the Table 2.1.

Rewriting Eq. (2.39), the following important set of equations can be derived:

$$T_0 = T + \frac{u^2}{2C_p} = T + \frac{(\gamma - 1)u^2}{2\gamma R} \tag{2.42}$$

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}\frac{u^2}{\gamma RT}$$

Introducing the Mach number as the ratio of velocity to speed of sound

$$M = \frac{u}{a} \tag{2.43}$$

It very useful to convert Eq. (2.42) into a dimensionless form and denote

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2}M^2 \tag{2.44a}$$

$$M = \sqrt{\frac{2}{\gamma - 1}\left(\frac{T_0}{T} - 1\right)} \tag{2.44b}$$

$$\frac{a_0}{a} = \sqrt{1 + \frac{\gamma - 1}{2}M^2} \tag{2.44c}$$

$$\frac{P_0}{P} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{\gamma}{\gamma - 1}} \tag{2.44d}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2}M^2\right)^{\frac{1}{\gamma - 1}} \tag{2.44e}$$

The mass flow per unit area is

$$\frac{\dot{m}}{A} = \rho u$$

Using Eqs. (2.43) and (2.44), the velocity may be expressed as

$$u = M\sqrt{\frac{\gamma R T_0}{1 + \frac{\gamma-1}{2}M^2}}$$

From the density relation, the *mass flow rate* parameter is expressed as

$$\frac{\dot{m}}{A} = \frac{P_0\sqrt{\gamma}}{\sqrt{RT_0}} M\left(\frac{1}{1 + \frac{\gamma-1}{2}M^2}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{2.45a}$$

For a given fluid (γ, R) and inlet state (P_0, T_0), it can be readily shown that the mass flow rate per unit area is maximum at $M = 1$. Denoting the properties of the flow at $M = 1$ with an asterisk, the maximum flow per unit area is

$$\frac{\dot{m}}{A^*} = \frac{P_0\sqrt{\gamma}}{\sqrt{RT_0}}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}} \tag{2.45b}$$

From the above two Eqs. (2.45a) and (2.45b), we get

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}} \tag{2.46}$$

Gas dynamics books ([4, 5] as examples) include in its appendices a set of tables for isentropic flow parameters defined by Eqs. (2.44) and (2.46) for specific heat (γ = 1.4). Table 2.2 illustrates few lines of such tables.

For a given isentropic flow and known (γ, R, P_0, T_0, $\dot{m}\cdot$), it is clear that A^* is a constant, so we can use these relations to plot the fluid properties versus Mach number (Fig. 2.11).

2.4.2 Sonic Conditions

If the local flow velocity is equal to the speed of sound ($M = 1$), then such sonic condition is referred to as the *critical state* and is designated by an asterisk (*). The temperature, pressure, and density attain the following values, which solely depend on the stagnation conditions of the gas. From Eq. (2.44), we get

Table 2.2 Isentropic flow parameters ($\gamma = 1.4$)

M	P/P_0	T/T_0	$A/A*$	PA/P_0A*
O	1.0	1.0	∞	∞
0.5	0.84302	0.95238	1.33984	1.12951
1.0	0.52828	0.83333	1.0	0.52828
5.0	0.00189	0.16667	25.0	0.04725

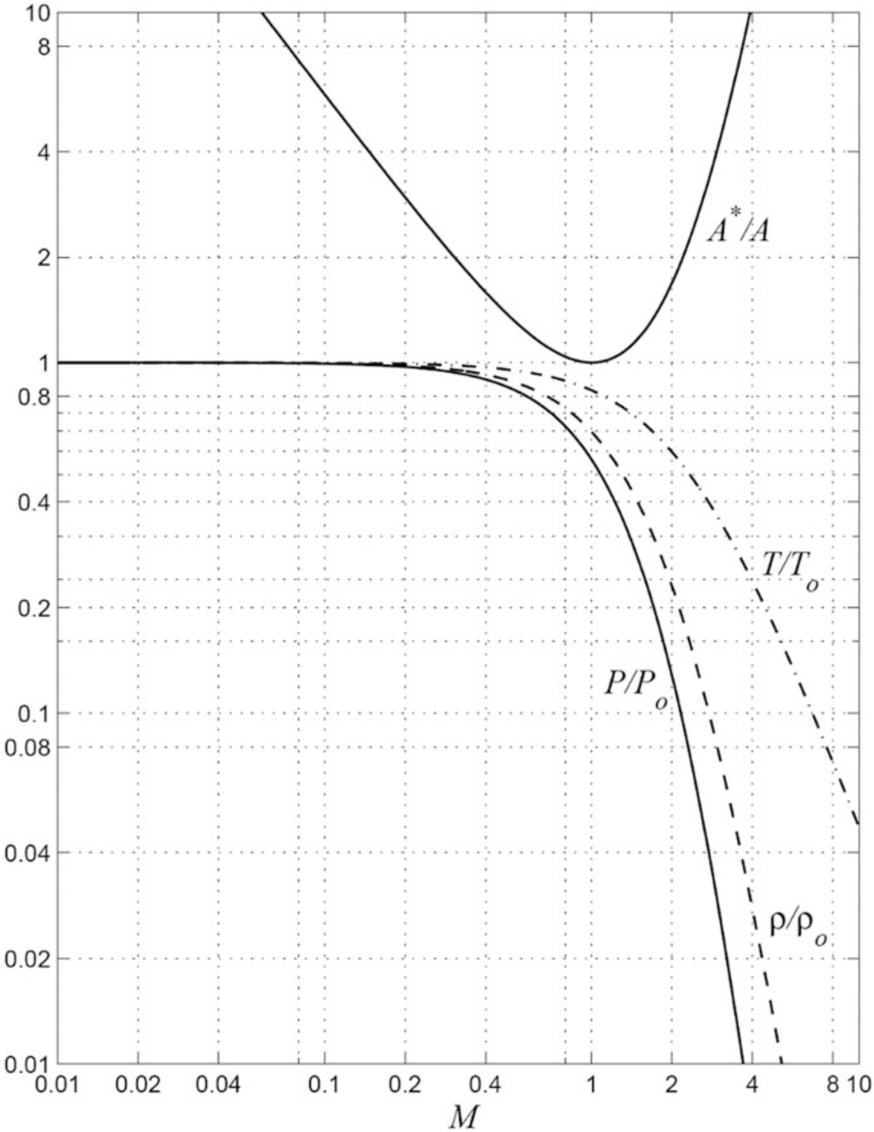


Fig. 2.11 One-dimensional isentropic flow of a perfect gas

Table 2.3 Critical ratios for different values of (γ)

γ	$\frac{T_0}{T^*}$	$\frac{P_0}{P^*}$	$\frac{\rho_0}{\rho^*}$	$\frac{a_0}{a^*}$
1.135	1.0675	1.7318	1.6223	1.0332
1.3	1.15	1.8324	1.5934	1.0723
1.4	1.2	1.8929	1.5774	1.095
1.667	1.335	2.0534	1.5429	1.155

$$\begin{aligned}\frac{T_0}{T^*} &= \frac{\gamma+1}{2}\\ \frac{a_0}{a^*} &= \sqrt{\frac{\gamma+1}{2}}\\ \frac{P_0}{P^*} &= \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}\\ \frac{\rho_0}{\rho} &= \left(\frac{\gamma+1}{2}\right)^{\frac{1}{\gamma-1}}\end{aligned} \tag{2.47}$$

For air with $\gamma = 1.4$, the critical values are as follows (Table 2.3):

$$\frac{T_0}{T^*} = 1.2, \qquad \frac{P_0}{P^*} = 1.8929, \qquad \frac{\rho_0}{\rho^*} = 1.5774, \qquad \frac{a_0}{a^*} = 1.095 \tag{2.48}$$

Instead of the local speed of sound (a), the critical speed of sound can be used to define a Mach number, which is called the critical Mach number:

$$M^* = u/a^* \tag{2.49}$$

The relation between the local Mach number ($M = u/a$) and the critical Mach number (M*) is derived from the relations (2.47) and (2.49), as

$$M^{*2} = \frac{\gamma+1}{\gamma-1+\frac{2}{M^2}} \tag{2.50}$$

For M $\rightarrow \infty$, the critical Mach number M* approaches the following limiting value:

$$\lim_{M\to\infty} M^* = \sqrt{\frac{\gamma+1}{\gamma-1}} \tag{2.51}$$

With these relations the ratios of the temperature, pressure, density, and speed of sound, referred to their stagnation values, can be expressed by the critical Mach number:

$$
\begin{aligned}
\frac{T_0}{T} &= \left(1 - \frac{\gamma-1}{\gamma+1} M^{*2}\right)^{-1} \\
\frac{a_0}{a} &= \left(1 - \frac{\gamma-1}{\gamma+1} M^{*2}\right)^{\frac{-1}{2}} \\
\frac{P_0}{P} &= \left(1 - \frac{\gamma-1}{\gamma+1} M^{*2}\right)^{\frac{-\gamma}{\gamma-1}} \\
\frac{\rho_0}{\rho} &= \left(1 - \frac{\gamma-1}{\gamma+1} M^{*2}\right)^{\frac{-1}{\gamma-1}}
\end{aligned}
\tag{2.52}
$$

2.4.3 Classification of Mach Regimes

Aerodynamicists often classify airflow regimes using Mach number values. Six flight regimes may be identified, namely, subsonic, transonic, supersonic, hypersonic, high hypersonic, and re-entry ones. Subsonic and supersonic speeds are associated with values of Mach number less or greater than unity, respectively. An in-between region defined as "transonic regime" where Mach number is around unity (from say 0.8 to 1.2). Mach values associated with supersonic regime vary from 1.2 to 5. For hypersonic regime Mach number ranges from 5 to 10. NASA defines "high" hypersonic when Mach number ranges from 10 to 25 and re-entry speeds as anything greater than Mach 25 (Space Shuttle as an application). Table 2.4 illustrates such a classification.

Table 2.4 Classification of flow regimes

Regime	Mach	General plane characteristics
Subsonic	<0.8	Propeller-driven and commercial turbofan aircrafts
Transonic	0.8–1.3	All present airliners (B777, 767,747 Airbus A320, A330, and A340) fly at the lowest transonic speeds (typical speeds are greater than 250 mph but less than 760 mph)
Supersonic	1.3–5.0	Modern combat aircrafts including Ilyushin IL-76TD, MIG31, F117 Night Hawk, F 22 Raptor
Hypersonic	5.0–10.0	Aircrafts have cooled nickel–titanium skin, highly integrated, small wings (X-51A WaveRider as an example)
High hypersonic	10.0–25.0	Vehicles are thermally controlled; its structure is protected by special silicate tiles or similar. They have blunt nose configurations to resist aerodynamic heating
Re-entry	>25.0	Vehicles have an ablative heat shield, no wings, blunt capsule shape

2.4.4 *Diffusers and Nozzles*

Diffusers and nozzles are commonly utilized in jet engines, rockets, and spacecrafts. A diffuser is a device that increases the pressure of a fluid by slowing it down, while a nozzle is a device that increases the velocity of a fluid at the expense of pressure. That is, diffusers and nozzles perform opposite tasks. Diffusers and nozzles involve no work ($\dot{W} \approx 0$) and negligible changes in potential energy ($\Delta PE \approx 0$). Moreover, the rate of heat transfer between the fluid flowing through a diffuser or a nozzle and the surroundings is usually very small ($\dot{Q} \approx 0$). This is due to the very short time air (or gas) spends in either duct (few or fraction of milliseconds) which is insufficient for a significant heat transfer to take place. However, fluid passing through diffusers and nozzles experiences large changes in velocity. Therefore, the kinetic energy changes must be accounted for ($\Delta KE \neq 0$). The shape of both diffuser and nozzle may be convergent or divergent depending on the velocity of flowing fluid. Rockets and military high supersonic aircrafts normally have *convergent–divergent or CD nozzles*. In a CD rocket nozzle, the hot exhaust leaves the combustion chamber and converges down to the minimum area, or *throat,* of the nozzle. The throat size is chosen to *choke* the flow and set the mass flow rate through the system. The flow in the throat is sonic which means the Mach number is equal to one in the throat. Downstream of the throat, the geometry diverges, and the flow is isentropically expanded to a supersonic Mach number that depends on the area ratio of the exit to the throat. The expansion of a supersonic flow causes the static pressure and temperature to decrease from the throat to the exit, so the amount of the expansion also determines the exit pressure and temperature. The exit temperature determines the exit speed of sound, which determines the exit velocity. The exit velocity, pressure, and mass flow through the nozzle determine the amount of thrust produced by the nozzle.

2.4.4.1 Variation of Fluid Velocity with Flow Area

We begin with the conservation of mass equation:

$$\dot{m} = \rho VA = \text{constant}$$

where $\dot{m}$ is the mass flow rate, ρ is the gas density, **V** is the gas velocity, and **A** is the cross-sectional flow area. If we differentiate this equation, we obtain

$$VAd\rho + \rho AdV + \rho VdA = 0$$

Divide by (ρVA) to get

$$\frac{d\rho}{\rho} + \frac{dV}{V} + \frac{dA}{A} = 0$$

Now we use the conservation of momentum equation:

$$\rho V dV = -dP$$

and an isentropic flow relation: $Tds = dh - vdP$

$$\frac{dP}{P} = \gamma \frac{\mathrm{d}\rho}{\rho}$$

where γ is the ratio of specific heats. Rewrite the above equation to obtain

$$dP = \gamma \frac{P}{\rho} d\rho$$

and use the equation of state ($\frac{P}{\rho} = RT$) to get

$$dP = \gamma RT d\rho$$

Since (γRT) is the square of the speed of sound (a), then

$$dP = a^2 d\rho$$

Combining this equation for the change in pressure with the momentum equation, we obtain

$$\rho V dV = - \ a^2 d\rho$$

$$\frac{\mathrm{V}}{a^2} dV = -\frac{\mathrm{d}\rho}{\rho}$$

using the definition of the Mach number $M = V/a$, then

$$-M^2 \frac{dV}{V} = \frac{\mathrm{d}\rho}{\rho} \quad (2.53)$$

Now we substitute this value of ($\mathrm{d}\rho/\rho$) into the mass flow equation to get

$$-M^2 \frac{dV}{V} + \frac{dV}{V} + \frac{dA}{A} = 0$$

$$\left(1 - M^2\right) \frac{dV}{V} = -\frac{dA}{A} \quad (2.54)$$

Equation (2.59) tells us how the velocity (V) changes when the area (A) changes and the results depend on the Mach number (M) of the flow.

If the flow is *subsonic* then ($M < 1.0$)—the term multiplying the velocity change is positive $[\left(1 - M^2\right) > 0]$—then an increase in the area $(dA > 0)$ produces a decrease in the velocity ($dV < 0$), which is the case of a diffuser. On the contrary a decrease in the area produces an increase in velocity, which is the case of a nozzle.

For a *supersonic* flow $(M > 1.0)$, the term multiplying velocity change is negative $[\left(1 - M^2\right) < 0]$. Then an increase in the area $(dA > 0)$ produces an increase in the velocity ($dV > 0$) or a nozzle. The decrease in the area leads to a decrease in velocity or a diffuser.

Table 2.5 summarizes this behavior.

Figure 2.12 illustrates the geometry of diffusers and nozzles in subsonic and supersonic speeds.

For the case of CD nozzle, if the flow in the throat is subsonic, the flow downstream of the throat will decelerate and stay subsonic. So if the converging section is too large and does not choke the flow in the throat, the exit velocity is very slow and does not produce much thrust. On the other hand, if the converging section is small enough so that the flow chokes in the throat, then a slight increase in area causes the flow to go supersonic. This is exactly the opposite of what happens subsonically.

Table 2.5 Variation of duct area with inlet Mach number

	Accelerated flow (nozzle)	Decelerated flow (diffuser)	
	$dV > 0$	$dV < 0$	Constant velocity
$M < 1.0$	$dA < 0$	$dA > 0$	$dA = 0$
$M > 1.0$	$dA > 0$	$dA < 0$	$dA = 0$

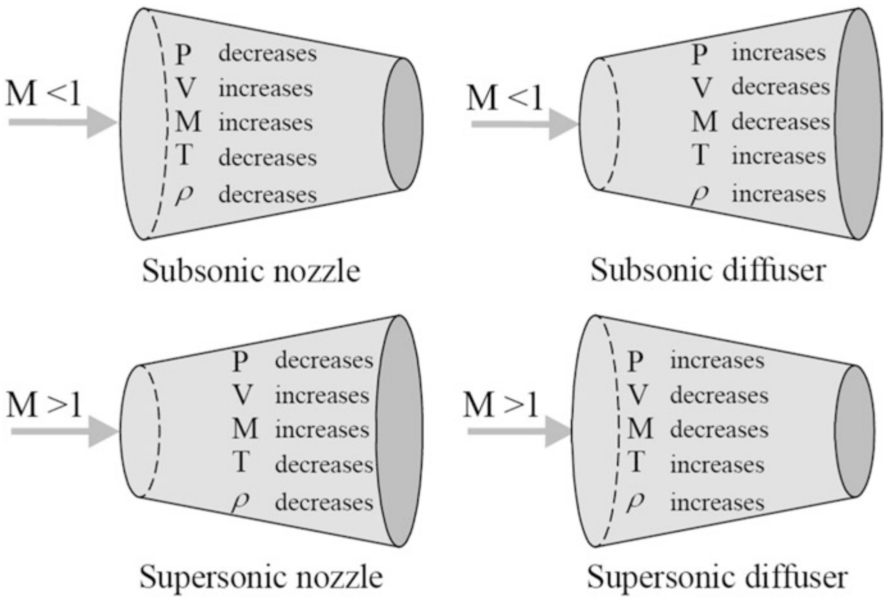


Fig. 2.12 Variation of flow properties in subsonic and supersonic nozzles and diffusers

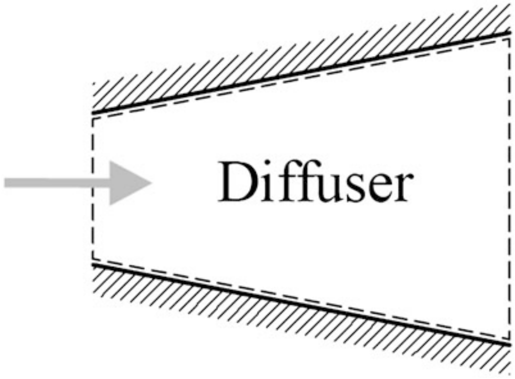


Fig. 2.13 Diffuser and control volume

Example 2.7 Air at 5 °C and 80 kPa enters the diffuser of a jet engine steadily with a velocity of 200 m/s. The inlet area of the diffuser is 0.4 m^2. The air leaves the diffuser with a velocity that is very small compared with the inlet velocity. Determine (*a*) the mass flow rate of the air and (*b*) the temperature of the air leaving the diffuser.

Solution

We take the *diffuser* as the system (Fig. 2.13). This is a *control volume* since mass crosses the system boundary during the process. We observe that there is only one inlet and one exit and thus $\dot{m}_1 = \dot{m}_2 = \dot{m}$.

(a) To determine the mass flow rate, we need to find the density of the air first. This is determined from the ideal gas relation at the inlet conditions:

$$\rho_1 = \frac{P_1}{RT_1} = \frac{80 \times 10^3}{287 \times (273 + 5)} = 1.0027 \ \text{kg/m}^3$$

$$\dot{m} = \rho_1 V_1 A_1 = 1.0027 \times 200 \times 0.4 = 79.8 \ \text{kg/s}$$

(b) From the energy equation

$$\frac{dQ}{dt} + \left(\frac{V_1^2}{2} + gz_1 + h_1\right)\dot{m} = \frac{dW_s}{dt} + \left(\frac{V_2^2}{2} + gz_2 + h_2\right)\dot{m}$$

With small exit velocity ($V_2 \approx 0$), negligible potential energy variation as well as heat and work exchange ($z_1 \approx z_2, \ dQ/dt = dW_s/dt = 0$), then, energy equation is reduced to

$$h_2 = h_1 + \frac{V_1^2}{2}$$

$$T_2 = T_1 + \frac{V_1^2}{2Cp} = 278 + \frac{200^2}{2 \times 1005} = 297.9 \ \text{K}$$

Example 2.8 Gas flows through a converging–diverging nozzle. Points G and H lie between the inlet and outlet of the nozzle. At a point "G," the cross-sectional area is 500 cm^2 and the Mach number was measured to be 0.4. At point "H" in the nozzle, the cross-sectional area is 400 cm^2. Find the Mach number at point H. Assume that the flow is isentropic and the gas-specific heat ratio is 1:3.

Solution

To obtain the Mach number at point G, apply Eq. (2.46) to find the ratio between the area (A_G) to the critical one (A^*)

$$\frac{A_G}{A^*} = \frac{1}{M_G}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}{M_G}^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{A_G}{A^*} = \left(\frac{1}{0.4}\right)\left[\left(\frac{2}{1.3+1}\right)\left(1+\frac{1.3-1}{2}(0.4)^2\right)\right]^{\frac{2.3}{2(0.3)}} = 1.6023$$

At point H, the area ratio is evaluated from the relation:

$$\frac{A_H}{A^*} = \frac{A_H}{A_G}\frac{A_G}{A^*} = \frac{400}{500} \times 1.6023 = 1.2818$$

Again from Eq. (2.46)

$$\frac{A_H}{A^*} = \frac{1}{M_H}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}{M_H}^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Rearranging to solve for the Mach number M_H,

$$\left(M_H\frac{A_H}{A^*}\right)^{\frac{2(\gamma-1)}{\gamma+1}} - \frac{\gamma-1}{\gamma+1}{M_H}^2 = \frac{2}{\gamma+1}$$

$$\left(M_H\frac{A_H}{A^*}\right)^{\frac{2(\gamma-1)}{\gamma+1}} - \frac{\gamma-1}{\gamma+1}{M_H}^2 = \frac{2}{\gamma+1}$$

$$1.0669{M_H}^{0.2609} - 0.1304{M_H}^2 = 0.8696$$

Solving the above equation by *trial and error*, we get either

$$M_H = 0.5374 \quad \text{or} \quad M_H = 1.612$$

Both solutions are possible, the first is still a subsonic Mach number which may be located in the convergent section, while the second one is supersonic which may be located in the divergent section if the speed at throat is sonic: $M_{\text{throat}} = 1.0$.

2.4.5 Shocks

A shock is an irreversible flow discontinuity in a (partly) supersonic flow fluid. It may be also defined as a pressure front which travels at speed through a gas. Upon crossing the shock waves, pressure, temperature, density, and entropy rise while the normal velocity decreases. There are two types of shocks, namely, normal and oblique.

2.4.5.1 Normal Shock Waves

Consider a plane supersonic flow with a normal compression shock in a channel with constant cross-sectional area (Fig. 2.14). The conditions upstream and downstream the shock are denoted by subscripts (1) and (2), respectively. Under the following assumptions—steady, one dimensional, adiabatic ($\delta q = 0$), no shaft work ($\delta w = 0$), negligible potential ($\delta z = 0$), constant area ($A_1 = A_2$) and negligible wall shear—then equations of state and integral forms of conservation equations will have the following forms:

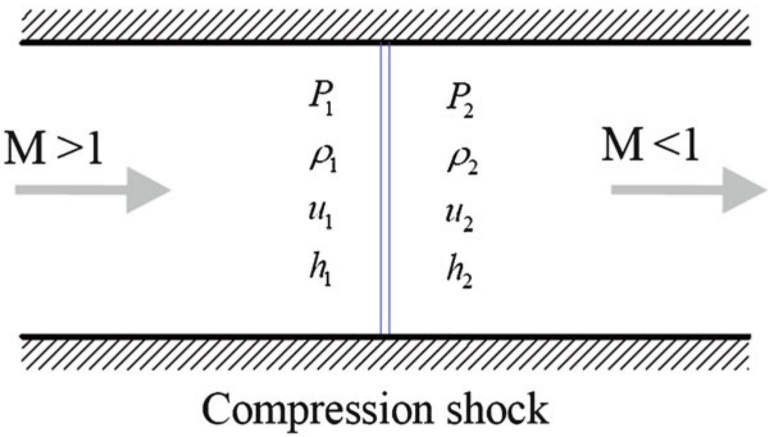


Fig. 2.14 One-dimensional shock waves

$$
\begin{aligned}
&\text{Continuity equation} && \frac{\dot{m}}{A} = \rho_1 \ u_1 = \rho_2 \ u_2 \\
&\text{Momentum equation} && p_1 + \rho_1 \ u_1{}^2 = p_2 + \rho_2 \ u_2{}^2 \\
&\text{Energy equation} && \mathrm{h}_{01} \equiv h_1 + u_1{}^2/2 = h_2 + u_2{}^2/2 = \mathrm{h}_{02} \\
&\text{Equation of state} && \frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2}
\end{aligned}
\tag{2.55}
$$

From the continuity equation, equation of state for perfect gas, and the velocity relation

$$u = M\sqrt{\gamma RT}$$

we arrive at the relation

$$\frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}}$$

Moreover, the energy equation together with the perfect gas relation (2.44a)

$$T_0 = T\left(1 + \frac{\gamma - 1}{2} M^2\right)$$

yields the following relation:

$$T_1\left(1 + \frac{\gamma - 1}{2} M_1{}^2\right) = T_2\left(1 + \frac{\gamma - 1}{2} M_2{}^2\right)$$

The momentum equation together with the equation of state provides the following relation:

$$p_1\left(1 + \gamma M_1{}^2\right) = p_2\left(1 + \gamma M_2{}^2\right)$$

The following relations give the downstream Mach number, static temperature, pressure, and density ratios as well as the total pressure and temperature ratios across the shock:

$$M_2{}^2 = \frac{(\gamma - 1)M_1{}^2 + 2}{2\gamma M_1{}^2 - (\gamma - 1)} \tag{2.56}$$

$$\frac{T_2}{T_1} = \frac{\left[2\gamma M_1{}^2 - (\gamma - 1)\right]\left[(\gamma - 1)M_1{}^2 + 2\right]}{(\gamma + 1)^2 M_1{}^2} \tag{2.57}$$

$$\frac{P_2}{P_1} = \left[\frac{2\gamma {M_1}^2 - (\gamma - 1)}{\gamma + 1}\right] \tag{2.58}$$

$$\frac{\rho_2}{\rho_1} = \left[\frac{(\gamma + 1){M_1}^2}{2 + (\gamma - 1){M_1}^2}\right] = \frac{u_1}{u_2} \tag{2.59}$$

$$\frac{P_{02}}{P_{01}} = \left[\frac{(\gamma + 1){M_1}^2}{2 + (\gamma - 1){M_1}^2}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{(\gamma + 1)}{2\gamma {M_1}^2 - (\gamma - 1)}\right]^{\frac{1}{\gamma-1}} \tag{2.60}$$

$$\frac{T_{02}}{T_{01}} = 1$$

$$u_1 u_2 = {a^*}^2$$

This means that $u_1 > a^* > u_2$. The critical sonic speed is expressed as

$${a^*}^2 = \frac{2\gamma R T_0}{(\gamma + 1)} = \frac{2(\gamma - 1)}{(\gamma + 1)} C_p T_0$$

From the entropy relation, the total pressure ratio can be also expressed as

$$\frac{p_{02}}{p_{01}} = e^{-\left(\frac{s_2 - s_1}{R}\right)}$$

Equations (2.56), (2.57), (2.58), (2.59), and (2.60) are plotted in Fig. 2.15.

We can state two simple rules of thumb:

1. A normal shock wave always forms between supersonic and subsonic flow.
2. The flow behind a normal shock wave is always subsonic.

Normal shock waves are encountered in the flow in intakes and nozzles as well as over aircraft wings. Figure 2.16 illustrates normal shock waves formed on the suction or both suction and pressure surfaces of wing sections.

It is obvious that a very useful table for fluid flow changes across a normal shock can be constructed using the above equations. This kind of table is available in all gas dynamics or compressible flow texts [4, 5]. Table 2.6 illustrates these relations. You are encouraged to complete missing data in Table 2.6.

Example 2.9 Air is flowing through normal shock. Flow conditions upstream of the shock are $\mathrm{u}_1 = 600\,\mathrm{m/s}$, $\mathrm{T}_{01} = 500\,\mathrm{K}$, $\mathrm{P}_{01} = 700\,\mathrm{kPa}$ It is required to calculate the downstream conditions M_2, u_2, T_2, P_2, P_{o2} and $(s_2 - s_1)$.Assume: calorically perfect ideal gas.

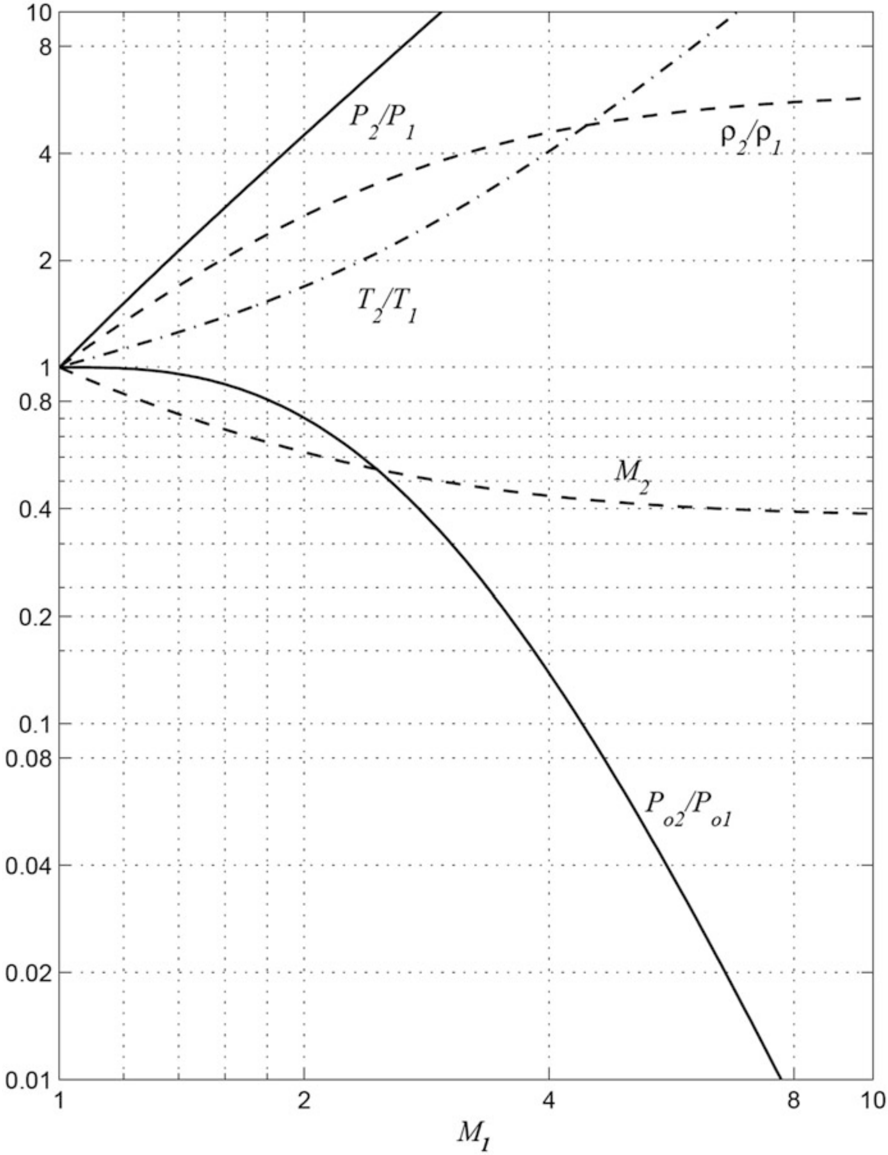


Fig. 2.15 Normal shock functions ($\gamma = 1.4$)

Solution

The upstream conditions (static temperature, pressure, and density as well as sonic speed and Mach numbers) can be calculated from the following relations:

$$T_1 = T_{01} - \frac{u_1{}^2}{2C_p} = 500 - \frac{(600)^2}{2 \times 1005} = 320.89 \quad \text{K}$$

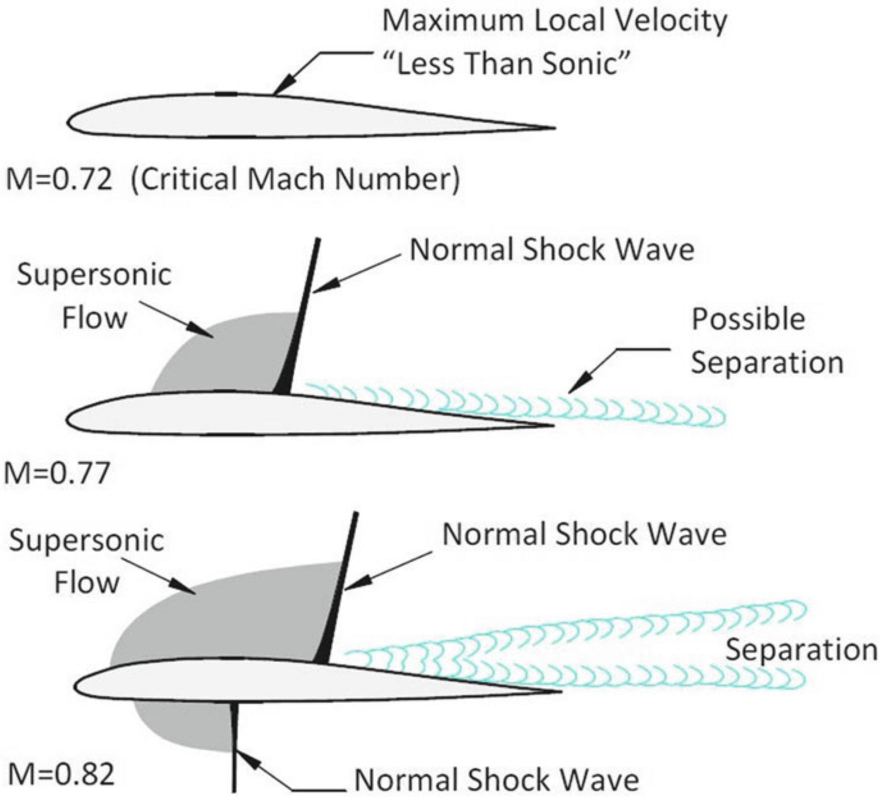


Fig. 2.16 Normal shock waves over either suction or suction/pressure sides of wing section

Table 2.6 Normal shock parameters ($\gamma = 1.4$)

M_1	M_2	P_2/P_1	T_2/T_1	P_{02}/P_{01}
3.0	0.47519	10.3333	2.6790	0.32834
2.5	?	?	?	?
2.0	0.57735	4.5000	1.6875	0.72089
1.5	?	?	?	?
1.00	1.00	1.00	1.00	1.00

$$P_1 = P_{01}/\left(1 + \frac{\gamma - 1}{2} M_1{}^2\right)^{\frac{\gamma}{\gamma-1}} = 700/4.7249 = 148.15 \quad \text{kPa}$$

$$\rho_1 = P_1/RT_1 = \frac{148.15}{0.287 \times 320.9} = 1.609 \quad \text{kg/m}^3$$

$$a_1 = \sqrt{\gamma RT_1} = \sqrt{1.4 \times 287 \times 320.89} = 359.0 \quad \text{m/s}$$

$$M_1 = u_1/a_1 = 600/359 = 1.671$$

Mach number downstream the shock wave (M_2) is evaluated from the relation:

$$M_2{}^2 = \frac{(\gamma-1)M_1{}^2+2}{2\gamma M_1{}^2-(\gamma-1)} = \frac{0.4\times(1.671)^2+2}{2\times1.4\times(1.671)^2-0.4} = \frac{3.1169}{7.418} = 0.34489$$
$$M_2 = 0.648$$

From known Mach number (M_2), the air properties downstream of the normal shock can be evaluated as follows:

$$\frac{T_2}{T_1} = \frac{\left[2\gamma M_1{}^2-(\gamma-1)\right]\left[(\gamma-1)M_1{}^2+2\right]}{(\gamma+1)^2M_1{}^2}$$
$$= \frac{\left[2\times1.4\times(1.671)^2-0.4\right]\left[0.4\times(1.671)^2+2\right]}{(2.4\times1.671)^2}$$
$$\frac{T_2}{T_1} = \frac{7.4183\times3.1169}{16.0833} = 1.4376$$
$$T_2 = 461.34 \quad \mathrm{K}$$
$$u_2 = M_2\times c_2 = M_2\times\sqrt{\gamma RT_2} = 0.648\times\sqrt{1.4\times287\times461.34} = 278.9\mathrm{m/s}$$
$$\rho_2 = \rho_1\,u_1/u_2 = 1.609\times600/278.9 = 3.4614 \quad \mathrm{kg/m^3}$$
$$p_2 = p_1 + \rho_1\,u_1^2 - \rho_2\,u_2^2$$
$$P_2 = 148.15\times10^3 + 1.609\times(600)^2 - 3.4614\times(278.9)^2$$
$$P_2 = 458{,}144 \quad Pa = 458.14 \quad \mathrm{kPa}$$

Since $$\frac{P_{02}}{P_{01}} = \left[\frac{(\gamma+1)M_1{}^2}{2+(\gamma-1)M_1{}^2}\right]^{\frac{\gamma}{\gamma-1}}\left[\frac{(\gamma+1)}{2\gamma M_1{}^2-(\gamma-1)}\right]^{\frac{1}{\gamma-1}}$$

Then $$\frac{P_{02}}{P_{01}} = \left[\frac{2.4\times(1.671)^2}{2+0.4(1.671)^2}\right]^{3.5}\left[\frac{2.4}{2.8\times(1.671)^2-0.4}\right]^{2.5} = 0.86759$$
$$P_{02} = 607.32 \quad \mathrm{kPa}$$

As a check, calculate the value of (T_{02}):

$$T_{02} = T_2\left(1+\frac{\gamma-1}{2}M_2{}^2\right) = 461.34\times\left[1+0.2\times(0.648)^2\right] = 500\ \mathrm{K}$$

This confirms the total temperature (or enthalpy) equality, $T_{02} = T_{01}$, as stated above.

$$\text{Since} \qquad \frac{P_{02}}{P_{01}} = e^{-\left(\frac{s_2 - s_1}{R}\right)}$$

$$\text{Then} \qquad s_2 - s_1 = -R\ln\left(\frac{P_{02}}{P_{01}}\right) = 40.764 \quad \text{J/kg K}$$

The temperature–entropy diagram is illustrated by Fig. 2.17, which shows the static and total conditions upstream and downstream of the shock. Note the entropy increased despite not including any entropy-generating mechanisms in this model. Why? First, the differential equations themselves required the assumption of continuous differentiable functions. Our shock violates this.

When one returns to the more fundamental control volume forms, it can be shown that the entropy-generating mechanism returns. From a continuum point of view, one can also show that the neglected terms, that momentum and energy diffusion, actually give rise to a *smeared* shock. These mechanisms generate just enough entropy to satisfy the entropy jump which was just calculated.

Another interpretation may be also given as follows: the assumption that the compression shock represents a discontinuity is only an approximation. In reality the shock has a thickness (δ) of the order of magnitude of several free mean paths. If the gas flowing through the shock can be assumed to be a continuum, the Navier–Stokes equations can be employed for the description of the flow between the upstream and downstream edge of the compression shock. The flow quantities do not change discontinuously in the form of a jump but in a continuous transition from the free-stream conditions to the flow conditions downstream from the shock. The increase of the entropy can now be explained as an action of the frictional forces and the heat conduction within the shock region of finite thickness.

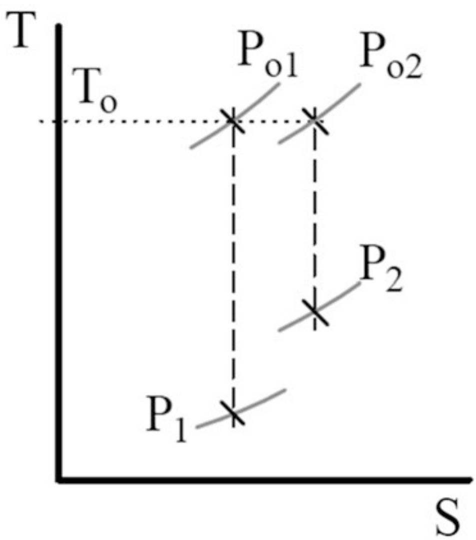


Fig. 2.17 Static and total conditions

2.4.5.2 Off Design and Normal Shock Waves in Nozzles

The objective of CD nozzle is to obtain supersonic flow. Thus, the design operating condition is to have a subsonic flow in the convergent section, a sonic condition at throat, and a supersonic flow in the divergent part. For off-design conditions, many possibilities for the speed at the nozzle exit may be encountered depending on the back pressure P_b. The fluid may find itself decelerating in the diverging section instead of accelerating. A detailed description is given with the aid of Fig. 2.18. When a fluid enters the nozzle with a low velocity at stagnation pressure P_0, the state of the nozzle flow is determined by the overall pressure ratio P_b/P_0. When the back pressure $P_b = P_0$ (case A), there will be no flow through the nozzle. This is expected since the flow in a nozzle is driven by the pressure difference between the nozzle inlet and the exit. Now let us examine what happens as the back pressure is lowered.

1. When $P_0 > P_b > P_C$ (critical pressure), the flow remains subsonic throughout the nozzle, and the mass flow is less than that for choked flow. The fluid velocity increases in the converging section and reaches a maximum at the throat (but still subsonic; $M < 1$). However, most of the gain in velocity is lost in the diverging section of the nozzle, which acts as a diffuser. The pressure decreases in the converging section, reaches a minimum at the throat, and increases at the expense of velocity in the diverging section.
2. When $P_b = P_C$, the throat pressure becomes P^* and the fluid achieves sonic velocity at the throat. But the diverging section of the nozzle still acts as a diffuser, slowing the fluid to subsonic velocities. The mass flow rate that was increasing with decreasing P_b also reaches its maximum value. Recall that P^* is the lowest pressure that can be obtained at the throat, and the sonic velocity is the highest velocity that can be achieved with a converging nozzle. Thus, lowering P_b further has no influence on the fluid flow in the converging part of the nozzle or the mass flow rate through the nozzle. However, it does influence the character of the flow in the diverging section. This mode of operation is frequently called the *first critical* [5].
3. When $P_C > P_b > P_E$, the fluid that achieved a sonic velocity at the throat continues accelerating to supersonic velocities in the diverging section as the pressure decreases. This acceleration comes to a sudden stop, however, as a *normal shock* develops at a section between the throat and the exit plane, which causes a sudden drop in velocity to subsonic levels and a sudden increase in pressure. This mode of operation is frequently called the *second critical* [5]. The fluid then continues to decelerate further in the remaining part of the converging–diverging nozzle. Flow through the shock is highly irreversible, and thus it cannot be approximated as isentropic. The normal shock moves downstream away from the throat as P_b is decreased, and it approaches the nozzle exit plane as P_b approaches P_E. When $P_b = P_E$, the normal shock forms at the exit plane of the nozzle. The flow is supersonic through the entire diverging section in this case, and it can be approximated as isentropic.

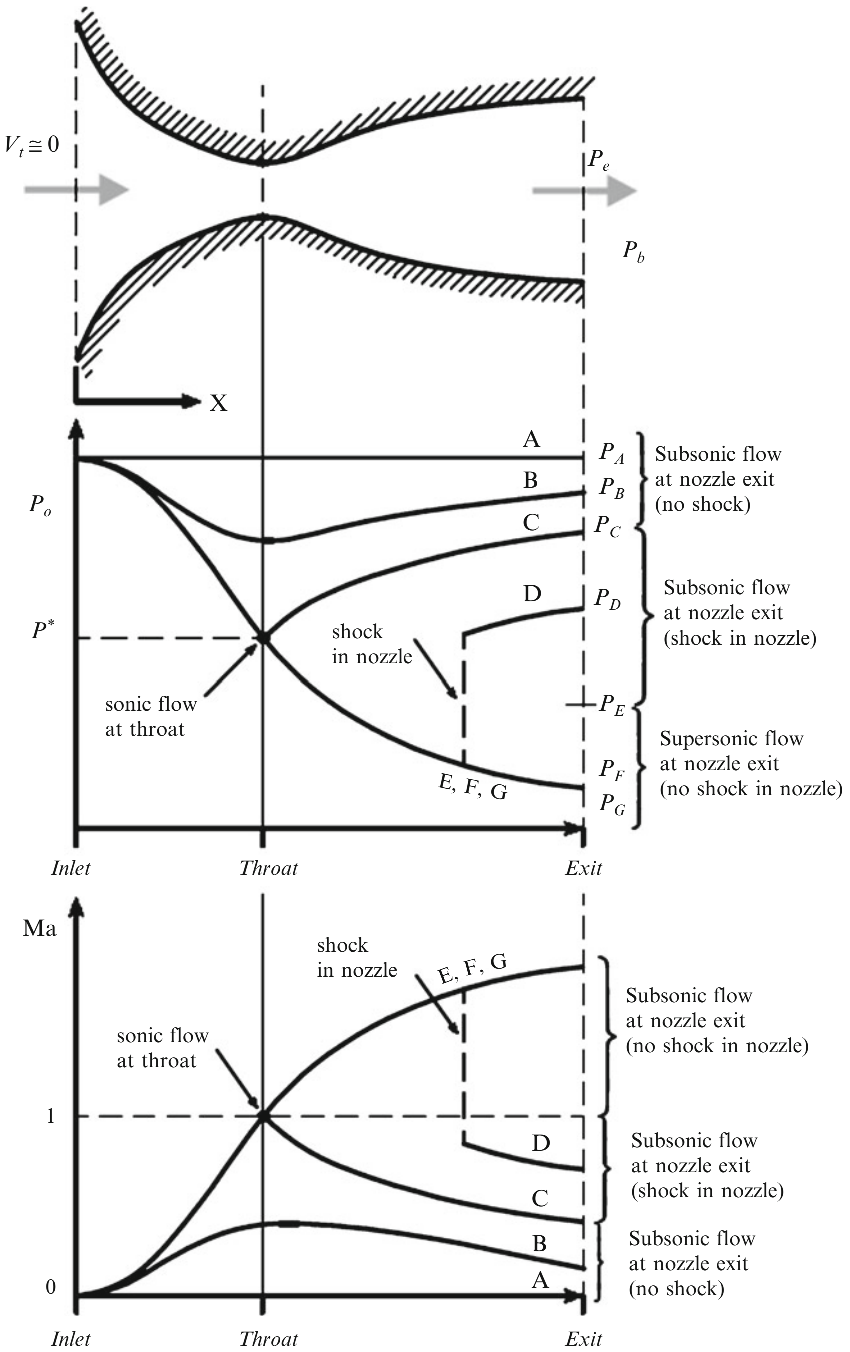


Fig. 2.18 The effects of back pressure on the flow through a converging–diverging nozzle

However, the fluid velocity drops to subsonic levels just before leaving the nozzle as it crosses the normal shock.

4. When $P_E > P_b > 0$, the flow in the diverging section is supersonic, and the fluid expands to P_F at the nozzle exit with no normal shock forming within the nozzle. Thus, the flow through the nozzle can be approximated as isentropic. When $P_b = P_F$, no shocks occur within or outside the nozzle. This mode of operation is frequently called the *third critical* [5]. When $P_b < P_F$ (*underexpanded case)*, irreversible mixing and expansion waves occur downstream of the exit plane of the nozzle. When $P_b > P_F$ (*overexpanded case*), however, the pressure of the fluid increases from P_F to P_b irreversibly in the wake of the nozzle exit, creating what are called *oblique shocks*.

Example 2.10 A large tank with compressed air is attached into a converging–diverging nozzle (Fig. 2.19) with pressure 8 bar and temperature of 327 °C. Nozzle throat area is 30 cm^2 and the exit area is 90 cm^2. The shock occurs in a location where the cross section area is 60 cm^2. Calculate the back pressure and the temperature of the flow. Also determine the critical subsonic and supersonic points for the back pressure (point "a" and point "b").

Solution

The stagnation temperature and pressure at the nozzle inlet are equal to the specified values in the tank.

$$P_{01} = 8 \text{ bar} \quad , \quad T_{01} = 327 + 273 = 600 \text{ K}$$

Since the star area (the throat area), A^*, and the area upstream of the shock are known, then this ratio is given as

$$\frac{A_x}{A^*} = \frac{60}{30} = 2$$

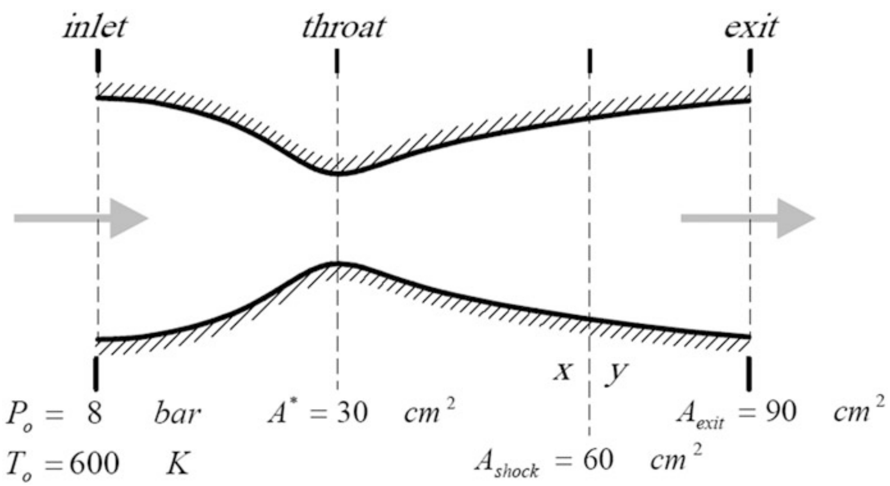


Fig. 2.19 Convergent-divergent nozzle

To evaluate the conditions upstream of the normal shock (state x), Eq. (2.46) is employed. It may be reduced to

$$\left(M_x \frac{A_x}{A^*}\right)^{\frac{2(\gamma-1)}{\gamma+1}} - \frac{\gamma-1}{\gamma+1} M_x{}^2 = \frac{2}{\gamma+1}$$

with $\gamma = 1.4$, then it is further simplified to

$$1.25992 M_x{}^{0.3333} - 0.16667 M_x{}^2 = 0.8333$$

Solve the above equation by trial and error to get $M_x = 2.1972$

From isentropic relations (2.44)

$$\frac{T_0}{T_x} = 1 + \frac{\gamma-1}{2} M_x{}^2 = 1.96554$$

$$T_x = 305.3 \ \text{K}$$

$$\frac{P_{0x}}{P_x} = \left(1 + \frac{\gamma-1}{2} M_x{}^2\right)^{\frac{\gamma}{\gamma-1}} = 10.64596$$

$$P_x = 0.7515 \quad bar$$

From normal shock relations (2.56), (2.57), (2.58), (2.59), and (2.60)

$$M_y{}^2 = \frac{(\gamma-1)M_x{}^2 + 2}{2\gamma M_x{}^2 - (\gamma-1)} = \frac{3.931075}{13.11752}$$

$$M_y = 0.54743$$

$$\frac{T_y}{T_x} = \frac{\left[2\gamma M_x{}^2 - (\gamma-1)\right]\left[(\gamma-1)M_x{}^2 + 2\right]}{(\gamma+1)^2 M_x{}^2} = \frac{13.117523 \times 3.931}{27.8074}$$

$$\frac{T_y}{T_x} = 1.8543$$

$$T_y = 566.1 \quad \text{K}$$

$$\frac{P_y}{P_x} = \left[\frac{2\gamma M_1{}^2 - (\gamma-1)}{\gamma+1}\right] = 5.46564$$

$$P_y = 4.1 \quad bar$$

$$\frac{P_{0y}}{P_{0x}} = \left[\frac{(\gamma+1){M_x}^2}{2+(\gamma-1){M_x}^2}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{(\gamma+1)}{2\gamma{M_x}^2-(\gamma-1)}\right]^{\frac{\gamma}{\gamma-1}}$$

$$\frac{P_{0y}}{P_{0x}} = \left[\frac{2.4\times(2.1972)^2}{2+0.4\times(2.1972)^2}\right]^{3.5} \left[\frac{2.4}{2.8\times(2.1972)^2-0.4}\right]^{2.5}$$

$$\frac{P_{0y}}{P_{0x}} = 43.95788\times 0.014318 = 0.62941$$

$$P_{0y} = 4.11 \ \text{ bar}$$

Again utilizing the isentropic relationship, the exit conditions can be evaluated. With known Mach number the new star area ratio (A_y/A^*) can be calculated from the relation:

$$\frac{A_y}{A^*} = \frac{1}{M_y}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}{M_y}^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\frac{A_y}{A^*} = \frac{1}{0.54743}\left[\frac{2}{2.4}\left(1+0.2(0.54743)^2\right)\right]^3 = 1.25883$$

From known exit area, then

$$\frac{A_e}{A^*} = \frac{A_e}{A_y}\frac{A_y}{A^*} = \frac{90}{60}\times 1.25883 = 1.88824$$

From this area ratio, then (M_e) can be calculated by trial and error using the relation

$$\left(M_e\frac{A_e}{A^*}\right)^{\frac{2(\gamma-1)}{\gamma+1}} - \frac{\gamma-1}{\gamma+1}{M_e}^2 = \frac{2}{\gamma+1}$$

$$1.236{M_x}^{0.3333} - 0.16667{M_x}^2 = 0.8333$$

$$M_e = 0.327$$

From isentropic relations (2.44)

$$\frac{T_0}{T_e} = 1+\frac{\gamma-1}{2}{M_e}^2 = 1.02138$$

$$\frac{P_{0e}}{P_e} = \frac{P_{0y}}{P_e} = \left(1+\frac{\gamma-1}{2}{M_e}^2\right)^{\frac{\gamma}{\gamma-1}} = 1.07687$$

$$P_e = \frac{P_e}{P_{0y}}\frac{P_{0y}}{P_{0x}}P_{0x} = \frac{1}{1.07687}\times 0.62941\times 8 = 4.6758 \quad \text{bar}$$

Table 2.7 Properties of air inside the nozzle

	Inlet	Upstream normal shock	Downstream normal shock	Exit
M	0	2.1972	0.54743	0.327
P_0 (bar)	8	8	4.11	4.11
P (bar)	–	0.7515	4.1	4.6758
T (K)	–	305.3	566.1	587.4

$$T_e = \frac{T_e}{T_0} T_0 = \frac{1}{1.02138} \times 600 = 587.4 \quad \text{K}$$

A summary of the above results is given here in Table 2.7.

The "critical" points "a" and "b" at nozzle exit resemble the subsonic and supersonic limits if no shock waves exist and the flow achieves a Mach equal of unity at the throat. The area ratio for both cases is calculated from the relation:

$$\frac{A_e}{A^*} = \frac{1}{M_e}\left[\frac{2}{\gamma+1}\left(1+\frac{\gamma-1}{2}{M_e}^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

Since $\frac{A_e}{A^*} = \frac{90}{30} = 3.0$, then the exit Mach number may be calculated by trial and error from the relation:

$$\left(M_e\frac{A_e}{A^*}\right)^{\frac{2(\gamma-1)}{\gamma+1}} - \frac{\gamma-1}{\gamma+1}{M_e}^2 = \frac{2}{\gamma+1}$$

$$1.4422{M_e}^{0.3333} - 0.16667{M_e}^2 = 0.8333$$

Two solutions are obtained, namely, $M_e = 0.19745$ and $M_e = 2.6374$. Both solutions are illustrated by points (a) and (b) in Fig. 2.20.

For point "a," $M_a = 0.19745$:

The exit pressure and temperature are determined as follows:

$$\frac{T_0}{T_a} = 1 + \frac{\gamma-1}{2}{M_a}^2 = 1.00779$$

$$T_a = 600/1.00779 = 595.3 \text{ K}$$

$$\frac{P_0}{P_a} = \left(1+\frac{\gamma-1}{2}{M_a}^2\right)^{\frac{\gamma}{\gamma-1}} = 1.027557$$

$$P_a = 8/1.027557 = 7.785 \text{ bar}$$

For point "b," $M_b = 2.6374$.

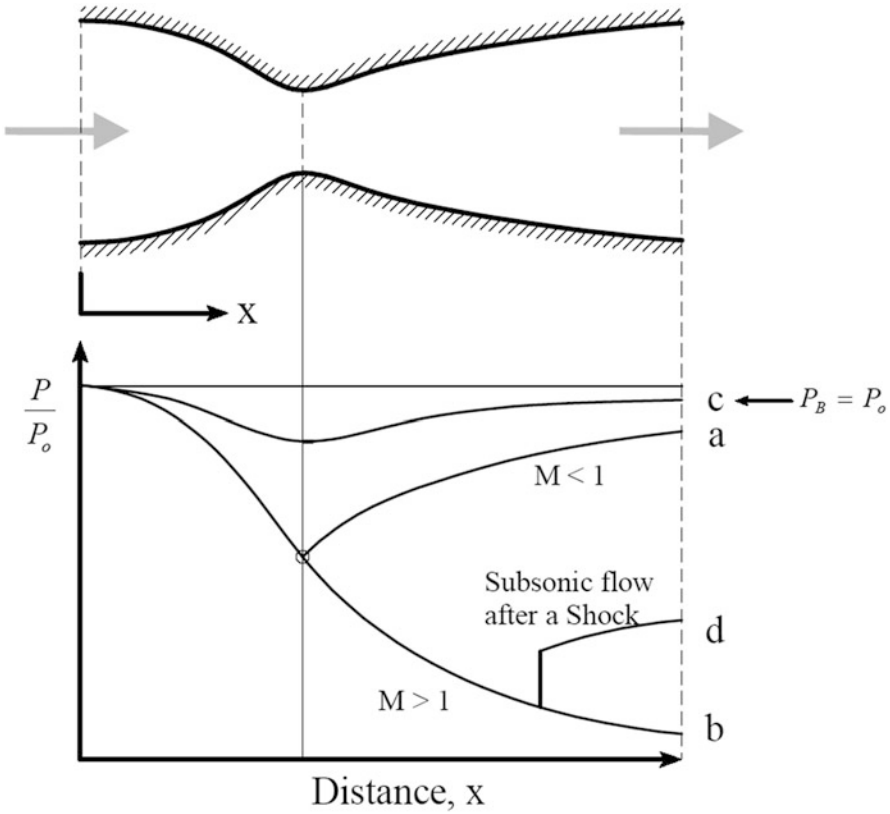


Fig. 2.20 Subsonic and supersonic solutions

Table 2.8 Subsonic and supersonic solutions

	Mach number	Static temperature (K)	Static pressure (bar)
Point (a) (subsonic solution)	0.19745	595.3	7.785
Point (b) (supersonic solution)	2.6374	250.92	0.3784

The exit pressure and temperature are determined as follows:

$$\frac{T_0}{T_b} = 1 + \frac{\gamma - 1}{2} M_b^2 = 2.39117$$

$$T_b = 600/2.39117 = 250.92 \quad \text{K}$$

$$\frac{P_0}{P_b} = \left(1 + \frac{\gamma - 1}{2} M_b^2\right)^{\frac{\gamma}{\gamma - 1}} = 21.14172$$

$$P_b = 0.3784 \text{ bar}$$

A summary of these results are given in Table 2.8.

2.4.5.3 Oblique Shock Wave Relations

Normal shock was examined in Sect. 2.4.4.2. Normal shock is a special case of the general inclined one. When shock is inclined to the flow direction, it is identified as an oblique shock. When a wedge-shaped object is placed in a two-dimensional supersonic flow, a plane-attached shock wave may emanate from the nose of the body at an angle (β) as long as shown in Fig. 2.21. The flow Mach number and the wedge angle (δ) together define the resulting attached or detached shock configuration. Similarly, when a supersonic flow encounters a concave corner with an angle (δ), two possibilities of attached or detached shock waves exist. Figure 2.22 illustrates the abovementioned four cases. There is a maximum deflection angle ($\delta_{\max}$) associated with any given Mach number. When the deflection angle exceeds $\delta_{\max}$, a detached shock forms which has a curved wave front. Behind this curved (or -bow-like) wave, we find all possible shock solutions associated with the initial Mach number M_1. At the center a normal shock exists, with subsonic flow resulting. As the wave front curves around, the shock angle decreases continually, with a resultant decrease in shock strength. Eventually, we reach a point where supersonic flow exists after the shock front.

Oblique shock waves are preferred predominantly in engineering applications compared with normal shock waves. This can be attributed to the fact that using one or a combination of oblique shock waves results in more favorable post-shock conditions (lower post-shock temperature and pressure) when compared to utilizing a single normal shock. An example of this technique can be seen in the design of supersonic aircraft engine inlets, which are wedge shaped to compress airflow into the combustion chamber while minimizing thermodynamic losses. Early supersonic aircraft jet engine inlets were designed using compression from a single normal

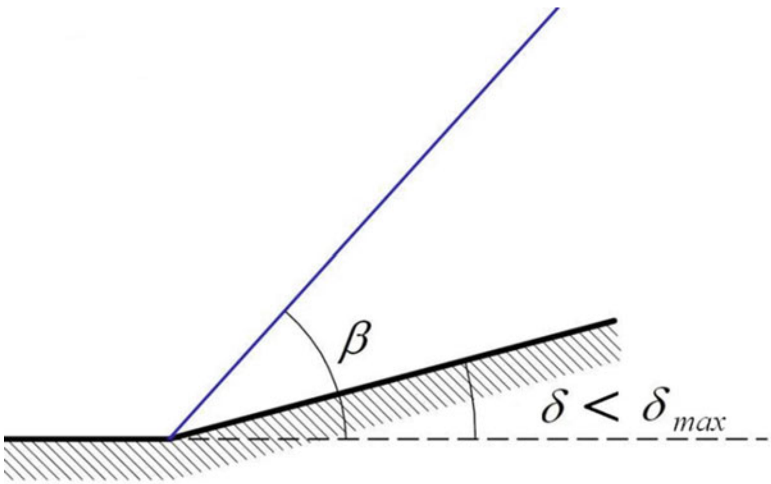


Fig. 2.21 Oblique shock wave applications

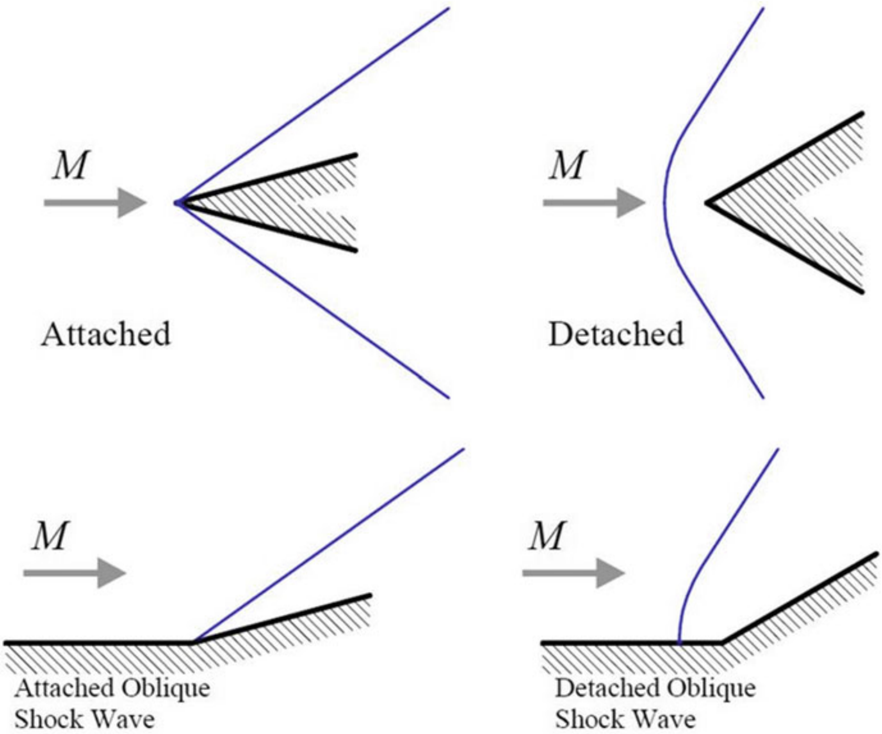


Fig. 2.22 Oblique shock wave applications

shock, but this approach caps the maximum achievable Mach number to roughly 1.6. The wedge-shaped inlets are clearly visible on the sides of the F-14 Tomcat, which has a maximum speed of Mach 2.34.

For analyzing oblique shock, consider Fig. 2.23 where the flow is deflected angle δ, and a shock generated inclined an angle β to the flow direction. The flow approaches the shock wave with a velocity V_1 and Mach number M_1 at an angle β with respect to the shock. It is turned through an angle δ as it passes through the shock, leaving with a velocity V_2 and a Mach number M_2 at an angle $(\beta - \delta)$ with respect to the shock. The inlet and exit velocities can be separated into tangential and normal components. The tangential velocity components upstream and downstream the shocks are equal. The normal velocity component may be treated as flow through a normal shock. This means that V_{1n} is supersonic and V_{2n} is subsonic, but still the downstream velocity V_2 is supersonic. The following relations define the normal and tangential velocity components and Mach number for both upstream and downstream conditions:

$$V_{1t} = V_{2t}$$

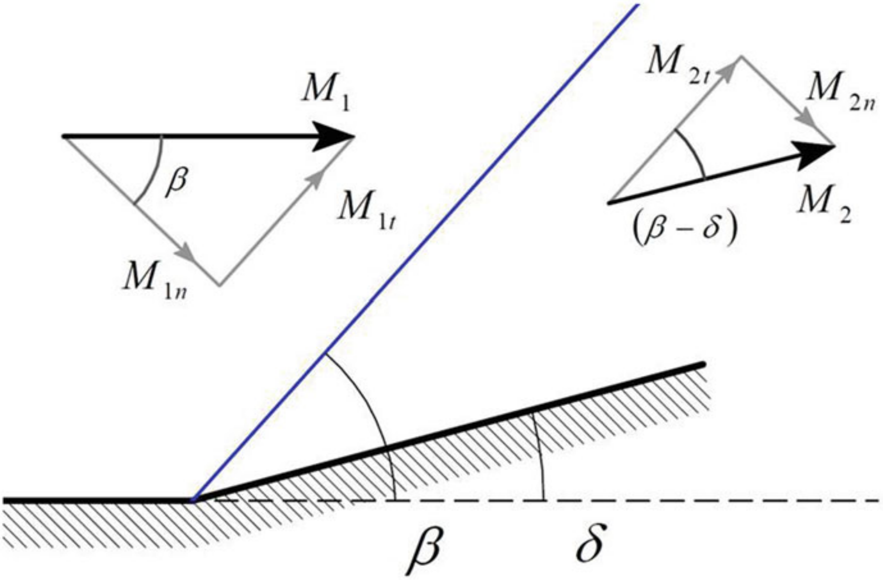


Fig. 2.23 Nomenclature of oblique shock wave

$$
\begin{aligned}
V_{1t} &= V_1 \cos\beta & V_{2t} &= V_2 \cos(\beta-\delta)\\
V_{1n} &= V_1 \sin\beta & V_{2n} &= V_2 \sin(\beta-\delta)\\
M_{1n} &= M_1 \sin\beta > 1.0 & M_{2n} &= M_2 \sin(\beta-\delta) < 1.0\\
M_{1t} &= M_1 \cos\beta & M_{2t} &= M_2 \cos(\beta-\delta)\\
M_1 &> 1.0, & M_2 &> 1.0
\end{aligned}
$$

Since the oblique shock can be treated as a normal shock having an upstream Mach number $M_{1n} = M_1 \sin\beta$ and a tangential component $M_{1t} = M_1 \cos\beta$, then using Eqs. (2.56), (2.57), (2.58), (2.59), and (2.60), the relations (2.62–2.66) can be deduced; [9] and [10]. The relation between (δ, β, M_1) is given by Eq. (2.61).

$$\tan\delta = \frac{2\cot\beta\left(M_1^2 \sin^2\beta - 1\right)}{(\gamma+1)M_1^2 - 2\left(M_1^2 \sin^2\beta - 1\right)} \tag{2.61a}$$

For $\gamma = 7/4$, then

$$\tan\delta = 5\,\frac{M_1^2 \sin 2\beta - 2\cot\beta}{10 + M_1^2(7 + 5\cos 2\beta)} \tag{2.61b}$$

$$M_{2n}{}^2 = \frac{(\gamma-1)M_{1n}{}^2 + 2}{2\gamma M_{1n}{}^2 - (\gamma-1)} \tag{2.62a}$$

$$M_2{}^2 \sin^2(\beta-\delta) = \frac{(\gamma-1)M_1{}^2 \sin^2\beta + 2}{2\gamma M_1{}^2 \sin^2\beta - (\gamma-1)} \tag{2.62b}$$

For $\gamma = 7/4$, then

$$M_2{}^2 = \frac{36 M_1{}^4 \sin^2\beta - 5\left(M_1{}^2 \sin^2\beta - 1\right)\left(7M_1{}^2 \sin^2\beta + 5\right)}{\left(7M_1{}^2 \sin^2\beta - 1\right)\left(M_1{}^2 \sin^2\beta + 5\right)} \tag{2.62c}$$

$$\frac{P_2}{P_1} = \left[\frac{2\gamma M_1{}^2 \sin^2\beta - (\gamma-1)}{\gamma+1}\right] \tag{2.63a}$$

For $\gamma = 7/4$, then

$$\frac{P_2}{P_1} = \left(\frac{7M_1{}^2 \sin^2\beta - 1}{6}\right) \tag{2.63b}$$

$$\frac{T_2}{T_1} = \frac{\left[2\gamma M_1{}^2 \sin^2\beta - (\gamma-1)\right]\left[(\gamma-1)M_1{}^2 \sin^2\beta + 2\right]}{(\gamma+1)^2 M_1{}^2 \sin^2\beta} \tag{2.64a}$$

For $\gamma = 7/4$, then

$$\frac{T_2}{T_1} = \frac{\left(7M_1{}^2 \sin^2\beta - 1\right)\left(M_1{}^2 \sin^2\beta + 5\right)}{36 M_1{}^2 \sin^2\beta} \tag{2.64b}$$

$$\frac{\rho_2}{\rho_1} = \left[\frac{(\gamma+1)M_1{}^2 \sin^2\beta}{2 + (\gamma-1)M_1{}^2 \sin^2\beta}\right] \tag{2.65a}$$

For $\gamma = 7/4$, then

$$\frac{\rho_2}{\rho_1} = \left[\frac{6M_1{}^2 \sin^2\beta}{M_1{}^2 \sin^2\beta + 5}\right] \tag{2.65b}$$

$$\frac{P_{02}}{P_{01}} = \left[\frac{(\gamma+1)M_1{}^2 \sin^2\beta}{(\gamma-1)M_1{}^2 \sin^2\beta + 2}\right]^{\frac{\gamma}{\gamma-1}} \left[\frac{\gamma+1}{2\gamma M_1{}^2 \sin^2\beta - (\gamma-1)}\right]^{\left(\frac{1}{\gamma-1}\right)} \tag{2.66a}$$

For $\gamma = 7/4$, then the relation for total pressure ratio will be

$$\frac{P_{02}}{P_{01}} = \left[\frac{6M_1{}^2 \sin^2\beta}{M_1{}^2 \sin^2\beta + 5}\right]^{\frac{7}{2}} \left[\frac{6}{7M_1{}^2 \sin^2\beta - 1}\right]^{\left(\frac{5}{2}\right)} \tag{2.66b}$$

Figure 2.24 illustrates the relation between M_1, β and δ for oblique shock wave for the case of $\gamma = 1.4$. Figure 2.25 illustrates the downstream Mach number M_2 for oblique shock wave also for $\gamma = 1.4$.

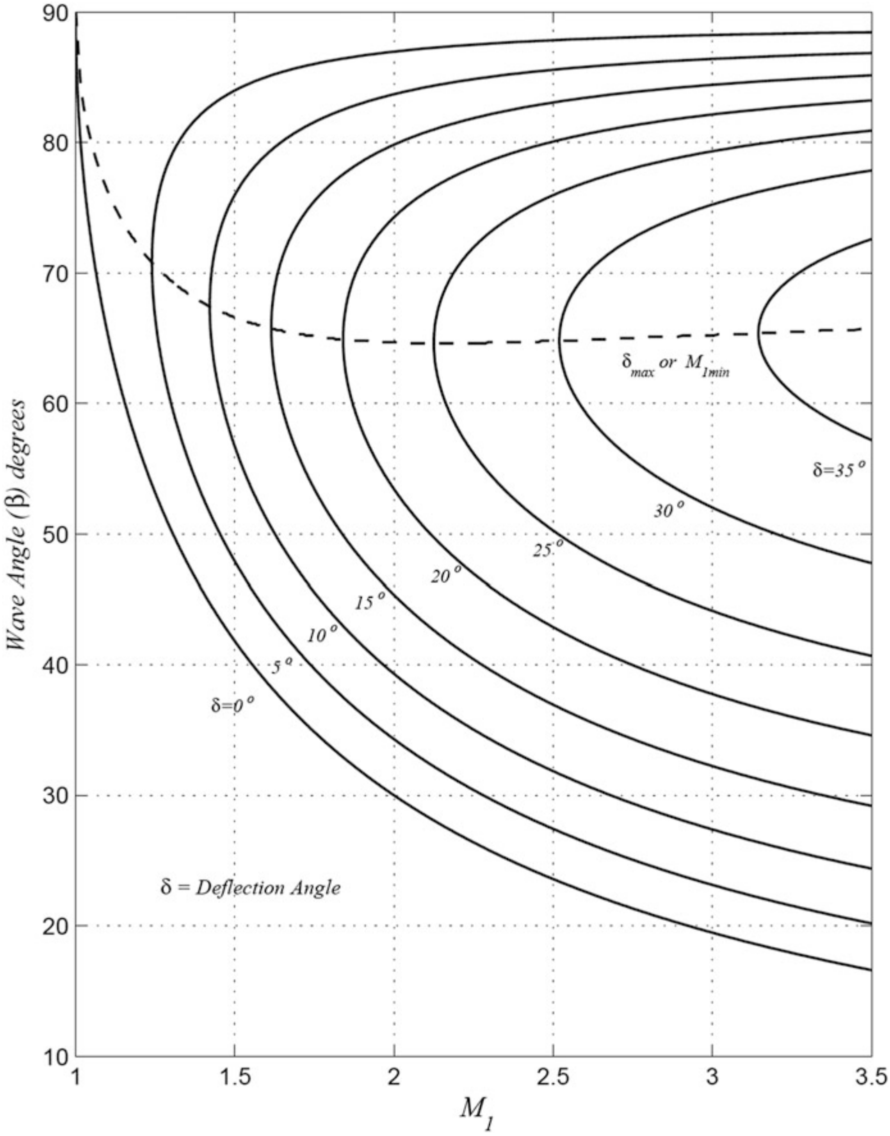


Fig. 2.24 Relation between M_1, β and δ for oblique shock wave $\gamma = 1.4$

Example 2.11 An oblique shock wave has the following upstream static conditions and Mach number: $P_1 = 150$ kPa, $T_1 = 500$ K, $M_1 = 1.605$ and a shock angle $\beta = 60^0$. It is required to calculate:

1. Upstream velocity (V_1), deflection angle (δ)
2. Downstream Mach number (M_2)

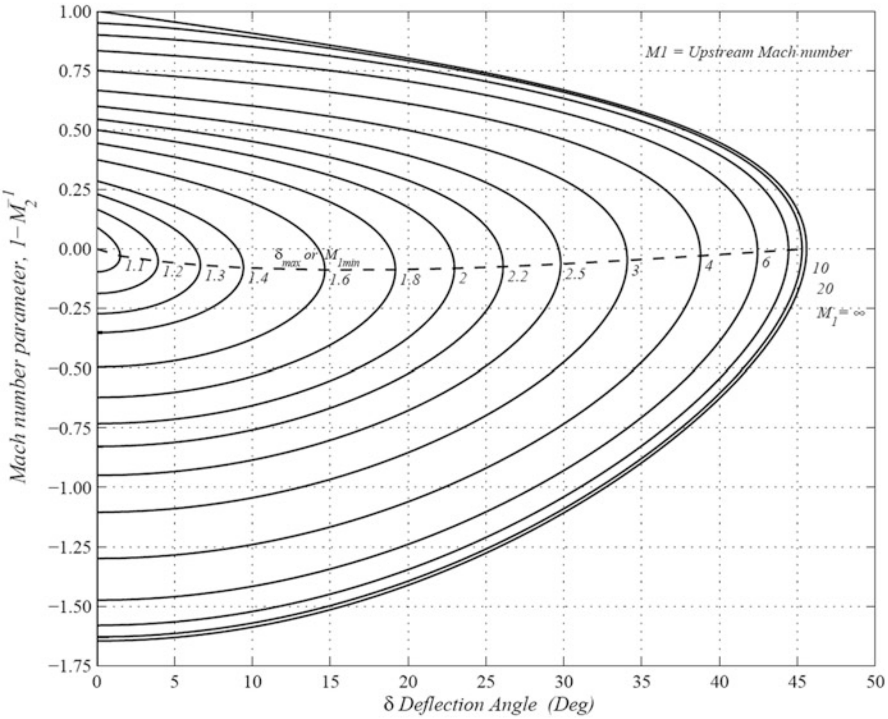


Fig. 2.25 Downstream Mach number M_2 for oblique shock wave $\gamma = 1.4$

3. Static and total temperatures (T_2, T_{01}, T_{02})
4. Normal and tangential velocity components upstream and downstream of the oblique shock $(V_{1n}, V_{1t}, V_{2n}, V_{2t})$
5. Static and total pressures (P_2, P_{01}, P_{02})

Solution

$$V_1 = M_1 a_1 = M_1\sqrt{\gamma R T_1} = 1.609 \times \sqrt{1.4 \times 287 \times 500}$$
$$V_1 = 719.4\ \ m/s$$

The deflection angle is calculated from Eq. (2.61a):

$$\tan\delta = \frac{2\cot 60\left[(1.605\sin 60)^2 - 1\right]}{(2.4)(1.605)^2 - 2\left[(1.605\sin 60)^2 - 1\right]} = 0.24916$$
$$\delta = 14^\circ$$

From Eq. (2.62b)

$$M_2{}^2 \sin^2(60-14) = \frac{0.4 \times (1.604 \sin 60)^2 + 2}{2.8 \times (1.604 \sin 60)^2 - 0.4}$$

$$M_2{}^2 = 1.06968$$

$$M_2 = 1.042$$

From Eq. (2.64a)

$$\frac{T_2}{T_1} = \frac{[2.8 \times 1.932 - 0.4][0.4 \times 1.932 + 2]}{(2.4)^2 \times 1.932} = 1.2482$$

$$T_2 = 624.1 \text{ K}$$

Isentropic relation at inlet gives

$$T_{01} = T_1\left(1 + \frac{\gamma - 1}{2} M_1{}^2\right) = 500 \times 1.5152 = 757.6 \ K = T_{02}$$

$$V_{1n} = V_1 \sin\beta = 719.4 \times \sin 60 = 623 \text{ m/s}$$

$$V_{1t} = V_1 \cos\beta = 359.7 \text{ m/s}$$

$$V_2 = M_2 a_2 = M_2\sqrt{\gamma R T_2 =} 1.034 \times \sqrt{1.4 \times 287 \times 624} = 517.8 \text{ m/s}$$

$$V_2 = 517.8 \text{ m/s}$$

$$V_{2n} = V_2 \sin(\beta - \delta) = 372.5 \text{ m/s}$$

$$V_{2t} = V_2 \cos(\beta - \delta) = 359.7 \equiv V_{1t}$$

From Eqs. (2.63a) and (2.63b),

$$\frac{P_2}{P_1} = \left[\frac{2\gamma M_1{}^2 \sin^2\beta - (\gamma - 1)}{\gamma + 1}\right] = \left(\frac{7M_1{}^2 \sin^2\beta - 1}{6}\right)$$

$$\frac{P_2}{P_1} = \frac{7 \times (1.605 \times \sin 60)^2 - 1}{6} = 2.0874$$

$$P_2 = 313.1 \quad \text{kPa}$$

$$P_{01} = P_1\left(1 + \frac{\gamma - 1}{2} M_1{}^2\right)^{\frac{\gamma}{\gamma-1}} = 150 \times \left[1 + 0.2 \times (1.605)^2\right]^{3.5} = 642.3 \text{ kPa}$$

From Eqs. (2.66a) and (2.65b),

$$\frac{P_{02}}{P_{01}} = \left[\frac{6M_1{}^2 \sin^2\beta}{M_1{}^2 \sin^2\beta + 5}\right]^{\frac{7}{2}} \left[\frac{6}{7M_1{}^2 \sin^2\beta - 1}\right]^{\left(\frac{5}{2}\right)}$$

$$\frac{P_{02}}{P_{01}} = \left[\frac{6 \times 1.932}{1.932 + 5}\right]^{\frac{7}{2}} \left[\frac{6}{7 \times 1.932 - 1}\right]^{\left(\frac{5}{2}\right)} = 0.96$$

$$P_{02} = 0.96 P_{01} = 616.6 \text{ kPa}$$

As a comment here, oblique shock wave has the following features:

- Downstream flow is maintained supersonic.
- Both downstream static pressure and temperature are increased.
- Total temperature is kept constant while downstream total pressure is slightly reduced.

2.5 Rayleigh Flow Equations

Rayleigh flow resembles the case of a steady one-dimensional flow with *heat transfer*. Thus, it is appropriate to treat the flow in combustion chambers as a Rayleigh flow case. Consider the fluid flow in Fig. 2.26. No work exchange while heat is added (${}_1Q_2$).

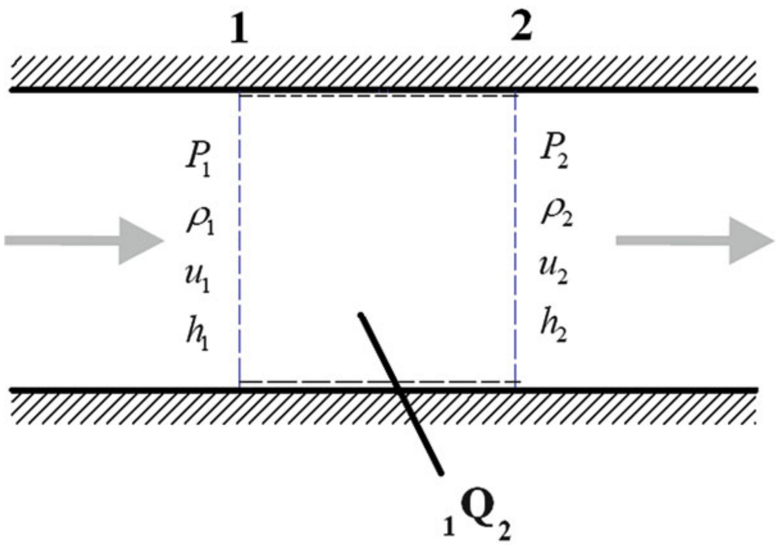


Fig. 2.26 Steady one-dimensional frictionless flow in a constant-area duct with heat transfer

The governing equations are

Continuity equation $\frac{\dot{m}}{A} = \rho_1 \ u_1 = \rho_2 \ u_2$

Momentum equation $p_1 + \rho_1 \ u_1{}^2 = p_2 + \rho_2 \ u_2{}^2$

Energy equation $\mathrm{h}_{02} \equiv h_2 + \frac{V_2{}^2}{2} = h_{01} + {}_1Q_2 = h_1 + \frac{V_1{}^2}{2} + {}_1Q_2$

Equation of state $\frac{p_1}{\rho_1 T_1} = \frac{p_2}{\rho_2 T_2}$

For a perfect gas, the momentum equation can be rewritten as

$$p_1\left(1 + \frac{\rho_1 \, u_1^2}{p_1}\right) = p_2\left(1 + \frac{\rho_2 \, u_2^2}{p_2}\right)$$

or

$$\frac{P_2}{P_1} = \left[\frac{1 + \gamma M_1{}^2}{1 + \gamma M_2{}^2}\right] \tag{2.67}$$

From continuity equation

$$\rho_1 \, u_1 = \rho_2 \, u_2$$
$$\frac{P_1 M_1}{\sqrt{T_1}} = \frac{P_2 M_2}{\sqrt{T_2}}$$

Thus

$$\frac{T_2}{T_1} = \left(\frac{1 + \gamma M_1{}^2}{1 + \gamma M_2{}^2}\right)^2 \frac{M_2{}^2}{M_1{}^2} \tag{2.68}$$

Since the total and static temperatures are related by the relation:

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$

Then

$$\frac{T_{02}}{T_{01}} = \left(\frac{1 + \gamma M_1{}^2}{1 + \gamma M_2{}^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \left(\frac{1 + \frac{\gamma - 1}{2} M_2{}^2}{1 + \frac{\gamma - 1}{2} M_1{}^2}\right) \tag{2.69}$$

Similarly, the total pressure and static density ratios may be expressed as

$$\frac{P_{02}}{P_{01}} = \frac{1 + \gamma {M_1}^2}{1 + \gamma {M_2}^2} \left(\frac{1 + \frac{\gamma - 1}{2} {M_2}^2}{1 + \frac{\gamma - 1}{2} {M_1}^2} \right)^{\frac{\gamma}{\gamma - 1}} \tag{2.70}$$

$$\frac{\rho_2}{\rho_1} = \left(\frac{M_1}{M_2} \right)^2 \left(\frac{1 + \gamma {M_2}^2}{1 + \gamma {M_1}^2} \right) \tag{2.71}$$

The downstream Mach number is expressed by the relation:

$$M_2^2 = \frac{(\gamma - 1) M_1^2 + 2}{2\gamma M_1^2 - (\gamma - 1)} \tag{2.72}$$

At critical conditions, these relations will be reduced to

$$\frac{P}{P*} = \left[\frac{\gamma + 1}{1 + \gamma M^2} \right] \tag{2.73}$$

$$\frac{T}{T*} = \frac{M^2 (\gamma + 1)^2}{\left(1 + \gamma M^2\right)^2} \tag{2.74}$$

$$\frac{T_0}{T_0^*} = \frac{2(1 + \gamma) M^2}{\left(1 + \gamma M^2\right)^2} \left(1 + \frac{\gamma - 1}{2} M^2 \right) \tag{2.75}$$

$$\frac{P_0}{P_0^*} = \frac{(1 + \gamma)}{\left(1 + \gamma M^2\right)} \left(\frac{1 + \frac{\gamma - 1}{2} M^2}{\frac{\gamma + 1}{2}} \right)^{\frac{\gamma}{\gamma - 1}} \tag{2.76}$$

$$\frac{\rho}{\rho^*} = \frac{1 + \gamma M^2}{(1 + \gamma) M^2} \tag{2.77}$$

Example 2.12 The combustion chamber in a ramjet engine has the following characteristics:

$$T_{01} = 360 \ \text{K}, \ \ T_{02} = 1440 \ \text{K}, \ \ M_2 = 0.9.$$

It is required to calculate:

- The inlet Mach number M_1
- The amount of heat added

Assume that $\gamma = 1.3$ and $R = 287$ J/kg.K

Solution
From Eq. (2.69)

$$M_1{}^2\left(1+\frac{\gamma-1}{2}M_1{}^2\right)=\frac{T_{01}}{T_{02}}\frac{M_2{}^2\left(1+\frac{\gamma-1}{2}M_2{}^2\right)}{\left(1+\gamma M_2{}^2\right)^2}\left(1+\gamma M_1{}^2\right)^2$$

$$M_1{}^4\left(\gamma^2 A-\frac{\gamma-1}{2}\right)+M_1{}^2(2\gamma A-1)+A=0$$

$$\text{Where}\quad A=\frac{T_{01}}{T_{02}}\frac{M_2{}^2\left(1+\frac{\gamma-1}{2}M_2{}^2\right)}{\left(1+\gamma M_2{}^2\right)^2}=\frac{360}{1440}\frac{(0.9)^2\left[1+0.15\times(0.9)^2\right]}{\left[1+1.3\times(0.9)^2\right]^2}=0.0538823$$

$$M_1{}^4(-0.0589389)+M_1{}^2(-.0859906)+0.0538823=0$$

$$M_1{}^2=0.0623939$$

$$\text{Thus}\qquad M_1=0.24979$$

Since

$$Cp=\frac{\gamma R}{\gamma-1}=\frac{1.3\times 287}{0.3}=1243.67\quad \text{J/kg.K}$$

The heat added is then from energy equation:

$$Q=h_{02}-h_{01}=Cp(T_{02}-T_{01})=1243.67(1440-360)=1{,}343{,}160\quad \text{J/kg.K}$$

$$Q=1343.16\text{kJ/kg.K}$$

Example 2.13 The combustion chamber in a turbojet engine has the following inlet conditions:

$T_{01}=500$ K, $P_{01}=15$ *bar*, $M_1=0.15, \dot{m}_1=200$ kg/s, the exit temperature is $T_{02}=1500$ K, and fuel-to-air ratio is $f=0.0273$. Calculate:

1. Inlet area of combustor (A_1)
2. The total pressure ratio across the combustor
3. Mach number and area at combustor outlet (M_2, A_2)

Assume that $\gamma=1.4$ and $R=287$ J/kg.K

Solution
The static temperature and pressure at inlet are

$$T_1=\frac{T_{01}}{1+\frac{\gamma-1}{2}M^2}=\frac{500}{1+0.2\times(0.15)^2}=497.8\ \text{K}$$

$$P_1 = \frac{P_{01}}{\left(1 + \frac{\gamma-1}{2}M^2\right)^{\frac{\gamma}{\gamma-1}}} = \frac{15}{\left(1 + 0.2 \times (0.15)^2\right)^{3.5}} = 14.766 \ \ bar$$

$$V_1 = M_1 a_1 = M_1\sqrt{\gamma RT_1} = 0.15 \times \sqrt{1.4 \times 287 \times 500} = 67.23 \ \ \mathrm{m/s}$$

$$\rho_1 = \frac{P_1}{RT_1} = 10.335 \ \ \mathrm{kg/m^3}$$

$$A_1 = \frac{\dot{m}_1}{\rho_1 V_1} = \frac{200}{10.335 \times 67.23} = 0.288 \ \ \mathrm{m^2}$$

To evaluate Mach number at combustor outlet, we can use Eq. (2.69):

$$\frac{T_{02}}{T_{01}} = 3 = \left(\frac{1 + 1.4 \times [0.15]^2}{1 + 1.4 \times M_2{}^2}\right)^2 \left(\frac{M_2}{0.15}\right)^2 \left(\frac{1 + 0.2M_2{}^2}{1 + 0.2 \times [0.15]^2}\right)$$

$$3 = 47.08 \times \frac{M_2{}^2 \times \left(1 + 0.2M_2{}^2\right)}{\left(1 + 1.4 \times M_2{}^2\right)^2}$$

The above equation is solved by trial and error to obtain $M_2 = 0.277$

Now the total pressure ratio across the combustor is obtained from Eq. (2.70); thus,

$$\frac{P_{02}}{P_{01}} = \frac{1 + 1.4 \times (0.15)^2}{1 + 1.4 \times (.0.277)^2} \left(\frac{1 + 0.2 \times (0.277)^2}{1 + 0.2 \times (0.15)^2}\right)^{3.5} = \frac{1.0315}{1.1074} \times 1.0383$$

$$\frac{P_{02}}{P_{01}} = 0.9671$$

The static properties at the outlet of combustion chamber are obtained from relations (2.67), (2.68), and (2.71).

$$\frac{P_2}{P_1} = \left[\frac{1 + 1.4 \times 0.15^2}{1 + 1.4 \times 0.277^2}\right] = 0.93144$$

$$\frac{T_2}{T_1} = \left(\frac{1 + 1.4 \times 0.15^2}{1 + 1.4 \times 0.277^2} \times \frac{0.277}{0.15}\right)^2 = 2.9586$$

$$\frac{\rho_2}{\rho_1} = \left(\frac{0.15}{0.277}\right)^2 \left(\frac{1 + 1.4 \times 0.277^2}{1 + 1.4 \times 0.15^2}\right) = 0.3148$$

$$\rho_2 = 3.2535 \ \ \mathrm{kg/m^3}, \quad T_2 = 1472.8\mathrm{K}$$

$$V_2 = M_2\sqrt{\gamma RT_2} = 213 \ \ \mathrm{m/s}$$

The outlet mass flow rate is $\dot{m}_2 = \dot{m}_1(1 + f) = 200(1 + 0.0273) = 205.46 \ \ \mathrm{kg/s}$

The outlet area of combustor is then

$$A_2 = \frac{\dot{m}_2}{\rho_2 V_2} = \frac{205.46}{3.2535 \times 213} = 0.2965 \ \text{m}^2$$

2.6 The Standard Atmosphere

For a fluid in rest without shearing stresses, any elementary fluid element will be subjected to two types of forces, namely, *surface forces* due to the pressure and a *body force* equal to the weight of the element. Force balance will yield the following relation:

$$\nabla p = -\gamma \overline{k}$$

where γ is the specific weight of fluid and $\overline{k}$ is the unit vector in the positive vertical direction (opposite to the gravitational force) and

$$\gamma = \rho g$$

Thus,

$$\frac{\partial p}{\partial x} = 0, \quad \frac{\partial p}{\partial y} = 0, \quad \frac{\partial p}{\partial z} = -\gamma = -\rho g \tag{2.78}$$

The first two derivatives in Eq. (2.78), show that the pressure does not depend on x or y. Thus, as we move from one point to another in a horizontal plane (any plane parallel to the x–y plane), the pressure does not change. Since p depends only on z, the last of Eq. (2.78) can be written as the ordinary differential equation

$$\frac{dp}{dz} = -\gamma = -\rho g \tag{2.79}$$

Equation (2.79) is the fundamental equation for fluids at rest and can be used to determine how pressure changes with elevation. This equation indicates that the pressure gradient in the vertical direction is negative; that is, the pressure decreases as we move upward in a fluid at rest. For the Earth's atmosphere where the variations in heights are large, on the order of thousands of feet, attention must be given to the variation in the specific weight (γ). Since air may be considered an ideal (or perfect) gas, its equation of state ($p = \rho RT$) is used.

This relationship can be combined with Eq. (2.79) to give

$$\frac{dp}{dz} = -\frac{gp}{RT}$$

and by separating variables

$$\int_{p_1}^{p_2} \frac{dp}{p} = \ln \frac{p_2}{p_1} = -\frac{g}{R} \int_{z_1}^{z_2} \frac{dz}{T} \tag{2.80}$$

where g and R are assumed to be constant over the range of elevation involved.

Equation (2.80) relates to the variation in pressure in the Earth's atmosphere.

Ideally, we would like to have measurements of pressure versus altitude over the specific range of altitude. However, this type of information is usually not available. Thus, a "standard atmosphere" has been determined that can be used in the design of aircraft and rockets. The concept of a standard atmosphere was first developed in the 1920s, and since that time many US and international committees and organizations have pursued the development of such a standard. The currently accepted standard atmosphere is based on a report published in 1962 and updated in 1976 [11, 12], defining the so-called *US standard atmosphere*, which is an idealized representation of middle-latitude, year-round mean conditions of the Earth's atmosphere. Figure 2.27 shows the temperature profile for the US standard atmosphere.

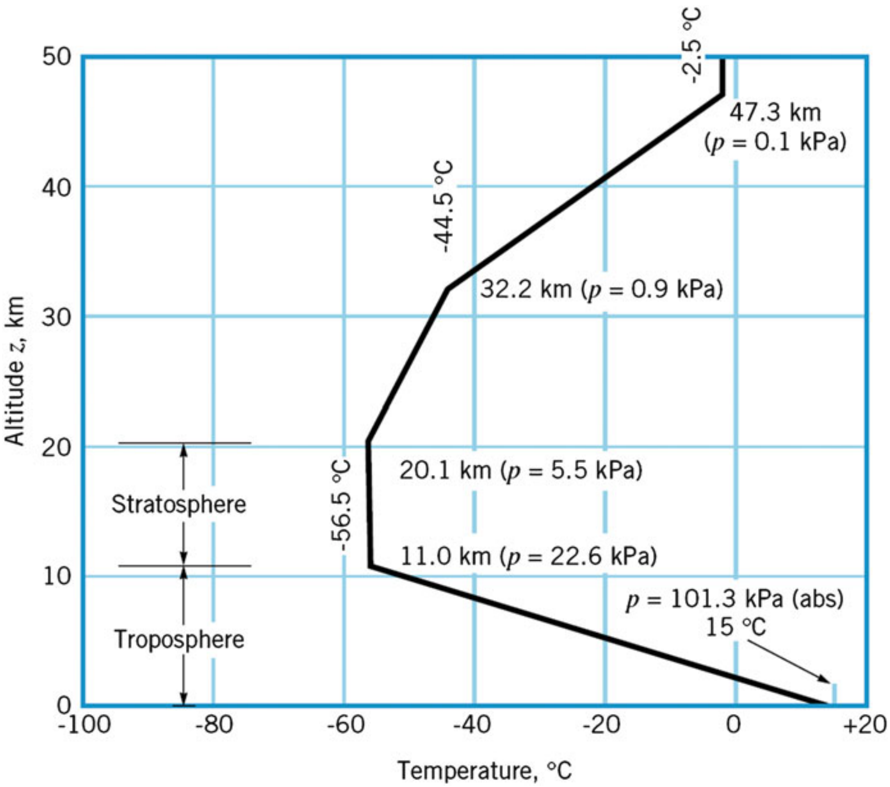


Fig. 2.27 Variation of temperature with altitude in the US Standard Atmosphere

As is shown in this figure, the temperature decreases with altitude in the region nearest the Earth's surface (*troposphere)*, then becomes essentially constant in the next layer (*stratosphere)*, and subsequently starts to increase in the next layer. Since the temperature variation is represented by a series of linear segments, it is possible to integrate Eq. (2.80) to obtain the corresponding pressure variation. For example, in the troposphere, which extends to an altitude of about 11 km ($\simeq 36,000\ ft$), the temperature variation is of the form

$$T = T_a - \beta z \tag{2.81a}$$

where T_a is the temperature at sea level ($z = 0$) and β is the *lapse rate (*the rate of change of temperature with elevation), 0.00356616 °F/ft, or 0.0019812 °C/ft. For the standard atmosphere in the troposphere, and if (z) represents altitude in feet, then Eq. (2.81a) may be further expressed as

$$T = 518.67 - 0.00356616\ z\,°\mathrm{R} \tag{2.81b}$$

$$T = 288.15 - 0.0019812\ z\,°\mathrm{K} \tag{2.81c}$$

$$t = 59 - 0.00356616\ z\,°\mathrm{F} \tag{2.81d}$$

$$t = 15 - 0.0019812\ z\ °\mathrm{C} \tag{2.81e}$$

Equation (2.81a) together with Eq. (2.80) yields

$$p = p_a\left(1 - \frac{\beta z}{T_a}\right)^{\frac{g}{\beta R}} \tag{2.82}$$

where (p_a) is the absolute pressure at $z = 0$. With $p_a = 101.33$ kPa, $T_a = 288.15$ K and $g = 9.807\ \mathrm{m/s^2}$, and with the gas constant $R = 286.9\mathrm{J/kg.K}$. The pressure variation throughout the troposphere can be determined from Eq. (2.82). This calculation shows that at the outer edge of the troposphere, where the temperature is −56.5 °C, the absolute pressure is about 23 kPa. It is to be noted that modern jetliners cruise at approximately this altitude.

For the stratosphere atmospheric layer (between 11.0 and 20.1 km), the temperature has a constant value (*isothermal* conditions) which is −56.5 °C (or −69.7 °F, 389.97 °R, 216.65°K).

It then follows from Eq. (2.80), the pressure-elevation relationship expressed as

$$p = p_0 \exp\left[-\frac{g(z - z_0)}{RT_0}\right] \tag{2.83}$$

where p_0, T_0, and z_0 are the pressure, temperature, and altitude of the lower edge of the stratosphere (23 kPa, −56.5 °C, 36,000 ft).

Figure 2.28 illustrates the variation of temperature with altitude. Figure 2.29 illustrates flight altitudes appropriate to different aircrafts. Table 2.9 defines the properties of the Earth standard atmosphere [12].

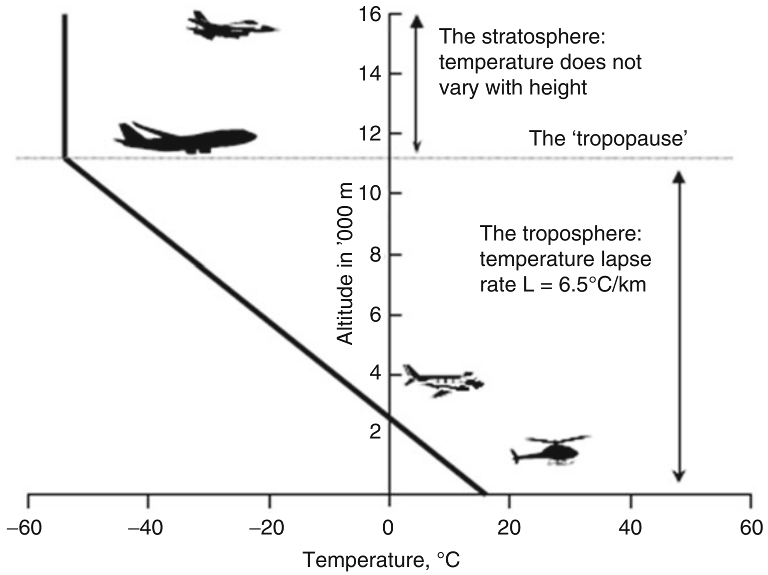


Fig. 2.28 The ISA: variation of temperature with altitude

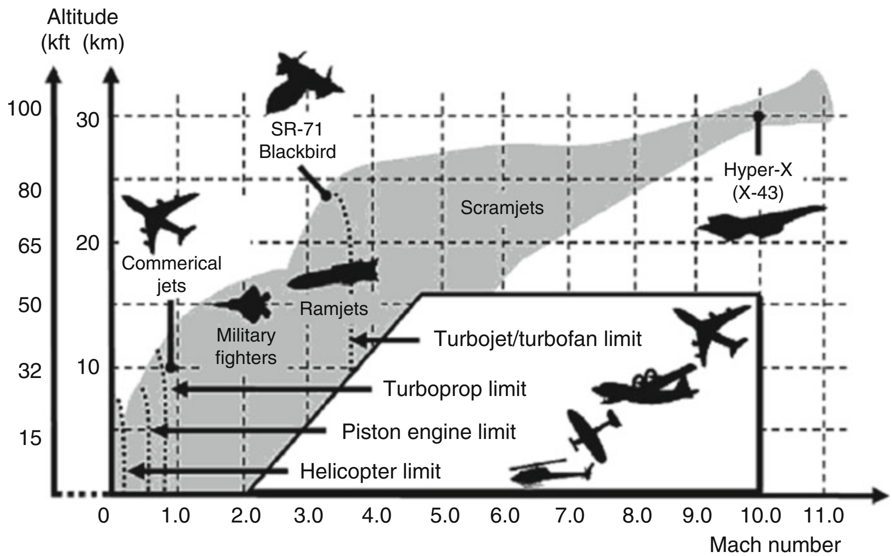


Fig. 2.29 Flight envelopes of aircraft with different engine types

Table 2.9 Properties of the Earth's standard atmosphere (Ref. [12])

Altitude (m)	Temperature (K)	Pressure ratio	Density (kg/m^3)
0 (sea level)	288.150	1.0000	1.2250
1000	281.651	8.87×10^{-1}	1.11117
3000	268.650	6.6919×10^{-1}	0.90912
5000	255.65	5.3313×10^{-1}	0.76312
10,000	223.252	2.6151×10^{-1}	4.1351×10^{-1}
25,000	221.552	2.5158×10^{-2}	4.0084×10^{-2}
50,000	270.650	7.8735×10^{-4}	1.0269×10^{-3}
75,000	206.650	2.0408×10^{-5}	3.4861×10^{-5}
100,000	195.08	3.1593×10^{-7}	5.604×10^{-7}
130,000	469.27	1.2341×10^{-8}	8.152×10^{-9}
160,000	696.29	2.9997×10^{-9}	1.233×10^{-9}
200,000	845.56	8.3628×10^{-10}	2.541×10^{-10}
300,000	976.01	8.6557×10^{-11}	1.916×10^{-11}
400,000	995.83	1.4328×10^{-11}	2.803×10^{-12}
600,000	999.85	8.1056×10^{-13}	2.137×10^{-13}
1,000,000	1000.00	7.4155×10^{-14}	3.561×10^{-15}

Problems

2.1 Calculate the Mach number for a flight vehicle flying at a speed of 10,000 km/h at the following altitudes:
sea level – 10,000 m –25,000 m – 50,000 m – 100,000 m – 200,000 m – 400,000 m – 1,000,000 m.

2.2 Describe probe-and-drogue air-to-air refueling system.

2.3 What are the advantages of refueling a military aircraft?

2.4 Tornado GR4 refueling from the drogue of an RAF VC10 tanker as shown in figure Problem 2.4 at the rate of 600 gal/min of fuel having a specific gravity of 0.75. The inside diameter of hose is 0.14 m. The fluid pressure at the entrance of the fighter plane is 40 kPa gage. What additional thrust does the plane need to develop to maintain the constant velocity it had before the hookup?

Figure (Problem 2.4)

2.5 Fighter airplane is *refueling from the DC-10 tanker* as shown in figure Problem 2.5 at the rate of 700 gal/min of fuel having a specific gravity of 0.72. The inside diameter of hose is 0.13 m. The fluid pressure at the entrance of the fighter plane is 45 kPa gage. What additional thrust does the plane need to develop to maintain the constant velocity it had before the hookup?

Figure (Problem 2.5)

2.6 A jet plane is on the runway after touching down. The pilot puts into play movable vanes to achieve a reverse thrust from his two engines. Each engine takes in 50 kg of air per second, The fuel-to-air ratio is 1–40. If the exit velocity of the combustion products is 800 m/s relative to the plane, what is the total reverse thrust of the airplane if it is moving at a speed of 180 km/h ? The exit jets are close to atmospheric pressure.

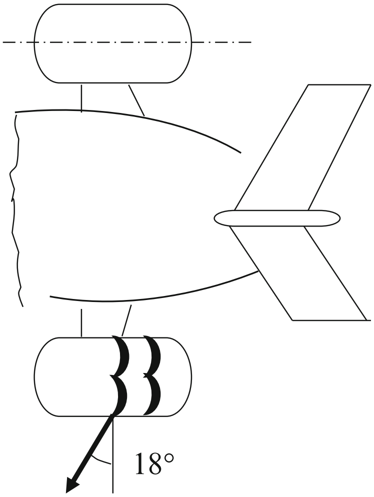


Figure (Problem 2.6)

2.7 A fighter plane is climbing at an angle θ of 60^o at a constant speed of 900 Km/h. The plane takes in air at a rate of 450 kg/s. The fuel-to-air ratio is 2 %. The exit speed of the combustion products is 1800 m/s relative to the plane. If the plane changes to an inclination angle θ of 20^o, what will be the speed of the plane when it reaches uniform speed? The new engine settings are such that the same amount of air taken in and the exhaust speed relative to the plane are the same. The plane weights 150 kN. The drag force is proportional to the speed squared of the plane.

2.8 If the fighter plane in problem (2.7) is climbing also at an angle $\theta = 60^0$ but at a constant acceleration (a). The weight, thrust, and drag forces are 150, 715, and 500 kN, respectively. Calculate the acceleration (a). Next, the plane changes its angle to 20^0, while the air mass flow rate is 450 kg/s, exhaust speed of gases is 1800 m/s, and fuel-to-air ratio is 2 %. For the same value of acceleration calculated above and if the drag force is proportional to the speed squared of the plane, what will be the aircraft velocity.

2.9 Figure Problem (2.9) illustrates supersonic jet fighter aircraft Mikoyan–Gurevich MiG-21. One type of its armament is Nudelman–Rikhter NR-30, twin-barrel 23 mm GSh-23 cannon. It had a muzzle velocity of 800 m/s. Each bullet (cartridge) is 30 × 155 mm and has a mass of 400 grams and a rate of fire of 900 cycles per minute. What is the additional thrust needed to keep a constant aircraft speed of 600 km/h? (Neglect the change of mass of aircraft.)

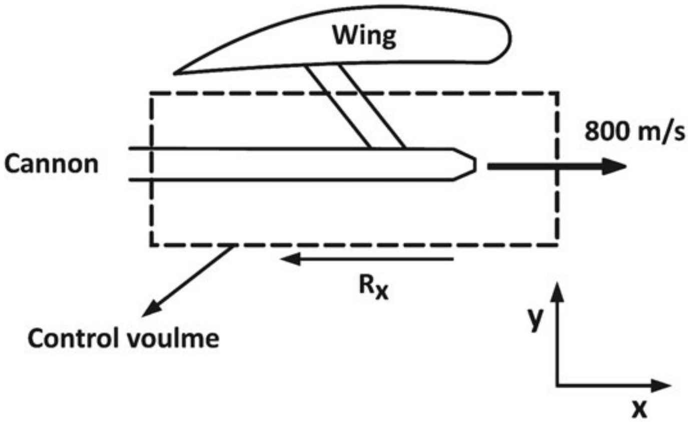


Figure (Problem 2.9)

2.10 If the specific heat at constant pressure is expressed by the relation

$$\frac{C_p}{R} = \frac{7}{2} + \left[\frac{T_R/(2T)}{\sinh\{T_R/(2T)\}}\right]^2$$

where the reference temperature $T_R = 3060\,\text{K}$, *plot* C_p, C_v, γ and h of air as a function of T/T_R over the range 300 to 3800 K.

2.11 A rocket engine uses nitric acid as oxidizer. The oxidizer flow rate is 2.60 kg/s and a fuel flow of 0.945 kg/s. Thus, the propellant flow rate is 3.545 kg/s. If the flow leaves the nozzle at 1900 m/s through an area of 0.012 m^2 with a pressure of 110 kPa, what is the thrust of the rocket motor?

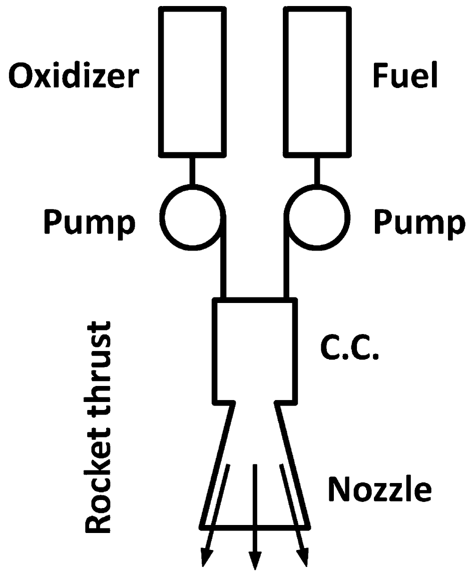


Figure (Problem 2.11)

2.12 A rocket is designed to have four nozzles, each canted at 30^o with respect to the rocket's centerline. The gases exit at 2200 m/s through the exit area of 1.2 m^2. The density of the exhaust gases is 0.3 kg/m^3, and the exhaust pressure is 55 kPa. The atmospheric pressure is 12 kPa. Determine the thrust on the rocket.

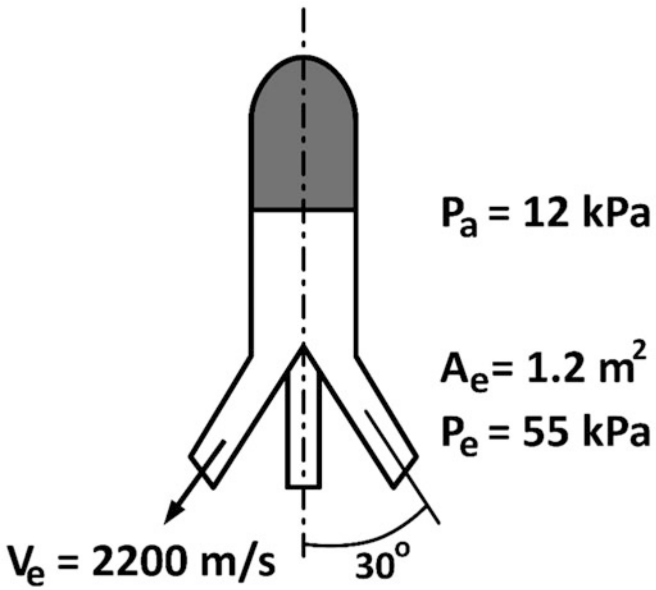


Figure (Problem 2.12)

2.13 A convergent nozzle has an exit area of 500 mm^2. Air enters the nozzle with a stagnation pressure of 1000 kPa and a stagnation temperature of 360 K. Determine the mass rate of flow for back pressures of 850, 528, and 350 kPa, assuming isentropic flow.

2.14 A converging–diverging nozzle has an exit area to throat area ratio of 2. Air enters this nozzle with a stagnation pressure of 1000 kPa and a stagnation temperature of 460 K.

The throat area is 500 mm^2. Determine the mass rate of flow, exit pressure, exit temperature, exit Mach number, and exit velocity for the following conditions:

(a) Sonic velocity at the throat, diverging section acting as a nozzle (corresponds to point G in Fig. 2.18)
(b) Sonic velocity at the throat, diverging section acting as a diffuser (corresponding to point C in Fig. 2.18)

2.15 An oblique shock wave has the following data

$$M_1 = 3.0,\ \ P_1 = 1\ \text{atm},\ \ \ T_1 = 288\ \ \text{K},\ \ \gamma = 1.4,\ \ \delta = 20^\circ$$

(a) Compute shock wave angle (weak)
(b) Compute P_{02}, T_{02}, P_2, T_2, M_2 behind shock

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Fundamentals of Aircraft and Rocket Propulsion
El-Sayed, A.F.
2016, XXI, 1010 p. 571 illus., 242 illus. in color. With online files/update., Hardcover
ISBN: 978-1-4471-6794-5