Chapter 2
The Linear Canonical Transformation: Definition and Properties

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Abstract In this chapter we introduce the class of linear canonical transformations, which includes as particular cases the Fourier transformation (and its generalization: the fractional Fourier transformation), the Fresnel transformation, and magnifier, rotation and shearing operations. The basic properties of these transformations—such as cascadability, scaling, shift, phase modulation, coordinate multiplication and differentiation—are considered. We demonstrate that any linear canonical transformation is associated with affine transformations in phase space, defined by time-frequency or position-momentum coordinates. The affine transformation is described by a symplectic matrix, which defines the parameters of the transformation kernel. This alternative matrix description of linear canonical transformations is widely used along the chapter and allows simplifying the classification of such transformations, their eigenfunction identification, the interpretation of the related Wigner distribution and ambiguity function transformations, among many other tasks. Special attention is paid to the consideration of one- and two-dimensional linear canonical transformations, which are more often used in signal processing, optics and mechanics. Analytic expressions for the transforms of some selected functions are provided.

2.1 Introduction

In this chapter we introduce the class of linear canonical transformations and study some of the basic properties of these transformations. In one dimension, this class forms a three-parameter class of linear integral transformations, and includes such operations as the well-known Fourier transformation, the Fresnel transformation

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(used in optics, for instance, to describe the paraxial propagation of light in free space), and simple operations like scaling and multiplication by a quadratic-phase function. In the D-dimensional case, the class has $D(2D + 1)$ free parameters and includes additional operations, like rotation and shearing. Although we focus on the two-dimensional case, with an occasional restriction to one dimension, many of the results will hold for the general D-dimensional case. And although the results are normally presented using higher-dimensional vectors and matrices, the reduction to one dimension is throughout straightforward.

After a formal definition of the linear canonical transformation in Sect. 2.2, we turn to a description in a so-called phase space in Sect. 2.3, where a matrix is introduced with which the transformation is parameterized, and we derive some of the transformation's basic properties; this section contains also a number of linear canonical transformations for the one-dimensional case, which are more easily to visualize. Special cases of the transformation in two (and more) dimensions are considered in Sect. 2.4, while decompositions of a general linear canonical transformation into cascades of simpler transformations are studied in Sect. 2.5. In Sect. 2.6 we derive the linear canonical transforms of some selected functions, like a Gaussian signal, a harmonic signal, a periodic signal (with a short detour to Talbot imaging) and the Hermite–Gauss modes; the transformation of these Hermite–Gauss modes leads to a general class of Hermite–Gaussian type modes, with the Laguerre–Gauss modes as a special case, which are very important in optics. The eigenvalues of the transformation matrix are discussed in Sect. 2.7; the possible distributions of these eigenvalues lead to a classification of the linear canonical transformation based on simple nuclei, from which eigenfunctions may be derived. The final Sect. 2.8, deals with the effect of the linear canonical transformation on the second-order moments in phase space; we discuss, in particular, moment combinations that are invariant under a linear canonical transformation.

We conclude this Introduction with some remarks about notation. We will throughout denote column vectors by bold-face, lower-case symbols like $\mathbf{r}$ and $\mathbf{q}$, which in the two-dimensional case $D = 2$ read $\mathbf{r} = [x, y]^t$ and $\mathbf{q} = [u, v]^t$, while matrices and submatrices are denoted by bold-face, upper-case symbols, like $\mathbf{T} = [\mathbf{A}, \mathbf{B}; \mathbf{C}, \mathbf{D}]$; transposition of vectors and matrices is denoted by the superscript ^{t}. The identity matrix is denoted by $\mathbf{I}$. Submatrices may carry subscripts, like in $\mathbf{M} = [\mathbf{M_{rr}}, \mathbf{M_{rq}}; \mathbf{M_{qr}}, \mathbf{M_{qq}}]$. Scalars, including the entries of a 2×2 matrix, will appear in normal face, like in $\mathbf{T} = [a, b; c, d]$ and $\mathbf{M_{rq}} = [m_{xu}, m_{xv}; m_{yu}, m_{yv}]$. Complex conjugation is denoted by the superscript *, and the combined action of conjugation and transposition for vectors and matrices is denoted by the superscript †: $\mathbf{U}^\dagger = \mathbf{U}^{*t}$. Operators appear in calligraphic style, like $\mathscr{L}$ and $\mathscr{F}$, and the symbols $\mathbb{M}$ and $\mathbb{D}$ are used to denote the special operators related to coordinate multiplication and differentiation, respectively; brackets in connection with operators will be used to avoid ambiguities, if necessary. While a normal face font is used for signals and functions, a sans serif style is used for the Hermite–Gauss modes $\mathsf{H}_{m,n}(\mathbf{r})$ and Laguerre–Gauss modes $\mathsf{L}_{m,n}(\mathbf{r})$. Unless otherwise stated, all integrations and summations extend from $-\infty$ to $+\infty$, and the short-hand notation d$\mathbf{r}$ in higher-dimensional integrals is used for dx dy. Subscripts $_i$ and $_o$, like in $f_i(\mathbf{r}_i)$, are throughout used to mark input and output signals and coordinates.

Finally: we will very often meet expressions of the form $(i\,\ell)^{1/2}$ or $(\det \mathrm{i}\,\mathbf{L})^{1/2}$. They shall be interpreted as $\exp[\mathrm{i}\,(\frac{1}{4}\pi)\,\mathrm{sgn}(\ell)]\,|\ell|^{1/2}$ or, with $\mathbf{L}$ a $D \times D$ matrix, as $\exp[\mathrm{i}\,(\frac{1}{4}\pi)D\ \mathrm{sgn}(\det \mathbf{L})]\,|\det \mathbf{L}|^{1/2}$.

2.2 Definition of the Linear Canonical Transformation

With the two-dimensional column vector $\mathbf{r}$ defined as $\mathbf{r} = [x, y]^t$, where ^{t} denotes transposition, and $\mathrm{d}\mathbf{r} = \mathrm{d}x\,\mathrm{d}y$, the *linear canonical transformation* of the two-dimensional signal $f(\mathbf{r}), f_i(\mathbf{r}_i) \to f_o(\mathbf{r}_o) = \mathscr{L} f_i(\mathbf{r}_i)$, is defined as

$$f_o(\mathbf{r}_o) = (\det \mathrm{i}^{-1}\mathbf{L}_{io})^{1/2} \int \exp[\mathrm{i}\,\pi(\mathbf{r}_o^t\mathbf{L}_{oo}\mathbf{r}_o - 2\mathbf{r}_i^t\mathbf{L}_{io}\mathbf{r}_o + \mathbf{r}_i^t\mathbf{L}_{ii}\mathbf{r}_i)] f_i(\mathbf{r}_i)\,\mathrm{d}\mathbf{r}_i\ , \quad (2.1)$$

where the 2×2 matrices $\mathbf{L}_{oo}$ and $\mathbf{L}_{ii}$ are symmetric: $\mathbf{L}_{oo} = \mathbf{L}_{oo}^t$ and $\mathbf{L}_{ii} = \mathbf{L}_{ii}^t$. We restrict ourselves to the case that the (not necessarily symmetric) 2×2 matrix $\mathbf{L}_{io}$ is non-singular. Although we will focus on the two-dimensional case, with an occasional restriction to one dimension, most of the results will hold for the general D-dimensional case. The reduction to one dimension is throughout straightforward; the one-dimensional version of Eq. (2.1), for instance, takes the form

$$f_o(x_o) = (\mathrm{i}^{-1}\ell_{io})^{1/2} \int \exp[\mathrm{i}\,\pi(\ell_{oo}x_o^2 - 2\ell_{io}x_i x_o + \ell_{ii}x_i^2)] f_i(x_i)\,\mathrm{d}x_i \quad (\ell_{io} \neq 0)\ . \quad (2.2)$$

Note that in the D-dimensional case, a symmetric $D \times D$ matrix has $\frac{1}{2}D(D+1)$ degrees of freedom, and that the three matrices $\mathbf{L}_{oo}$, $\mathbf{L}_{ii}$ and $\mathbf{L}_{io}$ together thus have $D(2D+1)$ degrees of freedom: 3 for $D = 1$ and 10 for $D = 2$. If $\mathbf{L}_{oo} = \mathbf{L}_{ii}$ and $\mathbf{L}_{io} = \mathbf{L}_{io}^t$, the transformation is *symmetric* and the roles of $\mathbf{r}_o$ and $\mathbf{r}_i$ can be interchanged, as will be discussed in more detail in Sect. 2.3.1.2.

The linear canonical transformation is a *unitary* transformation in the sense that

$$\int f_o(\mathbf{r})\,h_o^*(\mathbf{r})\,\mathrm{d}\mathbf{r} = \int f_i(\mathbf{r})\,h_i^*(\mathbf{r})\,\mathrm{d}\mathbf{r}, \quad (2.3)$$

where * denotes complex conjugation, which relation is known as *Parseval's theorem* for lossless transformations; it yields the energy preservation law for $f(\mathbf{r}) = h(\mathbf{r})$.

We finally remark that an additional phase factor $\exp(i\varphi)$ may be added to the definition (2.1) without changing the main properties of the transformation. This is sometimes done to get a better connection with the actual physical phenomenon that the linear canonical transformation is describing.

Among the many works on the linear canonical transformation, to which we refer for further reading, we mention [7, 26, 32, 44, 46, 47, 51, 71–75].

2.3 Representation of the Linear Canonical Transformation in Phase Space

To represent the linear canonical transformation in phase space, we use the *Wigner distribution* $W(\mathbf{r},\mathbf{q})$ of the signal $f(\mathbf{r})$, defined as [11, 13, 18, 19, 27, 28, 43, 66–69]

$$W(\mathbf{r},\mathbf{q}) = \int f\left(\mathbf{r}+\tfrac{1}{2}\mathbf{r}'\right) f^*\left(\mathbf{r}-\tfrac{1}{2}\mathbf{r}'\right) \exp[-\mathrm{i}\,2\pi\,\mathbf{q}^t\mathbf{r}']\,\mathrm{d}\mathbf{r}' \,. \tag{2.4}$$

The column vector $\mathbf{q}$ can be considered as the frequency variable associated with $\mathbf{r}$. Note that the Wigner distribution is real, $W(\mathbf{r},\mathbf{q}) = W^*(\mathbf{r},\mathbf{q})$, and that a constant phase factor in the signal is no longer visible in the Wigner distribution: the signals $f(\mathbf{r})$ and $f(\mathbf{r})\exp(\mathrm{i}\,\varphi)$ lead to the same Wigner distribution.

We derive the linear canonical transformation in terms of the *phase-space variables* $\mathbf{r}$ and $\mathbf{q}$ by substituting from (2.1) into (2.4) and get the relationship (see Appendix)

$$W_o(\mathbf{r}_o,\mathbf{q}_o) = W_o(\mathbf{A}\mathbf{r}_i+\mathbf{B}\mathbf{q}_i,\mathbf{C}\mathbf{r}_i+\mathbf{D}\mathbf{q}_i) = W_i(\mathbf{r}_i,\mathbf{q}_i)\,, \tag{2.5}$$

where the input variables $(\mathbf{r}_i,\mathbf{q}_i)$ and the output variables $(\mathbf{r}_o,\mathbf{q}_o)$ are related by the simple matrix relation

$$\begin{bmatrix}\mathbf{r}_o\\ \mathbf{q}_o\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix} \equiv \mathbf{T}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{q}_i\end{bmatrix}\,, \tag{2.6}$$

in which the *transformation matrix* $\mathbf{T} = [\mathbf{A},\mathbf{B};\mathbf{C},\mathbf{D}]$ has been introduced, and where the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are related to the matrices $\mathbf{L}_{oo}$, $\mathbf{L}_{io}$ and $\mathbf{L}_{ii}$ by

$$\begin{aligned}&\mathbf{A}=\mathbf{L}_{io}^{-1}\mathbf{L}_{ii}\,,\quad \mathbf{B}=\mathbf{L}_{io}^{-1}\,,\quad \mathbf{C}=\mathbf{L}_{oo}\mathbf{L}_{io}^{-1}\mathbf{L}_{ii}-\mathbf{L}_{io}^t\,,\quad \mathbf{D}=\mathbf{L}_{oo}\mathbf{L}_{io}^{-1}\,,\\ &\mathbf{L}_{oo}=\mathbf{D}\mathbf{B}^{-1}\,,\quad \mathbf{L}_{ii}=\mathbf{B}^{-1}\mathbf{A}\,,\quad \mathbf{L}_{io}=\mathbf{B}^{-1}\,.\end{aligned} \tag{2.7}$$

Note that we can formulate the relationship between the frequency variables $\mathbf{q}$ and the original variables $\mathbf{r}$ also in the form

$$\begin{bmatrix}-\mathbf{q}_i\\ \mathbf{q}_o\end{bmatrix} = \begin{bmatrix}\mathbf{L}_{ii} & -\mathbf{L}_{io}\\ -\mathbf{L}_{io}^t & \mathbf{L}_{oo}\end{bmatrix}\begin{bmatrix}\mathbf{r}_i\\ \mathbf{r}_o\end{bmatrix}\,. \tag{2.8}$$

In terms of the transformation matrix $\mathbf{T}$, the linear canonical transformation can be expressed in the form [26, 33, 40, 44, 74, 75]

$$f_o(\mathbf{r}_o) = \mathscr{L}(\mathbf{T})f_i(\mathbf{r}_i) = (\det \mathrm{i}\,\mathbf{B})^{-1/2} \times \int \exp[\mathrm{i}\,\pi(\mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o - 2\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o + \mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{A}\mathbf{r}_i)]f_i(\mathbf{r}_i)\,\mathrm{d}\mathbf{r}_i\ , \quad (2.9)$$

with $\det\mathbf{B} \neq 0$. In the limiting case $\mathbf{B} \Rightarrow \mathbf{0}$, for which $\mathbf{r}_o \Rightarrow \mathbf{A}\mathbf{r}_i$, see (2.6), we have

$$f_o(\mathbf{r}) = \mathscr{L}(\mathbf{T})f_i(\mathbf{r}) = |\det\mathbf{A}|^{-1/2}\exp[\mathrm{i}\,\pi\,\mathbf{r}^t\mathbf{C}\mathbf{A}^{-1}\mathbf{r}]f_i(\mathbf{A}^{-1}\mathbf{r})\ , \quad (2.10)$$

which follows readily when we write the exponent in (2.9) as

$$\exp[\mathrm{i}\,\pi\,\mathbf{r}_o^t\mathbf{C}\mathbf{A}^{-1}\mathbf{r}_o]\ \exp[\mathrm{i}\,\pi(\mathbf{r}_i - \mathbf{A}^{-1}\mathbf{r}_o)^t\mathbf{B}^{-1}\mathbf{A}(\mathbf{r}_i - \mathbf{A}^{-1}\mathbf{r}_o)]$$

and recall that $(\det \mathrm{i}\,\mathbf{B})^{-1/2}\exp[\mathrm{i}\,\pi\,\mathbf{r}^t\mathbf{B}^{-1}\mathbf{A}\mathbf{r}] \Rightarrow |\det\mathbf{A}|^{-1/2}\,\delta(\mathbf{r})$ when $\mathbf{B} \Rightarrow \mathbf{0}$. The singular case $\det\mathbf{B} = 0$, but $\mathbf{B} \neq \mathbf{0}$, will be dealt with in Sect. 2.4.4.1. For the one-dimensional versions of Eqs. (2.9) and (2.10), we refer to Sect. (2.3.2), where we will study one-dimensional transformations in more detail.

The symmetry of the matrices $\mathbf{L}_{oo}$ and $\mathbf{L}_{ii}$ reflects itself in the *symplecticity* of the transformation matrix $\mathbf{T}$:

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{D}^t & -\mathbf{B}^t \\ -\mathbf{C}^t & \mathbf{A}^t \end{bmatrix} = \mathbf{J}\mathbf{T}^t\mathbf{J} \quad \text{with} \quad \mathbf{J} \equiv \mathrm{i}\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}\ . \quad (2.11)$$

Note that $\mathbf{J} = \mathbf{J}^{-1} = -\mathbf{J}^t = \mathbf{J}^{*t} = \mathbf{J}^\dagger$.

The symplectic $2D \times 2D$ transformation matrix $\mathbf{T}$ has $D(2D+1)$ degrees of freedom (3 for the one-dimensional case and 10 for the two-dimensional case), the same number as in the three $D \times D$ matrices $\mathbf{L}_{oo}$, $\mathbf{L}_{ii}$ and $\mathbf{L}_{io}$ together. In the one-dimensional case, the symplecticity condition (2.11) reduces to the much simpler relation $\det\mathbf{T} = ad - bc = 1$.

The input–output relationship (2.5) implies that the Wigner distribution of the output signal is simply a linearly distorted form of the Wigner distribution of the input signal, with the value of the Wigner distribution at each point in phase space being mapped to another point in phase space, without the need to calculate an integral. Since the determinant of the transformation matrix $\mathbf{T}$ is equal to unity, this pointwise geometrical distortion or deformation is area preserving; it distorts but does not concentrate or deconcentrate the Wigner distribution.

An isolated distribution around a point (t, f) in a *time-frequency* phase space can be considered as the representation of a short *musical note* at a certain time t with a certain frequency f. The Wigner distribution can thus be considered as a *musical score*, which tells us how a time signal can be composed as a superposition of notes.

In optics, such an isolated distribution around a point $(\mathbf{r}, \mathbf{q})$ in a *position-direction* phase space can be considered as an *optical ray* at a certain position $\mathbf{r}$ with a certain direction (i.e., spatial frequency) $\mathbf{q}$, and the Wigner distribution then tells us how the optical signal can be composed as a superposition of rays. This simultaneous position-direction description closely resembles the ray concept in geometrical optics, where the position and direction of a ray are also given simultaneously. The Wigner distribution thus yields a *ray pattern* of the optical signal and, in a way, $W(\mathbf{r},\mathbf{q})$ is the amplitude of a ray, passing through the point $\mathbf{r}$ with a direction $\mathbf{q}$.

Simple optical systems like a thins lens, a section of free space in the Fresnel approximation, and cascades of such systems are described by an input–output relation of the form (2.1) and belong to the realm of what is called *first-order optics* [33, 46]. The propagation of an optical signal through first-order optical systems can most elegantly be described by the coordinate transformation (2.5) of the signal's Wigner distribution. It is thus obvious why phenomena in first-order optics are often treated in a phase space [12, 64–66, 75].

We finally note that the concept of the Wigner distribution can directly be applied to stochastic signals; we only have to replace in its definition (2.4) the product $f(\mathbf{r}_1)f^*(\mathbf{r}_2)$ by the two-point correlation function $\langle f(\mathbf{r}_1)f^*(\mathbf{r}_2)\rangle$, where $\langle\cdot\rangle$ denotes ensemble averaging. The phase-space description of the linear canonical transformation is therefore not restricted to deterministic signals, but applies immediately to stochastic signals, as well [13, 18, 19].

Instead of with the Wigner distribution $W(\mathbf{r},\mathbf{q})$, we could also have chosen to work with the other well-known phase-space description, the *ambiguity function* $A(\mathbf{r}',\mathbf{q}')$ defined as [76, Chap. 7]

$$A(\mathbf{r}',\mathbf{q}') = \int f\left(\mathbf{r}+\tfrac{1}{2}\mathbf{r}'\right) f^*\left(\mathbf{r}-\tfrac{1}{2}\mathbf{r}'\right) \exp[-\mathrm{i}\,2\pi\,\mathbf{r}^t\mathbf{q}']\,\mathrm{d}\mathbf{r}\;, \tag{2.12}$$

for which we have the input–output relation $A_o(\mathbf{A}\mathbf{r}'+\mathbf{B}\mathbf{q}',\mathbf{C}\mathbf{r}'+\mathbf{D}\mathbf{q}') = A_i(\mathbf{r}',\mathbf{q}')$, which is similar to (2.5).

2.3.1 *Basic Properties*

For easy reference, the basic properties of the linear canonical transformation and transforms, treated in this section, have been collected in Tables 2.1 and 2.2.

2.3.1.1 Cascadability

If two linear canonical transformations $\mathscr{L}(\mathbf{T}_1)$ and $\mathscr{L}(\mathbf{T}_2)$ are performed in cascade, $\mathscr{L}(\mathbf{T}_2)\,\mathscr{L}(\mathbf{T}_1)$, the resulting operation is again a linear canonical transformation $\mathscr{L}(\mathbf{T})$ with a transformation matrix $\mathbf{T}=\mathbf{T}_2\mathbf{T}_1$ that is the product of $\mathbf{T}_2$ and $\mathbf{T}_1$. An immediate consequence is that the *inverse* of the operation $\mathscr{L}(\mathbf{T})$ is parameterized by $\mathbf{T}^{-1}$: $\mathscr{L}^{-1}(\mathbf{T}) = \mathscr{L}(\mathbf{T}^{-1})$.

Table 2.1 Some basic properties of the linear canonical transformation

Operator		Transformation matrix	Remark
$\mathscr{L}(\mathbf{T})$		$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$	$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{D}^t & -\mathbf{B}^t \\ -\mathbf{C}^t & \mathbf{A}^t \end{bmatrix}$ symplecticity condition
$\mathscr{L}(\mathbf{T}_2)\mathscr{L}(\mathbf{T}_1)$	cascadability	$\mathbf{T}_2\mathbf{T}_1$	
$\mathscr{L}^{-1}(\mathbf{T}) = \mathscr{L}(\mathbf{T}^{-1})$	inverse	$\begin{bmatrix} \mathbf{D}^t & -\mathbf{B}^t \\ -\mathbf{C}^t & \mathbf{A}^t \end{bmatrix}$	
$\mathscr{L}(\hat{\mathbf{T}}) = \mathscr{L}^*(\mathbf{T}^{-1})$	reverse	$\begin{bmatrix} \mathbf{D}^t & \mathbf{B}^t \\ \mathbf{C}^t & \mathbf{A}^t \end{bmatrix}$	$\mathscr{L}(\mathbf{T})f^*(\mathbf{r}) = [\mathscr{L}(\mathbf{T}^{-1})f(\mathbf{r})]^*$
$\mathscr{F}$	Fourier transformation	$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}$	
$\mathscr{F}\mathscr{L}(\mathbf{T})\mathscr{F}^{-1} = \mathscr{L}(\mathbf{T}^{t-1})$ $\mathscr{F}^{-1}\mathscr{L}(\mathbf{T})\mathscr{F} = \mathscr{L}(\mathbf{T}^{t-1})$	duality	$\begin{bmatrix} \mathbf{D} & -\mathbf{C} \\ -\mathbf{B} & \mathbf{A} \end{bmatrix}$	$\mathbf{T} \Leftrightarrow \mathbf{T}^{t-1}$ implies $\begin{matrix} \mathbf{A} \Leftrightarrow \mathbf{D} \\ \mathbf{B} \Leftrightarrow -\mathbf{C} \end{matrix}$

The property of cascadability is true when we consider linear canonical transformations as simple coordinate transformations in phase space. If we treat the concatenation of operators in their integral representations (2.1) or (2.9), we will encounter the possibility of an additional minus sign. This problem is known as the *metaplectic sign problem* and is carefully studied in [74, Sect. 9.1.4, Composition of transforms] and [75, Sect. C2. Linear canonical transforms] by considering the integral (with complex-valued variables r and s)

$$\int_{\Re} \exp[\mathrm{i}\,\pi(r^2q^2 + 2sq)]\,\mathrm{d}q = \sigma(r)\,\exp(\mathrm{i}\,\tfrac{1}{4}\pi)r^{-1}\exp(-\mathrm{i}\,\pi\,s^2r^{-2})\,, \qquad (2.13\text{a})$$

$$\text{with} \quad \sigma(r) = \begin{cases} +1 \text{ when } & 0 \le \arg r \le \frac{1}{2}\pi\,, \\ -1 \text{ when } & -\pi \le \arg r \le -\frac{1}{2}\pi\,. \end{cases} \qquad (2.13\text{b})$$

In this chapter we will ignore the metaplectic sign problem.

2.3.1.2 The Reverse System

If a system performs a linear canonical transformation $\mathscr{L}(\mathbf{T})$, it is sometimes advantageous to consider the behaviour of the so-called *reverse* system $\mathscr{L}(\hat{\mathbf{T}})$, with transformation matrix $\hat{\mathbf{T}}$; note that this is not the same as the inverse system $\mathscr{L}(\mathbf{T}^{-1})$. In the reverse system, the signal phases are reversed and the signals

Table 2.2 Some basic properties of linear canonical transforms

Input signal	Output signal	Remark
$f_i(\mathbf{r})$	$f_o(\mathbf{r}) = \mathscr{L}(\mathbf{T})f_i(\mathbf{r}) \qquad \mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$	(2.1)(2.9)(2.10)
$\sum_n a_n f_n(\mathbf{r})$	$\sum_n a_n \mathscr{L} f_n(\mathbf{r})$	Linearity
$f_i(\mathbf{r}), h_i(\mathbf{r})$	$\int f_i(\mathbf{r})\, h_i^*(\mathbf{r})\, \mathrm{d}\mathbf{r} = \int f_o(\mathbf{r})\, h_o^*(\mathbf{r})\, \mathrm{d}\mathbf{r}$	Parseval's theorem (2.3)
$f_i^*(\mathbf{r})$	$[\mathscr{L}(\mathbf{T}^{-1})f_i(\mathbf{r})]^* \qquad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{D}^t & -\mathbf{B}^t \\ -\mathbf{C}^t & \mathbf{A}^t \end{bmatrix}$	Complex conjugation (2.15)
$\mathbb{M}^n f_i(\mathbf{r})$	$(\mathbf{D}^t\mathbb{M} - \mathbf{B}^t\mathbb{D})^n f_o(\mathbf{r}) \qquad \mathbb{M} = \mathbf{r}$	Multiplication (2.23a)
$\mathbb{D}^n f_i(\mathbf{r})$	$(-\mathbf{C}^t\mathbb{M} + \mathbf{A}^t\mathbb{D})^n f_o(\mathbf{r}) \qquad \mathbb{D} = (\mathrm{i}\,2\pi)^{-1}\nabla^t$	Derivation (2.23b)
$f_i(\mathbf{r}) \exp(\mathrm{i}\,2\pi\, \mathbf{k}^t\mathbf{r})$	$f_o(\mathbf{r} - \mathbf{Bk})\, \exp(\mathrm{i}\,2\pi\, \mathbf{k}^t\mathbf{D}^t\mathbf{r})\, \exp(-\mathrm{i}\,\pi\, \mathbf{k}^t\mathbf{B}^t\mathbf{Dk})$	Modulation (2.25a)
$f_i(\mathbf{r} - \mathbf{k})$	$f_o(\mathbf{r})\, \exp(\mathrm{i}\,2\pi\, \mathbf{k}^t\mathbf{C}^t\mathbf{r})\, \exp(-\mathrm{i}\,\pi\, \mathbf{k}^t\mathbf{C}^t\mathbf{Ak})$	Shift (2.25b)
$\lvert \det \mathbf{W}\rvert^{-1/2} f_i(\mathbf{W}^{-1}\mathbf{r})$	$\mathscr{L}(\tilde{\mathbf{T}})f_i(\mathbf{r}) \qquad \tilde{\mathbf{T}} = \mathbf{T}\begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{t\,-1} \end{bmatrix}$	Scaling (2.43)
$f_i(-\mathbf{r})$	$\mathscr{L}(-\mathbf{T})f_i(\mathbf{r}) = f_o(-\mathbf{r})$	Scaling by -1 (2.43) $\mathbf{W} = -\mathbf{I}$
$\exp(-\pi\, \mathbf{r}^t\mathbf{L}_i\mathbf{r})$	$[\det(\mathbf{A} + \mathrm{i}\,\mathbf{L}_i)]^{-1/2} \exp(-\pi\, \mathbf{r}^t\mathbf{L}_o\mathbf{r})$ with $\mathrm{i}\,\mathbf{L}_o = (\mathbf{C} + \mathrm{i}\,\mathbf{DL}_i)(\mathbf{A} + \mathrm{i}\,\mathbf{BL}_i)^{-1}$	Sect. 2.6.1 $\mathbf{L} = \mathbf{L}^t$
1	$(\det \mathbf{A})^{-1/2} \exp(\mathrm{i}\,\pi\, \mathbf{r}^t\mathbf{CA}^{-1}\mathbf{r})$	Sect. 2.6.1
$\exp(\mathrm{i}\,2\pi\, k^t\mathbf{r})$	$(\det \mathbf{A})^{-1/2} \exp(\mathrm{i}\,\pi\, \mathbf{r}^t\mathbf{CA}^{-1}\mathbf{r}) \times \exp(-\mathrm{i}\,\pi\, \mathbf{k}^t\mathbf{A}^{-1}\mathbf{Bk}) \exp(\mathrm{i}\,2\pi\, \mathbf{k}^t\mathbf{A}^{-1}\mathbf{r})$	Sect. 2.6.1

propagate in the opposite direction, which means that the frequency variable $\mathbf{q}$ has to be reversed, i.e., replaced by $-\mathbf{q}$, and that $f_o^*(\mathbf{r}_o)$ now acts as the input signal while $f_i^*(\mathbf{r}_i) = \mathscr{L}(\hat{\mathbf{T}})f_o^*(\mathbf{r}_o)$ is the output signal. The transformation matrix $\hat{\mathbf{T}}$ of the reverse system thus takes the form

$$\hat{\mathbf{T}} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} \mathbf{T}^{-1} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B} \\ -\mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{D}^t & \mathbf{B}^t \\ \mathbf{C}^t & \mathbf{A}^t \end{bmatrix}. \qquad (2.14)$$

Substituting $f_o(\mathbf{r}_o) = \mathscr{L}(\mathbf{T})f_i(\mathbf{r}_i)$ into $f_i^*(\mathbf{r}_i) = \mathscr{L}(\hat{\mathbf{T}})f_o^*(\mathbf{r}_o)$, we immediately get $f_i^*(\mathbf{r}_i) = \mathscr{L}(\hat{\mathbf{T}})\,[\mathscr{L}(\mathbf{T})f_i(\mathbf{r}_i)]^* = \mathscr{L}(\hat{\mathbf{T}})\,\mathscr{L}^*(\mathbf{T})f_i^*(\mathbf{r}_i)$ and we conclude that $\mathscr{L}(\mathbf{T}) = \mathscr{L}^*(\hat{\mathbf{T}}^{-1})$ and also

$$\mathscr{L}(\mathbf{T})f^*(\mathbf{r}) = [\mathscr{L}(\mathbf{T}^{-1})f(\mathbf{r})]^* \ . \tag{2.15}$$

The system is symmetric if $\hat{\mathbf{T}} = \mathbf{T}$, i.e., if $\mathbf{A} = \mathbf{D}^t$, $\mathbf{B} = \mathbf{B}^t$ and $\mathbf{C} = \mathbf{C}^t$. This corresponds to $\mathbf{L}_{oo} = \mathbf{L}_{ii}$ and $\mathbf{L}_{io} = \mathbf{L}_{io}^t$, as we already mentioned before.

2.3.1.3 Formulation in Terms of Fourier Transforms

For $\mathbf{A} = \mathbf{D} = \mathbf{0}$ and $\mathbf{B} = -\mathbf{C} = \mathbf{I}$, the linear canonical transformation reduces to the common *Fourier transformation* [apart from the phase factor $(\det \mathrm{i}\,\mathbf{I})^{-1/2}$]

$$f_o(\mathbf{r}_o) = (\det \mathrm{i}\,\mathbf{I})^{-1/2}\int \exp[-\mathrm{i}\,2\pi\,\mathbf{r}_i^t\mathbf{r}_o]f_i(\mathbf{r}_i)\,\mathrm{d}\mathbf{r}_i \equiv (\det \mathrm{i}\,\mathbf{I})^{-1/2}\bar{f}_i(\mathbf{r}_o) \equiv \mathscr{F}f_i(\mathbf{r}_i)\ , \tag{2.16}$$

whereas the inverse Fourier transformation arises for $\mathbf{B} = -\mathbf{C} = -\mathbf{I}$; note that $(\det \mathrm{i}\,\mathbf{I})^{-1/2} = \mathrm{i}^{-D/2}$ for a D-dimensional signal. We thus conclude that the transformation matrices

$$\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{2.17}$$

correspond to a Fourier transformation and its inverse, respectively.

Using the Fourier transformation, we can easily express the linear canonical transformation in terms of the Fourier transforms $\bar{f}_{i,o}(\mathbf{q})$ of the signals $f_{i,o}(\mathbf{r})$:

$$\bar{f}_o(\mathbf{q}_o) = \mathscr{F}f_o(\mathbf{r}_o) = \mathscr{F}\mathscr{L}(\mathbf{T})f(\mathbf{r}_i) = \mathscr{F}\mathscr{L}(\mathbf{T})\mathscr{F}^{-1}\bar{f}_i(\mathbf{q}_i) = \mathscr{L}(\mathbf{T}^{t^{-1}})\bar{f}_i(\mathbf{q}_i)\ . \tag{2.18}$$

The transformation matrix $\mathbf{T}^{t^{-1}}$ in the Fourier domain takes the *dual* form

$$\mathbf{T}^{t^{-1}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}\begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{D} & -\mathbf{C} \\ -\mathbf{B} & \mathbf{A} \end{bmatrix} \tag{2.19}$$

and is related to $\mathbf{T}$ by interchanging $\mathbf{A} \Leftrightarrow \mathbf{D}$ and $\mathbf{B} \Leftrightarrow -\mathbf{C}$. Note that interchanging $\mathscr{F}$ and $\mathscr{F}^{-1}$ leads to the same result.

2.3.1.4 Coordinate Multiplication and Differentiation

One of the main properties of the Fourier transformation is that a multiplication of $f(\mathbf{r})$ by its argument $\mathbf{r}$ corresponds to a differentiation of its Fourier transform $\bar{f}(\mathbf{q})$, and vice versa with a minus sign. With the operator $\mathbb{M}$ denoting multiplication by the argument and the operator $\mathbb{D}$ denoting differentiation with respect to the argument and dividing by $(\mathrm{i}\,2\pi)$, both resulting in a column vector, we can write

$$\mathbb{M}f(\mathbf{r}) = \mathbf{r}f(\mathbf{r}) \quad \Leftrightarrow \quad -\mathbb{D}\bar{f}(\mathbf{q}) = -(\mathrm{i}\,2\pi)^{-1}\nabla^t\bar{f}(\mathbf{q}) \;, \tag{2.20a}$$

$$\mathbb{M}\bar{f}(\mathbf{q}) = \mathbf{q}\bar{f}(\mathbf{q}) \quad \Leftrightarrow \quad \mathbb{D}f(\mathbf{r}) = (\mathrm{i}\,2\pi)^{-1}\nabla^t f(\mathbf{r}) \;, \tag{2.20b}$$

where the nabla operator ∇, which after transposition takes the form of a column vector, has been used to denote partial derivatives. We also define the operators $\mathbb{D}^t$ and $\mathbb{M}^t$ where the vectors are transposed, and easily verify that $\mathbb{D}^t\mathbb{M}f(\mathbf{r}) = (1 + \mathbb{M}^t\mathbb{D})f(\mathbf{r})$. The operator $\mathbb{D}^t\mathbb{M} - \mathbb{M}^t\mathbb{D}$ thus corresponds to the identity operator.

We now turn our attention to the linear canonical transformation $\mathscr{L}$ and determine the two transforms $\mathscr{L}[\mathbb{M}f(\mathbf{r})]$ and $\mathscr{L}[\mathbb{D}f(\mathbf{r})]$ in terms of $\mathscr{L}f(\mathbf{r})$. After some lengthy but straightforward calculation we get

$$\mathscr{L}[\mathbb{M}f(\mathbf{r})] = (\mathbf{D}^t\mathbb{M} - \mathbf{B}^t\mathbb{D})\mathscr{L}f(\mathbf{r}) \;, \tag{2.21a}$$

$$\mathscr{L}[\mathbb{D}f(\mathbf{r})] = (-\mathbf{C}^t\mathbb{M} + \mathbf{A}^t\mathbb{D})\mathscr{L}f(\mathbf{r}) \;, \tag{2.21b}$$

and we easily verify that for the Fourier transformation, i.e., $\mathbf{A} = \mathbf{D} = \mathbf{0}$ and $\mathbf{B} = -\mathbf{C} = \mathbf{I}$, we get $\mathscr{F}[\mathbb{M}f(\mathbf{r})] = -\mathbb{D}[\mathscr{F}f(\mathbf{r})]$ and $\mathscr{F}[\mathbb{D}f(\mathbf{r})] = \mathbb{M}[\mathscr{F}f(\mathbf{r})]$, see Eqs. (2.20). The operators $\mathbb{M}$ and $\mathbb{D}$ in the input domain thus lead to the operators $\mathbb{M}_\mathbf{T}$ and $\mathbb{D}_\mathbf{T}$ in the output domain, and the four operators are related by

$$\begin{bmatrix} \mathbb{M}_\mathbf{T} \\ \mathbb{D}_\mathbf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbb{M} \\ \mathbb{D} \end{bmatrix} \;, \tag{2.22}$$

in accordance with (2.6).

We also verify that $\mathbb{D}^t_\mathbf{T}\mathbb{M}_\mathbf{T} - \mathbb{M}^t_\mathbf{T}\mathbb{D}_\mathbf{T}$ is the identity operator. Indeed: $\mathbb{D}^t_\mathbf{T}\mathbb{M}_\mathbf{T} - \mathbb{M}^t_\mathbf{T}\mathbb{D}_\mathbf{T} = (-\mathbb{M}^t\mathbf{C}^t + \mathbb{D}^t\mathbf{A}^t)(\mathbf{D}\mathbb{M} - \mathbf{B}\mathbb{D}) - (\mathbb{M}^t\mathbf{D}^t - \mathbb{D}^t\mathbf{B}^t)(-\mathbf{C}\mathbb{M} + \mathbf{A}\mathbb{D}) = \mathbb{M}^t(-\mathbf{C}^t\mathbf{D} + \mathbf{D}^t\mathbf{C})\mathbb{M} + \mathbb{M}^t(\mathbf{C}^t\mathbf{B} - \mathbf{D}^t\mathbf{A})\mathbb{D} + \mathbb{D}^t(\mathbf{A}^t\mathbf{D} - \mathbf{B}^t\mathbf{C})\mathbb{M} + \mathbb{D}^t(-\mathbf{A}^t\mathbf{B} + \mathbf{B}^t\mathbf{A})\mathbb{D} = \mathbb{D}^t\mathbb{M} - \mathbb{M}^t\mathbb{D}$, where we have used the symplecticity conditions $\mathbf{C}^t\mathbf{D} = \mathbf{D}^t\mathbf{C}$, $\mathbf{A}^t\mathbf{D} - \mathbf{B}^t\mathbf{C} = \mathbf{I}$ and $\mathbf{A}^t\mathbf{B} = \mathbf{B}^t\mathbf{A}$. We note that an alternative development of linear canonical transformations may be found in [74, Sect. 9.1.1, Posing the operator problem], where Eq. (2.22) is taken as the defining characteristic of these transformations and the integral form is subsequently defined.

The operators $\mathbb{M}$ and $\mathbb{D}$ can be applied an arbitrary number of times in cascade, which leads to the following relations, cf. (2.21):

$$\mathscr{L}[\mathbb{M}^n f(\mathbf{r})] = (\mathbf{D}^t\mathbb{M} - \mathbf{B}^t\mathbb{D})^n\mathscr{L}f(\mathbf{r}) \;, \tag{2.23a}$$

$$\mathscr{L}[\mathbb{D}^n f(\mathbf{r})] = (-\mathbf{C}^t\mathbb{M} + \mathbf{A}^t\mathbb{D})^n\mathscr{L}f(\mathbf{r}) \;. \tag{2.23b}$$

2.3.1.5 Coordinate Shift and Modulation

One of the other main properties of the Fourier transformation is that a modulation of $f(\mathbf{r})$ by $\exp(\mathrm{i}\,2\pi\,\mathbf{k}^t\mathbf{r})$ corresponds to a shift of the argument of its Fourier transform $\bar{f}(\mathbf{q})$, and vice versa with a minus sign:

$$f(\mathbf{r})\,\exp(\mathrm{i}\,2\pi\,\mathbf{k}^t\mathbf{r}) \quad\Leftrightarrow\quad \bar{f}(\mathbf{q}-\mathbf{k})\,, \tag{2.24a}$$

$$f(\mathbf{r}-\mathbf{k}) \quad\Leftrightarrow\quad \bar{f}(\mathbf{q})\,\exp(-\mathrm{i}\,2\pi\,\mathbf{k}^t\mathbf{q})\,. \tag{2.24b}$$

Similar relations hold for the linear canonical transformations. With $f_o(\mathbf{r}) = \mathscr{L}f_i(\mathbf{r})$ we write

$$\mathscr{L}[f_i(\mathbf{r})\,\exp(\mathrm{i}\,2\pi\,\mathbf{k}^t\mathbf{r})] = f_o(\mathbf{r}-\mathbf{B}\mathbf{k})\,\exp(\mathrm{i}\,2\pi\,\mathbf{k}^t\mathbf{D}^t\mathbf{r})\,\exp(-\mathrm{i}\,\pi\,\mathbf{k}^t\mathbf{B}^t\mathbf{D}\mathbf{k})\,, \tag{2.25a}$$

$$\mathscr{L}[f_i(\mathbf{r}-\mathbf{k})] = f_o(\mathbf{r})\,\exp(\mathrm{i}\,2\pi\,\mathbf{k}^t\mathbf{C}^t\mathbf{r})\,\exp(-\mathrm{i}\,\pi\,\mathbf{k}^t\mathbf{C}^t\mathbf{A}\mathbf{k})\,, \tag{2.25b}$$

and we verify that for the Fourier transformation we get indeed Eqs. (2.24).

The important *convolution property* of the Fourier transformation,

$$\int f(\mathbf{r}-\mathbf{k})\,h(\mathbf{k})\,\mathrm{d}\mathbf{k} \equiv (f*h)(\mathbf{r}) \quad\Leftrightarrow\quad \bar{f}(\mathbf{q})\,\bar{h}(\mathbf{q})\,, \tag{2.26}$$

does not have a nice counterpart for a general linear canonical transformation; see [7], for instance. Nevertheless, for a transformation with the additional condition $\mathbf{A} = \mathbf{0}$, which is actually a Fourier transformation followed by a multiplication with a quadratic-phase function, we can write $f_o(\mathbf{r}_o) = \mathscr{L}f_i(\mathbf{r}_i) = (\det\mathrm{i}\,\mathbf{B})^{-1/2}\exp(\mathrm{i}\,\pi\,\mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o)\bar{f}_i(\mathbf{B}^{-1}\mathbf{r}_o)$ and thus

$$\begin{aligned}\mathscr{L}[(f_i*h_i)(\mathbf{r})] &= (\det\mathrm{i}\,\mathbf{B})^{-1/2}\exp(\mathrm{i}\,\pi\,\mathbf{r}^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r})\bar{f}_i(\mathbf{B}^{-1}\mathbf{r})\,\bar{h}_i(\mathbf{B}^{-1}\mathbf{r})\\ &= (\det\mathrm{i}\,\mathbf{B})^{1/2}\exp(-\mathrm{i}\,\pi\,\mathbf{r}^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r})f_o(\mathbf{r})\,h_o(\mathbf{r})\,.\end{aligned} \tag{2.27}$$

By analogy with the alternative representation of the convolution in the Fourier domein, $f_1 * f_2 = \mathscr{F}^{-1}\{[\mathscr{F}f_1(\mathbf{r})][\mathscr{F}f_2(\mathbf{r})]\}$, we can introduce the *generalized canonical convolution* as $\mathscr{L}(\mathbf{T}_3)\{[\mathscr{L}(\mathbf{T}_1)f_1(\mathbf{r})][\mathscr{L}(\mathbf{T}_2)f_2(\mathbf{r})]\}$, which resembles the common convolution and fractional convolution operations [4, 9, 39]. In particular, if the transformation matrices used in the latter expression correspond to the ones of the fractional Fourier transformation, we obtain the generalized fractional convolution, whose applications for shift-variant filtering, encryption, etc., have been proposed [4, 9, 31, 39, 77]. Since two-dimensional canonical transformations include such operations as rotation, scaling, shearing, and fractional Fourier transformation, the generalized canonical convolution can be helpful for resolving the problem of scale-, rotation- and shear-invariant (or partially invariant) filtering and pattern recognition.

2.3.2 Simple Transformations for the One-dimensional Case

In this section we consider some simple transformations in the one-dimensional case. We explicitly state the two governing equations for one dimension, cf. (2.9) and (2.10):

$$f_o(x_o) = (\mathrm{i}\, b)^{-1/2} \int \exp[\mathrm{i}\,\pi\, b^{-1}(dx_o^2 - 2x_o x_i + ax_i^2)]\, f_i(x_i)\, \mathrm{d}x_i \quad (b \neq 0)\ , \tag{2.28a}$$

$$f_o(x) = |a|^{-1/2} \exp[\mathrm{i}\,\pi\, ca^{-1}x^2] f_i(a^{-1}x) \quad (b = 0)\ , \tag{2.28b}$$

and we study the result of a linear canonical transformation in phase space for several simple cases. As an illustration, we consider the effect on a rectangularly shaped Wigner distribution, see the top figure in Table 2.3.

2.3.2.1 Fourier Transformer

We already met the special case of a Fourier transformation,

$$f_o(x_o) = \mathrm{i}^{-1/2} \int \exp(-\mathrm{i}\, 2\pi\, x_o x_i) f_i(x_i)\, \mathrm{d}x_i = \mathrm{i}^{-1/2} \bar{f}_i(x_o) = \mathscr{F} f_i(x_i)\ , \tag{2.29}$$

which is connected to the transformation matrix $[\,0, 1; -1, 0\,]$. Its effect on the Wigner distribution is a simple clockwise rotation in the xu plane through $\pi/2$. In other words: whatever happened in the original Wigner distribution on the u axis, now happens on the x axis, and whatever happened on the x axis, now happens on the $-u$ axis. We recall the well-known property of the Fourier transformation that $\mathscr{F}^4$ results in the identity operation.

2.3.2.2 Magnifier

Let us now turn our attention to some basic one-parameter transformations. We start with two examples of the $b = 0$ class. The matrix $[\,a, 0; 0, a^{-1}]$, associated with the transformation

$$f_o(x) = |a|^{-1/2} f_i(a^{-1}x) \equiv \mathscr{M}(a) f_i(x)\ , \tag{2.30}$$

leads to a scaling by a in the x direction and an inverse scaling by a^{-1} in the u direction. We denote this transformation by the operator $\mathscr{M}(a)$. In the example we have chosen $a = 2$. We recognize the well-known property of the Fourier transformation, that a scaling for the original variable x corresponds to an inverse scaling for its Fourier conjugate u.

2.3.2.3 Quadratic-Phase Modulator

Our second example for the $b = 0$ class is the transformation matrix $[1, 0; c, 1]$, with the input–output relation

$$f_o(x) = \exp(\mathrm{i}\,\pi\, cx_o^2) f_i(x) \equiv \mathscr{Q}(-c) f_i(x) \; . \tag{2.31}$$

The effect of this modulation by a quadratic-phase function leads to a shear in the u direction. We denote this transformation by the operator $\mathscr{Q}(-c)$. In the example we have chosen $-c = 1$.

2.3.2.4 Fresnel Transformer

The dual of the previous example arises when $a \Leftrightarrow d$ and $b \Leftrightarrow -c$; the resulting transformation matrix then reads $[1, b; 0, 1]$ and its effect is a shear in the x direction. We denote this transformation, which is also known as the *Fresnel transformation*,

$$f_o(x_o) = (\mathrm{i}\, b)^{-1/2} \int \exp[\mathrm{i}\,\pi\, b^{-1}(x_o - x_i)^2] f_i(x_i)\,\mathrm{d}x_i \equiv \mathscr{S}(b) f_i(x_i) \; , \tag{2.32}$$

by the operator $\mathscr{S}(b)$, expressible in terms of previously defined operators as $\mathscr{S}(b) = \mathscr{F}\mathscr{Q}(b)\mathscr{F}^{-1}$. In the example we have chosen $b = 1$. We note that the Fresnel transformation takes the form of a convolution with the quadratic-phase function $(\mathrm{i}\, b)^{-1/2} \exp(\mathrm{i}\,\pi\, b^{-1}x^2)$; in the Fourier domain such a convolution becomes a multiplication with the quadratic-phase function, $\bar{f}_o(u) = \exp(-\mathrm{i}\,\pi\, bu^2) \bar{f}_i(u)$, which shows the duality between the quadratic-phase modulation and the Fresnel transformation.

Now that we have introduced the Fresnel transformation, we can try and express a magnifier $\mathscr{M}(s)$ in terms of quadratic-phase modulators $\mathscr{Q}(-c)$ and Fresnel transformers $\mathscr{S}(b)$. We easily verify that

$$\mathscr{Q}(b_2^{-1} + b_1 b_2^{-2})\, \mathscr{S}(b_2)\, \mathscr{Q}(b_2^{-1} + b_1^{-1})\, \mathscr{S}(b_1) = \mathscr{M}(-b_2 b_1^{-1}) \; . \tag{2.33}$$

A decomposition of linear canonical transformations in terms of quadratic-phase modulators and Fresnel transformers is especially interesting in optics, where these two basic transformations correspond to a lens $\mathscr{Q}(f^{-1})$ with focal distance f and a section of free space $\mathscr{S}(z)$ with distance z, respectively. In (2.33), we then recognize the imaging condition $f^{-1} = b_1^{-1} + b_2^{-1}$, the magnification factor $s = -b_2 b_1^{-1}$ and a phase-compensating lens with focal distance $-sf$. We easily verify that the magnification factor is negative if b_1, b_2 and f are positive, and that we will see an inverted image of the object, as expected.

2.3.2.5 Fractional Fourier Transformer

When dealing with the Fourier transformation and its inverse, we saw that these transformations resulted in a clockwise rotation by $\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$, respectively. If the rotation takes a different value γ, say, (with γ typically between 0 and π), the transformation matrix reads $[\cos\gamma, \sin\gamma; -\sin\gamma, \cos\gamma]$. We are now in the realm of the *fractional Fourier transformation* [42, 45, 47, 57], denoted by the operator $\mathscr{F}(\gamma)$:

$$\begin{aligned} f_o(x_o) &= \frac{\exp(\mathrm{i}\,\frac{1}{2}\gamma)}{\sqrt{\mathrm{i}\,\sin\gamma}} \int \exp\left(\mathrm{i}\,\pi\,\frac{x_o^2\cos\gamma - 2x_ix_o + x_i^2\cos\gamma}{\sin\gamma}\right) f_i(x_i)\,\mathrm{d}x_i \\ &\equiv \mathscr{F}(\gamma) f_i(x_i) \equiv F_\gamma(x_o) \qquad\qquad (\gamma \neq n\pi)\;; \end{aligned} \tag{2.34}$$

we recall that $(\mathrm{i}\,\sin\gamma)^{1/2}$ is defined as $\exp[\mathrm{i}\,(\frac{1}{4}\pi)\,\mathrm{sgn}(\sin\gamma)]|\sin\gamma|^{1/2}$. Note the additional phase factor $\exp(\mathrm{i}\,\frac{1}{2}\gamma)$, which has been added to get a better correspondence to the physical process of Fourier transformation; in particular, the phase additivity is preserved, $\mathscr{F}(\gamma_1)\,\mathscr{F}(\gamma_2) = \mathscr{F}(\gamma_1+\gamma_2)$, and $\mathscr{F}(\frac{1}{2}\pi) f(x_i) = \bar{f}(x_o)$. With the *fractional angle* γ going from 0 via $\frac{1}{2}\pi$ to π, $F_\gamma(x)$ goes from $F_0(x) = f(x)$ via $F_{\pi/2}(x) = \bar{f}(x)$ to $F_\pi(x) = f(-x)$. In the example we have chosen $\gamma = \frac{1}{4}\pi$.

A fractional Fourier transformer can easily be realized as a cascade of quadratic-phase modulators and Fresnel transformers. We have, for instance, the relations [19, 38]

$$\mathscr{F}(\gamma) = \mathscr{S}(\tan\tfrac{1}{2}\gamma)\,\mathscr{Q}(\sin\gamma)\,\mathscr{S}(\tan\tfrac{1}{2}\gamma)\;, \tag{2.35a}$$

$$\mathscr{F}(\gamma) = \mathscr{Q}(\tan\tfrac{1}{2}\gamma)\,\mathscr{S}(\sin\gamma)\,\mathscr{Q}(\tan\tfrac{1}{2}\gamma)\;. \tag{2.35b}$$

In coherent optics (with optical wavelength $\lambda_\circ$, say) this corresponds to cascades of thin convex lenses (with focal length $f > 0$) and sections of free space (with distance $z > 0$). Since we now work with real-world coordinates that are no longer dimensionless, we need additional magnifiers $\mathscr{M}(w)$ and $\mathscr{M}(w^{-1})$, where w has the dimension [m]. The real-world fractional Fourier transformer can then be described as the cascade $\mathscr{M}(w)\,\mathscr{F}(\gamma)\,\mathscr{M}(w^{-1})$, and similar relations hold for the thin lens and the section of free space. In real-world coordinates, the two Eqs. (2.35) read

$$\mathscr{M}(w)\,\mathscr{F}(\gamma)\,\mathscr{M}(w^{-1}) = \mathscr{S}(w^2\tan\tfrac{1}{2}\gamma)\,\mathscr{Q}(w^{-2}\sin\gamma)\,\mathscr{S}(w^2\tan\tfrac{1}{2}\gamma)\;, \tag{2.36a}$$

$$\mathscr{M}(w)\,\mathscr{F}(\gamma)\,\mathscr{M}(w^{-1}) = \mathscr{Q}(w^{-2}\tan\tfrac{1}{2}\gamma)\,\mathscr{S}(w^2\sin\gamma)\,\mathscr{Q}(w^{-2}\tan\tfrac{1}{2}\gamma)\;, \tag{2.36b}$$

and the corresponding cascades are depicted in Fig. 2.1(a) and (b), respectively. The fractional angle γ is related to z and f through the relation $\sin^2(\frac{1}{2}\gamma) = z/2f$ in both cascades, and the scaling parameter w is related to γ and $\lambda_\circ z$ through the

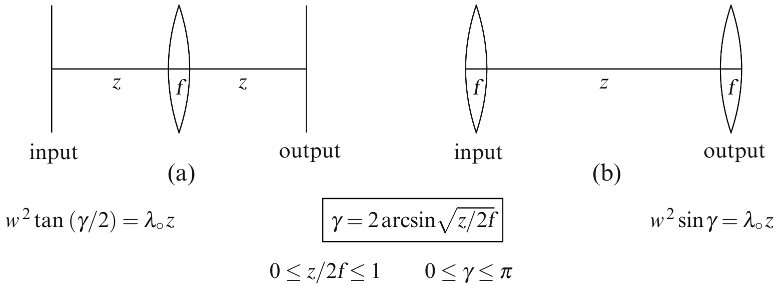


Fig. 2.1 Two optical realizations of the fractional Fourier transformer, consisting of lenses with focal distance f and sections of free space with a distance z

relations $w^2 \tan(\frac{1}{2}\gamma) = \lambda_\circ z$ in cascade (a) and $w^2 \sin\gamma = \lambda_\circ z$ in cascade (b). A common Fourier transformation $f_o(x) \propto w^{-1}\bar{f}_i(xw^{-2})$ occurs for $z = f$, while a simple coordinate reversion $f_o(x) \propto f_i(-x)$ occurs for $z = 2f$. The general formula (in real-world coordinates) reads

$$f_o(x_o) = \frac{\exp(\mathrm{i}\,\frac{1}{2}\gamma)}{w\sqrt{\mathrm{i}\,\sin\gamma}} \int \exp\left(\mathrm{i}\,\pi\,\frac{x_o^2\cos\gamma - 2x_i x_o + x_i^2\cos\gamma}{w^2\sin\gamma}\right) f_i(x_i)\,\mathrm{d}x_i\ . \tag{2.37}$$

2.3.2.6 Hyperbolic Expander

Our final example is the *hyperbolic expander* [75, p. 183, Example: Hyperbolic expanders] with transformation matrix $[\cosh\gamma, \sinh\gamma; \sinh\gamma, \cosh\gamma]$. We denote the corresponding operation by $\mathscr{H}(\gamma)$ and remark that it can be considered as a magnifier $\mathscr{M}(\exp\gamma)$ embedded in between two fractional Fourier transformers $\mathscr{F}(-\frac{1}{4}\pi)$ and $\mathscr{F}(\frac{1}{4}\pi)$. With this decomposition in mind, we readily recognize the effect of the hyperbolic expander in Table 2.3: the fractional Fourier transformer $\mathscr{F}(\frac{1}{4}\pi)$ rotates the Wigner distribution clockwise through $\frac{1}{4}\pi$, the magnifier $\mathscr{M}(\exp\gamma)$ scales by $\exp\gamma$ in the x direction and by $(\exp\gamma)^{-1}$ in the u direction, and the final fractional Fourier transformer $\mathscr{F}(-\frac{1}{4}\pi)$ rotates everything back to its original orientation. In the example we have chosen $\gamma = 0.5$, and thus $\exp\gamma = 1.64872$.

A hyperbolic expander can easily be realized again as a cascade of quadratic-phase modulators and Fresnel transformers. The result shows a nice resemblance to the fractional Fourier transformer; we only have to replace the quadratic-phase modulations by their inverses:

$$\mathscr{H}(\gamma) = \mathscr{S}(\tan\tfrac{1}{2}\gamma)\,\mathscr{Q}(-\sin\gamma)\,\mathscr{S}(\tan\tfrac{1}{2}\gamma)\ , \tag{2.38a}$$

$$\mathscr{H}(\gamma) = \mathscr{Q}(-\tan\tfrac{1}{2}\gamma)\,\mathscr{S}(\sin\gamma)\,\mathscr{Q}(-\tan\tfrac{1}{2}\gamma)\,. \tag{2.38b}$$

In optics this corresponds to replacing in Fig. 2.1 the convex lenses (with a positive focal length) by concave lenses (with a negative focal length).

2.3.2.7 Modified Iwasawa Decomposition

We end this section about the one-dimensional case with the important *modified Iwasawa decomposition* [55] (see also [75, Sects. 9.5 and 10.2]) of a linear canonical transformation $\mathscr{L}(\mathbf{T})$ as a cascade of a fractional Fourier transformer, a magnifier and a quadratic-phase modulator: $\mathscr{L}(\mathbf{T}) = \mathscr{Q}(g)\,\mathscr{M}(s)\,\mathscr{F}(\gamma)$, with

$$s = \sqrt{a^2 + b^2}\,, \quad g = -(ac + bd)/s^2 \quad \text{and} \quad \exp(\mathrm{i}\,\gamma) = (a + \mathrm{i}\,b)/s\,;$$

see also [47, Sect. 9.7.1, Quadratic-phase systems as fractional Fourier transforms]. Note that this decomposition holds for all values of the matrix entries a, b, c and d. We can formulate an Iwasawa-type decomposition in the reversed order of the operators, by finding the Iwasawa decomposition of the inverse transformation $\mathscr{L}(\mathbf{T}^{-1})$ in its regular order, reversing the order of the operators, and replacing each operator by its inverse. We then get $\mathscr{L}(\mathbf{T}) = \mathscr{F}(\hat{\gamma})\,\mathscr{M}(\hat{s})\,\mathscr{Q}(\hat{g})$, with

$$\hat{s} = 1/\sqrt{d^2 + b^2}\,, \quad \hat{g} = -(cd + ab)\,\hat{s}^2 \quad \text{and} \quad \exp(\mathrm{i}\,\hat{\gamma}) = (d + \mathrm{i}\,b)\,\hat{s}\,.$$

Many other possible decompositions, in particular as cascades of quadratic-phase modulators $\mathscr{Q}$, Fresnel transformers $\mathscr{S}$, magnifiers $\mathscr{M}$ and Fourier transformers $\mathscr{F}$, are known. If necessary, we can even restrict us to quadratic-phase modulators and Fresnel transformers. We refer to [47, Sect. 3.4, Linear canonical transforms], where some more detailed information about the linear canonical transformation for one-dimensional signals can be found.

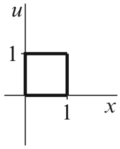
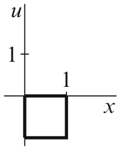
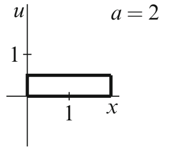
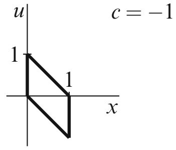
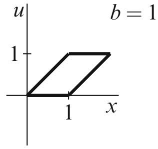
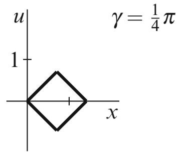
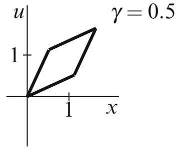
2.4 Special Cases of the Linear Canonical Transformation

In this section we consider some special cases of linear canonical transformers:

1. The case $\mathbf{B} = \mathbf{0}$ and consequently $\mathbf{A}^{-1} = \mathbf{D}^t$,
2. The case $\mathbf{B}^{-1} = -\mathbf{C}^t$ and consequently $\mathbf{A}\mathbf{D}^t = \mathbf{0}$,
3. The case $\mathbf{C} = \mathbf{0}$ and consequently $\mathbf{A}^{-1} = \mathbf{D}^t$, and
4. The case that $\mathbf{T}$ is not only symplectic but also orthogonal, $\mathbf{T}^{-1} = \mathbf{T}^t$.

The reduction from the higher-dimensional to the one-dimensional case is throughout straightforward by simply substituting the general matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ by their scalar versions $a\mathbf{I}$, $b\mathbf{I}$, $c\mathbf{I}$ and $d\mathbf{I}$, and leads to some additional formulas that are not included in Table 2.3. Matrix transposition is no longer relevant for the

Table 2.3 Some simple linear canonical transformations in one dimension

Operator	Transformation matrix	Example
$\mathscr{L}(\mathbf{T})$ $\quad \det \mathbf{T} = ad - bc = 1$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	
Fourier transformation $\mathscr{F}$ rotation in the xu plane through $\gamma = \frac{1}{2}\pi$	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$	
$\mathscr{F}\mathscr{L}(\mathbf{T})\mathscr{F}^{-1} = \mathscr{L}(\mathbf{T}^{t-1})$	$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$	
Magnification $\mathscr{M}(a)$ scaling in the x direction by a scaling in the u direction by a^{-1}	$\begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$	
Quadratic-phase modulation $\mathscr{Q}(-c)$ shear in the u direction by c	$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$	
Fresnel transformation $\mathscr{S}(b) = \mathscr{F}\mathscr{Q}(b)\mathscr{F}^{-1}$ shear in the x direction by b	$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$	
Fractional Fourier transformation $\mathscr{F}(\gamma) = \mathscr{S}(\tan\frac{1}{2}\gamma)\mathscr{Q}(\sin\gamma)\mathscr{S}(\tan\frac{1}{2}\gamma)$ $\mathscr{F}(\gamma) = \mathscr{Q}(\tan\frac{1}{2}\gamma)\mathscr{S}(\sin\gamma)\mathscr{Q}(\tan\frac{1}{2}\gamma)$	$\begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$	
Hyperbolic expansion $\mathscr{H}(\gamma) = \mathscr{F}(\frac{1}{4}\pi)\mathscr{M}(\exp\gamma)\mathscr{F}(-\frac{1}{4}\pi)$ $\mathscr{H}(\gamma) = \mathscr{S}(\tan\frac{1}{2}\gamma)\mathscr{Q}(-\sin\gamma)\mathscr{S}(\tan\frac{1}{2}\gamma)$ $\mathscr{H}(\gamma) = \mathscr{Q}(-\tan\frac{1}{2}\gamma)\mathscr{S}(\sin\gamma)\mathscr{Q}(-\tan\frac{1}{2}\gamma)$	$\begin{bmatrix} \cosh\gamma & \sinh\gamma \\ \sinh\gamma & \cosh\gamma \end{bmatrix}$	
Iwasawa decomposition $\mathscr{L}(\mathbf{T}) = \mathscr{Q}(g)\mathscr{M}(s)\mathscr{F}(\gamma)$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$s = \sqrt{a^2+b^2}$ $g = -(ac+bd)/s^2$ $\exp(\mathrm{i}\gamma) = (a+\mathrm{i}b)/s$

one-dimensional case. A similar remark applies to the case of diagonal matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, in which case we are actually dealing with a transformation that is separable in multiple one-dimensional transformations.

2.4.1 *The Case* $\mathbf{B} = \mathbf{0}$ *and Consequently* $\mathbf{A}^{-1} = \mathbf{D}^t$

From the case $\mathbf{B} = \mathbf{0}$, see (2.10), we first consider the particular case of a D-dimensional *magnifier*

$$f_o(\mathbf{r}_o) = |\det \mathbf{A}|^{-1/2} f_i(\mathbf{A}^{-1}\mathbf{r}_o) \equiv \mathscr{M}(\mathbf{A}) f_i(\mathbf{r}_i) \ , \tag{2.39}$$

which results for $\mathbf{C} = \mathbf{0}$. If, in the two-dimensional case, the 2×2 matrix $\mathbf{A}$ takes the special form

$$\mathbf{A} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \equiv \mathbf{U}_r(\alpha) = \mathbf{U}_r^{t-1}(\alpha) \ , \tag{2.40}$$

the magnifier $\mathscr{M}[\mathbf{U}_r(\alpha)]$ reduces to a *rotator*,

$$f_o(x_o, y_o) = f_i(x_o\cos\alpha - y_o\sin\alpha, x_o\sin\alpha + y_o\cos\alpha) \equiv \mathscr{R}(\alpha) f_i(x_i, y_i) \ . \tag{2.41}$$

Since an arbitrary 2×2 matrix $\mathbf{A}$ can always be expressed as a product of a positive-definite symmetric matrix $\mathbf{S} = (\mathbf{A}\mathbf{A}^t)^{1/2} = \mathbf{S}^t$ and a rotation matrix $\mathbf{U}_r(\alpha) = (\mathbf{A}\mathbf{A}^t)^{-1/2}\mathbf{A}$, we can easily consider the general magnifier as the cascade of a rotator and a *pure* magnifier: $\mathscr{M}(\mathbf{A}) = \mathscr{M}(\mathbf{S})\,\mathscr{R}(\alpha)$. The symmetric matrix $\mathbf{S}$ on its turn can be expressed in terms of its eigenvalues and eigenvectors as

$$\begin{aligned} \mathbf{S} = \begin{bmatrix} s_{xx} & s_{xy} \\ s_{xy} & s_{yy} \end{bmatrix} &= \begin{bmatrix} \cos\varphi_s & \sin\varphi_s \\ -\sin\varphi_s & \cos\varphi_s \end{bmatrix} \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} \cos\varphi_s & -\sin\varphi_s \\ \sin\varphi_s & \cos\varphi_s \end{bmatrix} \\ &\equiv \mathbf{U}_r(\varphi_s)\,\boldsymbol{\Lambda}(s_1, s_2)\,\mathbf{U}_r(-\varphi_s) \ , \end{aligned} \tag{2.42}$$

so that such a pure magnifier can be realized as a *separable* magnifier, i.e., a combination of two orthogonal one-dimensional magnifiers, oriented along the principal axes of the symmetry ellipse determined by $\mathbf{S}$ by embedding this combination in between two rotators: $\mathscr{M}(\mathbf{S}) = \mathscr{R}(\varphi_s)\,\mathscr{M}[\boldsymbol{\Lambda}(s_1, s_2)]\,\mathscr{R}(-\varphi_s)$. Each of the two one-dimensional magnifiers, with magnification factor s_1 and s_2, can then be decomposed, if necessary, in the form (2.33). The general magnifier $\mathscr{M}(\mathbf{A})$ can thus be decomposed as the cascade $\mathscr{M}(\mathbf{A}) = \mathscr{R}(\varphi_s)\,\mathscr{M}[\boldsymbol{\Lambda}(s_1, s_2)]\,\mathscr{R}(-\varphi_s + \alpha)$: a separable magnifier embedded in between two rotators (in general with different rotation angles). Of course, instead of $\mathscr{M}(\mathbf{A}) = \mathscr{M}(\mathbf{S})\,\mathscr{R}(\alpha)$, we may as well write $\mathscr{M}(\mathbf{A}) = \mathscr{R}(\alpha)\,\mathscr{M}(\mathbf{S}')$, where $\mathbf{S}' = (\mathbf{A}^t\mathbf{A})^{1/2}$.

Now that we have introduced the magnifier, we can easily derive the linear canonical transform of a scaled function and formulate the *scaling theorem*. We notice that scaling itself belongs to the class of linear canonical transformations, and we can thus use the cascadability property. We then observe that scaling of the input signal merely leads to a change of the parameterizing matrix. Indeed,

$$\mathscr{L}(\mathbf{T})\,|\det\mathbf{W}|^{-1/2}f(\mathbf{W}^{-1}\mathbf{r}) = \mathscr{L}(\mathbf{T})\,\mathscr{M}(\mathbf{W})f(\mathbf{r}) = \mathscr{L}(\tilde{\mathbf{T}})f(\mathbf{r})\,, \tag{2.43a}$$

where $\tilde{\mathbf{T}}$ takes the form

$$\tilde{\mathbf{T}} = \begin{bmatrix} \tilde{\mathbf{A}} & \tilde{\mathbf{B}} \\ \tilde{\mathbf{C}} & \tilde{\mathbf{D}} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{W} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}^{t-1} \end{bmatrix} = \begin{bmatrix} \mathbf{AW} & \mathbf{BW}^{t-1} \\ \mathbf{CW} & \mathbf{DW}^{t-1} \end{bmatrix}. \tag{2.43b}$$

It may be interesting to see under what conditions scaling of the input signal by a matrix $\mathbf{W}_i$ produces only a scaling of the output signal by a (possibly different) matrix $\mathbf{W}_o$, i.e., $\mathscr{L}(\mathbf{T})\,\mathscr{M}(\mathbf{W}_i) = \mathscr{M}(\mathbf{W}_o)\,\mathscr{L}(\mathbf{T})$. This has been extensively studied in [7].

The second particular case is the D-dimensional *quadratic-phase modulator*

$$f_o(\mathbf{r}) = \exp[\mathrm{i}\,\pi\,\mathbf{r}^t\mathbf{C}\mathbf{r}]f_i(\mathbf{r}) \equiv \mathscr{Q}(-\mathbf{C})f_i(\mathbf{r})\,, \tag{2.44}$$

which results for $\mathbf{A} = \mathbf{D} = \mathbf{I}$ (and hence also $\mathbf{C} = \mathbf{C}^t$). Since $\mathbf{C}$ is symmetric, it can again be expressed in terms of its eigenvalues and eigenvectors and realized as a separable magnifier embedded in between two rotators: $\mathscr{Q}(-\mathbf{C}) = \mathscr{R}(\varphi_c)\,\mathscr{Q}[-\boldsymbol{\Lambda}(c_1,c_2)]\,\mathscr{R}(-\varphi_c)$.

The general case, represented by Eq. (2.10), which may be called a *generalized magnifier*, results as the cascade of a magnifier $\mathscr{M}(\mathbf{A})$ and a quadratic-phase modulator $\mathscr{Q}(-\mathbf{CA}^{-1})$,

$$f_o(\mathbf{r}) = \mathscr{Q}(-\mathbf{CA}^{-1})\,\mathscr{M}(\mathbf{A})f_i(\mathbf{r}) \tag{2.45}$$

and the transformation matrix has been decomposed as

$$\begin{aligned} \mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{A}^{t-1} \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{CA}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^{t-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{CA}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{AA}^t)^{1/2} & \mathbf{0} \\ \mathbf{0} & (\mathbf{AA}^t)^{-1/2} \end{bmatrix} \begin{bmatrix} \mathbf{U}_r(\alpha) & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_r(\alpha) \end{bmatrix}. \end{aligned} \tag{2.46}$$

We can of course change the order and write $\mathscr{M}(\mathbf{A})\,\mathscr{Q}(-\mathbf{A}^t\mathbf{C})$ instead. In one dimension, the positive-definite matrix $(\mathbf{AA}^t)^{1/2}$ reduces to $|a|$, and the rotation matrix $\mathbf{U}_r(\alpha)$ reduces to $\operatorname{sgn} a$, which takes care of a possible negative sign in a. We thus have $\mathscr{Q}(-ca^{-1})\,\mathscr{M}(a) = \mathscr{Q}(-ca^{-1})\,\mathscr{M}(|a|)\,\mathscr{M}(\operatorname{sgn} a)$, but the decomposition of the magnifier $\mathscr{M}(a)$ into the cascade $\mathscr{M}(|a|)\,\mathscr{M}(\operatorname{sgn} a)$ is rather irrelevant.

For easy reference, some linear canonical transformations for the case $\mathbf{B} = \mathbf{0}$, treated in this section, have been collected in Table 2.4.

2.4.2 *The Case* $\mathbf{B}^{-1} = -\mathbf{C}^t$ *and Consequently* $\mathbf{A}\mathbf{D}^t = \mathbf{0}$

Let us first consider the special case that both $\mathbf{A}$ and $\mathbf{D}$ vanish, for which the integral (2.9) reduces to

$$f_o(\mathbf{r}_o) = (\det \mathrm{i}\,\mathbf{B})^{-1/2} \int \exp[-\mathrm{i}\,2\pi\,\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o] f_i(\mathbf{r}_i)\,\mathrm{d}\mathbf{r}_i\ . \tag{2.47}$$

We already studied the case $\mathbf{B} = \mathbf{I}$, which led to the common Fourier transformation. The general expression (2.47), with $\mathbf{B} \neq \mathbf{I}$, can be interpreted as a *scaled Fourier transformation*, i.e., a Fourier transformation with an additional scaling (and rotation if $\mathbf{B} \neq \mathbf{B}^t$):

$$f_o(\mathbf{r}_o) = (\det \mathrm{i}\,\mathbf{B})^{-1/2}\bar{f}_i(\mathbf{B}^{-1}\mathbf{r}_o) = \mathscr{M}(\mathbf{B})\,\mathscr{F} f_i(\mathbf{r}_i) = \mathscr{F}\mathscr{M}(\mathbf{B}^{t^{-1}}) f_i(\mathbf{r}_i)\ . \tag{2.48}$$

If, moreover, we allow $\mathbf{D} \neq \mathbf{0}$, we get an additional quadratic-phase modulation:

$$f_o(\mathbf{r}) = \mathscr{Q}(-\mathbf{D}\mathbf{B}^{-1})\,\mathscr{M}(\mathbf{B})\,\mathscr{F} f_i(\mathbf{r})\ . \tag{2.49}$$

The case $\mathbf{A} \neq \mathbf{0}$ can be considered as the dual of the case $\mathbf{D} \neq \mathbf{0}$, as described in Sect. 2.3.1.3; we only have to replace $\mathscr{F} f_i(\mathbf{r})$ by $f_i(\mathbf{r})$, $f_o(\mathbf{r})$ by $\mathscr{F}^{-1} f_o(\mathbf{r})$, $\mathbf{D}$ by $\mathbf{A}$ and $\mathbf{B}$ by $-\mathbf{C}$ in Eq. (2.49) and get

$$f_o(\mathbf{r}) = \mathscr{F}\mathscr{Q}(\mathbf{A}\mathbf{C}^{-1})\,\mathscr{M}(-\mathbf{C}) f_i(\mathbf{r})\ . \tag{2.50}$$

For easy reference, some linear canonical transformations for the case $\mathbf{B}^{-1} = -\mathbf{C}^t$, treated in this section, have been collected in Table 2.4.

2.4.3 *The Case* $\mathbf{C} = \mathbf{0}$ *and Consequently* $\mathbf{A}^{-1} = \mathbf{D}^t$

We consider the case $\mathbf{C} = \mathbf{0}$ as the dual of the case $\mathbf{B} = \mathbf{0}$, see Sect. 2.4.1. We thus start with the expression (2.45), replace the signals by their Fourier transforms, replace $\mathbf{A}$ by $\mathbf{D}$ and $\mathbf{C}$ by $-\mathbf{B}$, and get the general expression

$$\bar{f}_o(\mathbf{q}) = |\det \mathbf{D}|^{-1/2} \mathscr{Q}(\mathbf{B}\mathbf{D}^{-1})\bar{f}_i(\mathbf{D}^{-1}\mathbf{q})\ , \tag{2.51}$$

which is the dual of (2.45). The case $\mathbf{A} = \mathbf{D} = \mathbf{I}$ (and hence also $\mathbf{B} = \mathbf{B}^t$) is special again and leads to a mere multiplication (but now in the frequency domain) by a quadratic phase-function: $\bar{f}_o(\mathbf{q}) = \exp[-\mathrm{i}\,\pi\,\mathbf{q}^t\mathbf{B}\mathbf{q}]\bar{f}_i(\mathbf{q})$. This is the Fourier domain version of a D-dimensional *Fresnel transformation*, which in terms of the variables $\mathbf{r}_o$ and $\mathbf{r}_i$ reads

$$
\begin{aligned}
f_o(\mathbf{r}_o) &= (\det \mathrm{i}\,\mathbf{B})^{-1/2} \int \exp[\mathrm{i}\,\pi(\mathbf{r}_o - \mathbf{r}_i)^t \mathbf{B}^{-1} (\mathbf{r}_o - \mathbf{r}_i)] f_i(\mathbf{r}_i)\, \mathrm{d}\mathbf{r}_i \\
&= \mathscr{F}^{-1} \mathscr{Q}(\mathbf{B})\, \mathscr{F} f_i(\mathbf{r}_i) \equiv \mathscr{S}(\mathbf{B}) f_i(\mathbf{r}_i)
\end{aligned}
\tag{2.52}
$$

and takes the form of a *convolution* with a quadratic-phase function. The general case with an additional scaling by $\mathbf{A}$,

$$
\begin{aligned}
f_o(\mathbf{r}_o) &= (\det \mathrm{i}\,\mathbf{B})^{-1/2} \int \exp[\mathrm{i}\,\pi(\mathbf{r}_o - \mathbf{A}\mathbf{r}_i)^t (\mathbf{B}\mathbf{A}^t)^{-1} (\mathbf{r}_o - \mathbf{A}\mathbf{r}_i)] f_i(\mathbf{r}_i)\, \mathrm{d}\mathbf{r}_i \\
&= \mathscr{S}(\mathbf{B}\mathbf{A}^t)\, \mathscr{M}(\mathbf{A}) f_i(\mathbf{r}_i) = \mathscr{M}(\mathbf{A})\, \mathscr{S}(\mathbf{A}^{-1}\mathbf{B}) f_i(\mathbf{r}_i)\ ,
\end{aligned}
\tag{2.53}
$$

may be called a *scaled Fresnel transformation*.

For easy reference, some linear canonical transformations for the case $\mathbf{C} = \mathbf{0}$, treated in this section, have been collected in Table 2.4.

2.4.4 The Case $\mathbf{T}^{-1} = \mathbf{T}^t$*: Phase-Space Rotators*

We now concentrate on the important class of transformation matrices that are not only symplectic, $\mathbf{T}^{-1} = \mathbf{J}\mathbf{T}^t\mathbf{J}$, but also orthogonal, $\mathbf{T}^{-1} = \mathbf{T}^t$. We call such matrices *orthosymplectic*. We easily see that $\mathbf{A} = \mathbf{D}$ and $\mathbf{B} = -\mathbf{C}$, and that the combination $\mathbf{A} + \mathrm{i}\,\mathbf{B} = \mathbf{U}$ is a unitary matrix: $\mathbf{U}^{-1} = \mathbf{U}^\dagger$. We thus have

$$
\mathbf{T} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ -\mathbf{B} & \mathbf{A} \end{bmatrix} \quad \text{and} \quad (\mathbf{A} - \mathrm{i}\,\mathbf{B})^t = \mathbf{U}^\dagger = \mathbf{U}^{-1} = (\mathbf{A} + \mathrm{i}\,\mathbf{B})^{-1}\ . \tag{2.54}
$$

In the one-dimensional case, with the scalar matrix entries $a = d = \cos\gamma$ and $b = -c = \sin\gamma$, the matrix $\mathbf{T}$ reduces to the rotation matrix $\mathbf{U}_r(\gamma)$. Note that this represents a *rotation in phase space*,

$$
\begin{aligned}
x_o &= x_i \cos\gamma \quad + u_i \sin\gamma\ , \\
u_o &= -x_i \sin\gamma + u_i \cos\gamma\ ,
\end{aligned}
\tag{2.55}
$$

and that the corresponding operation is known as the *fractional Fourier transformation*, see Sect. 2.3.2. The extension from the one-dimensional case to a higher-dimensional *separable* fractional Fourier transformer (with *diagonal* matrices $\mathbf{A}$ and $\mathbf{B}$, and possibly different fractional angles γ for the different coordinates) is straightforward.

In the two-dimensional case, we observe three basic systems with an orthogonal transformation matrix: (1) the *separable fractional Fourier transformer* $\mathscr{F}(\gamma_x, \gamma_y)$, (2) the *rotator* $\mathscr{R}(\varphi)$, which we already met in Sect. 2.4.1, and (3) the *gyrator* $\mathscr{G}(\varphi)$. Their unitary representations $\mathbf{U}_f(\gamma_x, \gamma_y)$, $\mathbf{U}_r(\varphi)$ and $\mathbf{U}_g(\varphi)$ take the forms

Table 2.4 Some linear canonical transformations for the cases $\mathbf{B}=\mathbf{0}$, $\mathbf{B}^{-1}=-\mathbf{C}^t$ and $\mathbf{C}=\mathbf{0}$

Operator	Transformation matrix	Remark
Magnifier $\mathscr{M}(\mathbf{A})$	$\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}^{t-1}\end{bmatrix}$	
Rotator $\mathscr{R}(\alpha)$	$\begin{bmatrix}\mathbf{U}_r(\alpha) & \mathbf{0}\\ \mathbf{0} & \mathbf{U}_r(\alpha)\end{bmatrix}$	$\mathbf{U}_r(\alpha)=\begin{bmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{bmatrix}$
$\mathscr{M}(\mathbf{A})=\mathscr{M}(\mathbf{S}_1)\mathscr{R}(\alpha)=\mathscr{R}(\alpha)\mathscr{M}(\mathbf{S}_2)$	$\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{A}^{t-1}\end{bmatrix}$	$\mathbf{S}_1=(\mathbf{A}\mathbf{A}^t)^{1/2}=\mathbf{S}_1^t$ $\mathbf{U}_r(\alpha)=\mathbf{S}_1^{-1}\mathbf{A}=\mathbf{A}\mathbf{S}_2^{-1}$ $\mathbf{S}_2=(\mathbf{A}^t\mathbf{A})^{1/2}=\mathbf{S}_2^t$
Pure magnifier $\mathscr{M}(\mathbf{S})=\mathscr{R}(\varphi_s)\mathscr{M}(\mathbf{\Lambda}_s)\mathscr{R}(-\varphi_s)$	$\begin{bmatrix}\mathbf{S} & \mathbf{0}\\ \mathbf{0} & \mathbf{S}^{-1}\end{bmatrix}$	$\mathbf{S}=\mathbf{S}^t=\mathbf{U}_r(\varphi_s)\mathbf{\Lambda}_s\mathbf{U}(-\varphi_s)$
Quadratic-phase modulator $\mathscr{Q}(-\mathbf{C})=\mathscr{R}(\varphi_c)\mathscr{Q}(-\mathbf{\Lambda}_c)\mathscr{R}(-\varphi_c)$	$\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{C} & \mathbf{I}\end{bmatrix}$	$\mathbf{C}=\mathbf{C}^t=\mathbf{U}_r(\varphi_c)\mathbf{\Lambda}_c\mathbf{U}(-\varphi_c)$
Generalized magnifier $\mathscr{Q}(-\mathbf{C}\mathbf{A}^{-1})\mathscr{M}(\mathbf{A})=\mathscr{M}(\mathbf{A})\mathscr{Q}(-\mathbf{A}^t\mathbf{C})$	$\begin{bmatrix}\mathbf{A} & \mathbf{0}\\ \mathbf{C} & \mathbf{A}^{t-1}\end{bmatrix}$	
Fourier transformation $\mathscr{F}$	$\begin{bmatrix}\mathbf{0} & \mathbf{I}\\ -\mathbf{I} & \mathbf{0}\end{bmatrix}$	
Scaled Fourier transformation $\mathscr{M}(\mathbf{B})\mathscr{F}=\mathscr{F}\mathscr{M}(\mathbf{B}^{t-1})$	$\begin{bmatrix}\mathbf{0} & \mathbf{B}\\ -\mathbf{B}^{t-1} & \mathbf{0}\end{bmatrix}$	
$\mathscr{Q}(-\mathbf{D}\mathbf{B}^{-1})\mathscr{M}(\mathbf{B})\mathscr{F}$	$\begin{bmatrix}\mathbf{0} & \mathbf{B}\\ -\mathbf{B}^{t-1} & \mathbf{D}\end{bmatrix}$	
$\mathscr{F}\mathscr{Q}(\mathbf{A}\mathbf{C}^{-1})\mathscr{M}(-\mathbf{C})$	$\begin{bmatrix}\mathbf{A} & -\mathbf{C}^{t-1}\\ \mathbf{C} & \mathbf{0}\end{bmatrix}$	
Fresnel transformation $\mathscr{S}(\mathbf{B})=\mathscr{F}^{-1}\mathscr{Q}(\mathbf{B})\mathscr{F}=\mathscr{F}\mathscr{Q}(\mathbf{B})\mathscr{F}^{-1}$	$\begin{bmatrix}\mathbf{I} & \mathbf{B}\\ \mathbf{0} & \mathbf{I}\end{bmatrix}$	$\mathbf{B}=\mathbf{B}^t$
Scaled Fresnel transformation $\mathscr{S}(\mathbf{B}\mathbf{A}^t)\mathscr{M}(\mathbf{A})=\mathscr{M}(\mathbf{A})\mathscr{S}(\mathbf{A}^{-1}\mathbf{B})$	$\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{0} & \mathbf{A}^{t-1}\end{bmatrix}$	

$$\mathbf{U}_f(\gamma_x,\gamma_y) = \begin{bmatrix} \exp(\mathrm{i}\,\gamma_x) & 0 \\ 0 & \exp(\mathrm{i}\,\gamma_y) \end{bmatrix}, \quad \mathbf{U}_r(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix},$$
$$\text{and} \quad \mathbf{U}_g(\varphi) = \begin{bmatrix} \cos\varphi & \mathrm{i}\,\sin\varphi \\ \mathrm{i}\,\sin\varphi & \cos\varphi \end{bmatrix}. \tag{2.56}$$

Note that $\det[\mathbf{U}_f(\gamma_x,\gamma_y)] = \exp[\mathrm{i}\,(\gamma_x+\gamma_y)]$ and $\det[\mathbf{U}_r(\varphi)] = \det[\mathbf{U}_g(\varphi)] = 1$. These three basic systems are additive in their parameters and correspond to *rotations in phase space*: the rotator $\mathscr{R}(\varphi)$ performs a rotation in the xy and the uv planes, the gyrator $\mathscr{G}(\varphi)$ in the xv and the yu planes, and the separable fractional Fourier transformer $\mathscr{F}(\gamma_x,\gamma_y)$ in the xu plane (through an angle γ_x) and the yv plane (through an angle γ_y), see also [75, Sect. 10.3, $\mathsf{U}(2)$ fractional Fourier transformers]. We remark that the *symmetric* fractional Fourier transformer $\mathscr{F}(\gamma,\gamma)$ is described by a *scalar* matrix $\mathbf{U}_f(\gamma,\gamma) = \exp(\mathrm{i}\,\gamma)\,\mathbf{I}$ and that it commutes with any other phase-space rotator $\mathscr{O}(\mathbf{U})$.

We easily verify, for instance, by expressing the unitary matrix $\mathbf{U}$ in the form

$$\mathbf{U} = \begin{bmatrix} \exp(\mathrm{i}\,\gamma_x)\cos\varphi & -\exp[\mathrm{i}\,(\gamma_y+\gamma)]\sin\varphi \\ \exp[\mathrm{i}\,(\gamma_x-\gamma)]\sin\varphi & \exp(\mathrm{i}\,\gamma_y)\cos\varphi \end{bmatrix}, \tag{2.57}$$

that the input–output relation for a phase-space rotator can be expressed in the form

$$\mathbf{r}_o - \mathrm{i}\,\mathbf{q}_o = \mathbf{U}(\mathbf{r}_i - \mathrm{i}\,\mathbf{q}_i), \tag{2.58}$$

which is an easy alternative for (2.6).

Any two-dimensional phase-space rotator can be realized as a cascade of the three basic phase-space rotators, in which cascade only two different kinds are actually needed. The gyrator $\mathscr{G}(\varphi)$, for instance, can be realized as the cascade $\mathscr{R}(-\frac{1}{4}\pi)\,\mathscr{F}(\varphi,-\varphi)\,\mathscr{R}(\frac{1}{4}\pi)$, which represents in fact a separable fractional Fourier transformer oriented at an angle of $\frac{1}{4}\pi$. Some important decompositions of an arbitrary phase-space rotator are considered in the next section.

For easy reference, some linear canonical transformations for the case $\mathbf{T}^{-1} = \mathbf{T}^t$, treated in this section, have been collected in Table 2.5.

2.4.4.1 Decompositions of Phase-Space Rotators

If we start with the general expression (2.57) for the unitary matrix $\mathbf{U}$, we are immediately led to the decomposition

$$\mathscr{O}(\mathbf{U}) = \mathscr{F}\left(\tfrac{1}{2}\gamma, -\tfrac{1}{2}\gamma\right)\mathscr{R}(-\varphi)\,\mathscr{F}\left(\gamma_x - \tfrac{1}{2}\gamma, \gamma_y + \tfrac{1}{2}\gamma\right), \tag{2.59}$$

Table 2.5 Some linear canonical transformations for the case $\mathbf{T}^{-1} = \mathbf{T}^t$

Operator	Transformation matrix	Remark
General phase-space rotator $\mathscr{O}(\mathbf{U}) = \mathscr{O}(\mathbf{X} + \mathrm{i}\,\mathbf{Y})$	$\begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{bmatrix}$	$\mathbf{U} = \mathbf{X} + \mathrm{i}\,\mathbf{Y}\,;\mathbf{X} = \Re\mathbf{U}\,,\mathbf{Y} = \Im\mathbf{U}$ $\mathbf{U}^{-1} = \mathbf{U}^{\dagger}$
Rotator $\mathscr{R}(\varphi)$	$\begin{bmatrix} \mathbf{X}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_r \end{bmatrix}$	$\mathbf{U}_r(\varphi) = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} = \mathbf{X}_r$
Separable fractional FT $\mathscr{F}(\gamma_x, \gamma_y)$	$\begin{bmatrix} \mathbf{X}_f & \mathbf{Y}_f \\ -\mathbf{Y}_f & \mathbf{X}_f \end{bmatrix}$	$\mathbf{U}_f(\gamma_x, \gamma_y) = \begin{bmatrix} \exp(\mathrm{i}\,\gamma_x) & 0 \\ 0 & \exp(\mathrm{i}\,\gamma_y) \end{bmatrix}$
Gyrator $\mathscr{G}(\varphi) = \mathscr{R}(-\frac{1}{4}\pi)\mathscr{F}(\varphi, -\varphi)\mathscr{R}(\frac{1}{4}\pi)$	$\begin{bmatrix} \mathbf{X}_g & \mathbf{Y}_g \\ -\mathbf{Y}_g & \mathbf{X}_g \end{bmatrix}$	$\mathbf{U}_g(\varphi) = \begin{bmatrix} \cos\varphi & \mathrm{i}\,\sin\varphi \\ \mathrm{i}\,\sin\varphi & \cos\varphi \end{bmatrix}$
$\mathscr{O}(\mathbf{U})\,\mathscr{F}(\gamma, \gamma) = \mathscr{F}(\gamma, \gamma)\,\mathscr{O}(\mathbf{U})$		$\mathbf{U}_f(\gamma, \gamma) = \exp(\mathrm{i}\,\gamma)\,\mathbf{I}$

Table 2.6 Some useful decompositions of phase-space rotators

Cascade of basic phase-space rotators	Equation	Remark
$\mathscr{O}(\mathbf{U}) = \mathscr{F}(\frac{1}{2}\gamma, -\frac{1}{2}\gamma)\,\mathscr{R}(-\varphi)\,\mathscr{F}(\gamma_x - \frac{1}{2}\gamma, \gamma_y + \frac{1}{2}\gamma)$	(2.59)	$\begin{aligned} u_{xx} &= \exp(\mathrm{i}\,\gamma_x)\cos\varphi \\ u_{xy} &= -\exp[\mathrm{i}\,(\gamma_y + \gamma)]\sin\varphi \\ u_{yx} &= \exp[\mathrm{i}\,(\gamma_x - \gamma)]\sin\varphi \\ u_{yy} &= \exp(\mathrm{i}\,\gamma_y)\cos\varphi \end{aligned}$
$\mathscr{O}(\mathbf{U}) = \mathscr{R}(-\alpha)\,\mathscr{G}(-\beta)\,\mathscr{F}(-\psi, \psi)\,\mathscr{F}(\gamma_x, \gamma_y)$	(2.60)	$\begin{aligned} \sin 2\beta &= \sin 2\varphi \sin\gamma \\ \cos 2\alpha &= \cos 2\varphi / \cos 2\beta \\ \tan\psi &= \tan\alpha \tan\beta \end{aligned}$
$\mathscr{O}(\mathbf{U}) = \mathscr{R}(\varphi_2)\,\mathscr{F}(\gamma_1, \gamma_2)\,\mathscr{R}(\varphi_1)$	(2.61)	(2.61b) – (2.61g), see also [5]
$\mathscr{F}(\pm\frac{1}{4}\pi, \mp\frac{1}{4}\pi)\,\mathscr{G}(\pm\varphi) = \mathscr{R}(-\varphi)\,\mathscr{F}(\pm\frac{1}{4}\pi, \mp\frac{1}{4}\pi)$		
$\mathscr{F}(\pm\frac{1}{4}\pi, \mp\frac{1}{4}\pi)\,\mathscr{R}(\pm\varphi) = \mathscr{G}(\varphi)\,\mathscr{F}(\pm\frac{1}{4}\pi, \mp\frac{1}{4}\pi)$		
$\mathscr{R}(\pm\frac{1}{4}\pi)\,\mathscr{F}(\pm\varphi, \mp\varphi) = \mathscr{G}(-\varphi)\,\mathscr{R}(\pm\frac{1}{4}\pi)$		
$\mathscr{G}(\pm\frac{1}{4}\pi)\,\mathscr{F}(\pm\varphi, \mp\varphi) = \mathscr{R}(\varphi)\,\mathscr{G}(\pm\frac{1}{4}\pi)$		
$\mathscr{R}(\pm\frac{1}{4}\pi)\,\mathscr{G}(\pm\varphi) = \mathscr{F}(\varphi, -\varphi)\,\mathscr{R}(\pm\frac{1}{4}\pi)$		
$\mathscr{G}(\pm\frac{1}{4}\pi)\,\mathscr{R}(\pm\varphi) = \mathscr{F}(-\varphi, \varphi)\,\mathscr{G}(\pm\frac{1}{4}\pi)$		

where we recognize a rotator embedded in between two separable fractional Fourier transformers. This decomposition, along with the other two that are introduced in this section, have been collected in Table 2.6.

From the many other decompositions of a general phase-space rotator $\mathscr{O}(\mathbf{U})$ into the more basic ones, we mention in particular the cascade of a separable fractional Fourier transformer, a gyrator and a rotator [25],

$$\mathscr{O}(\mathbf{U}) = \mathscr{R}(-\alpha)\,\mathscr{G}(-\beta)\,\mathscr{F}(\gamma_x - \psi, \gamma_y + \psi)\ , \tag{2.60a}$$

which follows directly from the equality

$$\mathscr{F}\left(\tfrac{1}{2}\gamma, -\tfrac{1}{2}\gamma\right)\,\mathscr{R}(-\varphi)\,\mathscr{F}\left(-\tfrac{1}{2}\gamma, \tfrac{1}{2}\gamma\right) = \mathscr{R}(-\alpha)\,\mathscr{G}(-\beta)\,\mathscr{F}(-\psi, \psi);$$

the angles α, β and ψ in (2.60a) follow from γ and φ in (2.59) by

$$\sin 2\beta = \sin 2\varphi \sin\gamma\ , \tag{2.60b}$$

$$\cos 2\varphi = \cos 2\alpha \cos 2\beta\ , \tag{2.60c}$$

$$\tan\psi = \tan\alpha \tan\beta\ . \tag{2.60d}$$

We will use this decomposition later in Sect. 2.8.3.

Another useful decomposition of a general phase-space rotator $\mathscr{O}(\mathbf{U})$ takes the form of a separable fractional Fourier transformer embedded in between two rotators [5], see also [75, Sect. 10.3.3, $\mathsf{SU}(2)$-Fourier transformer]:

$$\mathscr{O}(\mathbf{U}) = \mathscr{R}(\varphi_2)\,\mathscr{F}(\gamma_1, \gamma_2)\,\mathscr{R}(\varphi_1)\ . \tag{2.61a}$$

Without loss of generality, we may choose $0 \le \gamma_2 \le \gamma_1 < \pi$ and $0 \le \varphi_1 < \pi$, after which the four angles are unambiguous [5]. With $\mathbf{U} = \mathbf{X} + \mathrm{i}\,\mathbf{Y}$, the two fractional angles γ_1 and γ_2 follow easily from the relations

$$\exp[\mathrm{i}\,(\gamma_1 + \gamma_2)] = \det\mathbf{U}\ , \tag{2.61b}$$

$$\cos(\gamma_1 - \gamma_2) = \det\mathbf{X} + \det\mathbf{Y}\ ; \tag{2.61c}$$

and with $\mathbf{X} = [x_{11}, x_{12}; x_{21}, x_{22}]$ and $\mathbf{Y} = [y_{11}, y_{12}; y_{21}, y_{22}]$, the rotation angles φ_1 and φ_2 follow from the equations

$$x_{11} + x_{22} - y_{12} + y_{21} = 2\cos(\varphi_1 + \varphi_2 + \tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2)\ \cos\tfrac{1}{2}(\gamma_1 - \gamma_2), \tag{2.61d}$$

$$x_{12} - x_{21} + y_{11} + y_{22} = 2\ \sin(\varphi_1 + \varphi_2 + \tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2)\ \cos\tfrac{1}{2}(\gamma_1 - \gamma_2), \tag{2.61e}$$

$$-x_{11} + x_{22} + y_{12} + y_{21} = 2\ \sin(\varphi_1 - \varphi_2 + \tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2)\ \sin\tfrac{1}{2}(\gamma_1 - \gamma_2), \tag{2.61f}$$

$$x_{12} + x_{21} + y_{11} - y_{22} = 2\cos(\varphi_1 - \varphi_2 + \tfrac{1}{2}\gamma_1 + \tfrac{1}{2}\gamma_2)\ \sin\tfrac{1}{2}(\gamma_1 - \gamma_2). \tag{2.61g}$$

For the details we refer to [5]. We will use this decomposition in Sect. 2.5 to treat the case of a transformation matrix with $\det\mathbf{B} = 0$ but $\mathbf{B} \neq \mathbf{0}$. For completeness we mention [22] that the eigenvalues $\exp(\mathrm{i}\,\vartheta_1)$ and $\exp(\mathrm{i}\,\vartheta_2)$, say, of the unitary matrix $\mathbf{U} = \mathbf{U}_r(\varphi_2)\,\mathbf{U}_f(\gamma_1, \gamma_2)\,\mathbf{U}_r(\varphi_1)$, are related to the rotation angles φ_1 and φ_2 and the fractional angles γ_1 and γ_2 by the relationship

$$\vartheta_{1,2} = \tfrac{1}{2}(\gamma_1 + \gamma_2) \pm \arccos\left\{\cos(\varphi_1 + \varphi_2)\cos\left[\tfrac{1}{2}(\gamma_1 - \gamma_2)\right]\right\}. \tag{2.62}$$

For easy reference, we have collected in Table 2.6 also some useful relations between the rotator $\mathscr{R}(\varphi)$, the gyrator $\mathscr{G}(\varphi)$, and the antisymmetric fractional Fourier transformer $\mathscr{F}(\varphi, -\varphi)$, where the argument of one of them takes the value $\pm\frac{1}{4}\pi$.

2.5 Modified Iwasawa Decomposition

Any symplectic matrix can be decomposed in the *modified Iwasawa form* [55] (see also [75, Sects. 9.5 and 10.2]) as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{G} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{bmatrix} \tag{2.63}$$

with

$$\mathbf{G} = -(\mathbf{C}\mathbf{A}^t + \mathbf{D}\mathbf{B}^t)(\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t)^{-1} = \mathbf{G}^t, \tag{2.64a}$$

$$\mathbf{S} = (\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t)^{1/2} = \mathbf{S}^t, \tag{2.64b}$$

$$\mathbf{X} + \mathrm{i}\,\mathbf{Y} = (\mathbf{A}\mathbf{A}^t + \mathbf{B}\mathbf{B}^t)^{-1/2}(\mathbf{A} + \mathrm{i}\,\mathbf{B}) = (\mathbf{X} + \mathrm{i}\,\mathbf{Y})^{\dagger}. \tag{2.64c}$$

The first matrix is the transformation matrix of a quadratic-phase modulator $\mathscr{Q}(\mathbf{G})$, the second one of a pure magnifier $\mathscr{M}(\mathbf{S})$, and the third one of a phase-space rotator $\mathscr{O}(\mathbf{U})$, with $\mathbf{U} \equiv \mathbf{X} + \mathrm{i}\,\mathbf{Y}$. The Iwasawa decomposition thus leads to the following cascade of any linear canonical transformer:

$$\mathscr{L}(\mathbf{T}) = \mathscr{Q}(\mathbf{G})\,\mathscr{M}(\mathbf{S})\,\mathscr{O}(\mathbf{U}). \tag{2.65}$$

We can formulate an Iwasawa-type decomposition $\mathscr{L}(\mathbf{T}) = \mathscr{O}(\hat{\mathbf{U}})\,\mathscr{M}(\hat{\mathbf{S}})\,\mathscr{Q}(\hat{\mathbf{G}})$ in the reversed order of the operators, by finding the Iwasawa decomposition of the inverse transformation $\mathscr{L}(\mathbf{T}^{-1})$ in its regular order, reversing the order of the operators, and replacing each operator by its inverse; see Table 2.7.

If we substitute the cascade (2.60a) into (2.65), the Iwasawa decomposition can be expressed in the more detailed form

$$\mathscr{L}(\mathbf{T}) = \mathscr{Q}(\mathbf{G})\,\mathscr{M}(\mathbf{S})\,\mathscr{R}(-\alpha)\,\mathscr{G}(-\beta)\,\mathscr{F}(\gamma_x - \psi, \gamma_y + \psi). \tag{2.66}$$

We will use this particular decomposition in Sect. 2.8.3.

If we substitute the cascade (2.61a) into (2.65), the Iwasawa decomposition reads

$$\mathscr{L}(\mathbf{T}) = \mathscr{Q}(\mathbf{G})\,\mathscr{M}(\mathbf{S})\,\mathscr{R}(\varphi_2)\,\mathscr{F}(\gamma_1, \gamma_2)\,\mathscr{R}(\varphi_1), \tag{2.67}$$

Table 2.7 Some useful decompositions of the linear canonical transformation

Operator	Remark
$\mathscr{L}(\mathbf{T}) \qquad \mathbf{T} = \begin{bmatrix} \mathbf{A}\ \mathbf{B} \\ \mathbf{C}\ \mathbf{D} \end{bmatrix}$	$\begin{bmatrix} \mathbf{A}\ \mathbf{B} \\ \mathbf{C}\ \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{D}^t & -\mathbf{B}^t \\ -\mathbf{C}^t & \mathbf{A}^t \end{bmatrix}$
Iwasawa decomposition $\mathscr{L}(\mathbf{T}) = \mathscr{Q}(\mathbf{G})\mathscr{M}(\mathbf{S})\mathscr{O}(\mathbf{U})$	$\mathbf{G} = -(\mathbf{CA}^t + \mathbf{DB}^t)(\mathbf{AA}^t + \mathbf{BB}^t)^{-1} = \mathbf{G}^t$ $\mathbf{S} = (\mathbf{AA}^t + \mathbf{BB}^t)^{1/2} = \mathbf{S}^t$ $\mathbf{U} = (\mathbf{AA}^t + \mathbf{BB}^t)^{-1/2}(\mathbf{A} + \mathrm{i}\,\mathbf{B}) = \mathbf{U}^\dagger$
Iwasawa decomposition in reversed order $\mathscr{L}(\mathbf{T}) = \mathscr{O}(\hat{\mathbf{U}})\mathscr{M}(\hat{\mathbf{S}})\mathscr{Q}(\hat{\mathbf{G}})$	$\hat{\mathbf{G}} = -(\mathbf{C}^t\mathbf{D} + \mathbf{A}^t\mathbf{B})(\mathbf{D}^t\mathbf{D} + \mathbf{B}^t\mathbf{B})^{-1} = \hat{\mathbf{G}}^t$ $\hat{\mathbf{S}} = (\mathbf{D}^t\mathbf{D} + \mathbf{B}^t\mathbf{B})^{-1/2} = \hat{\mathbf{S}}^t$ $\hat{\mathbf{U}} = (\mathbf{D} + \mathrm{i}\,\mathbf{B})(\mathbf{D}^t\mathbf{D} + \mathbf{B}^t\mathbf{B})^{-1/2} = \hat{\mathbf{U}}^\dagger$
$\mathscr{L}(\mathbf{T}) = \mathscr{Q}(-\mathbf{CA}^{-1})\mathscr{M}(\mathbf{A})\mathscr{S}(\mathbf{A}^{-1}\mathbf{B})$	$\det \mathbf{A} \neq 0$
$\mathscr{L}(\mathbf{T}) = \mathscr{S}(\mathbf{BD}^{-1})\mathscr{M}(\mathbf{D}^{t-1})\mathscr{Q}(-\mathbf{D}^{-1}\mathbf{C})$	$\det \mathbf{D} \neq 0$
$\mathscr{L}(\mathbf{T}) = \mathscr{Q}(-\mathbf{DB}^{-1})\mathscr{M}(\mathbf{B})\mathscr{F}\mathscr{Q}(-\mathbf{B}^{-1}\mathbf{A})$	$\det \mathbf{B} \neq 0$ see below (i)
$\mathscr{L}(\mathbf{T}) = \mathscr{Q}[(\mathbf{I}-\mathbf{D})\mathbf{B}^{-1}]\mathscr{S}(\mathbf{B})\mathscr{Q}[\mathbf{B}^{-1}(\mathbf{I}-\mathbf{A})]$	$\det \mathbf{B} \neq 0$ $\mathbf{B} = \mathbf{B}^t$
$\mathscr{L}(\mathbf{T}) = \mathscr{S}(\mathbf{AC}^{-1})\mathscr{M}(-\mathbf{C}^{t-1})\mathscr{F}\mathscr{S}(\mathbf{C}^{-1}\mathbf{D})$	$\det \mathbf{C} \neq 0$ see below (ii)
$\mathscr{L}(\mathbf{T}) = \mathscr{S}[(\mathbf{A}-\mathbf{I})\mathbf{C}^{-1}]\mathscr{Q}(-\mathbf{C})\mathscr{S}[\mathbf{C}^{-1}(\mathbf{D}-\mathbf{I})]$	$\det \mathbf{C} \neq 0$ $\mathbf{C} = \mathbf{C}^t$
$\mathscr{M}(\mathbf{B}^{-1})\mathscr{Q}(\mathbf{DB}^{-1})\mathscr{L}(\mathbf{T})\mathscr{Q}(\mathbf{B}^{-1}\mathbf{A}) = \mathscr{F}$	$\det \mathbf{B} \neq 0$ (i)
$\mathscr{M}(-\mathbf{C}^t)\mathscr{S}(-\mathbf{AC}^{-1})\mathscr{L}(\mathbf{T})\mathscr{S}(-\mathbf{C}^{-1}\mathbf{D}) = \mathscr{F}$	$\det \mathbf{C} \neq 0$ (ii)

which enables us to treat the case $\det \mathbf{B} = 0$. To do this, we write the explicit expression for the submatrix $\mathbf{B}$,

$$\mathbf{B} = \mathbf{S}\,\mathbf{U}_r(\varphi_2) \begin{bmatrix} \sin\gamma_1 & 0 \\ 0 & \sin\gamma_2 \end{bmatrix} \mathbf{U}_r(\varphi_1)\ , \tag{2.68}$$

and conclude that, since $\mathbf{S}$ and $\mathbf{U}_r$ are non-singular, the case $\det \mathbf{B} = 0$ arises only for $\sin\gamma_1 \sin\gamma_2 = 0$. The cascade (2.67) yields a clear physical interpretation of the linear canonical transformation. The cascade starts with a rotator $\mathscr{R}(\varphi_1)$ that rotates the coordinate system such that the new axes coincide with the axes of the separable fractional Fourier transformer $\mathscr{F}(\gamma_1, \gamma_2)$. This separable fractional Fourier transformer itself is responsible for a possible degeneration of the submatrix $\mathbf{B}$, but such a degeneration has a clear interpretation: it simply means that for one coordinate (or maybe even for both coordinates) the separable fractional Fourier transformer acts as an identity system. The cascade then continues with the rotator $\mathscr{R}(\varphi_2)$, followed by the pure magnifier $\mathscr{M}(\mathbf{S})$ and the quadratic-phase modulator $\mathscr{Q}(\mathbf{G})$. Equation (2.67) provides a useful representation of the linear canonical transformation, valid for any values of the transformation matrix. Note that this equation can also be used for numerical calculation of the canonical transform, using the algorithms developed for the fractional Fourier transformation [47, Sect. 6.7, Discrete computation of the fractional Fourier transform].

For completeness we recall that a different way to deal with a singular matrix $\mathbf{B}$ has been presented in [44]. It was shown that any symplectic matrix with a singular submatrix $\mathbf{B}$ can be decomposed as

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{B}' \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A}-\mathbf{B}'\mathbf{C} & \mathbf{B}-\mathbf{B}'\mathbf{D} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \tag{2.69}$$

in which $\mathbf{B}'$ is a non-singular *diagonal* matrix and $\det(\mathbf{B}-\mathbf{B}'\mathbf{D}) \neq 0$. The integral (2.9) can then be used for each of the two subsystems in this cascade separately, thus avoiding the singular case. The way to find the diagonal matrix $\mathbf{B}'$, however, is not easy.

2.5.1 Other Decompositions

The modified Iwasawa decomposition is valid for all values of the submatrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$. Many other decompositions are possible, where sometimes an additional condition need be satisfied. A few of them are collected in Table 2.7, see also [47, Sect. 3.4.4, Decompositions]. Two of these decompositions have been repeated at the bottom of the table in a form that shows the possibility to bring an operator $\mathscr{L}(\mathbf{T})$ into a simple Fourier transformer $\mathscr{F}$, for instance—if $\det \mathbf{B} \neq 0$—by a pre-modulation by $\mathscr{Q}(\mathbf{B}^{-1}\mathbf{A})$, a post-modulation by $\mathscr{Q}(\mathbf{D}\mathbf{B}^{-1})$, and a scaling by the final magnifier $\mathscr{M}(\mathbf{B}^{-1})$.

2.6 Linear Canonical Transforms of Selected Functions

In this section we study the linear canonical transforms of

1. A Gaussian signal and a harmonic signal;
2. A periodic signal, with a short detour to Talbot imaging; and
3. Hermite–Gauss modes, in two dimensions leading to Hermite–Laguerre–Gauss modes—with the Laguerre–Gauss modes as a special case—and to the design of mode converters.

2.6.1 Gaussian Signal and Harmonic Signal

Let us consider the *Gaussian signal*

$$f_i(\mathbf{r}) = \exp(-\pi\, \mathbf{r}^t \mathbf{L}_i \mathbf{r}) \tag{2.70}$$

and determine its linear canonical transform

$$f_o(\mathbf{r}_o) = \mathscr{L}(\mathbf{T})f_i(\mathbf{r}_i) = (\det \mathrm{i}\,\mathbf{B})^{-1/2} \exp(\mathrm{i}\,\pi\,\mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o)$$
$$\times \int \exp[-\pi\,\mathbf{r}_i^t(\mathbf{L}_i - \mathrm{i}\,\mathbf{B}^{-1}\mathbf{A})\mathbf{r}_i - \mathrm{i}\,2\pi\,\mathbf{r}_i^t\mathbf{B}^{-1}\mathbf{r}_o)]\,\mathrm{d}\mathbf{r}_i$$
$$= (\det \mathrm{i}\,\mathbf{B})^{-1/2}[\det(\mathbf{L}_i - \mathrm{i}\,\mathbf{B}^{-1}\mathbf{A})]^{-1/2} \exp(\mathrm{i}\,\pi\,\mathbf{r}_o^t\mathbf{D}\mathbf{B}^{-1}\mathbf{r}_o)$$
$$\times \exp[-\pi(\mathbf{B}^{-1}\mathbf{r}_o)^t(\mathbf{L}_i - \mathrm{i}\,\mathbf{B}^{-1}\mathbf{A})^{-1}(\mathbf{B}^{-1}\mathbf{r}_o)]\,, \tag{2.71}$$

where we have used the identity

$$\int \exp(-\pi\,\mathbf{s}^t\mathbf{P}\mathbf{s} - \mathrm{i}\,2\pi\,\mathbf{q}^t\mathbf{s})\,\mathrm{d}\mathbf{s} = (\det \mathbf{P})^{-1/2} \exp(-\pi\,\mathbf{q}^t\mathbf{P}^{-1}\mathbf{q})\,, \tag{2.72}$$

with $\mathbf{P}$ a symmetric matrix whose real part is positive definite. If we separate the part of the exponent that depends on $\mathbf{r}$ from the part that does not depend on $\mathbf{r}$, the transform can be written as [7]

$$\mathscr{L}(\mathbf{T}) \exp(-\pi\,\mathbf{r}^t\mathbf{L}_i\mathbf{r}) = [\det(\mathbf{A} + \mathrm{i}\,\mathbf{B}\mathbf{L}_i)]^{-1/2} \exp(-\pi\,\mathbf{r}^t\mathbf{L}_o\mathbf{r})\,, \tag{2.73}$$

where

$$\mathrm{i}\,\mathbf{L}_o = (\mathbf{C} + \mathrm{i}\,\mathbf{D}\mathbf{L}_i)(\mathbf{A} + \mathrm{i}\,\mathbf{B}\mathbf{L}_i)^{-1}\,. \tag{2.74}$$

We will meet the bilinear relationship (2.74) again in Sect. 2.8.4, cf. Eq. (2.128). For $\mathbf{A} = \mathbf{D} = \mathbf{0}$ and $\mathbf{B} = -\mathbf{C} = \mathbf{I}$, i.e., for a Fourier transformation, we get $\mathscr{F} \exp(-\pi\,\mathbf{r}^t\mathbf{L}_i\mathbf{r}) = (\det \mathrm{i}\,\mathbf{L}_i)^{-1/2} \exp(-\pi\,\mathbf{r}^t\mathbf{L}_i^{-1}\mathbf{r})$, as expected.

The linear canonical transform of the *constant signal* $f_i(\mathbf{r}) = 1$ arises for $\mathbf{L}_i = \mathbf{0}$, and reads $(\det \mathbf{A})^{-1/2} \exp(\mathrm{i}\,\pi\,\mathbf{r}^t\mathbf{C}\mathbf{A}^{-1}\mathbf{r})$; we have ignored convergence issues that may arise from the fact that we have lost the positive definiteness of the real part of $\mathbf{L}_i$. The transform of the *harmonic signal* $f_i(\mathbf{r}) = \exp(\mathrm{i}\,2\pi\,\mathbf{k}^t\mathbf{r})$ with frequency $\mathbf{k}$ then follows after applying the modulation property (2.25a) and takes the form

$$\mathscr{L}(\mathbf{T}) \exp(\mathrm{i}\,2\pi\,\mathbf{k}^t\mathbf{r}) = (\det \mathbf{A})^{-1/2} \exp(-\mathrm{i}\,\pi\,\mathbf{k}^t\mathbf{A}^{-1}\mathbf{B}\mathbf{k} + \mathrm{i}\,\pi\,\mathbf{r}^t\mathbf{C}\mathbf{A}^{-1}\mathbf{r}$$
$$+\mathrm{i}\,2\pi\,\mathbf{k}^t\mathbf{A}^{-1}\mathbf{r}). \tag{2.75}$$

In the limit $\mathbf{A} \Rightarrow \mathbf{0}$ and $\mathbf{B} = -\mathbf{C} = \mathbf{I}$, i.e., for a Fourier transformation, we get indeed the Dirac delta function $\delta(\mathbf{r}_o - \mathbf{k})$.

2.6.2 Periodic Signal and Talbot Imaging

Let us consider a *periodic signal* with periods p_x and p_y in the x and y directions. Such a signal can be represented as

$$f_i(\mathbf{r}) = \sum_{m,n=-\infty}^{\infty} a_{mn} \exp(\mathrm{i}\, 2\pi\, \mathbf{k}_{mn}^t \mathbf{r}) \ , \quad \mathbf{k}_{mn}^t = [m/p_x, n/p_y] \ . \tag{2.76}$$

The canonical transform of this periodic signal reads [7]

$$f_o(\mathbf{r}) = (\det \mathbf{A})^{-1/2} \sum_{m,n=-\infty}^{\infty} a_{mn} \exp(-\mathrm{i}\, \pi\, \mathbf{k}_{mn}^t \mathbf{A}^{-1} \mathbf{B} \mathbf{k}_{mn} + \mathrm{i}\, 2\pi\, \mathbf{k}_{mn}^t \mathbf{A}^{-1} \mathbf{r} + \mathrm{i}\, \pi\, \mathbf{r}^t \mathbf{C} \mathbf{A} \mathbf{r}) \ . \tag{2.77}$$

If $\mathbf{k}_{mn}^t \mathbf{A}^{-1} \mathbf{B} \mathbf{k}_{mn}$ is an even integer for all m and n, we get

$$f_o(\mathbf{r}) = (\det \mathbf{A})^{-1/2} \exp(\mathrm{i}\, \pi\, \mathbf{r}^t \mathbf{C} \mathbf{A}^{-1} \mathbf{r}) f_i(\mathbf{A}^{-1} \mathbf{r}) \ , \tag{2.78}$$

which corresponds to generalized *Talbot imaging*, cf. Eq. (2.10): an affine transformation of the input signal, possibly with an additional modulation by a quadratic-phase function.

Talbot imaging is well known in optics, where it appears for such a simple system as free space. Free space propagation is governed by the Fresnel transformation $\mathscr{S}(b\mathbf{I})$, and the imaging condition now requires that $\mathbf{k}_{mn}^t \mathbf{k}_{mn} b = (m^2/p_x^2 + n^2/p_y^2) b$ is an even integer. Since this equality has to hold for any integers m and n, we conclude that Talbot imaging appears for such a value of b that both b/p_x^2 and b/p_y^2 are even integers. In the case that $b/p_x^2 = \ell_x$ and $b/p_y^2 = \ell_y$ are integers, but not necessarily even, a kind of pseudo-imaging appears: the expansion coefficients a_{mn} are replaced by $-a_{mn}$ for those (m, n) combinations for which $m^2 \ell_x + n^2 \ell_y$ is an odd integer. Other simple examples of Talbot imaging are the separable fractional Fourier transformer $\mathscr{F}(\gamma_x, \gamma_y)$, for which the expression $\mathbf{k}_{mn}^t \mathbf{A}^{-1} \mathbf{B} \mathbf{k}_{mn}$ leads to the requirement that $\tan \gamma_x / p_x^2$ and $\tan \gamma_y / p_y^2$ should be even integers, and the gyrator $\mathscr{G}(\varphi)$, for which we get the condition that $\tan \varphi / p_x p_y$ is an integer [7].

2.6.3 Hermite–Gaussian-Type Modes and Mode Conversion

2.6.3.1 The One-dimensional Case

Based on the generating function $\exp(-s^2 + 2sz)$ for the Hermite polynomials $H_k(z)$, $k = 0, 1, \ldots$ [2, Sect. 22, Orthogonal polynomials; see Eq. (22.9.17)],

$$\sum_{k=0}^{\infty} H_k(z) \frac{s^k}{k!} = \exp(-s^2 + 2sz) \ , \tag{2.79}$$

the one-dimensional *Hermite–Gauss modes*

$$\mathsf{H}_k(x) = 2^{1/4} (2^k k!)^{-1/2} H_k(\sqrt{2\pi} x) \exp(-\pi\, x^2) \tag{2.80}$$

follow easily from their generating function

$$\sum_{k=0}^{\infty} \mathsf{H}_k(x) \left(\frac{2^k}{k!}\right)^{1/2} s^k = 2^{1/4} \exp(-s^2 + 2\sqrt{2\pi}\, sx - \pi\, x^2)\ . \tag{2.81}$$

We recall that the Hermite–Gauss modes $\mathsf{H}_k(x)$ form a complete orthonormal basis on the interval $-\infty \leq x \leq \infty$:

$$\int \mathsf{H}_m(x)\, \mathsf{H}_n(x)\, \mathrm{d}x = \delta_{m-n} = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n\ . \end{cases} \tag{2.82}$$

The linear canonical transforms $\mathscr{L}(\mathbf{T})\mathsf{H}_k(x) \equiv \mathsf{H}_k^{\mathbf{T}}(x)$ of these Hermite–Gauss modes follow from their generating function [6, 20, 22]

$$\sum_{k=0}^{\infty} \mathsf{H}_k^{\mathbf{T}}(x) \left(\frac{2^k}{k!}\right)^{1/2} s^k = \frac{2^{1/4}}{\sqrt{a+\mathrm{i}\,b}} \exp\left[-\pi\, \frac{d-\mathrm{i}\,c}{a+\mathrm{i}\,b}\, x^2\right]$$

$$\times \exp\left[-\left(s\sqrt{\frac{a-\mathrm{i}\,b}{a+\mathrm{i}\,b}}\right)^2 + 2\left(s\sqrt{\frac{a-\mathrm{i}\,b}{a+\mathrm{i}\,b}}\right)\frac{\sqrt{2\pi}\,x}{a^2+b^2}\right]\ , \tag{2.83}$$

which function can be found by applying the integral (2.9) to the generating function (2.81) and evaluating the integral using the identity (2.72). We compare the right-hand side of (2.83) with the generating function (2.79) of the Hermite polynomials and write

$$\sum_{k=0}^{\infty} \mathsf{H}_k^{\mathbf{T}}(x) \left(\frac{2^k}{k!}\right)^{1/2} s^k$$

$$= \frac{2^{1/4}}{\sqrt{a+\mathrm{i}\,b}} \exp\left[-\pi\, \frac{d-\mathrm{i}\,c}{a+\mathrm{i}\,b}\, x^2\right] \sum_{k=0}^{\infty} H_k\left(\frac{\sqrt{2\pi}\,x}{\sqrt{a^2+b^2}}\right)\left(\sqrt{\frac{a-\mathrm{i}\,b}{a+\mathrm{i}\,b}}\right)^k \frac{s^k}{k!}\ , \tag{2.84}$$

from which we conclude that the transformed modes $\mathsf{H}_k^{\mathbf{T}}(x) = \mathscr{L}(\mathbf{T})\mathsf{H}_k(x)$ take the form [75, p. 284, Example: Canonical transforms of Hermite functions]

$$\mathsf{H}_k^{\mathbf{T}}(x)$$

$$= \frac{2^{1/4}(2^k k!)^{-1/2}}{\sqrt{a+\mathrm{i}\,b}} \left(\sqrt{\frac{a-\mathrm{i}\,b}{a+\mathrm{i}\,b}}\right)^k H_k\left(\frac{\sqrt{2\pi}\,x}{\sqrt{a^2+b^2}}\right) \exp\left[-\pi\, \frac{d-\mathrm{i}\,c}{a+\mathrm{i}\,b}\, x^2\right]\ . \tag{2.85}$$

In the special case of a fractional Fourier transformer, with $a = d = \cos\gamma$ and $b = -c = \sin\gamma$, the right-hand side of (2.85) can be written as $\mathsf{H}_k(x)\,\exp[-\mathrm{i}\,(k+\frac{1}{2})\gamma]$, from which we conclude that the Hermite–Gauss modes $\mathsf{H}_k(x)$ are eigenfunctions of the fractional Fourier transformer with eigenvalues $\exp(-\mathrm{i}\,k\gamma)$ [57]; note that we have taken into account the additional constant phase factor $\exp(i\frac{1}{2}\gamma)$. The fact that the Hermite–Gauss modes $\mathsf{H}_k(x)$ are eigenfunctions of the fractional Fourier transformation becomes also apparent when we inspect their Wigner distributions $W_{\mathsf{H}_k}(x,u)$ [16, 35], which depend on the combination x^2+u^2 only:

$$W_{\mathsf{H}_k}(x,u) = 2(-1)^k \exp[-2\pi(x^2+u^2)]\, L_k[4\pi(x^2+u^2)]\ , \tag{2.86}$$

where $L_k(\cdot)$, $k = 0,1,\ldots$, are the *Laguerre polynomials* [2, Sect. 22, Orthogonal polynomials]. This Wigner distribution is indeed invariant under rotation in the xu plane.

2.6.3.2 The Two-dimensional Case

For the two-dimensional *separable Hermite–Gauss modes* $\mathsf{H}_{m,n}(\mathbf{r}) = \mathsf{H}_m(x)\mathsf{H}_n(y)$ we have, with $\mathbf{s} = [s_x, s_y]^t$, the generating function

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathsf{H}_{m,n}(\mathbf{r}) \left(\frac{2^{m+n}}{m!n!}\right)^{1/2} s_x^m s_y^n = 2^{1/2}\exp(-\mathbf{s}^t\mathbf{s} + 2\sqrt{2\pi}\,\mathbf{s}^t\mathbf{r} - \pi\,\mathbf{r}^t\mathbf{r})\ , \tag{2.87}$$

which is simply the two-dimensional version of (2.81). The linear canonical transforms $\mathscr{L}(\mathbf{T})\mathsf{H}_{m,n}(\mathbf{r}) \equiv \mathsf{H}^{\mathbf{T}}_{m,n}(\mathbf{r})$ of these two-dimensional Hermite–Gauss modes follow from the two-dimensional version of the generating function (2.83) [6, 20–22]

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \mathsf{H}^{\mathbf{T}}_{m,n}(\mathbf{r}) \left(\frac{2^{m+n}}{m!n!}\right)^{1/2} s_x^m s_y^n = \frac{2^{1/2}}{\sqrt{\det(\mathbf{A}+\mathrm{i}\,\mathbf{B})}} \exp[-\mathbf{s}^t(\mathbf{A}+\mathrm{i}\,\mathbf{B})^{-1}(\mathbf{A}-\mathrm{i}\,\mathbf{B})\mathbf{s}]$$
$$\times \exp[2\sqrt{2\pi}\,\mathbf{s}^t(\mathbf{A}+\mathrm{i}\,\mathbf{B})^{-1}\mathbf{r} - \pi\,\mathbf{r}^t(\mathbf{D}-\mathrm{i}\,\mathbf{C})(\mathbf{A}+\mathrm{i}\,\mathbf{B})^{-1}\mathbf{r}]\ . \tag{2.88}$$

Note that the complex symmetric matrix $(\mathbf{A}+\mathrm{i}\,\mathbf{B})^{-1}(\mathbf{A}-\mathrm{i}\,\mathbf{B})$ is unitary and that the real part $(\mathbf{AA}^t + \mathbf{BB}^t)^{-1}$ of the complex symmetric matrix $(\mathbf{D}-\mathrm{i}\,\mathbf{C})(\mathbf{A}+\mathrm{i}\,\mathbf{B})^{-1}$ is positive definite. From the generating function (2.88) we can derive derivative and recurrence relations for the transformed Hermite–Gauss modes [20] and also the direct expressions [6, 22]

$$\mathsf{H}^{\mathbf{T}}_{m,n}(\mathbf{r}) = \frac{2^{1/2}\,\mathscr{P}_x^m\,\mathscr{P}_y^n\,\exp[-\pi\,\mathbf{r}^t(\mathbf{D}-\mathrm{i}\,\mathbf{C})(\mathbf{A}+\mathrm{i}\,\mathbf{B})^{-1}\mathbf{r}]}{2^{m+n}\sqrt{\pi^{m+n}m!n!}\,\sqrt{\det(\mathbf{A}+\mathrm{i}\,\mathbf{B})}}\ , \tag{2.89}$$

$$\mathsf{H}^{\mathbf{T}}_{0,0}(\mathbf{r}) = \frac{2^{1/2}\,\exp[-\pi\,\mathbf{r}^t(\mathbf{D}-\mathrm{i}\,\mathbf{C})(\mathbf{A}+\mathrm{i}\,\mathbf{B})^{-1}\mathbf{r}]}{2^{m+n}\sqrt{\pi^{m+n}m!n!}\,\sqrt{\det(\mathbf{A}+\mathrm{i}\,\mathbf{B})}}\;, \tag{2.90}$$

where the operators $\mathscr{P}_x$ and $\mathscr{P}_y$ are determined by

$$\begin{bmatrix}\mathscr{P}_x\\ \mathscr{P}_y\end{bmatrix} = 2\pi(\mathbf{A}-\mathrm{i}\,\mathbf{B})^t[(\mathbf{D}-\mathrm{i}\,\mathbf{C})(\mathbf{A}+\mathrm{i}\,\mathbf{B})^{-1}]^*\begin{bmatrix}x\\ y\end{bmatrix} - (\mathbf{A}-\mathrm{i}\,\mathbf{B})^t\begin{bmatrix}\partial/\partial x\\ \partial/\partial y\end{bmatrix}\;, \tag{2.91}$$

with $\mathscr{P}^0_x$ and $\mathscr{P}^0_y$ the identity operators.

The generating functions (2.83) and (2.88) represent a general class of Hermite–Gaussian-type modes. If a member of this class, with defining matrix $\mathbf{T}_i = [\mathbf{A}_i, \mathbf{B}_i; \mathbf{C}_i, \mathbf{D}_i]$, say, undergoes a linear canonical transformation with transformation matrix $\mathbf{T}$, the input matrix $\mathbf{T}_i$ is transformed into the output matrix $\mathbf{T}_o = [\mathbf{A}_o, \mathbf{B}_o; \mathbf{C}_o, \mathbf{D}_o]$ by the simple matrix multiplication $\mathbf{T}_o = \mathbf{T}\mathbf{T}_i$.

2.6.3.3 Mode Conversion: Hermite–Laguerre–Gauss Modes

The transformation by phase-space rotators, for which $\mathbf{A} + \mathrm{i}\,\mathbf{B} = \mathbf{D} - \mathrm{i}\,\mathbf{C} = \mathbf{U}$, is important in optics for *mode conversion*. In that case, Eq. (2.91) reduces to $\left[\mathscr{P}_x, \mathscr{P}_y\right]^t = \mathbf{U}^{-1}(2\pi\,\mathbf{r} - \nabla^t)$, the transformed Hermite–Gauss modes $\mathsf{H}^{\mathbf{T}}_{m,n}(\mathbf{r})$ can be represented as [6]

$$\mathsf{H}^{\mathbf{T}}_{m,n}(\mathbf{r}) = \frac{2^{1/2}}{2^{m+n}\sqrt{\pi^{m+n}m!n!}\,\sqrt{\det\mathbf{U}}}\,\mathscr{P}^m_x(\mathbf{T})\,\mathscr{P}^n_y(\mathbf{T})\exp(-\pi\,\mathbf{r}^t\mathbf{r})\;, \tag{2.92}$$

and the generating function (2.88) reduces to the simpler form

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\mathsf{H}^{\mathbf{T}}_{m,n}(\mathbf{r})\left(\frac{2^{m+n}}{m!n!}\right)^{1/2} s^m_x s^n_y = \frac{2^{1/2}}{\sqrt{\det\mathbf{U}}}\exp[-(\mathbf{U}^*\mathbf{s})^t(\mathbf{U}^*\mathbf{s})]$$
$$\times\exp[2\sqrt{2\pi}\,(\mathbf{U}^*\mathbf{s})^t\mathbf{r} - \pi\,\mathbf{r}^t\mathbf{r}]\;; \tag{2.93}$$

note that in this case the Gaussian part $\exp(-\pi\,\mathbf{r}^t\mathbf{r})$ of the Hermite–Gaussian-type modes does not change. Moreover, since $(\mathrm{d}/\mathrm{d}t - 2\pi\,t)^k\exp(-\pi\,t^2) = \exp(\pi\,t^2)(\mathrm{d}/\mathrm{d}t)^k\exp(-2\pi\,t^2)$, the transformed modes can as well be expressed in the more direct form

$$\mathsf{H}^{\mathbf{T}}_{m,n}(\mathbf{r}) = \frac{2^{1/2}(-1)^{m+n}\exp(\pi\,\mathbf{r}^t\mathbf{r})}{2^{m+n}\sqrt{\pi^{m+n}m!n!}\,\sqrt{\det\mathbf{U}}}\,\mathscr{U}^m_x\,\mathscr{U}^n_y\,\exp(-2\pi\,\mathbf{r}^t\mathbf{r})\;, \tag{2.94}$$

with the operators $\mathscr{U}_x$ and $\mathscr{U}_y$ defined as $[\mathscr{U}_x, \mathscr{U}_y]^t = \mathbf{U}^{-1}\nabla^t$. As an example we mention the phase-space rotator with

$$\mathbf{U} = \frac{1}{\sqrt{2}} \begin{bmatrix} \cos 2\alpha + \mathrm{i} & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha + \mathrm{i} \end{bmatrix} \equiv \mathbf{U}_{\mathrm{HLG}}(\alpha) \ , \tag{2.95}$$

which generates the recently introduced *Hermite–Laguerre–Gauss modes* [1] from the separable Hermite–Gauss modes.

For the special case $\alpha = \frac{1}{4}\pi$, the matrix $\mathbf{U}_{\mathrm{HLG}}(\alpha)$ takes the form

$$\mathbf{U}_{\mathrm{HLG}}\left(\tfrac{1}{4}\pi\right) = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathrm{i} & 1 \\ 1 & \mathrm{i} \end{bmatrix} \equiv \mathbf{U}_{\mathrm{LG}} = \mathbf{U}_g\left(-\tfrac{1}{4}\pi\right) \mathbf{U}_f\left(\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\right), \tag{2.96}$$

and the Hermite–Laguerre–Gauss modes reduce to the *Laguerre–Gauss modes*, whose generating function can be written in the form

$$2^{1/2}\mathrm{i}^{-1} \exp\{-\pi(x^2+y^2) + 2\mathrm{i}\, s_x s_y - 2\sqrt{\pi}\,[\mathrm{i}\, s_x(x+\mathrm{i}\,y) - s_y(x-\mathrm{i}\,y)]\} \ . \tag{2.97}$$

We remark that the latter expression depends only on the combinations $x^2 + y^2$, $s_x(x + \mathrm{i}\,y)$ and $s_y(x - \mathrm{i}\,y)$, which shows the vortex behaviour of such modes. For completeness, we recall the explicit form of the Laguerre–Gauss modes [53, 60]

$$\begin{aligned} \mathsf{L}_{m,n}(r,\varphi) = 2^{1/2} &\left[\frac{(\min\{m,n\})!}{(\max\{m,n\})!}\right]^{1/2} (\sqrt{2\pi}\, r)^{|m-n|} \exp[\mathrm{i}\,(m-n)\varphi] \\ &\times L_{\min\{m,n\}}^{(|m-n|)}(2\pi\, r^2) \exp[-\pi\, r^2] \ , \end{aligned} \tag{2.98}$$

where $x = r\cos\varphi$ and $y = r\sin\varphi$, and where $L_n^{(\alpha)}(\cdot)$ are the *generalized* Laguerre polynomials [2, Sect. 22, Orthogonal polynomials]. Note that the vortex behaviour is clearly visible in the phase factor $\exp[\mathrm{i}\,(m-n)\varphi]$ and that the Laguerre–Gauss modes are eigenfunctions of the rotator: $\mathscr{R}(\alpha)\,\mathsf{L}_{m,n}(r,\varphi) = \mathsf{L}_{m,n}(r,\varphi-\alpha)$.

Other phase-space rotators exist that convert the separable Hermite–Gauss modes into Laguerre–Gauss modes [6, 20, 41, 53]. To find the operators that generate modes with a vortex behaviour, we require that in the generating function (2.88) the term $\mathbf{s}^t(\mathbf{A} + \mathrm{i}\,\mathbf{B})^{-1}\mathbf{r}$ depends only on the combinations $s_x(x + \mathrm{i}\,y)$ and $s_y(x - \mathrm{i}\,y)$, and the term $\mathbf{r}^t(\mathbf{D} - \mathrm{i}\,\mathbf{C})(\mathbf{A} + \mathrm{i}\,\mathbf{B})^{-1}\mathbf{r}$ only on the combination $x^2 + y^2$. These requirements lead to the class of linear canonical transformers that can be decomposed as $\mathscr{Q}(c\mathbf{I})\,\mathscr{R}(\alpha)\,\mathscr{G}(-\frac{1}{4}\pi)\,\mathscr{F}(\gamma_x,\gamma_y)$. The generating functions of the modes that arise at the output of this cascade have basically the same form as the generating function (2.97); Eq. (2.97) itself arises for the special choice $\gamma_x = \gamma_y = \frac{1}{2}\pi$ (and $\alpha = c = 0$) [1], while the case $\gamma_x = \gamma_y = 0$ has been reported in [53, Eq. (14)]. We remark that the important element in the cascade $\mathscr{Q}(c\mathbf{I})\,\mathscr{R}(\alpha)\,\mathscr{G}(-\frac{1}{4}\pi)\,\mathscr{F}(\gamma_x,\gamma_y)$ is the gyrator. Since the separable Hermite–Gauss modes are eigenfunctions of the separable fractional Fourier transformer, such a transformer that precedes the gyrator does not change the character of these modes. And once the gyrator has converted these modes to the Laguerre–Gauss mode, the succeeding rotator, for which the Laguerre–Gauss mode is an eigenfunction, and the final isotropic quadratic-phase modulator, which corresponds to a simple multiplication by $\exp(-\mathrm{i}\,\pi\, c\mathbf{r}^t\mathbf{r})$, do not destroy the vortex behaviour.

We remark that the Laguerre–Gauss modes $\mathsf{L}_{m,n}(\mathbf{r}) = \mathscr{G}(-\frac{1}{4}\pi)\,\mathsf{H}_{m,n}(\mathbf{r})$ are eigenfunctions of the symmetric fractional Fourier transformer $\mathscr{F}(\gamma,\gamma)$ with eigenvalue $\exp[-\mathrm{i}\,(m+n)\gamma]$,

$$\begin{aligned}\mathscr{F}(\gamma,\gamma)\,\mathsf{L}_{m,n}(\mathbf{r}) &= \mathscr{F}(\gamma,\gamma)\,\mathscr{G}\left(-\tfrac{1}{4}\pi\right)\,\mathsf{H}_{m,n}(\mathbf{r})\\ &= \mathscr{G}\left(-\tfrac{1}{4}\pi\right)\,\mathscr{F}(\gamma,\gamma)\,\mathsf{H}_{m,n}(\mathbf{r})\\ &= \mathscr{G}\left(-\tfrac{1}{4}\pi\right)\,\mathsf{H}_{m,n}(\mathbf{r})\,\exp[-\mathrm{i}\,(m+n)\gamma]\\ &= \mathsf{L}_{m,n}(\mathbf{r})\,\exp[-\mathrm{i}\,(m+n)\gamma]\;,\end{aligned} \tag{2.99}$$

where we have used the fact that the symmetric fractional Fourier transformer $\mathscr{F}(\gamma,\gamma)$ commutes with any other phase-space rotator $\mathscr{O}(\mathbf{U})$ and that the separable Hermite–Gauss modes $\mathsf{H}_{m,n}(\mathbf{r})$ are eigenfunctions of the separable fractional Fourier transformer $\mathscr{F}(\gamma_x,\gamma_y)$ with eigenvalue $\exp[-\mathrm{i}\,(m\gamma_x+n\gamma_y)]$.

2.7 Classification of the Linear Canonical Transformation Based on the Distribution of the Eigenvalues of Its Transformation Matrix

In this section we look for simple linear canonical transformations $\mathscr{L}(\mathbf{N})$, called nuclei, that are similar to a given transformation $\mathscr{L}(\mathbf{T})$ in the sense [37]

$$\mathscr{L}(\mathbf{T}) = \mathscr{L}(\mathbf{T}_\circ)\,\mathscr{L}(\mathbf{N})\,\mathscr{L}^{-1}(\mathbf{T}_\circ)\;. \tag{2.100}$$

An obvious guess would be to look for the eigenvalues and eigenvectors of the transformation matrix $\mathbf{T}$ and express it in its Jordan form [30], $\mathbf{T} = \mathbf{Q}\boldsymbol{\Lambda}_{\mathrm{J}}\mathbf{Q}^{-1}$, but it is not guaranteed that the matrices $\boldsymbol{\Lambda}_{\mathrm{J}}$ and $\mathbf{Q}$ are symplectic. Based on the eigenvalues of $\mathbf{T}$, we will be able to classify the linear canonical transformation $\mathscr{L}(\mathbf{T})$ and find a nucleus $\mathscr{L}(\mathbf{N})$ for each class [23]. A general proof for the existence of the decomposition (2.100) can be found in [37]; see, in particular, Theorem 41, which deals with the real symplectic Jordan form.

We remark that once we have found the representation (2.100), the eigenfunctions $\Phi(\mathbf{r})$ of the nucleus $\mathscr{L}(\mathbf{N})$, i.e., $\mathscr{L}(\mathbf{N})\,\Phi(\mathbf{r}) = \mu\Phi(\mathbf{r})$, immediately lead to the eigenfunctions $\Psi(\mathbf{r}) = \mathscr{L}(\mathbf{T}_\circ)\,\Phi(\mathbf{r})$ of the transformation $\mathscr{L}(\mathbf{T})$. Indeed,

$$\begin{aligned}\mathscr{L}(\mathbf{T})\Psi(\mathbf{r}) &= \mathscr{L}(\mathbf{T}_\circ)\,\mathscr{L}(\mathbf{N})\,\mathscr{L}^{-1}(\mathbf{T}_\circ)\,\Psi(\mathbf{r})\\ &= \mathscr{L}(\mathbf{T}_\circ)\,\mathscr{L}(\mathbf{N})\,\Phi(\mathbf{r}) = \mathscr{L}(\mathbf{T}_\circ)\,\mu\Phi(\mathbf{r}) = \mu\Psi(\mathbf{r}).\end{aligned}$$

The simpler the nucleus $\mathscr{L}(\mathbf{N})$, the simpler it is to find its eigenfunctions.

The different classes are defined by the possible distributions of the eigenvalues of the transformation matrix $\mathbf{T}$. We note that if λ is an eigenvalue of a real symplectic matrix $\mathbf{T}$, then λ^*, $1/\lambda$ and $1/\lambda^*$ are eigenvalues, too. Indeed, from the realness of $\mathbf{T}$, we conclude that the characteristic equation $\det(\mathbf{T}-\lambda\mathbf{I}) = 0$ has real coefficients and that the eigenvalues are thus real or come in complex conjugated pairs: if λ is an eigenvalue, then λ^* is an eigenvalue, too. Moreover, from the symplecticity condition we get

$$\det(\mathbf{T}^{-1}-\lambda\mathbf{I}) = \det(\mathbf{J}\mathbf{T}^t\mathbf{J}-\lambda\mathbf{I}) = \det[\mathbf{J}(\mathbf{T}^t-\lambda\mathbf{I})\mathbf{J}) = \det(\mathbf{T}^t-\lambda\mathbf{I}) = \det(\mathbf{T}-\lambda\mathbf{I})$$

and we conclude that if λ is an eigenvalue, then $1/\lambda$ is an eigenvalue, too. So, for real symplectic matrices and $D \geq 2$, the eigenvalues come in complex quartets (if they are not unimodular and not real), or in complex conjugated pairs (if they are unimodular, but not real), or in real pairs (in particular: double if they are equal to $+1$ or -1). For $D = 1$, the two eigenvalues can of course only come as a single pair, either unimodular or real.

2.7.1 Nuclei for the One-dimensional Case

Let us first consider the one-dimensional case, in which the two eigenvalues follow from the characteristic equation $\lambda^2 - (a+d)\lambda + 1 = 0$ and three different distributions of the eigenvalues arise [23, 49]:

1. A pair of real eigenvalues σ and σ^{-1} ($\sigma \neq \pm 1$),
2. Two real eigenvalues $\lambda = 1$ or $\lambda = -1$, with only one eigenvector, and
3. A pair of unimodular, complex conjugated eigenvalues $\exp(i\gamma)$ and $\exp(-i\gamma)$.

The magnifier $\mathscr{M}(\sigma)$ is an obvious nucleus for class 1. Note that we can restrict ourselves to the case $\sigma > 0$; if the real eigenvalues are negative, we simply add an additional coordinate reverter $\mathscr{M}(-1)$ to the nucleus. The quadratic-phase modulator $\mathscr{Q}(-c)$ and the Fresnel transformer $\mathscr{S}(b)$ are obvious nuclei for class 2. Class 3 needs some more careful consideration, because the matrix of eigenvalues, $\mathbf{\Lambda}[\exp(\mathrm{i}\,\gamma), \exp(-\mathrm{i}\,\gamma)]$, is not symplectic and thus cannot act as the transformation matrix of a nucleus. A proper nucleus for this class might be the fractional Fourier transformer $\mathscr{F}(\gamma)$ [22].

For completeness we remark that for class 3—determined by the condition $|a + d| \leq 2$—the transformation $\mathscr{L}(\mathbf{T})$ can be decomposed as

$$\mathscr{L}(\mathbf{T}) = \mathscr{Q}(g)\,\mathscr{M}(s)\,\mathscr{F}(\gamma)\,\mathscr{M}(s^{-1})\,\mathscr{Q}(-g)\,, \tag{2.101}$$

where γ, s and g follow from $a+d=2\cos\gamma$, $b=s\sin\gamma$ and $a-d=2gb$. The opposite case, $|a+d|\geq 2$, corresponds in fact to class 1. To stress the symmetry of the two classes 1 and 3, we recall that for $|a+d|\geq 2$, the transformation $\mathscr{L}(\mathbf{T})$ can be decomposed as

$$\mathscr{L}(\mathbf{T})=\mathscr{Q}(g)\,\mathscr{M}(s)\,\mathscr{H}(\gamma)\,\mathscr{M}(s^{-1})\,\mathscr{Q}(-g)\;, \tag{2.102}$$

where γ, s and g follow from $|a+d|=2\cosh\gamma$, $b\,\mathrm{sgn}(a+d)=s\sinh\gamma$ and $a-d=2gb$, and where $\mathscr{H}(\gamma)=\mathscr{F}(-\frac{1}{4}\pi)\,\mathscr{M}[\exp(\gamma)]\,\mathscr{F}(\frac{1}{4}\pi)$ is known as the *hyperbolic expander* [75, p. 183, Example: Hyperbolic expanders], see also Sect. 2.3.2. We conclude that not only the magnifier $\mathscr{M}(\sigma)$ but also the hyperbolic expander $\mathscr{H}(\ln\sigma)$ can be a proper nucleus for class 1 [22].

Class 2, for which $|a+d|=2$, has been extensively studied in [49, Sect. IV]. If $a=d=\pm 1$ (and consequently $bc=0$), the linear canonical transformer is obviously either an identity operator (for $b=c=0$), or a quadratic-phase modulator $\mathscr{Q}(-c)$ (for $b=0, c\neq 0$), or a Fresnel transformer $\mathscr{S}(b)$ (for $b\neq 0, c=0$), possibly with an additional coordinate reversion (if $\lambda=-1$). For $a\neq d$ (and consequently $bc<0$), the linear canonical transformation $\mathscr{L}(\mathbf{T})$ can always be decomposed with a quadratic-phase modulator or a Fresnel transformer as the nucleus $\mathscr{L}(\mathbf{N})$. For more details we refer to [23, Sect. 4] and [49, Sects. IV.D and IV.E].

2.7.2 *Nuclei for the Two-dimensional Case*

Concatenations of one-dimensional nuclei lead to *separable* two-dimensional nuclei. As a first example we mention the concatenation of a quadratic-phase modulator $\mathscr{Q}(-c_x)$ in the x direction and a fractional Fourier transformer $\mathscr{F}(\gamma_y)$ in the y direction. The corresponding two-dimensional nucleus has the transformation matrix

$$\begin{bmatrix} 1 & 0 \\ c_x & 1 \end{bmatrix} \oplus \begin{bmatrix} \cos\gamma_y & \sin\gamma_y \\ -\sin\gamma_y & \cos\gamma_y \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\gamma_y & 0 & \sin\gamma_y \\ c_x & 0 & 1 & 0 \\ 0 & -\sin\gamma_y & 0 & \cos\gamma_y \end{bmatrix} .$$

As a second example we consider the class of phase-space rotators, described by their unitary matrix $\mathbf{U}$. It is well known that a unitary matrix has unimodular eigenvalues λ and can be diagonalized [36, Chap. 13]: $\mathbf{U}=\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^{-1}$. Moreover, it is not difficult to show that the eigenvalue $\exp(\mathrm{i}\,\gamma)$ of $\mathbf{U}$ corresponds to the complex conjugated pair of eigenvalues $\exp(\pm\mathrm{i}\,\gamma)$ of the symplectic matrix $\mathbf{T}$. The matrix $\mathbf{P}$ that diagonalizes the unitary matrix $\mathbf{U}$ can itself be made unitary and then corresponds to a symplectic matrix that diagonalizes the symplectic matrix $\mathbf{T}$. We thus conclude that the separable fractional Fourier transformer $\mathscr{F}(\gamma_x, \gamma_y)$, which is clearly a concatenation of two one-dimensional fractional Fourier transformers with transformation matrix

$$\begin{bmatrix} \cos\gamma_x & \sin\gamma_x \\ -\sin\gamma_x & \cos\gamma_x \end{bmatrix} \oplus \begin{bmatrix} \cos\gamma_y & \sin\gamma_y \\ -\sin\gamma_y & \cos\gamma_y \end{bmatrix} = \begin{bmatrix} \cos\gamma_x & 0 & \sin\gamma_x & 0 \\ 0 & \cos\gamma_y & 0 & \sin\gamma_y \\ -\sin\gamma_x & 0 & \cos\gamma_x & 0 \\ 0 & -\sin\gamma_y & 0 & \cos\gamma_y \end{bmatrix},$$

is an obvious nucleus of a phase-space rotator [22]. We recall that we already met decompositions based on $\mathbf{U} = \mathbf{P\Lambda P}^{-1}$ in Table 2.6, where we find such cascades as $\mathscr{G}(\gamma) = \mathscr{R}(-\frac{1}{4}\pi)\,\mathscr{F}(\gamma,-\gamma)\,\mathscr{R}(\frac{1}{4}\pi)$ and $\mathscr{R}(\gamma) = \mathscr{G}(\frac{1}{4}\pi)\,\mathscr{F}(\gamma,-\gamma)\,\mathscr{G}(-\frac{1}{4}\pi)$. From these cascades we conclude that the antisymmetric separable fractional Fourier transformer $\mathscr{F}(\gamma,-\gamma)$ can be decomposed as $\mathscr{R}(\frac{1}{4}\pi)\,\mathscr{G}(\gamma)\,\mathscr{R}(-\frac{1}{4}\pi)$ and as $\mathscr{G}(-\frac{1}{4}\pi)\,\mathscr{R}(\gamma)\,\mathscr{G}(\frac{1}{4}\pi)$, and that the gyrator and the rotator can thus also act as a nucleus in the special case that $\gamma_x = -\gamma_y$.

The four additional—inherently two-dimensional—classes correspond to the four possible eigenvalue distributions that can only occur in two dimensions [23, Sect. 5]:

4. A complex quartet of eigenvalues $\sigma\,\exp(\mathrm{i}\,\gamma)$, $\sigma\,\exp(-\mathrm{i}\,\gamma)$, $\sigma^{-1}\exp(-\mathrm{i}\,\gamma)$ and $\sigma^{-1}\exp(\mathrm{i}\,\gamma)$ $(\sigma \neq \pm 1)$,
5. Two identical pairs of unimodular, complex conjugated eigenvalues $\exp(\mathrm{i}\,\gamma)$ and $\exp(-\mathrm{i}\,\gamma)$, with only two linearly independent eigenvectors,
6. Two identical pairs of real eigenvalues σ and σ^{-1} $(\sigma \neq \pm 1)$, with only two linearly independent eigenvectors, and
7. Four real eigenvalues $\lambda = 1$ or $\lambda = -1$, with only one eigenvector.

We mention possible nuclei for these four classes.

A possible nucleus for class 4 is the (commuting) combination of a rotator and a magnifier $\mathscr{M}(\sigma\mathbf{I})\,\mathscr{R}(\gamma) = \mathscr{R}(\gamma)\,\mathscr{M}(\sigma\mathbf{I})$ with transformation matrix $[\sigma\,\mathbf{U}_r(\gamma), \mathbf{0}; \mathbf{0}, \sigma^{-1}\mathbf{U}_r(\gamma)]$. The input–output relation for such a nucleus reads

$$\sigma f_o(\sigma x, \sigma y) = f_i(x\cos\gamma - y\sin\gamma, x\sin\gamma + y\cos\gamma) \tag{2.103}$$

or in polar coordinates (with $x = r\cos\varphi$ and $y = r\sin\varphi$): $\sigma f_o(\sigma r, \varphi) = f_i(r, \varphi + \gamma)$. Like for class 1, an alternative nucleus for class 4 is the combination of a hyperbolic expander (instead of a magnifier) and a rotator.

A possible nucleus for class 5 is the (commuting) combination of a rotator and a quadratic-phase modulator $\mathscr{Q}(-c\mathbf{I})\,\mathscr{R}(\gamma) = \mathscr{R}(\gamma)\,\mathscr{Q}(-c\mathbf{I})$ with transformation matrix $[\mathbf{U}_r(\gamma), \mathbf{0}; c\,\mathbf{U}_r(\gamma), \mathbf{U}_r(\gamma)]$. The input–output relation for this nucleus reads

$$f_o(x,y) = f_i(x\cos\gamma - y\sin\gamma, x\sin\gamma + y\cos\gamma)\,\exp[\mathrm{i}\,\pi\,c(x^2+y^2)] \tag{2.104}$$

or in polar coordinates again: $f_o(r,\varphi) = f_i(r,\varphi+\gamma)\ \exp(\mathrm{i}\,\pi\, cr^2)$. Like for class 2, an alternative nucleus for class 5 is the combination of a Fresnel transformer (instead of a quadratic-phase modulator) and a rotator.

For the remaining two classes we need a new element, the *shearer* $\mathscr{Z}$ (to be defined shortly). A possible nucleus for class 6 is then the (commuting) shearer-magnifier combination $\mathscr{M}(\sigma\mathbf{I})\mathscr{Z} = \mathscr{Z}\mathscr{M}(\sigma\mathbf{I})$ with transformation matrix

$$\begin{bmatrix} \sigma\,\mathbf{Z}_+ & \mathbf{0} \\ \mathbf{0} & \sigma^{-1}\mathbf{Z}_- \end{bmatrix}, \quad \text{where} \quad \mathbf{Z}_+ = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{Z}_- = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}; \tag{2.105}$$

the shearer itself results for $\sigma = 1$. The input–output relation for this nucleus reads

$$\sigma\, f_o(\sigma x, \sigma y) = f_i(x-y, y)\ , \tag{2.106}$$

which represents—apart from a magnification with σ—a simple shearing of the x coordinate.

A possible nucleus for the final class 7 is the (non-commuting!) combination of a shearer and a quadratic-phase modulator $\mathscr{Q}(-c\mathbf{I})\mathscr{Z}$ with transformation matrix $[\mathbf{Z}_+, \mathbf{0}; c\,\mathbf{Z}_+, \mathbf{Z}_-]$. The input–output relation for this nucleus reads

$$f_o(x,y) = f_i(x-y, y)\ \exp[\mathrm{i}\,\pi\, c(x^2+y^2)]\ . \tag{2.107}$$

Again, an alternative nucleus is the combination of a Fresnel transformer (instead of a quadratic-phase modulator) and a shearer.

The results of this section have been combined in Table 2.8, where for each of the seven classes the corresponding nucleus can be extracted. Obvious concatenations of one-dimensional nuclei have not been stated explicitly. Two examples in reading the Table: (1) the nucleus for the (one-dimensional) class 3, i.e., a pair of unimodular, complex conjugated eigenvalues $\exp(\pm\mathrm{i}\,\gamma)$, is a fractional Fourier transformer $\mathscr{F}(\gamma)$. (2) possible nuclei for the (two-dimensional) class 7, i.e., four

Table 2.8 Seven classes of eigenvalue distributions λ for linear canonical transformers and their corresponding nuclei, composed of magnifiers $\mathscr{M}(\sigma)$ or hyperbolic expanders $\mathscr{H}(\ln\sigma)$, quadratic-phase modulators $\mathscr{Q}(\cdot)$ or Fresnel transformers $\mathscr{S}(\cdot)$, fractional Fourier transformers $\mathscr{F}(\gamma)$, rotators $\mathscr{R}(\gamma)$, and shearers $\mathscr{Z}$

λ	$\mathscr{M}(\sigma)$ or $\mathscr{H}(\ln\sigma)$	$\mathscr{Q}(\cdot)$ or $\mathscr{S}(\cdot)$	$\mathscr{F}(\gamma)$
	Class 1 σ, σ^{-1}	Class 2 $1, 1$	Class 3 $\mathrm{e}^{\mathrm{i}\,\gamma}, \mathrm{e}^{-\mathrm{i}\,\gamma}$
$\mathscr{R}(\gamma)$	Class 4 $\sigma\mathrm{e}^{\mathrm{i}\,\gamma}, \sigma\mathrm{e}^{-\mathrm{i}\,\gamma}, \sigma^{-1}\mathrm{e}^{-\mathrm{i}\,\gamma}, \sigma^{-1}\mathrm{e}^{\mathrm{i}\,\gamma}$	Class 5 $\mathrm{e}^{\mathrm{i}\,\gamma}, \mathrm{e}^{\mathrm{i}\,\gamma}, \mathrm{e}^{-\mathrm{i}\,\gamma}, \mathrm{e}^{-\mathrm{i}\,\gamma}$	
$\mathscr{Z}$	Class 6 $\sigma, \sigma, \sigma^{-1}, \sigma^{-1}$	Class 7 $1, 1, 1, 1$ $\mathscr{Z}$ and $\{\mathscr{Q}, \mathscr{S}\}$ do not commute !	

eigenvalues equal to 1 and with only one eigenvector, are a shearer $\mathscr{X}$ followed by a quadratic-phase modulator $\mathscr{Q}(\cdot)$ or followed by a Fresnel transformer $\mathscr{S}(\cdot)$, and the two subsystems do not commute.

2.7.3 The Search for Eigenfunctions

The search for eigenfunctions $\Psi(\mathbf{r})$ of linear canonical transformations now reduces to the search for eigenfunctions $\Phi(\mathbf{r})$ of the simple nuclei. The Dirac delta function $\delta(x-\xi)$ is an eigenfunction for any multiplication operator, with eigenvalue $\exp(\mathrm{i}\,\pi\,cx^2)$ in the particular case of a quadratic-phase modulator $\mathscr{Q}(-c)$; and the harmonic signal $\exp(\mathrm{i}\,2\pi\,ux)$ is an eigenfunction for any convolution operator, with eigenvalue $\exp(\mathrm{i}\,\pi\,bu^2)$ in the particular case of a Fresnel transformer $\mathscr{S}(b)$ (class 2). We also recall that the Hermite–Gauss modes (2.80) are eigenfunctions of the fractional Fourier transformer $\mathscr{F}(\gamma)$, with eigenvalues $\exp(-\mathrm{i}\,k\gamma)$ (class 3). And while powers x^k are evidently eigenfunctions of the nucleus $\mathscr{M}(\sigma)$ with eigenvalues $|\sigma|^{-1/2}\sigma^{-k}$ (class 1), signals of the form $r^k\exp(\mathrm{i}\,m\varphi)$ are eigenfunctions of the nucleus $\mathscr{M}(\sigma\mathbf{I})\,\mathscr{R}(\gamma)$ with eigenvalues $\sigma^{-1/2}\sigma^{-k}\exp(\mathrm{i}\,m\gamma)$ (class 4), and signals of the form $\delta(r-\rho)\,\exp(\mathrm{i}\,m\varphi)$ are eigenfunctions of the nucleus $\mathscr{Q}(-c\mathbf{I})\,\mathscr{R}(\gamma)$ with eigenvalues $\exp(\mathrm{i}\,m\gamma)\,\exp(\mathrm{i}\,\pi\,c\rho^2)$ (class 5). Proper eigenfunctions of the nuclei $\mathscr{M}(\sigma\mathbf{I})\,\mathscr{X}$ and $\mathscr{Q}(-c\mathbf{I})\,\mathscr{X}$ for the classes 6 and 7 are still to be found.

From the eigenfunctions $\Phi(\mathbf{r})$ for a nucleus $\mathscr{L}(\mathbf{N})$, we can generate eigenfunctions $\Psi(\mathbf{r})$ for the corresponding class of transformations $\mathscr{L}(\mathbf{T}_\circ)\,\mathscr{L}(\mathbf{N})\,\mathscr{L}^{-1}(\mathbf{T}_\circ)$ by letting the eigenfunctions $\Phi(\mathbf{r})$ propagate through $\mathscr{L}(\mathbf{T}_\circ)$: $\mathscr{L}(\mathbf{T}_\circ)\,\Phi(\mathbf{r}) = \Psi(\mathbf{r})$. We already met the Hermite–Gaussian-type modes $\mathsf{H}^{\mathbf{T}_\circ}_{m,n}(\mathbf{r})$ for class 3, see Sect. 2.6.3. As an example, we will find eigenfunctions for the one-dimensional hyperbolic expander $\mathscr{H}(\gamma)$ (class 1), see [24].

Let us start with the powers x^k, which are eigenfunctions of the magnifier $\mathscr{M}(\cdot)$, and recall that $H(\gamma) = \mathscr{F}(-\frac{1}{4}\pi)\,\mathscr{M}(\exp\gamma)\,\mathscr{F}(\frac{1}{4}\pi)$. We thus have to calculate the integral

$$f_o(x_o) = \mathscr{F}\left(-\tfrac{1}{4}\pi\right)\,x_i^k = \mathrm{i}^{\,1/2}2^{-1/4}\int x_i^k\exp[-\mathrm{i}\,\pi(x_o^2-2\sqrt{2}x_ox_i+x_i^2)]\,\mathrm{d}x_i\;, \tag{2.108}$$

for which we use the relationships

$$(-\mathrm{i}\,2\pi)^k\sqrt{p}\int x^k\exp(-\pi\,px^2-\mathrm{i}\,2\pi\,ux)\,\mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}u^k}\exp(-\pi\,p^{-1}u^2) \tag{2.109}$$

and

$$\frac{\mathrm{d}^k}{\mathrm{d}u^k}\exp(-\pi\,p^{-1}u^2) = (-\sqrt{\pi\,p^{-1}})^k\exp(-\pi\,p^{-1}u^2)\,H_k(u\sqrt{\pi\,p^{-1}})\;, \tag{2.110}$$

in which we substitute $p = \mathrm{i}$, $x = x_i$ and $u = -x_o\sqrt{2}$. Note that Eq. (2.109) follows by differentiating the Fourier transform $\exp(-\pi\, p^{-1}u^2)$ of $\exp(-\pi\, px^2)$, and holds for $\Re p > 0$ and for $[\Re p = 0, \Im p \neq 0]$, see, for instance, [50, (2.3.15.4)] and also [75, Sect. C.2, p. 279, Remark: The integral of complex Gaussians]; we may also refer to [50, (2.5.22.5)] and [34, (3.691.5) and (3.691.7)] for $k = 0$, and to [50, (2.5.22.3)] and [34, (3.851.1) and (3.851.3)] for $k = 1$. Equation (2.110) can be found, for instance, in [2, (7.1.19)]; see also Rodrigues' formula for Hermite polynomials [2, (22.11.7)]. We readily conclude that $f_o(x)$ is proportional to $H_k(\sqrt{2\pi}x\,\mathrm{i}^{-1/2})\,\exp(\mathrm{i}\,\pi\,x^2)$.

We note the remarkable resemblance between these eigenfunctions and the eigenfunctions of the fractional Fourier transformer—i.e., the Hermite–Gauss modes $\mathsf{H}_k(x)$, which are proportional to $H_k(\sqrt{2\pi}x)\,\exp(-\pi\,x^2)$—and we conclude that we can directly go from the fractional-Fourier-transformer case to the hyperbolic-expander case by simply replacing x by $(x\,\mathrm{i}^{-1/2})$.

2.8 The Effect of a Linear Canonical Transformation on the Second-order Moments in Phase Space

With $E = \iint W(\mathbf{r},\mathbf{q})\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{q}$ denoting the total energy of a signal, the *normalized second-order moments* of its Wigner distribution are defined as

$$\frac{1}{E}\iint \begin{bmatrix}\mathbf{r}\\ \mathbf{q}\end{bmatrix}[\mathbf{r}^t,\mathbf{q}^t]\,W(\mathbf{r},\mathbf{q})\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{q} \equiv \begin{bmatrix}\mathbf{M_{rr}} & \mathbf{M_{rq}}\\ \mathbf{M_{rq}^t} & \mathbf{M_{qq}}\end{bmatrix} \equiv \mathbf{M} \tag{2.111}$$

and constitute a real positive-definite symmetric moment matrix $\mathbf{M}$. It can easily be shown that when a signal undergoes a linear canonical transformation, $f_o(\mathbf{r}) = \mathscr{L}(\mathbf{T})f_i(\mathbf{r})$, the moment matrices $\mathbf{M}_i$ and $\mathbf{M}_o$ are related by the relationship [12, 59]

$$\mathbf{M}_o = \mathbf{T}\mathbf{M}_i\mathbf{T}^t\ . \tag{2.112}$$

We can easily prove the positive definiteness of the $2D \times 2D$ moment matrix $\mathbf{M}$ with the help of the input–output relationship (2.112). We therefore construct the transformation matrix $\mathbf{T}$ as follows:

- We choose $\mathbf{C} = \mathbf{0}$, with the immediate consequence $\mathbf{D} = \mathbf{A}^{t-1}$ and $\mathbf{AB}^t = \mathbf{BA}^t$ to satisfy the symplecticity condition.
- The matrix $\mathbf{A}$ is chosen as an upper triangular matrix with $\frac{1}{2}D(D+1)$ non-vanishing entries that can be chosen arbitrarily.
- The matrix $\mathbf{B}$ is chosen as an upper triangular matrix with D arbitrarily chosen entries in its top row.
- The remaining $\frac{1}{2}D(D-1)$ non-vanishing entries of $\mathbf{B}$ are determined from the $\frac{1}{2}D(D-1)$ equations that follow from the required symmetry of the matrix $\mathbf{AB}^t$.

The top row $\mathbf{t}^t$ of the transformation matrix $\mathbf{T}$ can thus be constructed completely arbitrarily. We now consider the upper left entry $m_{xx,o}$ of the matrix $\mathbf{M}_o$ in the left-hand side of (2.112); this entry, which represents the square of an effective width, is positive:

$$(m_{xx})_o = \frac{1}{E}\int x^2 \left[\int W_o(\mathbf{r},\mathbf{q})\,\mathrm{d}\mathbf{q}\right]\mathrm{d}\mathbf{r} = \frac{1}{E}\int x^2\,|f_o(\mathbf{r})|^2\,\mathrm{d}\mathbf{r} > 0\ .$$

On the other hand, this entry equals $\mathbf{t}^t\mathbf{M}_i\mathbf{t}$, where the vector $\mathbf{t}$ can be chosen arbitrarily. We thus conclude that the quadratic form $\mathbf{t}^t\mathbf{M}_i\mathbf{t}$ is positive for any vector $\mathbf{t}$, with which we have proved that the moment matrix $\mathbf{M}_i$ is positive definite.

The moment matrix $2\pi\,\mathbf{M}$ can be represented in the form [18, Sect. 2.6, Second- and higher-order moments]

$$2\pi\,\mathbf{M} = 2\pi\begin{bmatrix}\mathbf{M_{rr}} & \mathbf{M_{rq}}\\ \mathbf{M}^t_{\mathbf{rq}} & \mathbf{M_{qq}}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\mathbf{G}_1^{-1} & \mathbf{G}_1^{-1}\mathbf{H}\\ \mathbf{H}^t\mathbf{G}_1^{-1} & \mathbf{G}_2+\mathbf{H}^t\mathbf{G}_1^{-1}\mathbf{H}\end{bmatrix}, \tag{2.113}$$

where the matrices $\mathbf{G}_1$, $\mathbf{G}_2$ and $\mathbf{H}$ follow directly from the submatrices $\mathbf{M_{rr}}$, $\mathbf{M_{rq}}$ and $\mathbf{M_{qq}}$:

$$\mathbf{G}_1 = (4\pi)^{-1}\mathbf{M}_{\mathbf{rr}}^{-1} = \mathbf{G}_1^t \tag{2.114a}$$

$$\mathbf{G}_2 = 4\pi\,(\mathbf{M_{qq}} - \mathbf{M}^t_{\mathbf{rq}}\mathbf{M}^{-1}_{\mathbf{rr}}\mathbf{M_{rq}}) = \mathbf{G}_2^t\ , \tag{2.114b}$$

$$\mathbf{H} = \mathbf{M}^{-1}_{\mathbf{rr}}\mathbf{M_{rq}}\ . \tag{2.114c}$$

The matrices $\mathbf{G}_1$ and $\mathbf{G}_2$ are positive definite, which follows immediately from the positive definiteness of the quadratic form $(\mathbf{q}+\mathbf{Hr})^t\mathbf{G}_1^{-1}(\mathbf{q}+\mathbf{Hr}) + \mathbf{r}^t\mathbf{G}_2\mathbf{r} = 2[\mathbf{q}^t,\mathbf{r}^t]\,\mathbf{M}\,[\mathbf{q}^t,\mathbf{r}^t]^t$, and a possible asymmetry of the matrix $\mathbf{H}$ is responsible for the *twist* of the signal [10, 17, 29, 54, 56, 60, 61, 63]. We will study the twist later in Sect. 2.8.2.

In the one-dimensional case, the twist is irrelevant and the 2×2 moment matrix takes the form

$$\mathbf{M} = \frac{1}{4\pi}\begin{bmatrix} g_1^{-1} & hg_1^{-1}\\ hg_1^{-1} & g_2+h^2g_1^{-1}\end{bmatrix} = \frac{1}{4\pi\sigma}\begin{bmatrix} g^{-1} & hg^{-1}\\ hg^{-1} & g+h^2g^{-1}\end{bmatrix}, \tag{2.115}$$

with $g = \sqrt{g_1 g_2} > 0$ and $\sigma = \sqrt{g_1/g_2} > 0$. Note that $\det\mathbf{M} = (4\pi\sigma)^{-2}$ and that σ is bounded by 1, $\sigma \le 1$, as a result of the uncertainty relation $m_{xx}m_{uu} \ge (4\pi)^{-2}$.

2.8.1 Moment Invariants for the Linear Canonical Transformation

Using the symplecticity condition $\mathbf{T}^{-1} = \mathbf{J}\mathbf{T}^t\mathbf{J}$, the moment relation (2.112) can be rewritten in the form of the similarity relation $\mathbf{M}_o\mathbf{J} = \mathbf{T}\mathbf{M}_i\mathbf{J}\mathbf{T}^{-1}$ [14], from which we conclude that the (real!) eigenvalues of the matrix $\mathbf{MJ}$ are invariant under a linear canonical transformation. The same holds, of course, for the coefficients of the characteristic equation $\det(\mathbf{MJ} - \lambda\mathbf{I}) = 0$, which appears to be an equation in λ^2. In the one-dimensional case this equation reads $\lambda^2 - \det\mathbf{M} = 0$, from which we conclude that $\det\mathbf{M}$ is the (only) invariant. In the two-dimensional case we have

$$\lambda^4 - [(m_{xx}m_{uu} - m_{xu}^2) + (m_{yy}m_{vv} - m_{yv}^2) + 2(m_{xy}m_{uv} - m_{xv}m_{yu})]\lambda^2 + \det\mathbf{M} = 0 \tag{2.116}$$

and we thus find the two (independent) invariants

$$I_1 = \sqrt{\det\mathbf{M}}\,, \tag{2.117a}$$

$$I_2 = (m_{xx}m_{uu} - m_{xu}^2) + (m_{yy}m_{vv} - m_{yv}^2) + 2(m_{xy}m_{uv} - m_{xv}m_{yu}). \tag{2.117b}$$

The latter moment combination is known in optics as the beam quality parameter [52]. Instead of $I_1^2 = \lambda_x^2\lambda_y^2$ and $I_2 = \lambda_x^2 + \lambda_y^2$, we might as well consider the eigenvalues $\pm\lambda_{x,y}$ of $\mathbf{MJ}$ themselves as invariants. We may arbitrarily choose $\lambda_x \geq \lambda_y > 0$, in which case $\lambda_x \pm \lambda_y = (I_2 \pm 2I_1)^{1/2}$. Note that $I_2 \geq 2I_1$ and that the equality sign arises for $\lambda_x = \lambda_y$.

In the special case of phase-space rotators, for which the symplectic transformation matrix is also orthogonal, $\mathbf{T}^t = \mathbf{T}^{-1}$, the relation $\mathbf{M}_o = \mathbf{T}\mathbf{M}_i\mathbf{T}^t = \mathbf{T}\mathbf{M}_i\mathbf{T}^{-1}$ between the moment matrices themselves takes the form of a similarity transformation, and we conclude that the eigenvalues of $\mathbf{M}$ (or the coefficients of its characteristic equation, like the determinant and the trace of $\mathbf{M}$) are invariant. Note, however, that some of these invariants are not new in the sense that they are identical to or depend on the ones that we already found.

Another way to find moment invariants for phase-space rotators is to consider the Hermitian matrix [8]

$$\mathbf{M}' = \frac{1}{E}\iint (\mathbf{r} - \mathrm{i}\,\mathbf{q})\,(\mathbf{r} - \mathrm{i}\,\mathbf{q})^\dagger\, W(\mathbf{r},\mathbf{q})\,\mathrm{d}\mathbf{r}\,\mathrm{d}\mathbf{q} = \mathbf{M}_{\mathbf{rr}} + \mathbf{M}_{\mathbf{qq}} + \mathrm{i}\,(\mathbf{M}_{\mathbf{rq}} - \mathbf{M}_{\mathbf{rq}}^t) \tag{2.118}$$

and to use (2.58) to get the relation $\mathbf{M}'_o = \mathbf{U}\mathbf{M}'_i\mathbf{U}^\dagger = \mathbf{U}\mathbf{M}'_i\mathbf{U}^{-1}$, which is again a similarity transformation. In the two-dimensional case, the matrix $\mathbf{M}'$ can be written as

$$\mathbf{M}' = \begin{bmatrix} Q_0 + Q_1 & Q_2 + \mathrm{i}\,Q_3 \\ Q_2 - \mathrm{i}\,Q_3 & Q_0 - Q_1 \end{bmatrix} = Q_0\mathbf{I} + Q\begin{bmatrix} \cos\vartheta & \exp(\mathrm{i}\,\gamma)\sin\vartheta \\ \exp(-\mathrm{i}\,\gamma)\sin\vartheta & -\cos\vartheta \end{bmatrix} \tag{2.119}$$

$$\text{with} \quad \begin{aligned} Q_0 &= \tfrac{1}{2}[(m_{xx} + m_{uu}) + (m_{yy} + m_{vv})] \ , \\ Q_1 &= \tfrac{1}{2}[(m_{xx} + m_{uu}) - (m_{yy} + m_{vv})] = Q\cos\vartheta \ , \\ Q_2 &= m_{xy} + m_{uv} = Q\sin\vartheta\cos\gamma \ , \\ Q_3 &= m_{xv} - m_{yu} = Q\sin\vartheta\sin\gamma \ , \end{aligned} \tag{2.120}$$

where the moment parameters Q_j $(j = 0, 1, 2, 3)$ are the expectation values of the Hermitian operators [58, 60, 62] associated with the symmetric and antisymmetric fractional Fourier transformer, the gyrator, and the rotator, respectively; moreover, the matrix $\mathbf{M}'$ resembles the one introduced in [3], which is based on the operator approach. We then find two invariants from the coefficients of the characteristic equation $\det(\mathbf{M}' - \nu\mathbf{I}) = 0 = \nu^2 - 2Q_0\nu + Q_0^2 - Q^2 = (\nu - Q_0)^2 - Q^2$: the two parameters Q_0 and $Q^2 = Q_1^2 + Q_2^2 + Q_3^2$ or the two eigenvalues $\nu_{1,2} = Q_0 \pm Q$. Note that $2Q_0$, the trace of $\mathbf{M}'$, is also the trace of $\mathbf{M}$, and that $Q_0^2 - Q^2$ equals the determinant of $\mathbf{M}'$.

From the invariance of Q we conclude that the three-dimensional vector $(Q_1, Q_2, Q_3) = (Q\cos\vartheta, Q\sin\vartheta\cos\gamma, Q\sin\vartheta\sin\gamma)$ lives on a sphere with radius Q, known as the Poincaré sphere [3, 8, 48]. A phase-space rotator will only change the values of the angles ϑ and γ, but does not change the invariants Q_0 and Q. To transform a diagonal matrix $\mathbf{M}'$, with $\gamma = \vartheta = 0$, into the general form (2.119), we can use, for instance, the cascade $\mathscr{F}(\frac{1}{2}\gamma, -\frac{1}{2}\gamma)\,\mathscr{R}(-\frac{1}{2}\vartheta)\,\mathscr{F}(-\frac{1}{2}\gamma, \frac{1}{2}\gamma)$; we easily verify

$$\begin{aligned} &\mathbf{U}_f\left(\tfrac{1}{2}\gamma, -\tfrac{1}{2}\gamma\right)\,\mathbf{U}_r\left(-\tfrac{1}{2}\vartheta\right)\,\mathbf{U}_f\left(-\tfrac{1}{2}\gamma, \tfrac{1}{2}\gamma\right)\begin{bmatrix} Q_0 + Q & 0 \\ 0 & Q_0 - Q \end{bmatrix}\mathbf{U}_f\left(\tfrac{1}{2}\gamma, -\tfrac{1}{2}\gamma\right) \\ &\qquad \times \mathbf{U}_f\left(-\tfrac{1}{2}\gamma, \tfrac{1}{2}\gamma\right) = \begin{bmatrix} Q_0 + Q\cos\vartheta & Q\exp(\mathrm{i}\,\gamma)\sin\vartheta \\ Q\exp(-\mathrm{i}\,\gamma)\sin\vartheta & Q_0 - Q\cos\vartheta \end{bmatrix} . \end{aligned} \tag{2.121}$$

In the special case that the phase-space rotator is a symmetric fractional Fourier transformer, with a scalar matrix $\mathbf{U}$, the matrix $\mathbf{M}'$ itself is invariant, and so is the complete vector (Q_1, Q_2, Q_3). For the (antisymmetric) fractional Fourier transformer, the gyrator, and the rotator, one of the moment parameters Q_j $(j = 1, 2, 3)$ is invariant, while the other two undergo a rotation-type transformation, see Table 2.9.

2.8.2 *The Twist As an Invariant for Transformations with* $\mathbf{B} = \mathbf{0}$

We will now consider an important parameter for two-dimensional signals that is invariant under a linear canonical transformation with $\mathbf{B} = \mathbf{0}$ (and $\mathbf{A}^{-1} = \mathbf{D}^t$), see (2.46). The invariant that we consider is known as the *twist* of the signal, which is

Table 2.9 Second-order moment invariants for linear canonical transformations

Transformation	Invariant	Remark
$\mathscr{L}(\mathbf{T})$	$I_1 = (\det \mathbf{M})^{1/2} = \lambda_x \lambda_y$ $I_2 = (m_{xx}m_{uu} - m_{xu}^2) + (m_{yy}m_{vv} - m_{yv}^2) + 2(m_{xy}m_{uv} - m_{xv}m_{yu}) = \lambda_x^2 + \lambda_y^2$	$\pm\lambda_{x,y}$ are the eigenvalues of $\mathbf{MJ}$
$\mathscr{L}(\mathbf{T})$ $\mathbf{B} = \mathbf{0}$	$T = [(m_{xu} - m_{yv})m_{xy} + m_{xv}m_{yy} - m_{xx}m_{yu}]/(m_{xx}m_{yy} - m_{xy}^2)^{1/2}$	
$\mathscr{O}(\mathbf{U})$	$Q_0 = \frac{1}{2}[(m_{xx} + m_{uu}) + (m_{yy} + m_{vv})]$ $Q^2 = Q_1^2 + Q_2^2 + Q_3^2$	
$\mathscr{F}(\varphi, -\varphi)$	$Q_1 = \frac{1}{2}[(m_{xx} + m_{uu}) - (m_{yy} + m_{vv})]$	$(Q_2 + \mathrm{i}\, Q_3)_o = \exp(\mathrm{i}\, 2\varphi)\, (Q_2 + \mathrm{i}\, Q_3)_i$
$\mathscr{G}(\varphi)$	$Q_2 = m_{xy} + m_{uv}$	$(Q_3 + \mathrm{i}\, Q_1)_o = \exp(\mathrm{i}\, 2\varphi)\, (Q_3 + \mathrm{i}\, Q_1)_i$
$\mathscr{R}(-\varphi)$	$Q_3 = m_{xv} - m_{yu}$	$(Q_1 + \mathrm{i}\, Q_2)_o = \exp(\mathrm{i}\, 2\varphi)\, (Q_1 + \mathrm{i}\, Q_2)_i$
$\mathscr{F}(\gamma, \gamma)$	Q_1, Q_2, Q_3	

connected to the asymmetry of the matrix $\mathbf{H} = \mathbf{M}_{\mathbf{rr}}^{-1}\mathbf{M}_{\mathbf{rq}}$, see (2.114c). To measure the degree of twist, we use the asymmetry of the normalized matrix

$$\mathbf{M}_{\mathbf{rr}}^{1/2}(\mathbf{M}_{\mathbf{rr}}^{-1}\mathbf{M}_{\mathbf{rq}})\mathbf{M}_{\mathbf{rr}}^{1/2} = \mathbf{M}_{\mathbf{rr}}^{-1/2}\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}}^{1/2} = \mathbf{M}_{\mathbf{rr}}^{-1/2}(\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}})\mathbf{M}_{\mathbf{rr}}^{-1/2}$$

and we define the twist parameter T via the skew-symmetric matrix

$$\frac{\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}} - (\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}})^t}{(\det \mathbf{M}_{\mathbf{rr}})^{1/2}} \equiv \begin{bmatrix} 0 & T \\ -T & 0 \end{bmatrix};$$

hence

$$T = \frac{(m_{xu} - m_{yv})m_{xy} + m_{xv}m_{yy} - m_{xx}m_{yu}}{(m_{xx}m_{yy} - m_{xy}^2)^{1/2}} . \tag{2.122}$$

Note that the numerator in the above expression corresponds to the asymmetry of $\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}}$, i.e., to the upper off-diagonal element of $\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}} - (\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}})^t$. Using the input–output relation $\mathbf{M}_o = \mathbf{T}\mathbf{M}_i\mathbf{T}^t$, we easily derive

$$(\mathbf{M}_{\mathbf{rr}})_o = \mathbf{A}(\mathbf{M}_{\mathbf{rr}})_i\,\mathbf{A}^t , \tag{2.123a}$$

$$(\mathbf{M}_{\mathbf{rq}})_o = \mathbf{A}(\mathbf{M}_{\mathbf{rr}})_i\,\mathbf{C}^t + \mathbf{A}(\mathbf{M}_{\mathbf{rq}})_i\,\mathbf{A}^{-1} , \tag{2.123b}$$

$$(\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}})_o = \mathbf{A}(\mathbf{M}_{\mathbf{rr}})_i\,\mathbf{C}^t\mathbf{A}(\mathbf{M}_{\mathbf{rr}})_i\,\mathbf{A}^t + \mathbf{A}(\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}})_i\,\mathbf{A}^t . \tag{2.123c}$$

From (2.123a) we see that

$$(\det \mathbf{M}_{\mathbf{rr}})_o^{1/2} = (\det \mathbf{M}_{\mathbf{rr}})_i^{1/2} \det \mathbf{A} .$$

As the matrix $\mathbf{A}(\mathbf{M}_{\mathbf{rr}})_i\,\mathbf{C}^t\mathbf{A}(\mathbf{M}_{\mathbf{rr}})_i\,\mathbf{A}^t$ in (2.123c) is symmetric, the asymmetry of $(\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}})_o$ is equal to the asymmetry of $\mathbf{A}(\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}})_i\,\mathbf{A}^t$, which on its turn is equal to the asymmetry of $(\mathbf{M}_{\mathbf{rq}}\mathbf{M}_{\mathbf{rr}})_i \det \mathbf{A}$. Both the numerator and the denominator in the expression for T, see (2.122), scale with the same factor $\det \mathbf{A}$ and we thus conclude that the twist is invariant under a linear canonical transformation with $\mathbf{B} = \mathbf{0}$.

2.8.3 Williamson's Theorem, Canonical Form and the Twist

An interesting property follows from *Williamson's theorem* [25, 62, 70]: for any real positive-definite symmetric matrix $\mathbf{M}$, there exists a real symplectic matrix $\mathbf{T}_\circ$ such that $\mathbf{M} = \mathbf{T}_\circ \boldsymbol{\Delta}_\circ \mathbf{T}_\circ^t$, where $\boldsymbol{\Delta}_\circ = \mathbf{T}_\circ^{-1} \mathbf{M} \mathbf{T}_\circ^{t\,-1}$ takes the *canonical form*

$$\boldsymbol{\Delta}_\circ = \begin{bmatrix} \boldsymbol{\Lambda}_\circ & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_\circ \end{bmatrix} \quad \text{with} \quad \boldsymbol{\Lambda}_\circ = \begin{bmatrix} \lambda_x & 0 \\ 0 & \lambda_y \end{bmatrix} \quad \text{and} \quad \lambda_x \geq \lambda_y > 0 \,. \tag{2.124}$$

From the similarity transformation $\mathbf{MJ} = \mathbf{T}_\circ(\boldsymbol{\Delta}_\circ \mathbf{J})\mathbf{T}_\circ^{-1}$, we conclude that $\boldsymbol{\Delta}_\circ$ follows directly from the eigenvalues $\pm\lambda_x$ and $\pm\lambda_y$ of $\mathbf{MJ}$ and that $\mathbf{T}_\circ$ follows from the eigenvectors of $(\mathbf{MJ})^2$: $(\mathbf{MJ})^2 \mathbf{T}_\circ = \mathbf{T}_\circ \boldsymbol{\Delta}_\circ^2$. Any moment matrix $\mathbf{M}$ can thus be brought into the diagonal form $\boldsymbol{\Delta}_\circ$ by means of a realizable canonical transformation with ray transformation matrix $\mathbf{T}_\circ^{-1}$. We remark that the determination of the canonical eigenvalues λ_x and λ_y is easy; they follow immediately from the two moment invariants (2.117) $I_1 = \lambda_x \lambda_y$ and $I_2 = \lambda_x^2 + \lambda_y^2$.

The system $\mathscr{L}(\mathbf{T}_\circ)$ that connects the moment matrix $\mathbf{M}$ with its canonical form $\boldsymbol{\Delta}_\circ$ through $\mathbf{M} = \mathbf{T}_\circ \boldsymbol{\Delta}_\circ \mathbf{T}_\circ^t$, can be reduced—with the cascade (2.66) in mind—to the cascade of a gyrator and a generalized magnifier $\mathscr{Q}(\mathbf{G}_\circ)\,\mathscr{M}[\mathbf{S}_\circ \mathbf{U}_r(-\alpha_\circ)]\,\mathscr{G}(-\beta_\circ)$; note that the separable fractional Fourier transformer in (2.66) can be omitted because it does not affect the canonical form and that we have combined the rotator with the pure magnifier. The system $\mathscr{Q}(\mathbf{G}_\circ)\,\mathscr{M}[\mathbf{S}_\circ \mathbf{U}_r(-\alpha_\circ)]$ thus connects the moment matrix $\mathbf{M}$ with its *generalized canonical form* $\mathbf{M}^\circ$,

$$\mathbf{M}^\circ = \begin{bmatrix} \mathbf{M}^\circ_{\mathbf{rr}} & \mathbf{M}^\circ_{\mathbf{rq}} \\ -\mathbf{M}^\circ_{\mathbf{rq}} & \mathbf{M}^\circ_{\mathbf{rr}} \end{bmatrix} , \quad \mathbf{M}^\circ_{\mathbf{rr}} + \mathrm{i}\,\mathbf{M}^\circ_{\mathbf{rq}} = \mathbf{U}_g(-\beta_\circ)\,\boldsymbol{\Lambda}_\circ\,\mathbf{U}_g(\beta_\circ) \,,$$

$$\mathbf{M}^\circ_{\mathbf{rr}} + \mathrm{i}\,\mathbf{M}^\circ_{\mathbf{rq}} = \begin{bmatrix} \lambda_x \cos^2\beta_\circ + \lambda_y \sin^2\beta_\circ & \mathrm{i}\,\frac{1}{2}(\lambda_x - \lambda_y)\sin 2\beta_\circ \\ -\mathrm{i}\,\frac{1}{2}(\lambda_x - \lambda_y)\sin 2\beta_\circ & \lambda_x \sin^2\beta_\circ + \lambda_y \cos^2\beta_\circ \end{bmatrix} . \tag{2.125}$$

Applying the definition of the twist (2.122) to the generalized canonical form $\mathbf{M}^\circ$, we readily conclude that the gyrator angle $\beta_\circ$ is completely determined by the twist T and the two canonical eigenvalues λ_x and λ_y,

$$(\lambda_x - \lambda_y)\sin 2\beta_\circ = \frac{2T\sqrt{\lambda_x \lambda_y}}{\sqrt{(\lambda_x + \lambda_y)^2 - T^2}} \,, \tag{2.126}$$

and that the same holds for the generalized canonical form itself. Note that these three parameters are indeed invariant under a linear canonical transformation of the form $\mathscr{Q}(-\mathbf{C}\mathbf{A}^{-1})\,\mathscr{M}(\mathbf{A})$, for which $\mathbf{B} = \mathbf{0}$.

Note that for the generalized canonical form $\mathbf{M}^\circ$, the moment vector (Q_1, Q_2, Q_3) reads $(Q \sin 2\beta_\circ, 0, Q \cos 2\beta_\circ)$ with $Q = \lambda_x - \lambda_y$, cf. (2.119), and that Q_3 corresponds to the left-hand side of (2.126). We easily verify that the maximum value of $|T|$ is reached for $\beta_\circ = \pm\frac{1}{4}\pi$ and that $|T|_{\max} = Q = \lambda_x - \lambda_y = \sqrt{I_2 - 2I_1}$. We also recall, see Sects. 2.6.3.3 and 2.8.1, that it is the gyrator $\mathscr{G}(\pm\frac{1}{4}\pi)$ that transforms the (vortex-free) Hermite–Gauss modes, with $(Q_1, Q_2, Q_3) = (Q, 0, 0)$, into the (maximum-vortex) Laguerre–Gauss modes, with $(Q_1, Q_2, Q_3) = (0, 0, \pm Q)$.

In the special case that $\lambda_x = \lambda_y = \lambda$, the canonical form $\mathbf{\Delta}_\circ = \lambda\mathbf{I}$ takes the form of a scalar matrix, and the system $\mathscr{L}(\mathbf{T}_\circ)$ that connects the moment matrix $\mathbf{M}$ to its canonical form reduces to the cascade $\mathscr{Q}(\mathbf{G}_\circ)\,\mathscr{M}(\mathbf{S}_\circ)$. Note that the Poincaré sphere for such a canonical form reduces to a single point: $Q = 0$. The case $\lambda_x = \lambda_y$ is known as the intrinsically isotropic case [25, 62] and the moment matrix $\mathbf{M}$ is now proportional to a symplectic matrix. Symplecticity of a moment matrix is preserved under a linear canonical transformation, and the moment relations $\mathbf{M}_o = \mathbf{T}\mathbf{M}_i\mathbf{T}^t$ and $(\mathbf{M}_o\mathbf{J}) = \mathbf{T}(\mathbf{M}_i\mathbf{J})\mathbf{T}^{-1}$, which deal with 4×4 matrices, can be replaced by an easier one that deals with 2×2 matrices; see Eq. (2.128) in the next section.

2.8.4 The Special Case of a Symplectic Moment Matrix

In this section we study the special case that in the moment representation (2.113) we have the additional conditions $\mathbf{H} = \mathbf{H}^t$, $\mathbf{G}_1 = \sigma\,\mathbf{G}$ and $\mathbf{G}_2 = \sigma^{-1}\mathbf{G}$, where $0 < \sigma \leq 1$; note that we are now dealing with a signal that has zero twist. The matrix moment then takes the special form [14, 15]

$$2\pi\,\mathbf{M} = \frac{1}{2\sigma}\begin{bmatrix} \mathbf{G}^{-1} & \mathbf{G}^{-1}\mathbf{H} \\ \mathbf{H}^t\mathbf{G}^{-1} & \mathbf{G} + \mathbf{H}^t\mathbf{G}^{-1}\mathbf{H} \end{bmatrix}, \tag{2.127}$$

and the input–output relations for the moments $\mathbf{M}_o = \mathbf{T}\mathbf{M}_i\mathbf{T}^t$ and $(\mathbf{M}_o\mathbf{J})\mathbf{T} = \mathbf{T}(\mathbf{M}_i\mathbf{J})$ can now be expressed as $[\mathbf{H}_o \pm \mathrm{i}\,\mathbf{G}_o][\mathbf{A} + \mathbf{C}(\mathbf{H}_i \pm \mathrm{i}\,\mathbf{G}_i)] = [\mathbf{C} + \mathbf{D}(\mathbf{H}_i \pm \mathrm{i}\,\mathbf{G}_i)]$, which leads to the *bilinear relationship*

$$\mathbf{H}_o \pm \mathrm{i}\,\mathbf{G}_o = [\mathbf{C} + \mathbf{D}(\mathbf{H}_i \pm \mathrm{i}\,\mathbf{G}_i)][\mathbf{A} + \mathbf{C}(\mathbf{H}_i \pm \mathrm{i}\,\mathbf{G}_i)]^{-1}\,. \tag{2.128}$$

This bilinear relationship, together with the invariance of $\det\mathbf{M} = (4\pi\,\sigma)^{-2}$, completely describes the propagation of a symplectic moment matrix $\mathbf{M}$ when the signal undergoes a linear canonical transformation. Note that the bilinear relationship is identical to the so-called **ABCD**-law for chirp-like signals of the form $\exp(\mathrm{i}\,\pi\,\mathbf{r}^t\mathbf{H}\mathbf{r})$; we have only replaced the (real) chirp matrix $\mathbf{H}$ by the (generally complex) matrix $\mathbf{H} \pm \mathrm{i}\,\mathbf{G}$, cf. (2.74). The bilinear relationship is also the basis for the treatment of complex Gaussian functions, $f(\mathbf{r}) = \exp[-\mathrm{i}\,\pi\,\mathbf{r}^t(\mathbf{G} - \mathrm{i}\,\mathbf{H})\mathbf{r}]$, under linear canonical transformations, see, for instance, [47, Sect. 3.4.6, Linear fractional transformations].

2.9 Conclusions

The mathematical formalism introduced in this chapter is used nowadays for numerous applications, including the description of paraxial light propagation through first-order optical systems, design and characterization of optical beams and systems, development of filtering and encryption techniques in signal processing.

For example, the linear canonical transformation's phase-space representation allows associating the transformation parameters with the ray transformation matrix known from geometrical optics and therefore establishing a relation between the ray and the wave description of light. The matrix description of linear canonical transformations drastically simplifies the design and analysis of the composed optical systems, as well as the calculation of the beam propagation through them. The use of the modified Iwasawa decomposition of the transformation matrix together with the detailed analysis of the phase-space rotator matrix provides a clear interpretation of the signal modification produced by the transformation. Thus, the central role of the fractional Fourier transformation among other linear canonical transformations is revealed. The affine transformation of the Wigner distribution and the ambiguity function produced by such transformations is the key for the establishing of phase-space tomography methods used for the characterization of classical and quantum light. The corresponding transformation of the second-order moments of the Wigner distribution, described in this chapter, is useful for global beam analysis.

The diversity of the linear canonical transformation parameters (ten in the two-dimensional case) is exploited, as it is discussed in the next chapters, in different phase retrieval, filtering and encryption techniques.

Appendix

Derivation of the Phase-Space Relation (2.5)

We start with (2.4) and substitute from (2.1):

$$\begin{aligned} W_o(\mathbf{r}_o,\mathbf{q}_o) &= \int f\left(\mathbf{r}_o+\tfrac{1}{2}\mathbf{r}'_o\right) f^*\left(\mathbf{r}_o-\tfrac{1}{2}\mathbf{r}'_o\right)\,\exp[-\mathrm{i}\,2\pi\;\mathbf{q}_o^t\mathbf{r}'_o]\,\mathrm{d}\mathbf{r}'_o \\ &= |\det\mathbf{L}_{io}|\iiint \exp[\mathrm{i}\,\pi(E_1-E_2-2\mathbf{q}_o^t\mathbf{r}'_o)]f_i(\mathbf{r}_1)f_i^*(\mathbf{r}_2)\,\mathrm{d}\mathbf{r}_1\,\mathrm{d}\mathbf{r}_2\,\mathrm{d}\mathbf{r}'_o \end{aligned}$$

$$\text{with}\quad \begin{cases} E_1 = \left(\mathbf{r}_o+\tfrac{1}{2}\mathbf{r}'_o\right)^t\mathbf{L}_{oo}\left(\mathbf{r}_o+\tfrac{1}{2}\mathbf{r}'_o\right) - 2\mathbf{r}_1^t\mathbf{L}_{io}\left(\mathbf{r}_o+\tfrac{1}{2}\mathbf{r}'_o\right) + \mathbf{r}_1^t\mathbf{L}_{ii}\mathbf{r}_1\ , \\ E_2 = \left(\mathbf{r}_o-\tfrac{1}{2}\mathbf{r}'_o\right)^t\mathbf{L}_{oo}\left(\mathbf{r}_o-\tfrac{1}{2}\mathbf{r}'_o\right) - 2\mathbf{r}_2^t\mathbf{L}_{io}\left(\mathbf{r}_o-\tfrac{1}{2}\mathbf{r}'_o\right) + \mathbf{r}_2^t\mathbf{L}_{ii}\mathbf{r}_2\ . \end{cases}$$

We reorder the exponent $E_1 - E_2$ to get

$$E_1 - E_2 = (\mathbf{r}'^{\,t}_o \mathbf{L}_{oo}\mathbf{r}_o + \mathbf{r}^t_o \mathbf{L}_{oo}\mathbf{r}'_o) - 2(\mathbf{r}_1 - \mathbf{r}_2)^t \mathbf{L}_{io}\mathbf{r}_o - (\mathbf{r}_1 + \mathbf{r}_2)^t \mathbf{L}_{io}\mathbf{r}'_o \\ +(\mathbf{r}^t_1 \mathbf{L}_{ii}\mathbf{r}_1 - \mathbf{r}^t_2 \mathbf{L}_{ii}\mathbf{r}_2)\ .$$

We substitute $\mathbf{r}_1 = \mathbf{r}_i + \frac{1}{2}\mathbf{r}'_i$ and $\mathbf{r}_2 = \mathbf{r}_i - \frac{1}{2}\mathbf{r}'_i$ and get

$$W_o(\mathbf{r}_o, \mathbf{q}_o) = |\det \mathbf{L}_{io}| \iiint \exp[\mathrm{i}\,\pi(E_1 - E_2 - 2\mathbf{q}^t_o \mathbf{r}'_o)] \\ \times f_i\left(\mathbf{r}_i + \tfrac{1}{2}\mathbf{r}'_i\right) f^*_i\left(\mathbf{r}_i - \tfrac{1}{2}\mathbf{r}'_i\right)\, \mathrm{d}\mathbf{r}_i\, \mathrm{d}\mathbf{r}'_i\, \mathrm{d}\mathbf{r}'_o$$

with $E_1 - E_2 = (\mathbf{r}'^{\,t}_o \mathbf{L}_{oo}\mathbf{r}_o + \mathbf{r}^t_o \mathbf{L}_{oo}\mathbf{r}'_o) - \mathbf{r}'^{\,t}_i \mathbf{L}_{io}\mathbf{r}_o - 2\mathbf{r}^t_i \mathbf{L}_{io}\mathbf{r}'_o + (\mathbf{r}'^{\,t}_i \mathbf{L}_{ii}\mathbf{r}_i + \mathbf{r}^t_i \mathbf{L}_{ii}\mathbf{r}'_i)$.

We substitute $f_i\left(\mathbf{r}_i + \frac{1}{2}\mathbf{r}'_i\right) f^*_i\left(\mathbf{r}_i - \frac{1}{2}\mathbf{r}'_i\right) = \int W_i(\mathbf{r}_i, \mathbf{q}_i)\, \exp[\mathrm{i}\, 2\pi\, \mathbf{r}'^{\,t}_i \mathbf{q}_i]\, \mathrm{d}\mathbf{q}_i$ and get

$$\begin{aligned} &W_o(\mathbf{r}_o, \mathbf{q}_o) \\ &\quad = |\det \mathbf{L}_{io}| \iiiint W_i(\mathbf{r}_i, \mathbf{q}_i) \exp[\mathrm{i}\,\pi(E_1 - E_2 - 2\mathbf{q}^t_o \mathbf{r}'_o] \\ &\qquad \times \exp[\mathrm{i}\, 2\pi\, \mathbf{r}'^{\,t}_i \mathbf{q}_i]\, \mathrm{d}\mathbf{r}'_i\, \mathrm{d}\mathbf{r}'_o\, \mathrm{d}\mathbf{r}_i\, \mathrm{d}\mathbf{q}_i \\ &\quad = |\det \mathbf{L}_{io}| \iint W_i(\mathbf{r}_i, \mathbf{q}_i) \left(\int \exp[\mathrm{i}\, 2\pi(\mathbf{L}_{oo}\mathbf{r}_o - \mathbf{L}^t_{io}\mathbf{r}_i - \mathbf{q}_o)^t \mathbf{r}'_o]\, \mathrm{d}\mathbf{r}'_o\right) \\ &\qquad \times \left(\int \exp[\mathrm{i}\, 2\pi(\mathbf{L}_{ii}\mathbf{r}_i - \mathbf{L}_{io}\mathbf{r}_o + \mathbf{q}_i)^t \mathbf{r}'_i]\, \mathrm{d}\mathbf{r}'_i\right) \mathrm{d}\mathbf{r}_i\, \mathrm{d}\mathbf{q}_i \\ &\quad = |\det \mathbf{L}_{io}| \iint W_i(\mathbf{r}_i, \mathbf{q}_i)\, \delta(\mathbf{L}_{oo}\mathbf{r}_o - \mathbf{L}^t_{io}\mathbf{r}_i - \mathbf{q}_o)\, \delta(\mathbf{L}_{ii}\mathbf{r}_i - \mathbf{L}_{io}\mathbf{r}_o + \mathbf{q}_i)\, \mathrm{d}\mathbf{r}_i\, \mathrm{d}\mathbf{q}_i \\ &\quad = |\det \mathbf{L}_{io}| \int W_i(\mathbf{r}_i, \mathbf{L}_{io}\mathbf{r}_o - \mathbf{L}_{ii}\mathbf{r}_i)\, \delta(\mathbf{L}_{oo}\mathbf{r}_o - \mathbf{L}^t_{io}\mathbf{r}_i - \mathbf{q}_o)\, \mathrm{d}\mathbf{r}_i \\ &\quad = W_i\left(\mathbf{L}^{t\,-1}_{io}\mathbf{L}_{oo}\mathbf{r}_o - \mathbf{L}^{t\,-1}_{io}\mathbf{q}_o, \mathbf{L}_{io}\mathbf{r}_o - \mathbf{L}_{ii}\mathbf{L}^{t\,-1}_{io}\mathbf{L}_{oo}\mathbf{r}_o + \mathbf{L}_{ii}\mathbf{L}^{t\,-1}_{io}\mathbf{q}_o\right)\ . \end{aligned}$$

After substituting from (2.7), we finally get

$$W_o(\mathbf{r}_o, \mathbf{q}_o) = W_i(\mathbf{D}^t\mathbf{r}_o - \mathbf{B}^t\mathbf{q}_o, -\mathbf{C}^t\mathbf{r}_o + \mathbf{A}^t\mathbf{q}_o)$$

and hence $W_o(\mathbf{Ar} + \mathbf{Bq}, \mathbf{Cr} + \mathbf{Dq}) = W_i(\mathbf{r}, \mathbf{q})$, which is identical to (2.5).

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