# Chapter 2
# Topologies and Metric Spaces

## 1 Topological Spaces

Let $X$ be a set. A collection $\mathcal{U}$ of subsets of $X$ defines a *topology* on $X$ if:

**i**. the empty set $\emptyset$ and $X$ belong to $\mathcal{U}$

**ii**. the union of any collection of sets in $\mathcal{U}$ is in $\mathcal{U}$

**iii**. the intersection of finitely many elements of $\mathcal{U}$ is in $\mathcal{U}$.

The pair $\{X;\mathcal{U}\}$, that is $X$ endowed with the topology generated by $\mathcal{U}$, is a topological space. The elements $\mathcal{O}$ of $\mathcal{U}$ are the *open* sets of $X$. A set $C$ in $X$ is *closed* if $X - C$ is open. The empty set $\emptyset$ and $X$ are both open and closed. It follows from the definitions that the finite union of closed sets is closed and the intersection of any collection of closed sets is closed.

An open neighborhood of a set $A \subset X$ is any open set that contains $A$. In particular a neighborhood of a singleton $x \in X$, is any open set $\mathcal{O}$ such that $x \in \mathcal{O}$. A subset $\mathcal{O} \subset X$ is open if and only if is an open neighborhood of any of its points.

A point $x \in A$ is an *interior* point of $A$ if there exists an open set $\mathcal{O}$ such that $x \in \mathcal{O} \subset A$. The interior of $A$ is the set of all its interior points. A set $A \subset X$ is open if and only if it coincides with its interior.

A point $x$ is a point of *closure* of $A$ if every open neighborhood of $x$ intersects $A$. The closure $\bar{A}$ of $A$ is the set of all the points of closure of $A$. A set $A$ is closed if and only if $A = \bar{A}$. It follows that $A$ is closed if and only if it is the intersection of all closed sets containing $A$.

Let $\{x_n\}$ be a sequence of elements of $X$. A point $x \in X$ is a *cluster* point for the sequence $\{x_n\}$ if every open set $\mathcal{O}$ containing $x$, contains infinitely many elements of $\{x_n\}$. The sequence $\{x_n\}$ converges to $x$ if for every open set $\mathcal{O}$ containing $x$, there exists a positive integer $m(\mathcal{O})$ depending on $\mathcal{O}$, such that $x_n \in \mathcal{O}$ for all $n \geq m(\mathcal{O})$. Thus limit points for $\{x_n\}$ are cluster points. The converse is false. Indeed, there exist



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sequences $\{x_n\}$ with a cluster point $x_o$ such that no subsequence of $\{x_n\}$ converges to $x_o$ (**1.11** of the Complements).

**Proposition 1.1** (Cauchy) *A sequence $\{x_n\}$ of elements of a topological space $\{X; \mathcal{U}\}$ converges to $x$ if and only if every subsequence $\{x_{n'}\} \subset \{x_n\}$ contains in turn a subsequence $\{x_{n''}\} \subset \{x_{n'}\}$ converging to $x$.*

Let $A$ and $B$ be subsets of $X$. The set $B$ is *dense* in $A$ if $A \subset \bar{B}$. If also $B \subset A$, then $\bar{A} = \bar{B}$. The space $\{X; \mathcal{U}\}$ is *separable* if it contains a countable dense set.

Let $X_o$ be a subset of $X$. The collection $\mathcal{U}$ induces a topology on $X_o$, by the family $\mathcal{U}_o = \{\mathcal{O} \cap X_o\}$. The pair $\{X_o; \mathcal{U}_o\}$ is a topological subspace of $\{X; \mathcal{U}\}$. A subspace of a separable topological space need not be separable (**4.9** of the Complements).

Let $\{X; \mathcal{U}\}$ and $\{Y; \mathcal{V}\}$ be any two topological spaces. A function $f : X \to Y$ is continuous at a point $x \in X$ if for every open set $O \in \mathcal{V}$ containing $f(x)$, there is an open set $\mathcal{O} \in \mathcal{U}$ containing $x$ and such that $f(\mathcal{O}) \subset O$. A function $f : X \to Y$ is continuous if it is continuous at every $x \in X$. This implies that $f$ is continuous if and only if the pre-image of every open set is open, that is if for every open set $O \in \mathcal{V}$, the set $f^{-1}(O)$ is an open set $\mathcal{O} \in \mathcal{U}$. Equivalently $f$ is continuous if and only if the pre-image of a closed set is closed.

The restriction of a continuous function $f : X \to Y$ to a subset $X_o \subset X$ is continuous with respect to the induced topology of $\{X_o; \mathcal{U}_o\}$.

An *homeomorphism* between $\{X; \mathcal{U}\}$ and $\{Y; \mathcal{V}\}$ is a continuous one-to-one function $f$ from $X$ onto $Y$, with continuous inverse $f^{-1}$. If $f : X \to Y$ is a homeomorphism then $f(\mathcal{O}) \in \mathcal{V}$ for all $\mathcal{O} \in \mathcal{U}$.

Two homeomorphic topological spaces are equivalent in the sense that the elements of $X$ are in one-to-one correspondence with the elements of $Y$ and the open sets making up the topology of $\{X; \mathcal{U}\}$ are in one-to-one correspondence with the open sets making up the topology of $\{Y; \mathcal{V}\}$.

The collection $2^X$ of all subsets of $X$ generates a topology on $X$ called the *discrete topology*. Every function $f$ from $\{X; 2^X\}$ into a topological space $\{Y; \mathcal{V}\}$ is continuous.

By a *real valued* function $f$ defined on some $\{X; \mathcal{U}\}$, we mean a function from $\{X; \mathcal{U}\}$ into $\mathbb{R}$ endowed with the Euclidean topology.

The *trivial topology* on $X$ is that for which the only open sets are $X$ and $\emptyset$. The closure of any point $x \in X$ is $X$. All the continuous real-valued functions defined on $X$ are constant.

As a short-hand notation, we denote by $X$ a topological space, whenever a topology $\mathcal{U}$ is clear from the context, or whenever the specification of a topology $\mathcal{U}$ is immaterial.

## *1.1 Hausdorff and Normal Spaces*

A topological space $\{X; \mathcal{U}\}$ is a *Hausdorff* space if it separates points, that is, if for any two distinct points $x, y \in X$, there exist disjoint open sets $\mathcal{O}_x$ and $\mathcal{O}_y$ such that $x \in \mathcal{O}_x$ and $y \in \mathcal{O}_x$.

**Proposition 1.2** *Let $\{X; \mathcal{U}\}$ be a Hausdorff topological space. Then, the points $x \in X$ are closed.*

*Proof* Every point $y \in (X - x)$ is contained in some open set contained in $X - x$. Since $X - x$ is the union of all such open sets, it is open. Thus $\{x\}$ is closed. ∎

*Remark 1.1* The converse is false. See § **4.2** of the Complements.

A topological space $\{X; \mathcal{U}\}$ is *normal* if it separates closed sets, that is, for any two disjoint closed sets $C_1$ and $C_2$, there exist disjoint open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $C_1 \subset \mathcal{O}_1$ and $C_2 \subset \mathcal{O}_2$.

A normal space need not be Hausdorff. For example the trivial topology is not Hausdorff but it is normal. However, if in addition the singletons $\{x\}$ are closed, then a normal space is Hausdorff. The converse is false as there exist Hausdorff spaces that do not separate closed sets (§ **1.19** of the Complements).

## 2 Urysohn's Lemma

**Lemma 2.1** (Uryson [166]) *Let $\{X; \mathcal{U}\}$ be normal. Given any two closed, disjoint sets $A$ and $B$ in $X$, there exist a continuous function $f : X \to [0, 1]$ such that $f = 0$ on $A$ and $f = 1$ on $B$.*

*Proof* We may assume that neither $A$ nor $B$ is empty. Indeed, for example, $A = \emptyset$, the function $f = 1$ satisfies the conclusion of the Lemma. Let $t$ denote nonnegative, rational dyadic numbers in $[0, 1]$, that is of the form

$$t = \frac{m}{2^n} \qquad m = 0, 1, \dots, 2^n; \qquad n = 0, 1, \dots.$$

For each such $t$ we construct an open set $\mathcal{O}_t$ in such a way that the family $\{\mathcal{O}_t\}$ satisfies

$$\mathcal{O}_o \supset A, \quad \mathcal{O}_1 = X - B, \quad \text{and} \quad \bar{\mathcal{O}}_\tau \subset \mathcal{O}_t \quad \text{whenever} \ \ \tau < t. \tag{2.1}$$

Since $\{X; \mathcal{U}\}$ is normal, there exists an open set $\mathcal{O}_o$ containing $A$ and whose closure is contained in $X - B$. For $n = 0$ and $m = 0$, select such an open set $\mathcal{O}_o$. For $n = 0$ and $m = 1$ select $\mathcal{O}_1$ as in (2.1). To $n = 1$ and $m = 0, 1, 2$, there correspond sets $\mathcal{O}_o$, $\mathcal{O}_{\frac{1}{2}}$, and $\mathcal{O}_1$. The first and last have been selected and we select set $\mathcal{O}_{\frac{1}{2}}$ so that

$$\bar{\mathcal{O}}_o \subset \mathcal{O}_{\frac{1}{2}} \subset \bar{\mathcal{O}}_{\frac{1}{2}} \subset \mathcal{O}_1.$$

To $n = 2$ and $m = 0, 1, 2, 3, 4$ there correspond open sets $\mathcal{O}_{\frac{m}{2^2}}$ of which only $\mathcal{O}_{\frac{1}{4}}$ and $\mathcal{O}_{\frac{3}{4}}$ have to be selected. Since $\{X; \mathcal{U}\}$ is normal, there exist open sets $\mathcal{O}_{\frac{1}{4}}$ and $\mathcal{O}_{\frac{3}{4}}$ such that

$$\bar{\mathcal{O}}_o \subset \mathcal{O}_{\frac{1}{4}} \subset \bar{\mathcal{O}}_{\frac{1}{4}} \subset \mathcal{O}_{\frac{1}{2}} \quad \text{and} \quad \bar{\mathcal{O}}_{\frac{1}{2}} \subset \mathcal{O}_{\frac{3}{4}} \subset \bar{\mathcal{O}}_{\frac{3}{4}} \subset \mathcal{O}_1.$$

Proceeding by induction, if the open sets $\mathcal{O}_{m/2^{n-1}}$ have been selected we choose the sets $\mathcal{O}_{m/2^n}$ by first observing that the ones corresponding to $m$ even, have already been selected. Therefore, we have only to choose those corresponding to $m$ odd. For any such $m$ fixed the open sets $\mathcal{O}_{(m-1)/2^n}$ and $\mathcal{O}_{(m+1)/2^n}$ have been selected. Since $\{X; \mathcal{U}\}$ is normal, there exists an open set $\mathcal{O}_{m/2^n}$ such that

$$\bar{\mathcal{O}}_{\frac{m-1}{2^n}} \subset \mathcal{O}_{\frac{m}{2^n}} \subset \bar{\mathcal{O}}_{\frac{m}{2^n}} \subset \mathcal{O}_{\frac{m+1}{2^n}}.$$

Define $f : X \to [0, 1]$ by setting $f(x) = 1$ for $x \in B$ and

$$f(x) = \inf\{t \bigm| x \in \mathcal{O}_t\} \quad \text{for } x \in X - B.$$

By construction $f(x) = 0$ on $A$ and $f(x) = 1$ on $B$. It remains to prove that $f$ is continuous. From the definition of $f$ it follows that for all $s \in (0, 1]$

$$[f < s] = \textstyle\bigcup_{t<s} \mathcal{O}_t \qquad \text{and} \qquad [f \le s] = \textstyle\bigcap_{t>s} \mathcal{O}_t.$$

Therefore $[f < s]$ is open. On the other hand by the last of (2.1) $\bar{\mathcal{O}}_\tau \subset \mathcal{O}_t$, whenever $\tau < t$. Therefore

$$[f \le s] = \textstyle\bigcap_{t>s} \mathcal{O}_t = \textstyle\bigcap_{t>s} \bar{\mathcal{O}}_t$$

and $[f \le s]$ is closed. ∎

**Corollary 2.1** *A Hausdorff space $\{X; \mathcal{U}\}$ is normal if and only if, for every pair of closed disjoint subsets $C_1$ and $C_2$ of $X$, there exists a continuous function $f : X \to \mathbb{R}$ such that $f = 1$ on $C_1$ and $f = 0$ on $C_2$.*

## 3 The Tietze Extension Theorem

**Theorem 3.1** (Tietze [158]) *Let $\{X; \mathcal{U}\}$ be normal. A continuous function $f$ from a closed subset $C$ of $X$ into $\mathbb{R}$ has a continuous extension on $X$, that is, there exists a continuous real-valued function $f_*$ defined on the whole $X$, such that $f = f_*$ on $C$. Moreover if $f$ is bounded, say*

$$|f(x)| \le M \quad \textit{for all } x \in C \textit{ for some } M > 0$$

*then $f_*$ satisfies the same bound.*

*Proof* Assume first that $f$ is bounded and that $M \le 1$. We will construct a sequence of real valued, continuous functions $\{g_n\}$, defined on the whole $X$, such that for all $n \in \mathbb{N}$

$$
\begin{aligned}
|g_n(x)| &\le \frac{1}{3}\Big(\frac{2}{3}\Big)^{n-1} && \text{for all } x \in X \\
\Big|f(x) - \sum_{j=1}^{n} g_j(x)\Big| &\le \Big(\frac{2}{3}\Big)^{n} && \text{for all } x \in C.
\end{aligned}
\tag{3.1}
$$

Assuming the sequence $\{g_n\}$ has been constructed, by virtue of the first of (3.1), the series $\sum g_n$ is uniformly convergent in $X$ and $|\sum g_n| \le 1$. Therefore, the functions $f_n = \sum_{j=1}^{n} g_j$ are continuous and form a sequence $\{f_n\}$, uniformly convergent on $X$ to a continuous function $f*$. From the second of (3.1) it follows that $f = f_*$ on $C$. It remains to construct the sequence $\{g_n\}$.

Since $f$ is continuous, the two sets $[f \le -\frac{1}{3}]$ and $[f \ge \frac{1}{3}]$ are closed and disjoint. By Urysohn's lemma, there exists a continuous function $g_1 : X \to [-\frac{1}{3}, \frac{1}{3}]$ such that

$$g_1 = \tfrac{1}{3} \quad \text{on} \quad [f \ge \tfrac{1}{3}] \quad \text{and} \quad g_1 = -\tfrac{1}{3} \quad \text{on} \quad [f \le -\tfrac{1}{3}].$$

By construction $|f(x) - g_1(x)| \le \frac{2}{3}$ for all $x \in C$. The function $h_1 = f - g_1$ is continuous and bounded on $C$. The two sets $[h_1 \le -\frac{2}{9}]$ and $[h_1 \ge \frac{2}{9}]$ are closed and disjoint. Therefore by Urysohn's lemma there exists a continuous function $g_2 : X \to [-\frac{2}{9}, \frac{2}{9}]$ such that

$$g_2 = \tfrac{2}{9} \quad \text{on} \quad [h_1 \ge \tfrac{2}{9}] \quad \text{and} \quad g_2 = -\tfrac{2}{9} \text{ on} \quad [h_1 \le -\tfrac{2}{9}].$$

By construction

$$|h_1 - g_2| = |f - (g_1 + g_2)| \le \tfrac{4}{9} \quad \text{on } C.$$

The sequence $\{g_n\}$ is constructed inductively by this procedure. This proves Tietze's Theorem if $f$ is bounded and $|f| \le 1$. If $f$ is bounded and $|f| \le M$, the conclusion follows by replacing $f$ with $f/M$. If $f$ is unbounded, set

$$f_o = \frac{f}{1 + |f|}.$$

Since $f_o : C \to \mathbb{R}$ is continuous and bounded it has a continuous extension $g_o : X \to [-1, 1]$. In particular

$$g_o(x) = \frac{f(x)}{1 + |f(x)|} \in (-1, 1) \quad \text{for all } x \in C.$$

This implies that the set $[|g_o| = 1]$ is closed and disjoint from $C$. By the Uryshon Lemma there exists a continuous function $\eta : X \to \mathbb{R}$ such that $\eta = 1$ on $C$ and $\eta = 0$ on $[|g_o| = 1]$. The function

$$g = \frac{\eta g_o}{1 - \eta |g_o|} : X \to \mathbb{R}$$

is continuous and coincides with $f$ on $C$. ∎

# 4 Bases, Axioms of Countability and Product Topologies

A family of open sets $\mathcal{B}$ is a *base* for the topology of $\{X; \mathcal{U}\}$, if for every open set $\mathcal{O}$ and every $x \in \mathcal{O}$, there exists a set $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{O}$. A collection $\mathcal{B}_x$ of open sets, is a base at $x$ if for each open set $\mathcal{O}$ containing $x$, there exists $B \in \mathcal{B}_x$ such that $x \in B \subset \mathcal{O}$. Thus if $\mathcal{B}$ is a base for the topology of $\{X; \mathcal{U}\}$, it is also a base for each of the points of $X$. More generally, $\mathcal{B}$ is a base if and only if it contains a base for each of the points of $X$. A set $\mathcal{O}$ is open if and only if for each $x \in \mathcal{O}$ there exists $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{O}$.

Let $\mathcal{B}$ be a base for $\{X; \mathcal{U}\}$. Then:

**i**. Every $x \in X$ belongs to some $B \in \mathcal{B}$
**ii**. For any two given sets $B_1$ and $B_2$ in $\mathcal{B}$ and every $x \in B_1 \cap B_2$, there exists some $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

The notion of base is induced by the presence of a topology generated by $\mathcal{U}$ on $X$. Conversely, if a collection of sets $\mathcal{B}$ satisfies (**i**) and (**ii**), then it permits one to construct a topology on $X$ for which $\mathcal{B}$ is a base.

**Proposition 4.1** *Let $\mathcal{B}$ be a collection of sets in $X$ satisfying (*i*)–(*ii*). There exists a collection $\mathcal{U}$ of subsets of $X$, which generates a topology on $X$, for which $\mathcal{B}$ is a base.*

*Proof* Let $\mathcal{U}$ consist of the empty set $\emptyset$ and the collection of all subsets $\mathcal{O}$ of $X$, such that for every $x \in \mathcal{O}$ there exists an element $B \in \mathcal{B}$ such that $x \in B \subset \mathcal{O}$. Such a collection is not empty since $X \in \mathcal{U}$.

It follows from the definition that the union of any collection of elements in $\mathcal{U}$ remains in $\mathcal{U}$. Moreover $\mathcal{U}$ contains the empty set $\emptyset$ and $X$.

Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be any two elements of $\mathcal{U}$ with nonempty intersection. For every $x \in \mathcal{O}_1 \cap \mathcal{O}_2$ there exist sets $B_1$, $B_2$ and $B_3$ in $\mathcal{B}$ such that $B_i \subset \mathcal{O}_i$, $i = 1, 2$ and

$$x \in B_3 \subset B_1 \cap B_2 \subset \mathcal{O}_1 \cap \mathcal{O}_2.$$

Therefore $\mathcal{O}_1 \cap \mathcal{O}_2 \in \mathcal{U}$. This implies that the collection $\mathcal{U}$ generates a topology on $X$ for which $\mathcal{B}$ is a base. ∎

A topological space $\{X; \mathcal{U}\}$ satisfies the *first axiom of countability* if each point $x \in X$ has a countable base. The space $\{X; \mathcal{U}\}$ satisfies the *second axiom of countability* if there exists a countable base for its topology.

**Proposition 4.2** *Every topological space satisfying the second axiom of countability is separable.*

*Proof* Let $\{\mathcal{O}\}$ be a countable base for the topology of $\{X; \mathcal{U}\}$. For each $i \in \mathbb{N}$ select an element $x_i \in \mathcal{O}_i$. This generates a countable, dense subset of $\{X; \mathcal{U}\}$. ∎

## 4.1 Product Topologies

Let $\{X_1; \mathcal{U}_1\}$ and $\{X_2; \mathcal{U}_2\}$ be two topological spaces. The *product topology* $\mathcal{U}_1 \times \mathcal{U}_2$ on the Cartesian product $X_1 \times X_2$ is constructed by considering the collection $\mathcal{B}$ of all products $\mathcal{O}_1 \times \mathcal{O}_2$ where $\mathcal{O}_i \in \mathcal{U}_i$ for $i = 1, 2$. These are called the *open rectangles* of the product topology. First, one verifies that they form a base in the sense of (**i**)-(**ii**). Then the product topological space $\{X_1 \times X_2; \mathcal{U}_1 \times \mathcal{U}_2\}$ is constructed by the procedure of Proposition 4.1. The symbol $\mathcal{U}_1 \times \mathcal{U}_2$ means the collection $\mathcal{U}$ of sets in $X_1 \times X_2$, constructed by the procedure of Proposition 4.1.

If $\{X_1, \mathcal{U}_1\}$ and $\{X_2; \mathcal{U}_2\}$ are Hausdorff spaces, then the topological product space is a Hausdorff space.

Let $X_1 \times X_2$ be endowed with the product topology $\mathcal{U}_1 \times \mathcal{U}_2$. Then the projections

$$\pi_j : X_1 \times X_2 \to X_j, \qquad j = 1, 2$$

are continuous. Moreover $\mathcal{U}_1 \times \mathcal{U}_2$ is the weakest topology on $X_1 \times X_2$ for which such projections are continuous. The procedure can be iterated to construct the product topology on the product of $n$ topological spaces $\{X_i; \mathcal{U}_i\}_{i=1}^n$. Such a topology is the weakest topology for which the projections

$$\pi_j : \prod_{i=1}^n X_i \to X_j, \qquad j = 1, \dots, n$$

are continuous. More generally, given an infinite family of topological spaces $\{X_\alpha; \mathcal{U}_\alpha\}_{\alpha \in A}$, the product topology $\prod \mathcal{U}_\alpha$ on the Cartesian product $\prod X_\alpha$ is constructed as the weakest topology for which the projections

$$\pi_\beta : \prod X_\alpha \to X_\beta$$

are continuous for all $\beta \in A$. Such a topology must contain the collection

$$\mathcal{B} = \left\{ \begin{array}{c} \text{finite intersections of the inverse images} \\ \pi_\alpha^{-1}(\mathcal{O}_\alpha) \ \text{ for } \alpha \in A \ \text{ and } \mathcal{O}_\alpha \in \mathcal{U}_\alpha \end{array} \right\}.$$

One verifies that $\mathcal{B}$ satisfies the conditions (**i**)–(**ii**) of a base. Then, the product topology is generated, starting from such a base, by the procedure of Proposition 4.1.

Let $f$ be a function defined on $A$ such that $f(\alpha) \in X_\alpha$ for all $\alpha \in A$.

The collection $\{f(\alpha)\}$ can be identified with a point in $\prod X_\alpha$. Conversely any point $x \in \prod X_\alpha$ can be identified with one such function. If $X_\alpha = X$ for some set $X$ and all $\alpha \in A$, then $\prod X_\alpha$ is denoted by $X^A$ and it is identified with the set of all functions defined on $A$ and with values in $X$. If $A = \mathbb{N}$ then $X^{\mathbb{N}}$ is the set of all sequences $\{x_n\}$ of elements of $X$.

## 5 Compact Topological Spaces

A collection $\mathcal{F}$ of open sets $\mathcal{O}$ is an *open covering* of $X$ if every $x \in X$ is contained in some $\mathcal{O} \in \mathcal{F}$. The covering is countable if it consists of countably many elements, and it is finite if it consists of a finite number of open sets.

It might occur that $X$ is covered by a subfamily $\mathcal{F}'$ of elements of $\mathcal{F}$. Such a subfamily, if it exists, is an open sub-covering of $X$, relative to $\mathcal{F}$.

The topological space $\{X; \mathcal{U}\}$ is *compact* if every open covering $\mathcal{F}$ contains a finite sub-covering $\mathcal{F}'$. A set $X_o \subset X$ is compact if $\{X_o; \mathcal{U}_o\}$ is a compact topological space.

A collection $\mathcal{G}$ of closed subsets of $X$ has the *finite intersection property* if the elements of any finite subcollection have nonempty intersection. Let $\mathcal{F}$ be an open covering for $X$ and let $\mathcal{G}$ be the collection of the complements of the elements in $\mathcal{F}$. The elements of $\mathcal{G}$ are closed and if $X$ is compact, $\mathcal{G}$ does not have the finite intersection property. More generally $X$ is compact if and only if every collection of closed subsets of $X$ with empty intersection, does not have the finite intersection property.

**Proposition 5.1** *(i)* $\{X; \mathcal{U}\}$ *is compact if and only if every collection* $\mathcal{G}$ *of closed sets with the finite intersection property has nonempty intersection.*

*(ii) Let* $E$ *be a closed subset of a compact space* $\{X; \mathcal{U}\}$*. Then* $E$ *is compact.*

*(iii) Let* $E$ *be a compact subset of a Hausdorff topological space. Then* $E$ *is closed.*

*(iv) Let* $f$ *from* $\{X; \mathcal{U}\}$ *into* $\{Y; \mathcal{V}\}$ *be continuous. If* $\{X; \mathcal{U}\}$ *is compact then* $f(X) \subset Y$ *is compact. If in addition* $f$ *is one-to-one, and* $\{Y; \mathcal{V}\}$ *is Hausdorff, then* $f$ *is a homeomorphism between* $\{X; \mathcal{U}\}$ *and* $\{Y; \mathcal{V}\}$*.*

*Proof* Part (i) follows from the previous remarks. To prove (ii), let $\mathcal{F}$ be any open covering for $E$. Then the collection $\{\mathcal{F}, (X - E)\}$ is an open covering for $X$. From this we may extract a finite sub-covering $\mathcal{F}'$ for $X$ which, by possibly removing $X - E$, gives a finite sub-covering for $E$.

Turning to (iii), let $y \in (X - E)$ be fixed. Since $X$ is a Hausdorff space, for every $x \in E$ there exist disjoint open sets $\mathcal{O}_x$ and $\mathcal{O}_{x,y}$ separating $x$ and $y$. The collection $\{\mathcal{O}_x\}$ forms an open cover for $E$, from which we extract a finite one $\{\mathcal{O}_{x_1}, \ldots, \mathcal{O}_{x_n}\}$ for some positive integer $n$. The intersection $\bigcap_{j=1}^n \mathcal{O}_{x_j,y}$ is open and does not intersect

$E$. Therefore every element $y$ of the complement of $E$ contains an open neighborhood not intersecting $E$. Thus $E$ is closed.

To establish (iv), let $\{X; \mathcal{U}\}$ be compact and let $\{Y; \mathcal{V}\}$ be the image of a continuous function $f$ from $X$ onto $Y$. Given an open covering $\{\Phi\}$ of $Y$, the collection $\mathcal{F} = \{f^{-1}(\Phi)\}$ is an open covering for $X$ from which we may extract a finite one $\{f^{-1}(\Phi_1), \ldots, f^{-1}(\Phi_n)\}$. Then the finite collection $\{\Phi_1, \ldots, \Phi_n\}$ covers $Y$.

Let $f : X \to Y$ be continuous and one-to-one and let $\{Y; \mathcal{V}\}$ be Hausdorff. A closed subset $E \subset X$ is compact, and its image $f(E)$ is compact in $Y$ and hence closed. Therefore $f^{-1}$ is continuous. ∎

*Remark 5.1* In (iii), the assumption that $\{X; \mathcal{U}\}$ be Hausdorff cannot be removed. Indeed if $\mathcal{U}$ is the trivial topology on $X$, every proper subset of $X$ is compact and not closed.

A topological space $\{X; \mathcal{U}\}$ is *locally compact* if for each $x \in X$ there exists an open set $\mathcal{O}$ containing $x$ and such that $\bar{\mathcal{O}}$ is compact. For example, $\mathbb{R}^N$ endowed with the Euclidean topology is locally compact but not compact.

## 5.1 Sequentially Compact Topological Spaces

A topological space $\{X; \mathcal{U}\}$ is *countably compact* if every countable open covering of $X$ contains a finite sub-covering. If $\{X; \mathcal{U}\}$ is compact it is also countably compact. The converse is false (**5.7** of the Complements).

A topological space $\{X; \mathcal{U}\}$ has the Bolzano–Weierstrass property if every infinite sequence $\{x_n\}$ of elements of $X$ has at least one cluster point.

A topological space $\{X; \mathcal{U}\}$ is *sequentially compact* if every infinite sequence $\{x_n\}$ of elements of $X$ has a convergent subsequence.

Thus if $\{X; \mathcal{U}\}$ is sequentially compact it has the Bolzano–Weierstrass property. The converse is false.[1]

**Proposition 5.2** *(i) The continuous image of a countably compact space is countably compact.*

*(ii) $\{X; \mathcal{U}\}$ is countably compact if and only if every countable family $\mathcal{G}$ of closed sets with the finite intersection property has nonempty intersection.*

*(iii) $\{X; \mathcal{U}\}$ has the Bolzano–Weierstrass property if and only if it is countably compact.*

*(iv) If $\{X; \mathcal{U}\}$ is sequentially compact then it is countably compact.*

*(v) If $\{X; \mathcal{U}\}$ is countably compact and if it satisfies the first axiom of countability, then it is sequentially compact.*

*Proof* Parts (i)–(ii) follows from the definitions and the proof of Proposition 5.1. The proof of (iii) uses the characterization of countable compactness stated in (ii).

[1]By part (v) of Proposition 5.2 a counterexample can be constructed starting from a space $\{X; \mathcal{U}\}$ that does not satisfy the first axiom of countability. See **5.7** of the Complements.

Let $\{X; \mathcal{U}\}$ be countably compact and let $\{x_n\}$ be a sequence of elements of $X$. The closed sets

$$B_n = \text{closure of } \{x_n, x_{n+1}, \ldots, \}$$

satisfy the finite intersection property and therefore have nonempty intersection. Any element $x \in \cap B_n$ is a cluster point for $\{x_n\}$.

Conversely let $\{X; \mathcal{U}\}$ satisfy the Bolzano–Weierstrass property and let $\{B_n\}$ be a countable collection of closed subsets of $X$ with the finite intersection property. Since for all $n \in \mathbb{N}$ the intersection $\bigcap_{j=1}^{n} B_j$ is nonempty we may select an element $x_n$ out of it. The sequence $\{x_n\}$ has at least one cluster point $x$, which by construction, belongs to the intersection of all $B_n$.

Part (iv) follows from (iii), since sequential compactness implies the Bolzano–Weierstrass property.

To prove (v) let $\{x_n\}$ be an infinite sequence of elements of $X$ and let $x$ be in the closure of $\{x_n\}$. Since $\{X; \mathcal{U}\}$ satisfies the first axiom of countability, there exist a nested countable collection of open sets $\mathcal{O}_m \supset \mathcal{O}_{m+1}$, each containing $x$ and each containing an element $x_m$ out of the sequence $\{x_n\}$. The subsequence $\{x_m\}$ converges to $x$. ■

**Proposition 5.3** *Let* $\{X; \mathcal{U}\}$ *satisfy the second axiom of countability. Then every open covering of X contains a countable sub-covering.*

*Proof* Let $\{B_n\}$ a countable collection of open sets that forms a base for the topology of $\{X; \mathcal{U}\}$ and let $\mathcal{F}$ be an open covering of $X$. To each $B_n$ we associate one and only one open set $\mathcal{O}_n \in \mathcal{F}$ that contains it. The countable collection $\mathcal{O}_n$ is a countable sub-covering of $\mathcal{F}$. ■

**Corollary 5.1** *For spaces satisfying the second axiom of countability, compactness, countable compactness and sequential compactness, are equivalent.*

## 6 Compact Subsets of $\mathbb{R}^N$

Let $E$ be a subset of $\mathbb{R}^N$. We regard $E$ as a topological space with the topology inherited from the Euclidean topology of $\mathbb{R}^N$.

**Proposition 6.1** *The closed interval* $[0, 1]$ *is compact.*

*Proof* Let $\{I_\alpha\}$ be a collection of open intervals covering $[0, 1]$ and set

$$\mathcal{E} = \bigcup \left\{ \begin{array}{l} x \in [0, 1] \text{ such that the closed interval } [0, x] \text{ is} \\ \text{covered by finitely many elements out of } \{I_\alpha\} \end{array} \right\}.$$

Let $c = \sup\{x | x \in \mathcal{E}\}$. Since 0 belongs to some $I_\alpha$ we have $0 < c \leq 1$. Such an element $c$ it is covered by some open set $I_{\alpha_c} \in \{I_\alpha\}$, and therefore, there exist

$\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subset I_{\alpha_c}$. By the definition of $c$, the interval $[0, c - \varepsilon]$ is covered by finitely many open sets $\{I_1, \ldots, I_n\}$ out of $\{I_\alpha\}$. Augmenting such a finite collection with $I_{\alpha_c}$ gives a finite covering of $[0, c + \varepsilon]$. Thus if $c < 1$, it is not the supremum of the set $\mathcal{E}$. ∎

**Proposition 6.2** *The closed interval* $[0, 1]$ *has the Bolzano–Weierstrass property.*

*Proof* Let $\{x_n\}$ be an infinite sequence of elements of $[0, 1]$ without a cluster point in $[0, 1]$. Then, each of the open intervals $(x - \varepsilon, x + \varepsilon)$, for $x \in [0, 1]$ and $\varepsilon > 0$, contains at most finitely many elements of $\{x_n\}$. The collection of all such intervals forms an open covering of $[0, 1]$, from which we may select a finite one. This would imply $\{x_n\}$ is finite. ∎

**Corollary 6.1** *Every sequence in* $[0, 1]$ *has a convergent subsequence.*

**Corollary 6.2** *A bounded, closed subset* $E \subset \mathbb{R}^N$ *has the Bolzano–Weierstrass property.*

*Proof* Let $\{x_n\}$ be a sequence of elements in $E$ and represent each of the $x_n$ in terms of its coordinates, that is, $x_n = (x_{1,n}, \ldots, x_{N,n})$. Since $\{x_n\}$ is bounded, each of the sequences $\{x_{j,n}\}$ is contained in some closed interval $[a_j, b_j]$. Out of $\{x_{1,n}\}$ we extract a convergent subsequence $\{x_{1,n_1}\}$. Then out of $\{x_{2,n_1}\}$ we extract a convergent subsequence $\{x_{2,n_2}\}$. Proceeding in this fashion, the sequence $\{x_{n_N}\}$ has a limit. Since $E$ is closed, such a limit is in $E$. ∎

**Proposition 6.3** (Borel-Riesz ([18, 124])) *Let* $E$ *be a bounded, closed subset of* $\mathbb{R}^N$. *Then, every open covering* $\mathcal{U}$ *of* $E$ *contains a finite sub-covering* $\mathcal{U}'$.

*Proof* By Proposition 5.3, may assume the covering is countable, say $\mathcal{U} = \{\mathcal{O}_n\}$. We claim that $E \subset \bigcup_{i=1}^m \mathcal{O}_i$ for some $m \in \mathbb{N}$. Indeed if not, we may select for each positive integer $n$, an element $x_n \in E - \bigcup_{i=1}^n \mathcal{O}_n$ and select, out of the sequence $\{x_n\}$, a subsequence $\{x_{n'}\}$ convergent to some $x \in E$. Since the collection $\{\mathcal{O}_n\}$ covers $E$, there exists an index $m$ such that $x \in \mathcal{O}_m$. Thus $x_{n'} \in \mathcal{O}_m$ for infinitely many $n'$. ∎

**Proposition 6.4** (Heine-Borel) *Every compact subset of* $\mathbb{R}^N$, *endowed with the Euclidean topology, is closed and bounded.*

*Proof* Let $E \subset \mathbb{R}^N$ be compact. Since $\mathbb{R}^N$, endowed with the Euclidean topology, is a Hausdorff space, $E$ is closed by (iii) of Proposition 5.1. The collection of balls $\{B_n\}$ centered at the origin and radius $n \in \mathbb{N}$ is an open covering for $E$. Since $E$ is contained in the union of a finite sub-covering, it is bounded. ∎

**Theorem 6.1** *A subset* $E$ *of* $\mathbb{R}^N$ *is compact if and only if is bounded and closed.*

## 7 Continuous Functions on Countably Compact Spaces

Let $f$ be a map from a topological space $\{X; \mathcal{U}\}$ into $\mathbb{R}$ and for $t \in \mathbb{R}$ set $[f < t] = \{x \in X | f(x) < t\}$. The sets $[f \le t]$, $[f \ge t]$, and $[f > t]$ are defined analogously. A map $f$ from a topological space $\{X; \mathcal{U}\}$ into $\mathbb{R}$, is *upper semi-continuous* if $[f < t]$ is open for all $t \in \mathbb{R}$, and it is *lower semi-continuous* if $[f > t]$ is open for all $t \in \mathbb{R}$. A map $f : X \to \mathbb{R}$ is continuous if and only if is both upper and lower semi-continuous.

**Theorem 7.1** (Weierstrass-Baire) *Let $\{X; \mathcal{U}\}$ be countably compact and let $f : X \to \mathbb{R}$ be upper semi-continuous. Then $f$ is bounded above in $X$ and it achieves its maximum in $X$.*

*Proof* The collection of sets $\{[f < n]\}$ is a countable open covering of $X$, from which we extract a finite one, say, for example, $[f < n_1], \dots, [f < n_N]$. Then $f \le \max\{n_1, \dots, n_N\}$. Thus $f$ is bounded above. Next, let $f_o$ denote the supremum of $f$ on $X$. The sets $[f \ge f_o - \frac{1}{n}]$ are closed and form a family with the finite intersection property. Since $\{X; \mathcal{U}\}$ is countably compact, their intersection is nonempty. Therefore, there is an element $x_o \in [f \ge f_o - \frac{1}{n}]$ for all $n \in \mathbb{N}$. By construction $f(x_o) = f_o$. ∎

**Corollary 7.1** *(i) A continuous real-valued function from a countably compact topological space $\{X; \mathcal{U}\}$ takes its maximum and minimum in $X$.*
*(ii) A continuous real-valued function from a countably compact topological space $\{X; \mathcal{U}\}$ is uniformly continuous.*

**Theorem 7.2** (Dini) *Let $\{X; \mathcal{U}\}$ be countably compact and let $\{f_n\}$ be a sequence of real-valued, upper semi-continuous functions such that $f_{n+1} \le f_n$ for all $n \in \mathbb{N}$, and converging pointwise in $X$ to a lower semi-continuous function $f$. Then $\{f_n\} \to f$ uniformly in $X$.*

*Proof* By possibly replacing $f_n$ with $f_n - f$, we may assume that $\{f_n\}$ is a decreasing sequence of upper semi-continuous functions converging to zero pointwise in $X$. For every $\varepsilon > 0$, the collection of open sets $[f_n < \varepsilon]$ covers $X$ and we extract a finite cover, say for example, up to a possible reordering

$$[f_{n_1} < \varepsilon], \ [f_{n_2} < \varepsilon], \cdots, [f_{n_\varepsilon} < \varepsilon], \quad n_1 < n_2 \cdots < n_\varepsilon.$$

Since $\{f_n\}$ is decreasing $[f_{n_\varepsilon} < \varepsilon] = X$. Thus $f_n(x) < \varepsilon$ for all $x \in X$ and all $n \ge n_\varepsilon$. ∎

## 8 Products of Compact Spaces

**Theorem 8.1** (Tychonov ([165])) *Let $\{X_\alpha; \mathcal{U}_\alpha\}$ be a family of compactspaces. Then $\prod X_\alpha$ endowed with the product topology, is compact.*

The proof is based on showing that every collection of closed sets with the finite intersection property, has nonempty intersection.

**Lemma 8.1** *Let $\{X; \mathcal{U}\}$ be a topological space and let $\mathcal{G}_o$ be a collection of subsets of $X$ with the finite intersection property. There exists a maximal collection $\mathcal{G}$ of subsets of $X$ with the finite intersection property and containing $\mathcal{G}_o$, that is, if $\mathcal{G}'$ is another collection of subsets of $X$ with the finite intersection property and containing $\mathcal{G}$, then $\mathcal{G}' = \mathcal{G}$. Moreover, the finite intersection of elements in $\mathcal{G}$ is in $\mathcal{G}$ and every subset of $X$ that intersects each set of $\mathcal{G}$ is in $\mathcal{G}$.*[2]

*Proof* The family of all collections of sets with the finite intersection property and containing $\mathcal{G}_o$ is partially ordered by inclusion, so that by the Hausdorff principle, there is a maximal linearly ordered subfamily $\mathcal{F}$. We claim that $\mathcal{G}$ is the union of all the collections in $\mathcal{F}$.

Any $n$-tuple $\{E_1, \dots, E_n\}$ of elements of $\mathcal{G}$ belongs to at most $n$ collections $\mathcal{G}_j$. Since $\{\mathcal{G}_j\}$ is linearly ordered there is a collection $\mathcal{G}_n$ that contains the others. Therefore, $E_i \in \mathcal{G}_n$ for all $i = 1, \dots, n$ and since $\mathcal{G}_n$ has the finite intersection property, $\cap E_i \neq \emptyset$. Thus $\mathcal{G}$ has the finite intersection property. The maximality of $\mathcal{G}$ follows by its construction.

The collection $\mathcal{G}'$ of all finite intersections of sets in $\mathcal{G}$ contains $\mathcal{G}$ and has the finite intersection property. Therefore $\mathcal{G}' = \mathcal{G}$ by maximality.

Let $E$ be a subset of $X$ that intersects all the sets in $\mathcal{G}$. Then, the collection $\mathcal{G} \bigcup \{E\}$ has the finite intersection property and contains $\mathcal{G}$. Therefore $E \in \mathcal{G}$, by maximality. ■

*Proof* (*of Tychonov's Theorem*) Let $\mathcal{G}_o$ be a collection of closed sets in $\prod_\alpha X_\alpha$, with the finite intersection property and let $\mathcal{G}$ be the maximal collection constructed in Lemma 8.1. While the sets in $\mathcal{G}_o$ are closed, the elements of $\mathcal{G}$ need not be closed. We will establish that the intersection of the closure of all elements in $\mathcal{G}$ is not empty. For each $\alpha \in A$ let $\mathcal{G}_\alpha$ be the collection of the projection of $\mathcal{G}$ into $X_\alpha$, that is

$$\mathcal{G}_\alpha = \{\text{collection of } \ \pi_\alpha(E) | \text{ for } E \in \mathcal{G}\}.$$

The sets in $\mathcal{G}_\alpha$ need not be closed nor open. However since $\mathcal{G}$ has the finite intersection property in $\prod X_\alpha$, the collection $\mathcal{G}_\alpha$ has the finite intersection property in $X_\alpha$. Therefore the collection of their closures in $\{X_\alpha, \mathcal{U}_\alpha\}$

$$\overline{\mathcal{G}}_\alpha = \{\text{collection of } \ \overline{\pi_\alpha(E)} | \text{ for } E \in \mathcal{G}\}$$

has nonempty intersection, since each of $\{X_\alpha; \mathcal{U}_\alpha\}$ is compact. Select an element

$$x_\alpha \in \bigcap \{\overline{\pi_\alpha(E)} | \text{ for } E \in \mathcal{G}\} \subset X_\alpha.$$

We claim that the element $x \in \prod X_\alpha$, whose $\alpha$-coordinate is $x_\alpha$, belongs to the closure of all sets in $\mathcal{G}$. Let $\mathcal{O}$ be a set, open in the product topology that contains

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[2] It is not claimed here that the elements of $\mathcal{G}$ are closed.

$x$. By the construction of the product topology, there exists finitely many indices $\alpha_1, \dots, \alpha_n$ and finitely many sets $\mathcal{O}_{\alpha_j}$, open in $X_{\alpha_j}$, such that

$$x \in \bigcap_{j=1}^{n} \pi_{\alpha_j}^{-1}(\mathcal{O}_{\alpha_j}) \subset \mathcal{O}.$$

For each $j$ the projection $x_{\alpha_j}$ belongs to $\mathcal{O}_{\alpha_j}$. Since $x_{\alpha_j}$ belongs to the closure of all sets in $\mathcal{G}_{\alpha_j}$, the open set $\mathcal{O}_{\alpha_j}$ intersects all the sets in $\mathcal{G}_{\alpha_j}$. Therefore, $\pi_{\alpha_j}^{-1}(\mathcal{O}_{\alpha_j})$ intersects all the sets in $\mathcal{G}$ and by Lemma 8.1 it belongs to $\mathcal{G}$. Likewise the finite intersection $\bigcap_{j=1}^{n} \pi_{\alpha_j}^{-1}(\mathcal{O}_{\alpha_j})$ intersects every element in $\mathcal{G}$ and therefore it belongs to $\mathcal{G}$. Thus, an arbitrary open set $\mathcal{O}$ containing $x$, intersects all the sets in $\mathcal{G}$ and therefore $x$ belongs to the closure of all such sets. ∎

*Remark 8.1* Tychonov's theorem provides a motivation for defining the topology on a product space $\prod X_\alpha$ as the weakest topology for which all the projection maps $\pi_\alpha$ are continuous. Indeed if all the topological spaces $\{X_\alpha; \mathcal{U}_\alpha\}$ are Hausdorff, the product topology is also a Hausdorff topology. But then any topology stronger than the product topology would violate Tychonov's theorem. This follows from Proposition 5.1c and **5.3** of the Complements.

## 9 Vector Spaces

A *linear space* consists of a set $X$, whose elements are called *vectors*, and a field $\mathcal{F}$, whose elements are called scalars, endowed with operations of sum $+ : X \times X \to X$, and multiplication by scalars $\bullet : \mathcal{F} \times X \to X$ satisfying the addition laws

$$\begin{aligned} x + y &= y + x \\ (x+y)+z &= x+(y+z), \end{aligned} \qquad \text{for all } \; x, y, z \in X$$

$$\text{there exists } \Theta \in X \text{ such that } x + \Theta = x \text{ for all } x \in X$$

$$\text{for all } x \in X \text{ there exists } -x \in X \text{ such that } x + (-x) = \Theta$$

and the scalar multiplication laws

$$\begin{aligned} \lambda(x+y) &= \lambda x + \lambda y && \text{for all } \; x, y \in X \\ \lambda(\mu x) &= (\lambda\mu)x && \text{for all } \; \lambda, \mu \in \mathcal{F} \\ (\lambda+\mu)x &= \lambda x + \mu x && \text{for all } \; \lambda, \mu \in \mathcal{F} \\ 1x &= x && \text{where 1 is the unit element of } \mathcal{F}. \end{aligned}$$

It follows that $\lambda\Theta = \Theta$ for all $\lambda \in \mathcal{F}$ and if 0 is the zero-element of $\mathcal{F}$, then $0x = \Theta$ for all $x \in X$. Also, for all $x, y \in X$ and $\lambda \in \mathcal{F}$

$$(-1)x = -x, \qquad x - y = x + (-y), \qquad \lambda(x-y) = \lambda x - \lambda y.$$

A nonempty subset $X_o \subset X$ is a linear *subspace* of $X$ if it is closed under the inherited operations of sum and multiplication by scalars. The largest linear subspace of $X$ is $X$ itself and the smallest is the null space $\{\Theta\}$. A *linear combination* of an $n$-tuple of vectors $\{x_1, \dots, x_n\}$, is an expression of the form

$$y = \sum_{j=1}^{n} \lambda_j x_j \quad \text{where} \quad \{\lambda_1, \dots, \lambda_n\} \text{ is an } n\text{-tuple of scalars.}$$

If $X_o \subset X$, the *linear span* of $X_o$ is the set of all linear combinations of elements of $X_o$. It is a linear space, and it is the smallest linear subspace of $X$ containing $X_o$, or *spanned* by $X_o$. An $n$-tuple $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ of vectors is *linearly independent* if

$$\sum_{j=1}^{n} \lambda_j \mathbf{e}_j = 0 \quad \text{implies that} \quad \lambda_j = 0 \quad \text{for all } j = 1, \dots, n.$$

A linear space $X$ is of dimension $n$ if it contains an $n$-tuple of linearly independent vectors whose span is the whole $X$. Any such $n$-tuple, say for example $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a *basis* in the sense that given $x \in X$ there exists an $n$-tuple of scalars $\{\lambda_1, \dots, \lambda_n\}$ such that $x = \sum_{j=i}^{n} \lambda_j \mathbf{e_j}$. For each $x \in X$, the $n$-tuple $\{\lambda_1, \dots, \lambda_n\}$ is uniquely determined by the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. While $\mathcal{F}$ could be any field we will consider $\mathcal{F} = \mathbb{R}$ and call $X$ vector space over the reals.

Let $A$ and $B$ be subsets of a linear space $X$ and let $\alpha, \beta \in \mathbb{R}$. Define the set operation

$$\alpha A + \beta B = \cup\{\alpha a + \beta b | a \in A,\ b \in B\}.$$

One verifies that the sum is commutative and associative, that is,

$$A + B = B + A \quad \text{and} \quad A + (B + C) = (A + B) + C.$$

Moreover

$$A + (B \cup C) = (A + B) \cup (A + C).$$

However $A + A \neq 2A$ and $A - A \neq \{\Theta\}$.

## 9.1 Convex Sets

A convex combination of two elements $x, y \in X$ is an element of the form $tx + (1-t)y$ where $t \in [0, 1]$. As $t$ ranges over $[0, 1]$ this describes the *line segment* of extremities $x$ and $y$. The convex combination of $n$ elements $\{x_1, \dots, x_n\}$ of $X$ is an element of the form

$$\sum_{j=1}^{n} \alpha_j x_j \quad \text{where } \alpha_j \geq 0 \text{ and } \sum_{j=1}^{n} \alpha_j = 1.$$

A set $A \subset X$ is convex if for any pair $x, y \in A$ the elements $tx + (1-t)y$ belong to $A$ for all $t \in [0, 1]$. Alternatively, if the line segment of extremities $x$ and $y$ belongs to $A$.

The *convex hull* $c(A)$ of a set $A \subset X$ is the smallest convex set containing $A$. It can be characterized as either the intersection of all the convex sets containing $A$, or as the set of all convex combinations of $n$-tuples of elements in $A$, for any $n$.

The intersection of convex sets is convex; the union of convex sets need not be convex. Linear subspaces of $X$ are convex.

### 9.2 Linear Maps and Isomorphisms

Let $X$ and $Y$ be linear spaces over $\mathbb{R}$. A map $T : X \to Y$ is linear if

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y) \quad \text{for all } x, y \in X \text{ and } \lambda, \mu \in \mathbb{R}.$$

The image of $T$ is $T(X) \subset Y$ and the kernel of $T$ is $\ker\{T\} = T^{-1}\{0\}$. The image $T(X)$ is a linear subspace of $Y$ and the kernel $\ker\{T\}$ is a linear subspace of $X$. A linearmap $T : X \to Y$ is an *isomorphism* between $X$ and $Y$ if it is one-to-one and onto. The inverse of an isomorphism is an isomorphism and the composition of two isomorphisms is an isomorphism. If $X$ and $Y$ are finite-dimensional and are isomorphic, then they have the same dimension.

## 10 Topological Vector Spaces

A vector space $X$ endowed with a topology $\mathcal{U}$ is a *topological vector space* over $\mathbb{R}$, if the operations of sum $+ : X \times X \to X$, and multiplication by scalars $\bullet : \mathbb{R} \times X \to X$ are continuous with respect to the product topologies of $X \times X$ and $\mathbb{R} \times X$.

Fix $x_o \in X$. The translation by $x_o$ is defined by $T_{x_o}(x) = x_o + x$ for all $x \in X$. For a fixed $\lambda \in \mathbb{R} - \{0\}$, the dilation by $\lambda$ is defined by $D_\lambda(x) = \lambda x$ for all $x \in X$. If $\{X; \mathcal{U}\}$ is a topological vector space, the maps $T_{x_o}$ and $D_\lambda$ are homeomorphisms from $\{X; \mathcal{U}\}$ onto itself. In particular if $\mathcal{O}$ is open then $x + \mathcal{O}$ is open for all fixed $x \in X$. Any topology with such a property is *translation invariant*.

*Remark 10.1* This notion can be used to construct a vector topological space $\{X; \mathcal{U}\}$ for which the sum is not continuous. It suffices to construct a vector space endowed with a topology which is not translation invariant. For an example of a linear, topological vector space for which the product by scalars is not continuous, see **10.4** and **10.5** of the Complements.

Let $\{X; \mathcal{U}\}$ be a topological vector space. If $\mathcal{B}_\Theta$ is a base at the zero element $\Theta$ of $X$, then for any fixed $x \in X$, the collection $\mathcal{B}_x = x + \mathcal{B}_\Theta$ forms a base for the topology $\mathcal{U}$ at $x$. Thus a base $\mathcal{B}_\Theta$ at $\Theta$ determines the topology $\mathcal{U}$ on $X$. If the elements of the base $\mathcal{B}_\Theta$ are convex, the topology of $\{X; \mathcal{U}\}$ is called *locally convex*. An example of a topological vector space with a nonlocally convex topology, is in § 3.5c of the Complements of Chap. 6.

An open neighborhood of the origin $\mathcal{O}$ is symmetric if $\mathcal{O} = -\mathcal{O}$.

The next remarks imply that the topology of a topological vector space, while non-necessarily locally convex is, roughly speaking, ball-like and, while not necessarily Hausdorff is roughly speaking close to being Hausdorff.

**Proposition 10.1** *Let $\{X; \mathcal{U}\}$ be a topological vector space. Then:*

*(i)* *The topology $\mathcal{U}$ is generated by a symmetric base $\mathcal{B}_\Theta$.*
*(ii)* *If $\mathcal{O}$ is an open neighborhood of the origin, then $X = \bigcup_{\lambda \in \mathbb{R}} \lambda \mathcal{O}$.*
*(iii)* *$\{X; \mathcal{U}\}$ is Hausdorff if and only if the points are closed.*
*(iv)* *$\{X; \mathcal{U}\}$ is Hausdorff if and only if $\bigcap \{\mathcal{O} \in \mathcal{B}_\Theta\} = \{\Theta\}$.*

*Proof* The continuity of the multiplication by scalars implies that if $\mathcal{O}$ is open, also $\lambda\mathcal{O}$ is open for all $\lambda \in \mathbb{R} - \{0\}$. If $\Theta \in \mathcal{O}$, then $\Theta \in \lambda\mathcal{O}$ for all $|\lambda| \le 1$. In particular if $\mathcal{O}$ is a neighborhood of the origin also $-\mathcal{O}$ is a neighborhood of the origin. The set $A = -\mathcal{O} \cap \mathcal{O}$ is an open neighborhood the origin, and is symmetric since $A = -A$. One verifies that the collection of such symmetric sets is a base $\mathcal{B}_\Theta$ at the origin, for the topology of $\{X; \mathcal{U}\}$.

To prove (ii) fix $x \in X$ and let $\mathcal{O}$ be an open neighborhood of $\Theta$. Since $0 \cdot x = \Theta$, by the continuity of the product by scalars, there exist $\varepsilon > 0$ and an open neighborhood $\mathcal{O}_x$ of $x$ such that $\lambda \cdot y \in \mathcal{O}$ for all $|\lambda| < \varepsilon$ and all $y \in \mathcal{O}_x$. Thus $\delta \cdot x \in \mathcal{O}$ for some $0 < |\delta| < \varepsilon$ and $x \in \delta^{-1}\mathcal{O}$.

The direct part of (iii) follows from Proposition 1.1. For the converse, assume that $\Theta$ and $x \in (X - \Theta)$ are closed. Then there exists an open set $\mathcal{O}$ containing the origin $\Theta$ and not containing $x$. Since $\Theta + \Theta = \Theta$ and the sum $+ : (X \times X) \to X$ is continuous, there exists two open sets $\mathcal{O}_1$ and $\mathcal{O}_2$ such that $\mathcal{O}_1 + \mathcal{O}_2 \subset \mathcal{O}$. Set

$$\mathcal{O}_o = \mathcal{O}_1 \cap \mathcal{O}_2 \cap (-\mathcal{O}_1) \cap (-\mathcal{O}_2).$$

Then

$$\mathcal{O}_o + \mathcal{O}_o \subset \mathcal{O} \quad \text{and} \quad \mathcal{O}_o \cap (x + \mathcal{O}_o) = \emptyset.$$

The last statement is a consequence of (iii). ∎

**Proposition 10.2** *Let $\{X; \mathcal{U}\}$ and $\{Y; \mathcal{V}\}$ be topological vector spaces. A linear map $T : X \to Y$ is continuous if and only if is continuous at the origin $\Theta$ of $X$.*

*Proof* Since $T$ is linear, $T(\Theta) = \theta \in Y$, where $\theta$ is the origin of $Y$. Let $O \in \mathcal{V}$ be an open set containing $\theta$. By assumption $T^{-1}(O)$ is an open set containing $\Theta$. Let $x \in X$ be fixed. An open set in $Y$ that contains $T(x)$ is of the form $T(x) + O$, where $O$ is an open set containing $\theta$. The pre-image $T^{-1}(T(x) + O)$ contains the open set $x + T^{-1}(O)$. ∎

### 10.1 Boundedness and Continuity

Let $\{X; \mathcal{U}\}$ be a topological vector space. A subset $E \subset X$ is bounded if for every open neighborhood $\mathcal{O}$ of the origin $\Theta$, there exists $\mu > 0$ such that $E \subset \lambda\mathcal{O}$ for all $\lambda > \mu$. A map $T$ from a topological vector space $\{X; \mathcal{U}\}$ into a topological vector space $\{Y; \mathcal{V}\}$ is *bounded* if it maps bounded subsets of $X$ into bounded subsets of $Y$.

**Proposition 10.3** *A linear, continuous map $T$ from a topological vector space $\{X; \mathcal{U}\}$ into a topological vector space $\{Y; \mathcal{V}\}$, is bounded.*

*Proof* Let $E \subset X$ be bounded. For every neighborhood $O$ of the origin $\theta$ of $Y$, open in the topology of $\{Y; \mathcal{V}\}$, the inverse image $T^{-1}(O)$ is a neighborhood of the origin $\Theta$, open in the topology of $\{X; \mathcal{U}\}$. Since $E$ is bounded, there exists some $\delta > 0$ such that $E \subset \delta T^{-1}(O)$. Therefore, $T(E) \subset \delta O$. ■

*Remark 10.2* Linearity alone does not imply boundedness. An example of unbounded linear map between two topological vector spaces is in § 15. Further examples are in **3.4** and **3.5** of the Complements of Chap. 7.

*Remark 10.3* For general topological vector spaces $\{X; \mathcal{U}\}$ and $\{Y; \mathcal{V}\}$, the converse of Proposition 10.3 is false; that is, linearity and boundedness do not imply continuity of $T$. See **10.3** of the Complements for a counterexample. However, the converse is true for linear, bounded maps $T$ between *metric* vector spaces, as stated in Proposition 14.2.

## 11 Linear Functionals

If the target space $Y$ is the field $\mathbb{R}$ endowed with the Euclidean topology, the linear map $T : X \to \mathbb{R}$ is called a *functional* on $\{X; \mathcal{U}\}$. A linear functional $T : \{X; \mathcal{U}\} \to \mathbb{R}$ is bounded in a neighborhood of $\Theta$, if there exists an open set $\mathcal{O}$ containing $\Theta$ and a positive number $k$ such that $|T(x)| < k$ for all $x \in \mathcal{O}$.

**Proposition 11.1** *Let $T : \{X; \mathcal{U}\} \to \mathbb{R}$ be a not identically zero, linear functional on $X$. Then:*

*(i)* *If $T$ is bounded in a neighborhood of the origin, then $T$ is continuous.*
*(ii)* *If* $\ker\{T\}$ *is closed then $T$ is bounded in a neighborhood of the origin.*
*(iii)* *$T$ is continuous if and only if* $\ker\{T\}$ *is closed.*
*(iv)* *$T$ is continuous if and only if it is bounded in a neighborhood of the origin.*

*Proof* Let $\mathcal{O}$ be an open neighborhood of the origin such that $|T(x)| \le k$ for all $x \in \mathcal{O}$. For every $\varepsilon \in (0, k)$ the pre-image of the open interval $(-\varepsilon, \varepsilon)$, contains the open sets $\lambda\mathcal{O}$ for all $0 < \lambda < \varepsilon/k$. Thus $T$ is continuous at the origin and therefore continuous by Proposition 10.2.

Turning to (ii), if $\ker\{T\}$ is closed there exist $x \in X$ and some open neighborhood $\mathcal{O}$ of $\Theta$, such that $(x+\mathcal{O}) \cap \ker\{T\} = \emptyset$. By (i) of Proposition 10.1, we may assume, that $\mathcal{O}$ is symmetric and that $\lambda\mathcal{O} \subset \mathcal{O}$ for all $|\lambda| \le 1$. This implies that $T(\mathcal{O})$ is a symmetric interval about the origin of $\mathbb{R}$. If such an interval is bounded, there is nothing to prove. If such an interval coincides with $\mathbb{R}$, then there exist $y \in \mathcal{O}$ such that $T(y) = T(x)$. Thus $(x - y) \in \ker\{T\}$ and $(x + \mathcal{O}) \cap \ker\{T\}$ is not empty since $y \in \mathcal{O}$. The contradiction proves (ii).

To prove (iii) observe that the origin $\{0\}$ of $\mathbb{R}$ is closed. Therefore if $T$ is continuous $T^{-1}(0) = \ker\{T\}$ is closed.

The remaining statements follow from (i)–(ii). ∎

**Proposition 11.2** *Let $\{T_1, \dots, T_n\}$ be a finite collection of bounded linear functionals on a Hausdorff, linear, topological vector space $\{X; \mathcal{U}\}$, and set*

$$K = \bigcap_{j=1}^{n} \ker\{T_j\}.$$

*If $T$ is a bounded linear functional on $\{X; \mathcal{U}\}$ vanishing on $K$, then there exists a $n$-tuple $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that*

$$T = \sum_{j=1}^{n} \alpha_j T_j.$$

*Proof* The map

$$X \ni x \to \big(T_1(x), \dots, T_n(x)\big) \in \mathbb{R}^n$$

is bounded and linear, and its image $\mathbb{R}^n_o$ is a closed subspace $\mathbb{R}^n$. The map

$$\mathbb{R}^n_o \ni \big(T_1(x), \dots, T_n(x)\big) \to \mathcal{T}_o\big(T_1(x), \dots, T_n(x)\big) \stackrel{\text{def}}{\longrightarrow} T(x) \in \mathbb{R}$$

is well defined since $T$ vanishes on $K$. The map $\mathcal{T}_o$ is bounded and linear, and it extends to a bounded linear map $\mathcal{T} : \mathbb{R}^n \to \mathbb{R}$, defined in the whole $\mathbb{R}^n$. The latter must be of the form

$$\mathbb{R}^n \ni (y_1, \dots, y_n) \to \mathcal{T}(y_1, \dots, y_n) = \sum_{j=1}^{n} \alpha_j y_j$$

for a fixed $n$-tuple $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$. Since $\mathcal{T}$ agrees with $\mathcal{T}_o$ on $\mathbb{R}^n_o$

$$X \ni x \to T(x) = \sum_{j=1}^{n} \alpha_j T_j(x).$$ ∎

## 12 Finite Dimensional Topological Vector Spaces

The next proposition asserts that a $n$-dimensional Hausdorff topological vector space, can only be given, up to a homeomorphism, the Euclidean topology of $\mathbb{R}^n$.

**Proposition 12.1** *Let $\{X; \mathcal{U}\}$ be a n-dimensional Hausdorff topological vector space over $\mathbb{R}$. Then $\{X; \mathcal{U}\}$ is homeomorphic to $\mathbb{R}^n$ equipped with the Euclidean topology.*

*Proof* Given a basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ for $\{X; \mathcal{U}\}$, the representation map

$$\mathbb{R}^n \ni (\lambda_1, \dots, \lambda_n) \to T(\lambda_1, \dots, \lambda_n) = \sum_{i=1}^{n} \lambda_i \mathbf{e}_i \in X$$

is linear, one-to-one, and onto. Let $\mathcal{O}$ be a neighborhood of the origin in $X$, which we may assume to be symmetric and such that $\alpha\mathcal{O} \subset \mathcal{O}$ for all $|\alpha| \le 1$. By the continuity of the sum and multiplication by scalars the pre-image $T^{-1}(\mathcal{O})$ contains an open ball about the origin of $\mathbb{R}^n$. Thus $T$ is continuous at the origin and therefore continuous.

To show that $T^{-1}$ is continuous assume first that $n = 1$. In such a case $T(\lambda) = \lambda\mathbf{e}$ for some $\mathbf{e} \in (X - \Theta)$. The kernel of the inverse map $T^{-1} : X \to \mathbb{R}$ consist only of the zero element $\{\Theta\}$, which is closed since $X$ is Hausdorff. Therefore $T^{-1}$ is continuous by (iii) of Proposition 11.1. Proceeding by induction, assume that the representation map $T$ is a homeomorphism between $\mathbb{R}^m$ and any $m$-dimensional Hausdorff space, for all $m = 1, \dots, n-1$. Thus in particular any $(n-1)$-dimensional Hausdorff space is closed.

The inverse of the representation map has the form

$$X \ni x \to T^{-1}(x) = (\lambda_1(x), \dots, \lambda_{n-1}(x), \lambda_n(x)).$$

Each of the $n$ maps $\lambda_j(\cdot) : X \to \mathbb{R}$ is a linear functional on $X$, whose null-space is a $(n-1)$-dimensional subspace of $X$. Such a subspace is closed by the induction hypothesis. Thus each of the $\lambda_j(\cdot)$ is continuous. ∎

**Corollary 12.1** *Every finite dimensional subspace of a Hausdorff topological vector space is closed.*

If $\{X; \mathcal{U}\}$ is $n$-dimensional and not Hausdorff, it is not homeomorphic to $\mathbb{R}^n$. An example is $\mathbb{R}^N$ with the trivial topology.

### *12.1 Locally Compact Spaces*

A topological vector space $\{X; \mathcal{U}\}$ is locally compact if there exist an open neighborhood of the origin whose closure is compact.

**Proposition 12.2** *Let $\{X; \mathcal{U}\}$ be a Hausdorff, locally compact topological vector space. Then $X$ is of finite dimension.*

*Proof* Let $\mathcal{O}$ be a neighborhood of the origin, whose closure is compact. We may assume that $\mathcal{O}$ is symmetric and $\lambda\mathcal{O} \subset \mathcal{O}$ for all $|\lambda| \le 1$. There exist at most finitely many points $x_1, \dots, x_n \in \mathcal{O}$, such that

$$\overline{\mathcal{O}} \subset (x_1 + \tfrac{1}{2}\mathcal{O}) \cup (x_2 + \tfrac{1}{2}\mathcal{O}) \cup \dots \cup (x_n + \tfrac{1}{2}\mathcal{O}).$$

The space $Y = \text{span}\{x_1, \dots, x_n\}$, is a closed, finite dimensional subspace of $X$. From the previous inclusion, $\frac{1}{2}\mathcal{O} \subset Y + \frac{1}{4}\mathcal{O}$. Therefore

$$\mathcal{O} \subset Y + \tfrac{1}{2}\mathcal{O} \subset 2Y + \tfrac{1}{4}\mathcal{O} = Y + \tfrac{1}{4}\mathcal{O}.$$

Thus, by iteration

$$\mathcal{O} \subset \bigcap \big(Y + \tfrac{1}{2^n}\mathcal{O}\big) = \bar{Y} = Y.$$

This implies that $\lambda\mathcal{O} \subset Y$ for all $\lambda \in \mathbb{R}$. Thus, by (ii) of Proposition 10.1

$$X = \bigcup \lambda\mathcal{O} \subset Y \subset X.$$ ■

The assumption that $\{X; \mathcal{U}\}$ be Hausdorff cannot be removed. Indeed, any $\{X; \mathcal{U}\}$ with the trivial topology is compact, and hence locally compact. However, it is not Hausdorff and, in general, it is not of finite dimension.

## 13 Metric Spaces

A *metric* on a nonvoid set $X$ is a function $d : X \times X \to \mathbb{R}$ satisfying the properties:

(i) $d(x, y) \ge 0$ for all pairs $(x, y) \in X \times X$
(ii) $d(x, y) = 0$ if and only if $x = y$
(iii) $d(x, y) = d(y, x)$ for all pairs $(x, y) \in X \times X$
(iv) $d(x, y) \le d(x, z) + d(y, z)$ for all $x, y, z \in X$.

This last requirement is called the *triangle inequality*. The pair $\{X; d\}$ is a metric space. Denote by $B_\rho(x) = \{y \in X | d(y, x) < \rho\}$, the open ball centered at $x$ and of radius $\rho > 0$. The collection $\mathcal{B}$ of all such balls, satisfies the conditions (i)–(ii) of § 4 and therefore, by Proposition 4.1, generates a topology $\mathcal{U}$ on $\{X; d\}$, called metric topology, for which $\mathcal{B}$ is a base. The notions of open or closed sets can be given in terms of the elements of $\mathcal{B}$. In particular, a set $\mathcal{O} \subset X$ is open if for every $x \in \mathcal{O}$ there exists some $\rho > 0$ such that $B_\rho(x) \subset \mathcal{O}$.

A point $x$ is a point of closure for a set $E \subset X$ if $B_\varepsilon(x) \cap E \ne \emptyset$ for all $\varepsilon > 0$. A set $E$ is closed if and only if it coincides with the set of all its points of closure. In particular points are closed.

Let $\{x_n\}$ be a sequence of elements of $X$. A point $x \in X$ is a cluster point for $\{x_n\}$ if for all $\varepsilon > 0$, the open ball $B_\varepsilon(x)$ contains infinitely many elements of $\{x_n\}$. The sequence $\{x_n\}$ converges to $x$ if for every $\varepsilon > 0$ there exists $n_\varepsilon$ such that $d(x, x_n) < \varepsilon$ for all $n \ge n_\varepsilon$. The sequence $\{x_n\}$ is a Cauchy sequence if for every $\varepsilon > 0$ there exists an index $n_\varepsilon$, such that $d(x_n, x_m) \le \varepsilon$, for all $m, n \ge n_\varepsilon$.

A metric space $\{X; d\}$ is *complete* if every Cauchy sequence $\{x_n\}$ of elements of $X$ converges to some element $x \in X$.

## 13.1 Separation and Axioms of Countability

The distance between two subsets $A, B$ of $X$ is defined by

$$d(A, B) = \inf_{x \in A; y \in B} d(x, y).$$

**Proposition 13.1** *Let $A$ be a subset of $X$. The function $x \to d(A, x)$ is continuous in $\{X; d\}$.*

*Proof* Let $x, y \in X$ and $z \in A$. By the requirement (iv) of a metric

$$d(z, x) \le d(x, y) + d(z, y).$$

Taking the infimum of both sides for $z \in A$ gives

$$d(A, x) \le d(x, y) + d(A, y).$$

Interchanging the role of $x$ and $y$ yields

$$|d(A, x) - d(A, y)| \le d(x, y). \qquad \blacksquare$$

If $E_1$ and $E_2$ are two disjoint closed subsets of $\{X; d\}$, then the two sets

$$\mathcal{O}_1 = \{x \in X \mid d(x, E_1) < d(x, E_2)\}$$
$$\mathcal{O}_2 = \{x \in X \mid d(x, E_2) < d(x, E_1)\}$$

are open and disjoint. Moreover $E_1 \subset \mathcal{O}_1$ and $E_2 \subset \mathcal{O}_2$. Thus every metric space is normal. In particular every metric space is Hausdorff.

Every metric space satisfies the first axiom of countability. Indeed the collection of balls $B_\rho(x)$ as $\rho$ ranges over the rational numbers of $(0, 1)$, is a countable base for the topology at $x$.

**Proposition 13.2** *A metric space $\{X; d\}$ is separable if and only if it satisfies the second axiom of countability.*[3]

*Proof* Let $\{X; d\}$ be separable and let $A$ be a countable, dense subset of $\{X; d\}$. The collection of balls centered at points of $A$ and with rational radius forms a countable base for the topology of $\{X; d\}$. The converse follows from Proposition 4.2. ■

**Corollary 13.1** *Every subset of a separable metric space is separable.*

*Proof* Let $\{x_n\}$ be a countable dense subset. For a pair of positive integers $(m, n)$, consider the balls $B_{1/m}(x_n)$ centered at $x_n$ and radius $1/m$. If $Y$ is a subset of $X$, the ball $B_{1/m}(x_n)$ must intersect $Y$ for some pair $(m, n)$. For any such pair, select an element $y_{n,m} \in B_{1/m}(x_n) \cap Y$. The collection of such $y_{m,n}$ is a countable, dense subset of $Y$. ■

## *13.2 Equivalent Metrics*

From a given metric $d$ on $X$, one can generate other metrics. For example, for a given $d$, set

$$d_o(x, y) = \frac{d(x, y)}{1 + d(x, y)}. \tag{13.1}$$

One verifies that $d_o$ satisfies the requirements (i)–(iii). To verify that $d_o$ satisfies (iv) it suffice to observe that the function

$$t \to \frac{t}{1+t} \qquad \text{for} \qquad t \ge 0$$

is nondecreasing. Thus $d_o$ is a new metric on $X$ and generates the metric spaces $\{X; d_o\}$. Starting from the Euclidean metric in $\mathbb{R}^N$, one may introduce a new metric by

$$d_*(x, y) = \left| \frac{x}{1 + |x|} - \frac{y}{1 + |y|} \right| \qquad x, y \in \mathbb{R}^N. \tag{13.2}$$

More generally, the same set $X$ can be given different metrics, say for example $d_1$ and $d_2$, to generate metric spaces $\{X; d_1\}$ and $\{X; d_2\}$.

Two metrics $d_1$ and $d_2$ on the same set $X$ are equivalent if they generate the same topology. Equivalently $d_1$ and $d_2$ are equivalent if they define the same open sets. In such a case, the identity map between $\{X; d_1\}$ and $\{X; d_2\}$ is a homeomorphism.

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[3] An example of non separable metric space is in § 15.1 of Chap. 6. See also **15.2.** of the Complements of Chap. 6.

### *13.3 Pseudo Metrics*

A function $d : (X \times X) \to \mathbb{R}$ is a pseudometric if it satisfies all but (ii) of the requirements of being a metric. For example $d(x, y) = ||x| - |y||$ is a pseudometric on $\mathbb{R}$. The open balls $B_\rho(x)$ are defined as for metrics and generate a topology on $X$, called the pseudometric topology. The space $\{X; d\}$ is a pseudo-metric space. The statements of Propositions 13.1, 13.2 and Corollary 13.1 continue to hold for pseudo-metric spaces.

## 14 Metric Vector Spaces

Let $\{X; d\}$ and $\{Y; \eta\}$ be metric spaces. The notion of continuity of a function from $X$ into $Y$ can be rephrased in terms of the metrics $\eta$ and $d$. Precisely, a function $f : \{X; d\} \to \{Y; \eta\}$ is continuous at some $x \in X$, if and only if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, x)$ such that $\eta\{f(x), f(y)\} < \varepsilon$ whenever $d(x, y) < \delta$. The function $f$ is continuous if it is continuous at each $x \in X$ and it is uniformly continuous if the choice of $\delta$ depends on $\varepsilon$ and is independent of $x$.

A homeomorphism $f$ between two metric spaces $\{X; d\}$ and $\{Y; \eta\}$ is *uniform* if the map $f : X \to Y$ is one-to-one and onto, and if it is *uniformly* continuous and has uniformly continuous inverse.

An isometry between $\{X; d\}$ and $\{Y; \eta\}$ is a homeomorphism $f$ between $\{X; d\}$ and $\{Y; \eta\}$ that preserves distances, that is, such that

$$\eta\{f(x), f(y)\} = d(x, y) \qquad \text{for all } x, y \in X.$$

Thus an isometry is a uniform homeomorphism between $\{X; d\}$ and $\{Y; \eta\}$.

Let $\{X_1; d_1\}$ and $\{X_2; d_2\}$ be metric spaces. The product metric $(d_1 \times d_2)$ on the Cartesian product $(X_1 \times X_2)$ is defined by

$$(d_1 \times d_2)\{(x_1, x_2), (y_1, y_2)\} = d_1(x_1, y_1) + d_2(x_2, y_2)$$

for all $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$. One verifies that the topology generated by $(d_1 \times d_2)$ on $(X_1 \times X_2)$ coincides with the product topology of $\{X_1; d_1\}$ and $\{X_2; d_2\}$.

If $X$ is a *vector* space, then $\{X; d\}$ is a topological vector space if the operations of sum $+ : X \times X \to X$ and product by scalars $\bullet : \mathbb{R} \times X \to X$, are continuous with respect to the topology generated by $d$ on $X$ and the topology generated by $(d \times d)$ on $X \times X$.

A metric $d$ on a vector space $X$ is translation invariant if

$$d(x + z, y + z) = d(x, y) \qquad \text{for all } x, y, z \in X.$$

If $d$ is translation invariant, then the metric $d_o$ in of (13.1) is translation invariant. The metric $d_*$ in (13.2) is not translation invariant.

**Proposition 14.1** *If $d$ on a vector space $X$ is translation invariant then the sum $+ : (X \times X) \to X$ is continuous.*

*Proof* It suffices to show that $X \times X \ni (x, y) \to x + y$ is continuous at an arbitrary point $(x_o, y_o) \in X \times X$. From the definition of product topology

$$\begin{aligned} d(x + y, x_o + y_o) &= d(x - x_o, y_o - y) \\ &\le d(x - x_o, \Theta) + d(y_o - y, \Theta) \\ &= d(x, x_o) + d(y, y_o) \\ &= (d \times d)\{(x, y), (x_o, y_o)\}. \end{aligned}$$ ■

Translation invariant metrics generate translation invariant topologies. There exist nontranslation invariant metrics that generate translation invariant topologies.

*Remark 14.1* The topology generated by a metric on a vector space $X$, need not be locally convex. A counterexample is in Corollary 3.1c of the Complements of Chap. 6.

*Remark 14.2* In general the notion of a metric on a vector space $X$ does not imply, alone, any continuity statement of the operations of sum or product by scalars. Indeed there exists metric spaces for which both operations are discontinuous.

To construct examples, let $\{X; d\}$ be a metric vector space and let $h$ be a discontinuous bijection from $X$ onto itself. Setting

$$d_h(x, y) \stackrel{\text{def}}{=} d(h(x), h(y)) \tag{14.1}$$

defines a metric in $X$. The bijection $h$ can be chosen in such a way that for the metric vector space $\{X; d_h\}$ the sum and the multiplication by scalars are both discontinuous. One such a choice is in § 14c of the Complements.[4]

## 14.1 Maps Between Metric Spaces

The notion of maps between metric spaces and their properties, is inherited from the corresponding notions between topological vector spaces. In particular Propositions 10.1–10.3 and 11.1, continue to hold in the context of metric spaces. However for metric spaces Proposition 10.3 admits a converse.

**Proposition 14.2** *Let $\{X; d\}$ and $\{Y; \eta\}$ be metric vector spaces. A bounded linear map $T : X \to Y$ is continuous.*

[4]This construction was suggested by Ethan Devinatz.

*Proof* For any ball $\mathcal{B}_r$ in $\{Y; \eta\}$, of radius $r$ centered at the origin of $Y$, there exists a ball $B_\rho$ in $\{X; d\}$, centered at origin of $X$ such that $B_\rho \subset T^{-1}(\mathcal{B}_r)$. If not, for all $\delta > 0$ the ball $\delta^{-1}B_1$ is not contained in $T^{-1}(\mathcal{B}_r)$. Thus $T(B_1)$ is not contained in $\delta\mathcal{B}_r$ for any $\delta > 0$ against the boundedness of $T$. The contradiction implies $T$ is continuous at the origin and, by linearity $T$ is continuous everywhere. ■

## 15 Spaces of Continuous Functions

Let $E$ be a subset of $\mathbb{R}^N$, denote by $C(E)$ the collection of all continuous functions $f : E \to \mathbb{R}$ and set

$$d(f, g) = \sup_{x\in E} |f(x) - g(x)| \qquad f, g \in C(E). \tag{15.1}$$

If $E$ is compact, this defines a metric in $C(E)$ by which $C(E)$ turns into a metric vector space. The metric in (15.1) generates a topology in $C(E)$ called the topology of *uniform convergence*. Cauchy sequences in $C(E)$ converge uniformly to a continuous function in $E$. In this sense $C(E)$ is *complete*.

If $E$ is compact, $C(E)$ is separable (Corollary 16.1 of Chap. 5).

If $E$ is open, a function $f \in C(E)$, while bounded on every compact subset of $E$, in general is not bounded in $E$.

If $E \subset \mathbb{R}^N$ is compact, then any $f \in C(E)$ is uniformly continuous in $E$. If $E$ is open then $f \in C(E)$ does not imply uniform continuity even if $f$ is bounded in $E$.

Let $\{E_n\}$ be a collection of bounded open sets invading $E$, that is, $\bar{E}_n \subset E_{n+1}$ for all $n$, and $E = \cup E_n$. For every $f, g \in C(E)$ set

$$d_n(f, g) = \sup_{x\in \bar{E}_n} |f(x) - g(x)|.$$

Each $d_n$, while a metric in $C(\bar{E}_n)$, is a pseudometric in $C(E)$. Setting

$$d(f, g) = \sum \frac{1}{2^n} \frac{d_n(f, g)}{1 + d_n(f, g)} \tag{15.2}$$

defines a metric in $C(E)$ by which $\{C(E); d\}$ is a metric vector space.

A sequence $\{f_n\}$ of functions in $C(E)$ converges to $f \in C(E)$, in the metric (15.2), if and only if $\{f_n\} \to f$ uniformly on every compact subset of $E$. Cauchy sequences in $C(E)$ converge uniformly over compact subsets of $E$, to a function in $C(E)$. In this sense, the space $C(E)$ with the topology generated by the metric (15.2) is complete.

Denote by $\mathcal{L}^1(E)$ the collection of functions in $C(E)$ whose Riemann integral over $E$ is finite. Since $\mathcal{L}^1(E)$ is a linear subspace of $C(E)$, it can be given the metric (15.2) and the corresponding topology. This turns $\mathcal{L}^1(E)$ into a metric vector space. The linear functional

$$T(f) = \int_E f dx \ : \mathcal{L}^1(E) \to \mathbb{R}$$

is unbounded and hence discontinuous. As an example let $E = (0, 1)$. The functions $f_n(t) = t^{\frac{1}{n}-1}$ are all in $\mathcal{L}^1(0, 1)$ and the sequence $\{f_n\}$ is bounded in the topology of (15.2) since $d(f_n, 0) \le 1$. However $T(f_n) = n$. If $E$ is bounded, the linear functional

$$T(f) = \int_E f dx \ : C(\bar{E}) \to \mathbb{R}$$

is bounded and hence continuous.

## 15.1 Spaces of Continuously Differentiable Functions

Let $E$ be an open subset of $\mathbb{R}^N$ and denote by $C^1(E)$ the collection of all continuously differentiable functions $f : E \to \mathbb{R}$. Denote by $C^1(\bar{E})$ the collection of functions in $C^1(E)$ whose derivatives $f_{x_j}$ for $j = 1, \dots, N$ admit a continuous extension to $\bar{E}$, which we continue to denote by $f_{x_j}$.

For $f, g \in C^1(\bar{E})$ set formally

$$d(f, g) = \sup_{x \in \bar{E}} |f(x) - g(x)| + \sum_{j=1}^{N} \sup_{x \in \bar{E}} |f_{x_j}(x) - g_{x_j}(x)|. \tag{15.3}$$

If $E$ is bounded, so that $\bar{E}$ is compact, this defines a metric in $C^1(\bar{E})$ by which $C^1(\bar{E})$ turns into a metric vector space. Cauchy sequences in $C^1(\bar{E})$ converge to functions in $C^1(\bar{E})$. Therefore $C^1(\bar{E})$ is complete.

The space $C^1(\bar{E})$ can also be given the metric (15.1). This turns $C^1(\bar{E})$ into a metric space. The topology generated by such a metric in $C^1(\bar{E})$ is the same as the topology that $C^1(\bar{E})$ inherits as a subspace of $C(\bar{E})$. With respect to such a topology $C^1(\bar{E})$ is not complete. The linear map

$$T(f) = f_{x_j} : C^1(\bar{E}) \to C(\bar{E}) \quad \text{for some fixed } j \in \{1, \dots, N\}$$

is bounded, and hence continuous, provided $C^1(\bar{E})$ is given the metric (15.3). It is unbounded, and hence discontinuous if $C^1(\bar{E})$ is given the metric (15.1).

As an example, let $\bar{E} = [0, 1]$. The functions $f_n(t) = t^n$ are in $C^1[0, 1]$ for all $n \in \mathbb{N}$ and the sequence $\{f_n\}$ is bounded in $C[0, 1]$ since $d(f_n, 0) = 1$. However $T(f_n) = nt^{n-1}$ is unbounded in $C[0, 1]$.

If $E$ is open and $f \in C^1(E)$, the functions $f$ and $f_{x_j}$ for $j = 1, \dots, N$, while bounded on every compact subset of $E$, in general are not bounded in $E$. A metric in $C^1(E)$ can be introduced along the lines of (15.2).

### *15.2 Spaces of Hölder and Lipschitz Continuous Functions*

Let $E$ be an open set in $\mathbb{R}^N$ and let $\alpha \in (0, 1]$ be fixed. A function bounded$f : E \to \mathbb{R}$ is said to be *Hölder continuous* with exponent $\alpha$ if there exists a constant $L_\alpha > 0$ such that

$$|f(x) - f(y)| \le L_\alpha |x - y|^\alpha \quad \text{for all pairs } x, y \in E. \tag{15.4}$$

The best constant $L_\alpha$ is given by

$$[f]_{\alpha,E} = \sup_{x,y\in E} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \tag{15.5}$$

The collection of all Hölder continuous functions with exponent $\alpha \in (0, 1)$ is denoted by $C^\alpha(E)$. If $\alpha = 1$ these functions are called *Lipschitz continuous*, and their collection is denoted by $\mathrm{Lip}(E)$. Setting

$$d(f, g) = \sup_{x\in E} |f(x) - g(x)| + [f - g]_{\alpha,E}. \tag{15.6}$$

defines a translation invariant metric in $C^\alpha(E)$ or $\mathrm{Lip}(E)$ which turns these into metric topological vector spaces.

## 16 On the Structure of a Complete Metric Space

Let $\{X; \mathcal{U}\}$ be a topological space. A set $E \subset X$ is *nowhere dense* in $X$, if $\bar{E}^c$ is dense in $X$. If $E$ is nowhere dense, then also $\bar{E}$ is nowhere dense. A closed set $E$ is nowhere dense, if and only if it does not contain any open set. If $E$ is nowhere dense, for any open set $\mathcal{O}$ the complement $\mathcal{O} - \bar{E}$ must contain an open set. Indeed if not $\bar{E}$ would contain the open set $\mathcal{O}$. If $E$ is nowhere dense and open, $\bar{E} - E$ is nowhere dense. If $E$ is nowhere dense and closed, $E - \overset{\circ}{E}$ is nowhere dense.

A finite subset of $[0, 1]$ is nowhere dense in $[0, 1]$. The Cantor set is nowhere dense in $[0, 1]$. Such a set is the *uncountable* union of if $\bar{E}^c$ is dense in $X$. If $E$ is nowhere if $\bar{E}^c$ is dense in $X$. If $E$ is nowhere nowhere dense sets. The rationals are not nowhere dense in $[0, 1]$. However, they are the *countable* union of nowhere dense sets in $[0, 1]$. Thus, the uncountable union of nowhere dense sets might be nowhere dense and the countable union of nowhere dense sets, might be dense.

A set $E \subset X$ is said to be *meager*, or of *first category* if is the countable union of nowhere dense sets. A set that is not of first category, is said to be of *second category*.

The complement of a set of first category is called a *residual* or *non-meager* set. The rationals in $[0, 1]$ are a set of first category. The Cantor set is of first category in $[0, 1]$.

A metric space $\{X; d\}$ is *complete* if every Cauchy sequence $\{x_n\}$ of elements of $X$ converges to some element $x \in X$. An example of noncomplete metric space is the set of the rationals in [0, 1] with the Euclidean metric. Every metric space can be completed as indicated in § 16.3c of the Complements. The completion of the rationals are the real numbers.

The *Baire Category Theorem* asserts that a complete metric space cannot be the countable union of nowhere dense sets, much the same way as [0, 1] is not the union of the rationals.

**Theorem 16.1** (Baire [7]) *A complete metric space is of second category.*

*Proof* If not, there exist a countable collection $\{E_n\}$ of nowhere dense subsets of $X$, such that $X = \cup E_n$. Pick $x_o \in X$ and consider the open ball $B_1(x_o)$ centered at $x_o$ and radius one. Since $E_1$ is nowhere dense in $X$, the complement $B_1(x_o) - \bar{E}_1$ contains an open set. Select an open ball $B_{r_1}(x_1)$, such that

$$\bar{B}_{r_1}(x_1) \subset B_1(x_o) - \bar{E}_1 \subset \bar{B}_1(x_o).$$

The selection can be done so that $r_1 < \frac{1}{2}$. Since $E_2$ is nowhere dense the complement $B_{r_1}(x_1) - \bar{E}_2$ contains an open set so that we may select an open ball $B_{r_2}(x_2)$ such that

$$\bar{B}_{r_2}(x_2) \subset B_{r_1}(x_1) - \bar{E}_2 \subset \bar{B}_{r_1}(x_1).$$

The selection can be done so that $r_2 < \frac{1}{3}$. Proceeding in this fashion generates a sequence of points $\{x_n\}$ and a family of balls $\{B_{r_n}(x_n)\}$ such that

$$\bar{B}_{r_{n+1}}(x_{n+1}) \subset \bar{B}_{r_n}(x_n) \quad r_n \leq \frac{1}{n+1} \qquad \text{and } \bar{B}_{r_n}(x_n) \bigcap \bigcup_{j=1}^{n} \bar{E}_n = \emptyset$$

for all $n$. The sequence $\{x_n\}$ is Cauchy and we let $x$ denote its limit. Now the element $x$ must belong to all the closed ball $\bar{B}_{r_n}(x_n)$ and it does not belong to any of the $\bar{E}_n$. Thus $x \notin \cup \bar{E}_n$ and $X \neq \cup E_n$. ∎

**Corollary 16.1** *A complete metric space $\{X; d\}$ does not contain open subsets of first category.*

## *16.1 The Uniform Boundedness Principle*

**Theorem 16.2** (Banach-Steinhaus [15]) *Let $\{X; d\}$ be a complete metric space and let $\mathcal{F}$ be a family of continuous, real-valued functions defined in $X$. Assume that the functions $f \in \mathcal{F}$ are pointwise equi-bounded, that is, for all $x \in X$, there exists a positive number $F(x)$ such that*

$$|f(x)| \leq F(x) \quad \textit{for all} \quad f \in \mathcal{F}. \tag{16.1}$$

*Then, there exists a nonempty open set $\mathcal{O} \in X$ and a positive number $F$, such that*

$$|f(x)| \le F \quad \textit{for all} \quad f \in \mathcal{F} \quad \textit{and all} \quad x \in \mathcal{O}. \tag{16.2}$$

Thus if the functions of the family $\mathcal{F}$ are pointwise equibounded in $X$, they are uniformly equibounded within some open subset of $X$. For this reason the theorem is also referred to as the *Uniform Boundedness Principle*.

*Proof* For $n \in \mathbb{N}$, let $E_{n,f}$ and $E_n$ be subsets of $X$ defined by

$$E_{n,f} = \{x \in X \mid |f(x)| \le n\}, \qquad E_n = \bigcap_{f \in \mathcal{F}} E_{n,f}.$$

The sets $E_{n,f}$ are closed, since the functions $f$ are continuous. Therefore also the sets $E_n$ are closed. Since the functions $f$ are pointwise equi-bounded, for each $x \in X$ there exists some integer $n$ such that $|f(x)| \le n$ for all $f \in \mathcal{F}$. Therefore each $x \in X$ belongs to some $E_n$, that is, $X = \cup E_n$.

Since $\{X; d\}$ is complete, by the Baire category theorem, at least one of the $E_n$ must not be nowhere dense. Since $E_n$ is closed, it must contain a nonempty open set $\mathcal{O}$. Such a set satisfies (16.2) with $F = n$. ∎

The Baire category theorem, and related category arguments, are remarkable, as they afford function-theoretical conclusions from purely topological information.

## 17 Compact and Totally Bounded Metric Spaces

Since a metric space satisfies the first axiom of countability, sequential compactness, countable compactness and the Bolzano–Weierstrass property all coincide (Proposition 5.2).

A metric space $\{X; d\}$ is *totally bounded* if for each $\varepsilon > 0$ there exists a finite collection of elements of $X$, say $\{x_1, \ldots, x_m\}$ for some positive integer $m$ depending upon $\varepsilon$, such that $X$ is covered by the union of the balls $B_\varepsilon(x_i)$ of radius $\varepsilon$ and centered at $x_i$. A finite sequence $\{x_1, \ldots, x_m\}$ with such a property is called a finite $\varepsilon$-net for $X$.

**Proposition 17.1** *A countably compact metric space $\{X; d\}$ is totally bounded.*

*Proof* Proceeding by contradiction, assume that there exists some $\varepsilon > 0$ for which there is no finite $\varepsilon$-net. Then, for a fixed $x_1 \in X$ the ball $B_\varepsilon(x_1)$ does not cover $X$ and we choose $x_2 \in X - B_\varepsilon(x_1)$. The union of the two balls $B_\varepsilon(x_1)$ and $B_\varepsilon(x_2)$ does not cover $X$ and we select $x_3 \in X - B_\varepsilon(x_1) \cup B_\varepsilon(x_2)$. Proceeding in this fashion generates a sequence of points $\{x_n\}$ at mutual distance of at least $\varepsilon$. Such a sequence cannot have a cluster point, thus contradicting the Bolzano–Weierstrass property. ∎

**Corollary 17.1** *A countably compact metric space is separable.*

*Proof* For positive integers $m$ and $n$, let $E_{n,m}$ be the finite $\varepsilon$-net of $X$ corresponding to $\varepsilon = \frac{1}{n}$. The union $\cup E_{n,m}$ is a countable subset of $X$ which is dense in $X$. ■

If $\{X; d\}$ is separable, every open covering of $X$ has a countable sub-covering. Therefore, countable compactness implies compactness (Proposition 5.3 and Corollary 5.1). Thus for separable metric spaces, all the various notions of compactness are equivalent.

We next examine the relation between compactness and total boundedness.

If $\{X; d\}$ is compact it is also totally bounded. Indeed having fixed $\varepsilon > 0$, the balls $B_\varepsilon(x)$ centered at all points of $X$, form an open covering of $X$, from which one may extract a finite one. It turns out that total boundedness implies compactness, provided the metric space $\{X; d\}$ is complete.

**Proposition 17.2** *A totally bounded and complete metric space* $\{X; d\}$ *is sequentially compact.*

*Proof* Let $\{x_n\}$ be a sequence of elements of $X$. The proof consists of selecting a Cauchy sequence $\{x_{n'}\} \subset \{x_n\}$. Since $\{X; d\}$ is complete, such a Cauchy subsequence would then converge to some $x \in X$ thereby establishing that $\{X; d\}$ is sequentially compact. Fix $\varepsilon = \frac{1}{2}$ and determine a corresponding $\frac{1}{2}$-net $\{y_{1,1}, \dots, y_{1,m_1}\}$ for some positive integer $m_1$. The union of the balls $B_{\frac{1}{2}}(y_{1,j})$ for $j = 1, \dots, m_1$, covers $X$. Therefore at least one of these balls, say for example $B_{\frac{1}{2}}(y_{1,j})$ contains infinitely many elements of $\{x_n\}$. Select these elements and relabel them, to form a sequence $\{x_{n_1}\}$. These elements satisfy $d(x_{n_1}, x_{m_1}) < 1$.

Next, let $\varepsilon = \frac{1}{2^2}$ and determine a corresponding $\frac{1}{4}$-net $\{y_{2,1}, \dots, y_{2,m_2}\}$ for some positive integer $m_2$. There exist a ball $B_{\frac{1}{4}}(y_{2,j})$, for some $j \in \{1, \dots, m_2\}$ that contains infinitely many elements of $\{x_{n_1}\}$. Select these elements and relabel them to form a sequence $\{x_{n_2}\}$. Thy satisfy $d(x_{n_2}, x_{m_2}) < \frac{1}{2}$.

Let $h \geq 2$ be a positive integer. If the subsequence $\{x_{n_{h-1}}\}$ has been selected, we let $\varepsilon = 2^{-h}$, and determine a corresponding $2^{-h}-$net, say for example $\{y_{h,1}, \dots, y_{h,m_h}\}$ for some positive integer $m_h$. There exist a ball $B_{\frac{1}{2^h}}(y_{h,j})$, for some $j \in \{1, \dots, m_h\}$ that contains infinitely many elements of $\{x_{n_{h-1}}\}$. We select these elements and relabel them to form a sequence $\{x_{n_h}\}$, whose elements satisfy

$$d(x_{n_h}, x_{m_h}) < \frac{1}{2^{h+1}}.$$

The Cauchy subsequence $\{x_{n'}\}$ is selected by diagonalization, out of the sequences $\{x_{n_h}\}$. ■

**Theorem 17.1** *A metric space* $\{X; d\}$ *is compact if and only if is totally bounded and complete.*

## *17.1 Pre-Compact Subsets of X*

The various notions of compactness and their characterization in terms of total boundedness, do not require that $\{X; d\}$ be a vector space. Thus, in particular they apply to any subset $K \subset X$, endowed with the metric $d$ inherited from $\{X; d\}$, by regarding $\{K; d\}$ as a metric space in its own right.

**Proposition 17.3** *A subset $K \subset X$ of a metric space $\{X; d\}$ is compact if and only if it is sequentially compact.*

A subset $K \subset X$ is *pre-compact* if its closure $\bar{K}$ is compact.

**Proposition 17.4** *A subset $K$ of a complete metric space $\{X; d\}$ is pre-compact if and only if is totally bounded.*

# Problems and Complements

## 1c Topological Spaces

**1.1**. A countable union of open sets is open. A countable union of closed sets need not be closed.

**1.2**. A countable intersection of closed sets is closed. A countable intersection of open sets need not be open.

**1.3**. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be topologies on $X$. Then $\mathcal{U}_1 \cap \mathcal{U}_2$ is a topology on $X$; however $\mathcal{U}_1 \cup \mathcal{U}_2$ need not be a topology on $X$.

**1.4**. The Euclidean topology on $\mathbb{R}$ induces a relative topology on $[0, 1)$. The sets $[0, \varepsilon)$ for $\varepsilon \in (0, 1)$ are open in the relative topology of $[0, 1)$ and not in the original topology of $\mathbb{R}$.

**1.5**. Let $X = \mathbb{N} \cup \{\omega\}$, where $\omega$ is the first infinite ordinal. A set $\mathcal{O} \subset X$ is open if either is any subset of $\mathbb{N}$, or if it contains $\{\omega\}$ and all but finitely many elements of $\mathbb{N}$. The collection of all such sets, complemented with $\emptyset$ and $X$ defines a topology on $X$. A function $f : X \to \mathbb{R}$ is continuous with respect to such a topology if and only if $\lim f(n) = f(\omega)$.

**1.6**. Linear combinations of continuous functions are continuous. Let $g : \{X; \mathcal{U}\} \to \{Y; \mathcal{V}\}$ and $f : \{Y; \mathcal{V}\} \to \{Z; \mathcal{Z}\}$ be continuous. Then $f(g) : \{X; \mathcal{U}\} \to \{Z; \mathcal{Z}\}$ is continuous. The maximum or minimum of two real valued, continuous functions is continuous.

**1.7**. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be topologies on $X$. The topology $\mathcal{U}_1$ is stronger or finer than $\mathcal{U}_2$ if $\mathcal{U}_2 \subset \mathcal{U}_1$, that is, roughly speaking, if $\mathcal{U}_1$ contains more open sets than $\mathcal{U}_2$. Equivalently if the identity map from $\{X; \mathcal{U}_1\}$ onto $\{X; \mathcal{U}_2\}$ is continuous.

**1.8**. Let $\{f_n\}$ be a sequence of real valued, continuous functions from $\{X; \mathcal{U}\}$ into $\mathbb{R}$. If $\{f_n\} \to f$ uniformly, then $f$ is continuous.

**1.9**. Let $C \subset X$ be closed and let $\{x_n\}$ be a sequence of points in $C$. Every cluster point of $\{x_n\}$ belongs to $C$.

**1.10**. Let $f : X \to Y$ be continuous and let $\{x_n\}$ be a sequence in $X$. If $x$ is a cluster point of $\{x_n\}$, then $f(x)$ is a cluster point of $\{f(x_n)\}$.

**1.11**. Let $X$ be the collection of pairs $(m, n)$ of nonnegative integers. Any subset of $X$ that does not contain $(0, 0)$ is declared to be open. A set $\mathcal{O}$ that contains $(0, 0)$ is open if and only if for all but a finite number of integers $m$, the set $\{n \in \mathbb{N} \cup \{0\} | (m, n) \notin \mathcal{O}\}$ is finite. For a fixed $m$ the collection of $(m, n)$ as $n$ ranges over $\mathbb{N} \cup \{0\}$ can be regarded as a *column*. With this terminology, a set $\mathcal{O}$ containing $(0, 0)$ is open if and only if it contains all but a finite number of elements for all but a finite number of columns. This defines a Hausdorff topology on $X$. No sequence in $X$ can converge to $(0, 0)$. The sequence $(n, n)$ has $(0, 0)$ as a cluster point, but no subsequence of $(n, n)$ converges to $(0, 0)$. This example is in [4].

## *1.12c Connected Spaces*

A topological space $\{X; \mathcal{U}\}$ is connected if it is not the union of two disjoint open sets. A subset $X_o \subset X$ is connected if the space $\{X_o; \mathcal{U}_o\}$ is connected.

**1.13**. The continuous image of a connected space is connected.

**1.14**. Let $\{A_\alpha\}$ be a family of connected subsets of $\{X; \mathcal{U}\}$ with nonempty intersection. Then $\cup A_\alpha$ is connected.

**1.15**. **(Intermediate Value Theorem)**] Let $f$ be a real valued continuous function on a connected space $\{X; \mathcal{U}\}$. Let $a, b \in X$ such that $f(a) < z < f(b)$ for some real number $z$. There exists $c \in X$ such that $f(c) = z$.

**1.16**. The discrete topology is a Hausdorff topology. If $X$ is finite, then the discrete topology is the only one for which $\{X; \mathcal{U}\}$ is Hausdorff.

**1.17**. Let $\{X; \mathcal{U}\}$ be Hausdorff. Then $\{X; \mathcal{U}_1\}$ is Hausdorff for any stronger topology $\mathcal{U}_1$.

**1.18**. A Hausdorff space $\{X; \mathcal{U}\}$ is normal if and only if, for any closed set $C$ and any open set $\mathcal{O}$ such that $C \subset \mathcal{O}$, there exists an open set $O$ such that $C \subset O \subset \overline{O} \subset \mathcal{O}$.

## *1.19c Separation Properties of Topological Spaces*

A topological space $\{X; \mathcal{U}\}$ is said to be *regular* if points are separated from closed sets, that is, for a given closed set $C \subset X$ and $x$ not in $C$, there exist disjoint open sets $\mathcal{O}_C$ and $\mathcal{O}_x$ such that $C \subset \mathcal{O}_C$ and $x \in \mathcal{O}_x$.

A Hausdorff space is said to be of type $(T_2)$. A regular space for which the singletons $\{x\}$ are closed is said to be of type $(T_3)$. A normal space for which the singletons $\{x\}$ are closed is said to be of type $(T_4)$. The separation properties of a topological space $\{X; \mathcal{U}\}$ are classified as follows:

$T_o$: Points are separated by open sets, that is, for any two given points $x, y \in X$ there exists an open set containing one of the two points, say for example $y$, but not the other.
$T_1$: The singletons $\{x\}$ are closed.
$T_2$: Hausdorff spaces.
$T_3$: Regular $+T_1$.
$T_4$: Normal $+T_1$.

From the definitions it follows that $(T_4) \Longrightarrow (T_3) \Longrightarrow (T_2) \Longrightarrow (T_1) \Longrightarrow ((T_o)$. The converse implications are false in general. In particular $(T_o)$ does not imply $(T_1)$. For example the space $X = \{x, y\}$ with the open sets $\{\emptyset, X, y\}$ is $(T_o)$ and not $(T_1)$. We have already observed that $(T_1)$ does not imply Hausdorff. Hausdorff in turn does not imply normal. Counterexamples are rather specialized and can be found in [150, 41].

We will be concerned only with Hausdorff and normal spaces.

## 4c Bases, Axioms of Countability and Product Topologies

**4.1**. Let $\{X; \mathcal{U}\}$ satisfy the first axiom of countability and let $A \subset X$. For every $x \in \bar{A}$, there exists a sequence $\{x_n\}$ of elements of $A$ converging to $x$. For every cluster point $y$ of a sequence $\{x_n\}$ of points in $X$, there exists a subsequence $\{x_{n'}\} \to y$.

**4.2**. Let $X$ be infinite and let $\mathcal{U}$ consist of the empty set and the collection of all subsets of $X$ whose complement is finite. Then $\mathcal{U}$ is a topology on $X$. If $X$ is uncountable $\{X; \mathcal{U}\}$ does not satisfy the first axiom of countability. The points are closed but $\{X; \mathcal{U}\}$ is not Hausdorff.

**4.3**. Let $\mathcal{B}$ be the collection of all intervals of the form $[\alpha, \beta)$. Then $\mathcal{B}$ is a base for a topology $\mathcal{U}$ on $\mathbb{R}$, constructed as in Proposition 4.1. The set $\mathbb{R}$ endowed with such a topology satisfies the first but not the second axiom of countability. The intervals $[\alpha, \beta)$ are both open and closed. This is called the *half-open interval* topology. The sequence $\{1 - \frac{1}{n}\}$ converges to 1 in the Euclidean topology and not in the half-open interval topology.

**4.4**. Let $X$ be an uncountable set, well ordered by $\prec$ and let $\Omega$ be the first uncountable. Set $X_o = \{x \in X | x \prec \Omega\}$, $X_1 = X_o \cup \Omega$, and consider the collection $\mathcal{B}_o$ of sets

$$\begin{aligned} \{x \in X_o | x \prec \alpha\} &\quad \text{for some } \ \alpha \in X_o \\ \{x \in X_o | \beta \prec x\} &\quad \text{for some } \ \beta \in X_o \\ \{x \in X_o | \alpha \prec x \prec \beta\} &\quad \text{for } \ \alpha, \beta \in X_o. \end{aligned}$$

Define similarly a collection of sets $\mathcal{B}_1$ where the various elements are taken out of $X_1$.

- **i.** The collection $\mathcal{B}_o$ forms a base for a topology $\mathcal{U}_o$ on $X_o$. The resulting space $\{X_o; \mathcal{U}_o\}$ satisfies the first but not the second axiom of countability. Moreover $\{X_o; \mathcal{U}_o\}$ is separable.
- **ii.** The collection $\mathcal{B}_1$ forms a base for a topology $\mathcal{U}_1$ on $X_1$. The resulting space $\{X_1; \mathcal{U}_1\}$ does not satisfy the first axiom of countability and is not separable.

**4.5**. The product of two connected topological spaces is connected.
**4.6**. The product of a family $\{X_\alpha; \mathcal{U}_\alpha\}$ of Hausdorff spaces is Hausdorff.
**4.7**. A sequence $\{x_n\}$ of elements of $\prod X_\alpha$ converges to some $x \in \prod X_\alpha$ if and only if the sequences of the projections $\{x_{\alpha,n}\}$ converge to the projections $x_\alpha$ of $x$.
**4.8**. The *countable* product of separable topological spaces is separable.
**4.9**. Let $\{X; \mathcal{U}\}$ satisfy the second axiom of countability. Every topological subspace of $X$ is separable. If $\{X; \mathcal{U}\}$ is separable but it does not satisfy the second axiom of separability, a topological subspace of $X$ might not be separable. The interval $[0, 1]$ with the half-open interval topology is separable. The Cantor set $\mathcal{C} \subset [0, 1]$ with the inherited topology is not separable.

## *4.10c The Box Topology*

Let $\{X_\alpha; \mathcal{U}_\alpha\}$ be a family of topological spaces and set

$$\mathcal{B} = \bigcup\{\prod \mathcal{O}_\alpha | \mathcal{O}_\alpha \in \mathcal{U}_\alpha\}.$$

Each set in $\mathcal{B}$ is an open rectangle, since it is the Cartesian product of open sets in $\mathcal{U}_\alpha$. The collection $\mathcal{B}$ forms a base for a topology in $\prod_\alpha X_\alpha$, called the box-topology. While the projections $\pi_\alpha$ are continuous with respect to such a topology, the box topology contains, roughly speaking, too many open sets.

As an example let $[0, 1]$ be endowed with the topology inherited from the Euclidean topology on $\mathbb{R}$. Then the Hilbert box $[0, 1]^{\mathbb{N}}$ can be endowed with either the product topology or the box-topology. The sequence $\{x_n\} = \{\frac{1}{n}, \dots, \frac{1}{n}, \dots\}$ converges to zero in the product topology and not in the box-topology. Indeed the neighborhood of the origin $\mathcal{O} = \prod[0, \frac{1}{j})$ does not contain any of the elements of $\{x_n\}$.

## 5c Compact Topological Spaces

**5.1**. A Hausdorff and compact topological space is regular and normal.
**5.2**. If $\{X; \mathcal{U}\}$ is compact, then $\{X; \mathcal{U}_o\}$ is compact for any weaker topology $\mathcal{U}_o \subset \mathcal{U}$. The converse is false.

**Proposition 5.1c** *Let $f : \{X; \mathcal{U}\} \to \{Y; \mathcal{V}\}$ be continuous, one-to-one and onto. If $\{X; \mathcal{U}\}$ is compact and $\{Y; \mathcal{V}\}$ is Hausdorff then $f$ is a homeomorphism.*

*Proof* The inverse $f^{-1}$ is one-to-one and onto. It is continuous if for every closed set $C \subset X$, the image $f(C)$ is closed. If $C \subset X$ is closed, it is compact and its continuous image $f(C)$ is compact and hence closed since $\{Y; \mathcal{V}\}$ is Hausdorff. ∎

**5.3.** Let $\{X; \mathcal{U}\}$ be Hausdorff and compact. Then the previous Proposition implies that:

**i.** $\{X; \mathcal{U}_1\}$ is not compact for any stronger topology $\mathcal{U}_1$.
**ii.** $\{X; \mathcal{U}_o\}$ is not Hausdorff for any weaker topology $\mathcal{U}_o$.
**iii.** If $\{X; \mathcal{U}_1\}$ is compact for a stronger topology $\mathcal{U}_1 \supset \mathcal{U}$, then $\mathcal{U} = \mathcal{U}_1$.

Thus, the topological structure of a compact Hausdorff space $\{X; \mathcal{U}\}$ is rigid in the sense that one cannot strengthen its topology without loosing compactness and cannot weaken it without loosing the separation property.

**5.4.** Let $\|x\|$ be the Euclidean norm in $\mathbb{R}^N$ and consider the function

$$f(x) = \begin{cases} \dfrac{\max\{|x_1|, \dots, |x_N|\}}{\|x\|} x & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases}$$

The function $f$ maps cubes of wedge $2\rho$ in $\mathbb{R}^N$, onto balls of radius $\rho$ in $\mathbb{R}^N$, it is continuous, one-to-one and onto. Thus $f$ is a homeomorphism between $\mathbb{R}^N$ equipped with the topology generated by the cubes with faces parallel to the coordinate planes, and $\mathbb{R}^N$ equipped with the topology generated by the balls.

**5.5.** A space $X$ consisting of more than one point and equipped with the trivial topology is compact and not Hausdorff.

**5.6.** Let $\{X; \mathcal{U}\}$ be locally compact. A subset $C \subset X$ is closed if and only if $C \cap K$ is closed, for every closed compact subset $K \subset X$.

**5.7.** Let $\{X_o; \mathcal{U}_o\}$ and $\{X_1; \mathcal{U}_1\}$ be the spaces introduced in **4.4**. The space $\{X_o; \mathcal{U}_o\}$ is sequentially compact but not compact. The space $\{X_1; \mathcal{U}_1\}$ is compact.

## *5.8c The Alexandrov One-Point Compactification of $\{X; \mathcal{U}\}$ ([3])*

Let $\{X; \mathcal{U}\}$ be a noncompact Hausdorff topological space. Having fixed $x_* \notin X$ consider the set $X_* = X \cup \{x_*\}$ and define a collection of sets $\mathcal{U}_*$ consisting of $\mathcal{U}$, $X_*$, and all subsets $\mathcal{O}_* \subset X_*$ containing $x_*$ and such that $X_* - \mathcal{O}_*$ is compact in $\{X; \mathcal{U}\}$. Then $\mathcal{U}_*$ is a Hausdorff topology on $X_*$. Moreover $\{X_*; \mathcal{U}_*\}$ is compact, $X$ is dense in $X_*$ and the restriction of $\mathcal{U}_*$ to $X$, coincides with the original topology $\mathcal{U}$ on $X$.

**5.9.** The topological space of **1.5** is compact. It can be regarded as the Alexandrov one-point compactification of $\mathbb{N}$ equipped with the discrete topology.

**5.10**. The one-point compactification of $\mathbb{R}^N$ with its Euclidean topology, is homeomorphic to the unit sphere in $\mathbb{R}^{N+1}$ by stereographic projection.

**5.11**. The one-point compactification of $\{X_o; \mathcal{U}_o\}$ in **4.4** is $\{X_1; \mathcal{U}_1\}$.

# 7c Continuous Functions on Countably Compact Spaces

## *7.1c Upper-Lower Semi-continuous Functions*

**7.1**. Characteristic functions of open(closed) sets in $\mathbb{R}^N$ are lower(upper) semi-continuous. A function $f$ for an open set $E \subset \mathbb{R}^N$ into $\mathbb{R}^*$ is upper(lower) semi-continuous if and only is for each $x \in E$

$$\limsup_{y \to x} f(y) \le f(x) \qquad \big(\liminf_{y \to x} f(y) \ge f(x)\big).$$

**7.2**. Let $\{f_\alpha\}$ be a collection of upper(lower) semi-continuous functions on a topological space $\{X; \mathcal{U}\}$. Then $\inf(\sup) f_\alpha$ is upper(lower) semi-continuous.

**7.3**. The finite sum of nonnegative upper(lower) semi-continuous functions is upper(lower) semi-continuous.

**7.4**. Let $\{f_n\}$ be a sequence of nonnegative, lower semi-continuous function on $\{X; \mathcal{U}\}$. Then $\sum f_n$ is lower semi-continuous.

**7.5**. Let $\{f_n\}$ be a sequence of nonnegative, upper semi-continuous function on $\{X; \mathcal{U}\}$. Then $\sum f_n$ need not be upper semi-continuous. Give a counterexample.

**7.6**. **Modulus of Continuity**: For an arbitrary real valued function $f$ defined on an open set $E \subset \mathbb{R}^N$, and for $\varepsilon > 0$, set

$$\begin{aligned} \eta(x, \varepsilon) &= \sup\{|f(y) - f(z)| : y, z \in B_\varepsilon(x) \cap E\} \\ \eta(x) &= \inf_\varepsilon \eta(x, \varepsilon). \end{aligned}$$

Prove that $\eta(\cdot)$ is upper semi-continuous. Prove that $f$ is continuous at $x$ if and only if $\eta(x) = 0$ and therefore the points of continuity of any $f : E \to \mathbb{R}$ are countable intersection of open sets. The function

$$E \times \mathbb{R}^+ \ni (x, \varepsilon) \to \eta(x, \varepsilon)$$

is the *modulus of continuity* of $f$ at $x$. The function $f$ is Hölder continuous at $x$, with Hölder exponent $\alpha \in (0, 1]$, if there exists positive constant $\delta = \delta(x)$ and $C = C(x)$ depending upon such that $\eta(x, \varepsilon) \le C(x)\varepsilon^\alpha$ for all $0 < \varepsilon \le \delta(x)$. In such a case

$$|f(x) - f(y)| \le C(x)|x - y|^\alpha \quad \text{for all } |x - y| \le \delta(x).$$

The function $f$ is Hölder continuous in $E$ with exponent $\alpha$, if the constants $\delta(x)$ and $C(x)$ are independent of $x \in E$. If $\alpha = 1$ then $f$ is Lipschitz continuous at $x$ and respectively in $E$.

**7.7**. **Upper and Lower Envelope of a Function**: For an arbitrary real valued function $f$ defined on an open set $E \subset \mathbb{R}^N$, and for $\varepsilon > 0$, set

$$\varphi(x) = \sup_{\varepsilon>0} \inf_{|x-y|<\varepsilon} f(y) \quad \text{lower envelope of } f \text{ at } x$$

$$\psi(x) = \inf_{\varepsilon>0} \sup_{|x-y|<\varepsilon} f(y) \quad \text{upper envelope of } f \text{ at } x.$$

Prove that $\varphi$ is lower semi-continuous and $\psi$ is upper semi-continuous; moreover $\varphi \le f \le \psi$.

## *7.2c Characterizing Lower-Semi Continuous Functions in $\mathbb{R}^N$*

**Proposition 7.1c** *Let $E$ be an open subset of $\mathbb{R}^N$. A function $f : E \to \mathbb{R}^+$ is lower semi-continuous if and only if it is the pointwise limit of an increasing sequence of continuous functions defined in $E$.*

*Proof* *($\Longrightarrow$)* For $n \in \mathbb{N}$ and $x \in E$ set

$$f_n(x) = \inf\{f(z) + n|x-z| : z \in E\}.$$

Prove that $|f_n(x) - f_n(y)| \le n|x-y|$. ∎

## *7.3c On the Weierstrass-Baire Theorem*

**7.8**. The set of discontinuities of a real valued function could be as diverse as possible. As an example consider the functions

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in [0,1] - \mathbb{Q} \end{cases} \qquad g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \in [0,1] - \mathbb{Q}. \end{cases}$$

The first is everywhere discontinuous but its absolute value is continuous. The second is continuous only at $x = 0$.

**7.9**. There exists a function $f : (0, 1) \to \mathbb{R}$ continuous at the irrationals and discontinuous at the rationals of $(0, 1)$. To construct an example recall that a rational number $r \in (0, 1]$ can be written as the ratio $m/n$ of two positive integers in *lowest terms*. That is, $m$ and $n$ are the smallest integers for which $r = \frac{m}{n}$. A rational number $r$ is an equivalence class of ratios of the form $\frac{m}{n}$. Out of such an equivalence class we select the representative in *lowest terms*. Set

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x \in \mathbb{Q} \cap (0, 1] \\ \\ 0 & \text{if } x \in (0, 1] - \mathbb{Q}. \end{cases} \tag{7.1c}$$

However there exists no function $f : (0, 1] \to \mathbb{R}$ continuous at the rationals and discontinuous at the irrationals of $(0, 1]$ (see Corollary 16.1c of the Complements).
The function in (7.1c) is everywhere upper semi-continuous in $(0, 1]$ since, for every $y \in (0, ]$

$$\limsup_{x \to y} f(x) = 0 \le f(y).$$

**7.10**. There exists functions that are everywhere finite in their domain of definition and not bounded in every subset of their domain of definition. Continue to represent a rational number $r \in (0, 1)$ as the ratio $m/n$ of two positive integers in *lowest terms*. Then set

$$f(x) = \begin{cases} n & \text{if } x \in \mathbb{Q} \cap (0, 1) \\ \\ 0 & \text{if } x \in (0, 1) - \mathbb{Q}. \end{cases} \tag{7.2c}$$

Such a function is everywhere finite in $[0, 1]$ and unbounded in every subinterval of $[0, 1]$. Indeed let $I \subset [0, 1]$ be an interval. If $f$ were bounded in $I$, then the denominator $n$ of all rational numbers $\frac{m}{n} \in I$, would be bounded. This would imply that there are only finitely many rationals in $I$.

**7.11**. There exist real valued, bounded functions, defined on a compact set that do not take neither maxima or minima.
Continue to represent a rational number $r \in (0, 1)$ as the ratio $m/n$ of two positive integers in *lowest terms*. Then set

$$f(x) = \begin{cases} (-1)^n \frac{n}{n+1} & \text{if } x \in \mathbb{Q} \cap (0, 1) \\ \\ 0 & \text{if } x \in (0, 1) - \mathbb{Q}. \end{cases} \tag{7.3c}$$

About any point of $(0, 1)$ the values of $f$ are arbitrarily close to $\pm 1$. The function in (7.3c) is nowhere upper semi-continuous in $[0, 1]$. Indeed for every $y \in [0, 1]$

$$\limsup_{x \to y} f(x) = 1 > f(y).$$

Thus the assumption that $f$ be upper semi-continuous cannot be relaxed in the Weierstrass-Baire Theorem. The function in (7.3c) is also nowhere monotone in $[0, 1]$.

### 7.4c On the Assumptions of Dini's Theorem

**7.12**. The assumption that the limit function $f$ be lower semi-continuous cannot be removed from Dini's theorem. Indeed the sequence $\{x^n\}$ for $x \in [0, 1]$ is decreasing, each $x^n$ is continuous in $[0, 1]$ but the limit $f$ is not lower semi-continuous. Accordingly, the convergence $\{x^n\} \to f$ is not uniform in $[0, 1]$.

**7.13**. The assumption that each of the $f_n$ be upper semi-continuous, cannot be removed from Dini's theorem. Set

$$f_n(x) = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } 0 < x < \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} \le x \le 1. \end{cases}$$

The sequence $\{f_n\}$ is decreasing, it converges to zero pointwise in $[0, 1]$, but the convergence is not uniform.

**7.14**. The requirement that the sequence $\{f_n\}$ be decreasing cannot be removed from Dini's Theorem. Set

$$f_n(x) = \begin{cases} 2n^2 x & \text{for } 0 \le x \le \frac{1}{2n} \\ n - 2n^2(x - \frac{1}{2n}) & \text{for } \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} \le x \le 1. \end{cases}$$

The functions $f_n$ are continuous in $[0, 1]$ and converge to zero pointwise in $[0, 1]$. However the convergence is not uniform.

## 9c Vector Spaces

**9.1**. The element $\Theta \in X$ is unique.

**9.2**. Let $A$, $B$ and $C$ be subsets of a vector space $X$. Then:

(i) $A \cap B \neq \emptyset$ if and only if $\Theta \in A - B$.

(ii) $A \cap (B + C) \neq \emptyset$ if and only if $B \cap (A - C) \neq \emptyset$. Equivalently if and only if $C \cap (A - B) \neq \emptyset$.

(iii) $X_o$ is a subspace of $X$ if and only if $\alpha X_o = X_o$ for all $\alpha \in \mathbb{R} - \{0\}$ and $x + X_o = X_o$ for all $x \in X_o$.

(iv) If $X_o$ and $X_1$ are linear subspaces of $X$ then $\alpha X_o + \beta X_1$ is a subspace of $X$.

(v) If $A$ and $B$ are convex, then $A + B$ is convex and $\lambda A$ is convex for all $\lambda \in \mathbb{R}$.

### *9.3c Hamel Bases*

A collection $\{x_\alpha\}$ of elements of a vector space $X$ is linearly independent if any *finite* subcollection of elements $\{x_\alpha\}$ is linearly independent.

A linearly independent collection $\{x_\alpha\}$ is a *Hamel basis* for a vector space $X$ if $\text{span}\{x_\alpha\} = X$. Equivalently, if every $x \in X$ has a unique representation as a *finite* linear combination of elements of $\{x_\alpha\}$, that is

$$x = \sum_{j=1}^{m} c_j x_{\alpha_j} \quad \text{for some finite } m, \quad c_j \in \mathbb{R}.$$

**Proposition 9.1c** *Every vector space X has a Hamel basis.*

*Proof* Let $\mathcal{L}$ be the collection of all subsets of $X$ whose elements are linearly independent. This collection is partially ordered by inclusion. Every linearly ordered subcollection $\{B_\sigma\}$ of $\mathcal{L}$ has an upper bound given by $B = \cup B_\sigma$. Indeed the elements of $B$ are linearly independent since any finitely many of them must belong to some $B_\sigma$, and the elements of $B_\sigma$ are linearly independent. Therefore by Zorn's lemma $\mathcal{L}$ has a maximal element $\{x_\alpha\}$. The elements of $\{x_\alpha\}$ are linearly independent and every $x \in X$ can be written as a finite linear combination of them. Indeed if not, the collection $\{x_\alpha, x\}$ belongs to $\mathcal{L}$, contradicting the maximality of $\{x_\alpha\}$. ■

**9.4**. A Hamel basis for $\mathbb{R}^N$ is the usual Euclidean basis.

**9.5**. Let $\ell$ denote the collection of all sequences $\{c_n\}$ of real numbers and consider the countable subcollection of $\ell$

$$\begin{aligned} \mathbf{e}_1 &= \{1, 0, 0, \dots, 0_m, 0, \dots\} \\ \mathbf{e}_2 &= \{0, 1, 0, \dots, 0_m, 0, \dots\} \\ \cdots &= \cdots \quad \cdots \quad \cdots \quad \cdots \\ \mathbf{e}_m &= \{0, 0, 0, \dots, 1_m, 0, \dots\} \\ \cdots &= \cdots \quad \cdots \quad \cdots \quad \cdots \end{aligned} \tag{9.1c}$$

Every $x \in \ell$ can be written as $x = \sum c_n \mathbf{e}_n$. However $\{\mathbf{e}_n\}$ is not a Hamel basis for $\ell$.

### 9.6c On the Dimension of a Vector Space

If the Hamel basis of a vector space $X$ is of the form $\{x_n\}$ for $n \in \mathbb{N}$, the dimension of $X$ is $\aleph_o$, that is, the cardinality of $\mathbb{N}$. More generally, if $\{x_\alpha\}$ for $\alpha \in A$ is a Hamel basis for a vector space $X$, then the dimension of $X$ is the cardinality of $A$. This definition of dimension of $X$ is independent of the choice of the Hamel basis.

**9.7**. Let $\ell_o$ denote the collection of all sequences of real numbers $\{c_n\}$ with only finitely many non zero elements. Then (9.1c) is a Hamel basis for $\ell_o$ and the dimension of $\ell_o$ is $\aleph_o$.

**9.8**. Let $\ell[0, 1]$ denote the collection of all sequences $\{c_n\}$ of real numbers in $[0, 1]$. The dimension of $\ell[0, 1]$ is no less than the cardinality of $\mathbb{R}$, since the collection $x_\alpha = \{\alpha, \alpha^2, \dots, \alpha^n, \dots\}$ for $\alpha \in (0, 1)$ is linearly independent.

**9.9**. A vector space with a countable Hamel basis is separable.

**9.10**. The pair $\{\mathbb{R}; \mathbb{Q}\}$, that is, the reals $\mathbb{R}$ over the field of the rationals $\mathbb{Q}$, is a vector space. If $x \in \mathbb{R}$ is not an algebraic number, then the elements $\{1, x, x^2, \dots\}$ are linearly independent. The dimension of $\{\mathbb{R}; \mathbb{Q}\}$ is not less than the cardinality of $\mathbb{R}$.

## 10c Topological Vector Spaces

**10.1**. Let $A$ and $B$ be subsets of a topological vector space $\{X; \mathcal{U}\}$. Then:

(i) If $A$ and $B$ are open, the $\alpha A + \beta B$ is open.

(ii) $\bar{A} + \bar{B} \subset \overline{A + B}$. The inclusion is in general strict unless either one of $\bar{A}$ or $\bar{B}$ is compact.

(iii) If $A \subset X$ is convex then $\bar{A}$ and $\overset{\text{o}}{A}$ are convex.

(iv) The convex hull of an open set is open.

**10.2**. If $x \in \mathcal{O} \in \mathcal{B}_\Theta$, there exists an open set $A$ such that $x + A \subset \mathcal{O}$.

**10.3**. The identity map from $\mathbb{R}$ equipped with the Euclidean topology, onto $\mathbb{R}$ equipped with the half-open interval topology of **4.3**, is bounded, linear but not continuous.

**10.4**. Let $E$ be an open set in $\mathbb{R}^N$ and denote by $C(E)$ the linear vector space of all real valued continuous functions defined in $E$. In $C(E)$ introduce a topology as follows. For $g \in C(E)$ and $\rho > 0$, stipulate that the set,

$$\mathcal{O}_{g,\rho} = \Big\{ f \in C(E) \;\Big|\; \sup_E |f - g| < \rho \Big\}$$

is an open neighborhood of $g$. The collection of such $\mathcal{O}_{g,\rho}$ is a base for a topology in $C(E)$. The sum $+ : C(E) \times C(E) \to C(E)$ is continuous with respect to such a topology. However the multiplication by scalars $\bullet : \mathbb{R} \times C(E) \to C(E)$, is not continuous.

**10.5**. For $x, y \in \mathbb{R}$ set

$$d(x,y) = \begin{cases} |x-y| + 1 \text{ if either } x = 0 \text{ or } y = 0 \text{ but not both} \\ \\ |x-y| \qquad\quad \text{otherwise.} \end{cases}$$

Prove that $d(\cdot,\cdot)$ is a distance in $\mathbb{R}$ and that the resulting topological space, is not a linear topological vector space. The ball $B_{\frac{1}{2}}(0) = \{O\}$ is open, and its pre-image under translation need not be open.

## 13c Metric Spaces

**13.1**. Properties (i) and (iii) in the definition of a metric, follow from (ii) and (iv). Setting $x = y$ in the triangle inequality (iv), and using (ii) gives $2d(x,z) \ge 0$ for all $x, z \in X$. Setting $z = y$ in (iv) gives $d(x,y) \le d(y,x)$, and by symmetry $d(y,x) \le d(x,y)$. Thus a metric could be defined as a function $d : (X \times X) \to \mathbb{R}$ satisfying (ii) and (iv).

**13.2**. The identically zero pseudo-metric generates the trivial topology on $X$. The function $d(x,y) = 1$ if $x \ne y$ and $d(x,y) = 0$ if $x = y$ is the *discrete* metric on $X$ and generates the discrete topology. With respect to such a metric the open balls $B_1(x)$ contain only the element $x$ and their closure still coincides with $x$. Thus $\bar{B}_1(x) \ne \{y \in X | d(x,y) \le 1\}$.

**13.3**. The function $(x,y) \to \min\{1; |x-y|\}$ is a metric on $\mathbb{R}$.

**13.4**. Let $A \subset X$. Then $\bar{A} = \cup\{x | d(A,x) = 0\}$.

**13.5**. A function $f : \{X; d\} \to \{Y; \eta\}$ is continuous at $x \in X$ if and only if $\{f(x_n)\} \to f(x)$, for every sequence $\{x_n\} \to x$.

**13.6**. Two metrics $d_1$ and $d_2$ on $X$ are equivalent if and only if:

(i) For every $x \in X$ and every ball $B^1_\rho(x)$ in the metric $d_1$, there exists a radius $r = r(\rho, x)$ such that the ball $B^2_r(x)$ in the metric $d_2$ is contained in $B^1_\rho(x)$.

(ii) For every $x \in X$ and every ball $B^2_r(x)$ in the metric $d_2$, there exists a radius $\rho = \rho(r, x)$ such that the ball $B^1_\rho(x)$ in the metric $d_1$ is contained in $B^2_r(x)$.

The two metrics are uniformly equivalent if the choices of $r$ in (i) and the choice of $\rho$ in (ii) are independent of $x \in X$. Equivalently, $d_1$ and $d_2$ are uniformly equivalent if and only if the identity map between $\{X; d_1\}$ and $\{X; d_2\}$ is a uniform homeomorphism.

**13.7**. In $\mathbb{R}^N$ the following metrics are uniformly equivalent

$$d_p(x,y) = \begin{cases} \left( \sum_{i=1}^N |x_i - y_i|^p \right)^{\frac{1}{p}} & \text{for} \quad p \in [1, \infty) \\ \\ \max_{1 \le i \le N} |x_i - y_i| & \text{for} \quad p = \infty. \end{cases}$$

The discrete metric in $\mathbb{R}^N$ is not equivalent to any of the metrics $d_p$.

**13.8**. The metric $d_o$ in (13.1) is equivalent, but not uniformly equivalent, to the original metric $d$.

**13.9**. A metric space $\{X; d\}$ is bounded if there exists and element $\Theta \in X$ and a number $M > 0$ such that $d(x, \Theta) < M$ for all $x \in X$. Boundedness depends only on the metric and it is neither an intrinsic property of $X$ nor a topological property. In particular the same set $X$ can be endowed with two equivalent metrics $d$ and $d_o$ in such a way that $\{X; d\}$ is not bounded and $\{X; d_o\}$ is bounded.

### *13.10c The Hausdorff Distance of Sets*

Let $\{X; d\}$ be a metric space. For $A \subset X$ and $\sigma > 0$ set

$$A_\sigma = \{x \in X \bigm| d(x, A) < \sigma\}.$$

The Hausdorff distance of two sets $A$ and $B$ in $X$ is ([72], Chap. VIII)

$$d_{\mathcal{H}}(A, B) = \inf\{\sigma > 0 \text{ such that } A \subset B_\sigma \text{ and } B \subset A_\sigma\}.$$

If $A$ and $B$ have nonempty intersection their distance is zero but their Hausdorff distance might be positive. There exist distinct subsets $A$ and $B$ of $X$ whose Hausdorff distance is zero. Thus $d_{\mathcal{H}}$ is a pseudo-metric on $2^X$ and generates the pseudo-metric space $\{2^X; d_{\mathcal{H}}\}$.

The identity map from $\{X; d\}$ to $\{2^X; d_{\mathcal{H}}\}$ is an isometry.

The topology on $\{2^X; d_{\mathcal{H}}\}$ is generated only by the original metric $d$, via the definition of $d_{\mathcal{H}}$, and not by the topology of $\{X; d\}$. Indeed there might exist metrics $d_1$ and $d_2$ that generate the same topology on $X$ and such that the corresponding Hausdorff distances $d_{1,\mathcal{H}}$ and $d_{2,\mathcal{H}}$ generate different topologies on $2^X$. As an example let $X = \mathbb{R}^+$ endowed with the two equivalent metrics

$$d_1(x, y) = \left|\frac{x}{1+x} - \frac{y}{1+y}\right|, \qquad d_2(x, y) = \min\{1; |x - y|\}.$$

The topologies of $\{2^{\mathbb{R}^+}; d_{1,\mathcal{H}}\}$ and $\{2^{\mathbb{R}^+}; d_{2,\mathcal{H}}\}$ are different. The set of natural numbers $\mathbb{N}$ is a point in $2^{\mathbb{R}^+}$. The ball $B^1_\varepsilon(\mathbb{N})$ centered at $\mathbb{N}$ and of radius $\varepsilon \in (0, 1)$, in the topology of $\{2^{\mathbb{R}^+}; d_{1,\mathcal{H}}\}$ contains infinitely many finite subsets of $\mathbb{R}^+$. The ball $B^2_\varepsilon(\mathbb{N})$ in the topology of $\{2^{\mathbb{R}^+}; d_{2,\mathcal{H}}\}$ does not contain any finite subset of $\mathbb{R}^+$.

## *13.11c Countable Products of Metric Spaces*

Let $\{X_n; d_n\}$ be a countable collection of metric spaces. Then the product topology on $\prod X_n$ coincides with the topology generated by the metric

$$d(x, y) = \sum \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}. \tag{13.1c}$$

This will follow from the two inclusions:

(i) Every neighborhood $\mathcal{O}_y$ of a point $y \in \prod X_n$ open in the product topology, contains a ball $B_\varepsilon(y)$ with respect to the metric in (13.1c).
(ii) Every ball $B_\varepsilon(y)$ with respect to the metric in (13.1c) contains a neighborhood of $y$, open in the product topology.

Elements $y \in \prod X_n$ are sequences $\{y_n\}$ such that $y_n \in X_n$. For a fixed $y \in \prod X_n$ a neighborhood $\mathcal{O}_y$ of $y$, open in the product topology contains an open set of the form

$$\mathcal{O}_{y,k} = \prod_{n=1}^{k} B_\varepsilon^{(n)}(y_n) \quad \text{for some finite } k \tag{13.2c}$$

where $B_\varepsilon^{(n)}(y_n)$ is the ball in $\{X_n; d_n\}$, centered at $y_n$ and of radius $\varepsilon$.

There exists $\delta > 0$ sufficiently small depending on $\varepsilon$ and $k$, such that the ball $B_\delta(y)$ in $\prod X_n$ is contained in $\mathcal{O}_y$. Indeed from

$$\sum \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} < \delta$$

it follows that, the number $\delta$ can be chosen so small that

$$d_n(x_n, y_n) < 2^{n+1}\delta \le \varepsilon \quad \text{for all } n = 1, \dots, k.$$

Thus $B_\delta(y) \subset \mathcal{O}_y$. Conversely, every ball $B_\delta(y)$ in $\prod X_n$ contains an open set of the form (13.2c). Indeed let $k$ be a positive integer so large that

$$\sum_{n=k}^{\infty} \frac{1}{2^n} < \frac{\delta}{2}.$$

For such a $k$ fixed the open set in (13.2c) with $\varepsilon = \frac{1}{2}\delta$ is contained in $B_\delta(y)$.

**13.12**. The countable product of complete metric spaces is complete.

## 14c Metric Vector Spaces

Referring back to (14.1), the discontinuity of the bijection $h$ is meant with respect to the topology generated by the original metric, whereas the discontinuity of the sum $+ : X \times X \to X$, or the product by scalars $\bullet : \mathbb{R} \times X \to X$, should be proved with respect to the new metric $d_h$.

**14.1**. Let $X = \mathbb{R}$ and let $d$ be the usual Euclidean metric. Define

$$h(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \\ x & \text{otherwise} \end{cases}$$

and let $d_h = d(h)$ be defined as in (14.1). For $\varepsilon \in (0, 1)$ the ball $\mathcal{B}_\varepsilon(0)$, centered at 0 and radius $\varepsilon$, in the new metric $d_h$, consists of the singleton $\{0\}$ and the open ball $B_\varepsilon(1)$, in the original Euclidean metric, of radius $\varepsilon$ and centered at 1 from which the singleton $\{1\}$ has been removed. Likewise the ball $\mathcal{B}_\varepsilon(1)$ of radius $\varepsilon$ and centered at 1, in the new metric $d_h$, consists of the singleton $\{1\}$ and the and the open ball $B_\varepsilon(0)$, in the original Euclidean metric, of radius $\varepsilon$ and centered at 0 from which the singleton $\{0\}$ has been removed. For such a metric $d_h$, both the sum and the multiplication by scalars are discontinuous.

**14.2**. The half-open interval topology of **4.3** is not metrizable, that is, there exists no metric on $\mathbb{R}$ that generates the half-open interval topology. Combine **10.3** with Proposition 14.2.

**14.3**. Let $\ell_\infty$ be the collection of all sequences $x = \{x_n\}$ or real numbers such that $\sup |x_n| < \infty$, endowed with with the metric

$$d(x, y) = \sup_n |x_n - y_n|. \tag{14.1c}$$

Show that $\ell_\infty$ is not complete nor separable.

**14.4**. Let $t : \mathbb{R} \to [0, 1]$ be the *tent function* defined by

$$t(s) = \begin{cases} 1 - 2|s| & \text{if } |s| \le \frac{1}{2}; \\ 0 & \text{otherwise.} \end{cases} \tag{14.2c}$$

For $x \in \ell_\infty$ define

$$T(x) = \sum x_i t(s - i).$$

Prove that $T$ is an isometry between $\ell_\infty$ and $T(\ell_\infty)$.

## 15c Spaces of Continuous Functions

### *15.1c Spaces of Hölder and Lipschitz Continuous Functions*

**15.1**. Prove that the quantity $[f - g]_{\alpha,E}$ is a pseudo-metric in $C^\alpha(E)$.
**15.2**. Prove that $C^\alpha(E)$ is complete for the metric in (15.6).
**15.3**. Prove that $C^\beta(E) \subset C^\alpha(E)$ for all $\beta > \alpha$.
**15.4**. Prove that Hölder continuous functions in $E$ are uniformly continuous.
**15.5**. Let $E$ be a subset of $\mathbb{R}^N$ containing the origin, and let $\alpha \in (0, 1)$ be fixed. Prove that $E \ni x \to |x|^\alpha$ is in $C^\alpha(E)$. Prove also that for any $g \in C^1(\bar{E})$

$$d(|x|^\alpha, g) \ge 1 \quad \text{with respect to the metric } d(\cdot, \cdot) \text{ in (15.6).}$$

**15.6**. Prove that $C^\alpha(E)$ is not separable in its own metric topology.
**15.7**. Prove that $E \ni x \to |x| \in \mathrm{Lip}(E)$. Prove also that for any $g \in C^1(\bar{E})$, such that $g_{x_j}(x_o) = 0$ for $j = 1, \dots, N$, for some $x_o \in E$,

$$d(|x|, g) \ge 1 \quad \text{with respect to the metric } d(\cdot, \cdot) \text{ in (15.6)}$$

**15.8**. In $\mathrm{Lip}(0, 1)$ consider the functions

$$(0, 1) \ni x \to |a - x|, \ |b - x| \quad \text{for fixed } \ a, b \in (0, 1).$$

Prove that if $a \ne b$ then

$$d(|a - x|, |b - x|) \ge 2 \quad \text{with respect to the metric } d(\cdot, \cdot) \text{ in (15.6).}$$

Deduce that $\mathrm{Lip}(E)$ is not separable.

## 16c On the Structure of a Complete Metric Space

**16.1**. Let $d_1$ and $d_2$ be two equivalent metrics on the same vector space $X$. The two metric spaces $\{X; d_1\}$ and $\{X; d_2\}$ have the same topology and the identity map is a homeomorphism. However, the identity map does not preserve completeness. As an example consider $\mathbb{R}$ with the Euclidean metric and the metric $d_o$ given in (13.1) corresponding to the Euclidean metric.
**16.2**. Intersection Properties of a Complete Metric Space

**Proposition 16.1c** (Cantor) *Let $\{X; d\}$ be a complete metric space, and let $\{E_n\}$ be a countable collection of closed subsets of $X$ such that $E_{n+1} \subset E_n$ and $\mathrm{diam}\{E_n\} \to 0$. Then $\cap E_n \ne \emptyset$.*

### *16.3c Completion of a Metric Space*

Every metric space $\{X; d\}$ can be completed by the following procedure.

(a) First, one defines $X'$ as the set of all the Cauchy sequences $\{x_n\}$ of elements in $X$ and verifies that such a set has the structure of a linear space. Then on $X'$ one defines a distance function

$$(\{x_n\}; \{y_n\}) \to d'(\{x_n\}; \{y_n\}) = \lim d(x_n, y_n).$$

Since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $X$, the sequence $\{d(x_n, y_n)\}$ is a Cauchy sequence in $\mathbb{R}^+$. Thus the indicated limit exists. Since several pairs of Cauchy sequences might generate the same limit, this is not a metric on $X'$. One verifies however that it is a pseudo-metric.

(b) In $X'$ introduce an equivalence relation by which two sequences $\{x_n\}$ and $\{y_n\}$ are equivalent if $d'(\{x_n\}; \{y_n\}) = 0$. One verifies that such a relation is symmetric, reflexive and transitive and therefore generates equivalence classes. Define $X^*$ as the set of equivalence classes of all Cauchy sequences of $\{X; d\}$. Any such class, contains only sequences at zero mutual pseudo-distance. For any two such equivalence classes $x^*$ and $y^*$ choose representatives $\{x_n\} \in x^*$ and $\{y_n\} \in y^*$ and set

$$d^*(x^*, y^*) = d'(\{x_n\}; \{y_n\}).$$

One verifies that the definition is independent of the choices of the representatives and that $d^*$ defines a metric in $X^*$. The original metric space $\{X; d\}$ is embedded into $\{X^*; d^*\}$ by identifying elements of $X$ with elements of $X^*$ as constant Cauchy sequences. Such an embedding is an isometry.

(c) The metric space $\{X^*; d^*\}$ is complete. Let $\{x_j^*\}$ be a Cauchy sequence in $\{X^*; d^*\}$ and select a representative $\{x_{j,n}\}$ out of each equivalence class $x_j^*$. By construction any such a representative is a Cauchy sequence in $\{X; d\}$. Therefore for each $j \in \mathbb{N}$, there exists an index $n_j$ such that $d(x_{j,n}, x_{j,n_j}) \le \frac{1}{j}$ for all $n \ge n_j$. By diagonalization select now the sequence $\{x_{j,n_j}\}$ and verify that itself is a Cauchy sequence in $\{X; d\}$. Thus $\{x_{j,n_j}\}$ identifies an equivalence class $x^* \in X^*$. The Cauchy sequence $\{x_n^*\}$ converges to $x^*$ in $\{X^*; d^*\}$. Finally the original metric space $\{X; d\}$, with the indicated embedding, is dense in $\{X^*; d^*\}$.

*Remark 16.1c* While every metric space can be completed, a deeper problem is that of characterizing the elements of the new space and its metric. A typical example is the completion of the rational numbers into the real numbers.

## 16.4c Some Consequences of the Baire Category Theorem

The category theorem is equivalent to the following:

**Proposition 16.2c** *Let* $\{X; d\}$ *be a complete metric space. Then a countable collection* $\{\mathcal{O}_n\}$ *of open dense subsets of* $X$*, has nonempty intersection.*

**16.5**. Let $\{X; d\}$ be a complete metric space. Then every closed, proper subset of $X$ of first category, is nowhere dense.

**16.6**. The countable union of sets of first category is of first category. However the countable union of nowhere dense sets need not be nowhere dense. Give an example.

**16.7**. Let $\{E_n\}$ be a countable collection of closed subsets of a complete metric space $\{X; d\}$, such that $\cup E_n = X$. Then $\cup \overset{\circ}{E}_n$ is dense in $X$.

**16.8**. The rational numbers $\mathbb{Q}$ cannot be expressed as the countable intersection of open intervals.

**16.9**. An infinite dimensional complete metric space $\{X; d\}$ cannot have a countable Hamel basis.

**Proposition 16.3c** *Let* $f : \mathbb{R} \to \mathbb{R}$ *be continuous on a dense subset* $E_o$ *of* $\mathbb{R}$*. Then* $f$ *is continuous on a set* $E$ *of the second category.*

*Proof* For $x \in (0, 1)$

$$f'(x) = \sup_{\varepsilon} \inf_{|x-y|<\varepsilon} f(y), \qquad f''(x) = \inf_{\varepsilon} \sup_{|x-y|<\varepsilon} f(y).$$

For $n \in \mathbb{N}$ set also

$$\mathcal{O}_n = \left\{x \in \mathbb{R} \,\middle|\, f''(x) - f'(x) < \tfrac{1}{n}\right\}.$$

The set of continuity of $f$ is the intersection of the $\mathcal{O}_n$. The sets $\mathcal{O}_n$ are open and dense in $\mathbb{R}$ and their complements $\mathcal{O}_n^c$ are nowhere dense in $\mathbb{R}$. Therefore $\cup\mathcal{O}_n^c$ is of the first category in $\mathbb{R}$ and $\cap\mathcal{O}_n$ is of the second category in $\mathbb{R}$. ∎

**Corollary 16.1c** *There exist no function* $f : [0, 1] \to \mathbb{R}$ *continuous only at the rationals of* $[0, 1]$.

See also the construction in **7.2** of the Complements.

## 17c Compact and Totally Bounded Metric Spaces

**Lemma 17.1c** (The Lebesgue Number Lemma) *Let* $\{X; d\}$ *be a sequentially compact topological space. For every finite open covering* $\{\mathcal{O}_m\}_{m=1}^k$ *for some* $k \in \mathbb{N}$, *there exists a positive number $\sigma$ such that every ball $B_\sigma(y) \subset X$ is contained in some $\mathcal{O}_m$. The number $\sigma$ is called the Lebesgue number of the covering.*

*Proof* If such a $\sigma > 0$ does not exist, there exists a sequence of balls $\{B_{\frac{1}{n}}(x_n)\}_{n\in\mathbb{N}}$, of centers $\{x_n\}$ and radii $\frac{1}{n}$ each not contained in any of the open sets $\mathcal{O}_m$. There exists a subsequence $\{x_{n'}\} \subset \{x_n\}$ and $y \in X$ such that $\{x_{n'}\} \to y$. Since $\{\mathcal{O}_m\}$ is a covering $y \in \mathcal{O}_m$ for some $m$. Since $\mathcal{O}_m$ is open there exists $\sigma_m > 0$ such that $B_{\sigma_m}(y) \subset \mathcal{O}_m$. Then

$$B_{\frac{1}{n'}}(x_{n'}) \subset B_{\sigma_m}(y) \subset \mathcal{O}_m \quad \text{for } n' > \frac{2}{\sigma_m}. \qquad \blacksquare$$

## 17.1c An Application of the Lebesgue Number Lemma

Let $\mathcal{C}_\delta$ be the Cantor set constructed in § 2.1c–§ 2.2c of Chap. 1, and let $J_{n,j}$ be the closed intervals left after the $n$-stage of removal of the middle open intervals $I_{n,j}$. Then for any given open covering $\{I_m\}$ of $\mathcal{C}_\delta$ there exists $n$ sufficiently large such that each $J_{n,j}$ is contained in some $\mathcal{I}_m$.

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