Preface to the Second Edition

This is a revised and expanded version of my 2002 book on real analysis. Some topics and chapters have been rewritten (i.e., Chaps. 7–10) and others have been expanded in several directions by including new topics and, most importantly, considerably more practice problems. Noteworthy is the collection of problems in calculus with distributions at the end of Chap. 8. These exercises show how to solve algebraic equations and differential equations in the sense of distributions, and how to compute limits and series in $\mathcal{D}'$. Distributional calculations in most texts are limited to computing the fundamental solution of some linear partial differential equations. We have sought to give an array of problems to show the wide applicability of calculus in $\mathcal{D}'$. I must thank U. Gianazza and V. Vespri for providing me with most of these problems, taken from their own class notes. Chapter 9 has been expanded to include a proof of the Riesz convolution rearrangement inequality in N-dimensions. This is preceded by the topics on Steiner symmetrization as a supporting background. Chapter 11 is new, and it goes more deeply in the local fine properties of weakly differentiable functions by using the notion of p-capacity of sets in $\mathbb{R}^N$. It clarifies various aspects of Sobolev embedding by means of the isoperimetric inequality and the co-area formula (for smooth functions). It also links to measure theory in Chaps. 3 and 4, as the p-capacity separates the role of measures versus outer measure. In particular, while Borel sets are p-capacitable, Borel sets of positive and finite capacity are not measurable with respect to the measure generated by the outer measure of p-capacity. Thus, it also provides an example of nonmetric outer measures and non-Borel measure. As it stands, this book provides a background to more specialized fields of analysis, such as probability, harmonic analysis, functions of bounded variation in several dimensions, partial differential equations, and functional analysis. A brief connection to BV functions in several variables is offered in Sect. 7.2c of the Complements of Chap. 5.

The numbering of the sections of the Problems and Complements of each chapter follow the numbering of the section in that chapter. Exceptions are Chaps. 6 and 8. Most of the Problems and Complements of Chap. 8 are devoted to calculus with distributions, not directly related to the sections of that chapter.

Sections 20c–23c of the Complements of Chap. 6 are devoted to present the Vitali–Saks–Hahn theorem. The relevance of the theorem is in that it gives sufficient conditions on a set of integrable functions to be *uniformly integrable*. This in turn it permits one to connect the notions of weak and strong convergence to convergence in measure. In particular, as a consequence it gives necessary and sufficient conditions for a weakly convergent sequence in L^1 to be strongly convergent in L^1. As an application, in ℓ_1, weak and strong convergence coincide (Sects. 22c–23c of Chap. 6).

Over the years, I have benefited from comments and suggestions from several collaborators and colleagues including U. Gianazza, V. Vespri, U. Abdullah, Olivier Guibé, A. Devinatz†, J. Serrin†, J. Manfredi, and several current and former students, including Naian Liao, Colin Klaus Stockdale, Jordan Nikkel, and Zach Gaslowitz Special thanks go to Ugo Gianazza and Olivier Guibé for having read in detail large parts of the manuscript and for pointing out imprecise statements and providing valuable suggestions. To all of them goes my deep gratitude.

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Preface to the First Edition

This book is a self-contained introduction to real analysis assuming only basic notions on limits of sequences in $\mathbb{R}^N$, manipulations of series, their convergence criteria, advanced differential calculus, and basic algebra of sets.

The passage from the setting in $\mathbb{R}^N$ to abstract spaces and their topologies is gradual. Continuous reference is made to the $\mathbb{R}^N$ setting where most of the basic concepts originated.

The first eight chapters contain material forming the backbone of a basic training in real analysis. The remaining three chapters are more topical, relating to maximal functions, functions of bounded mean oscillation, rearrangements, potential theory and the theory of Sobolev functions. Even though the layout of the book is theoretical, the entire book and the last chapters in particular have in mind applications of mathematical analysis to models of physical phenomena through partial differential equations.

The preliminaries contain a review of the notions of countable sets and related examples. We introduce some special sets, such as the Cantor set and its variants and examine their structure. These sets will be a reference point for a number of examples and counterexamples in measure theory (Chapter 3) and in the Lebesgue differentiability theory of absolute continuous functions (Chapter 5). This initial Chapter contains a brief collection of the various notions of *ordering*, the Hausdorff maximal principle, Zorn's Lemma, the well-ordering principle, and their fundamental connections.

These facts keep appearing in measure theory (Vitali's construction of a Lebesgue non-measurable set), topological facts (Tychonov's Theorem on the compactness of the product of compact spaces; existence of Hamel bases) and functional analysis (Hahn-Banach Theorem; existence of maximal orthonormal bases in Hilbert spaces).

Chapter 2 is an introduction to those basic topological issues that hinge upon analysis or that are, one way or another, intertwined with it. Examples include Uhryson's Lemma and the Tietze Extension Theorem, characterization of compactness and its relation to the Bolzano-Weierstrass property, structure of the compact sets in $\mathbb{R}^N$, and various properties of semi-continuous functions defined on compact sets. This analysis of compactness has in mind the structure of the compact subsets of the space of continuous functions (Chapter 5) and the characterizations of the compact subsets of the spaces $L^p(E)$ for all $1 \le p < \infty$ (Chapter 6).

The Tychonov Theorem is proved with its application in mind in the proof of the Alaoglu Theorem on the weak* compactness of closed balls in a linear, normed space.

We introduce the notion of linear, topological vector spaces and that of linear maps and functionals and their relation to boundedness and continuity.

The discussion turns quickly to metric spaces, their topology, and their structure. Examples are drawn mostly from spaces of continuous or continuously differentiable functions or integrable functions. The notions and characterizations of compactness are rephrased in the context of metric spaces. This is preparatory to characterizing the structure of compact subsets of $L^p(E)$.

The structure of complete metric spaces is analyzed through Baire's Category Theorem. This plays a role in subsequent topics, such as an indirect proof of the existence of nowhere differentiable functions (Chapter 5), in the structure of Banach spaces (Chapter 6), and in questions of completeness and non-completeness of various topologies on $C_o^\infty(E)$ (Chapter 8).

Chapter 3 is a modern account of measure theory. The discussion starts from the structure of open sets in $\mathbb{R}^N$ as sequential coverings to construct measures and a brief introduction to the algebra of sets. Measures are constructed from outer measure by the Charathéodory process. The process is implemented in specific examples such as the Lebesgue-Stiltjes measures in $\mathbb{R}$ and the Hausdorff measure. The latter seldom appears in introductory textbooks in Real Analysis. We have chosen to present it in some detail because it has become, in the past two decades, an essential tool in studying the fine properties of solutions of partial differential equations and systems. The Lebesgue measure in $\mathbb{R}^N$ is introduced directly starting from the Euclidean measure of cubes rather than regarding it, more or less abstractly, as the N-product of the Lebesgue measure on $\mathbb{R}$. In $\mathbb{R}^N$, we distinguish between Borel sets and Lebesgue measurable sets, by cardinality arguments and by concrete counterexamples.

For general measures, emphasis is put on necessary and sufficient criteria of measurability in terms of $\mathcal{G}_\delta$ and $\mathcal{F}_\sigma$. In this, we have in mind the operation of measuring a set as an approximation process. From the applications point of view, one would like to approximate the measure of a set by the measure of measurable sets containing it and measurable sets contained into it. The notion is further expanded in the theory of Radon measures and their regularity properties.

It is also further expanded into the covering theorems, even though these represent an independent topic in their own right. The Vitali Covering Theorem is presented by the proof due to Banach. The Besicovitch covering is presented by emphasizing its value for general Radon measures in $\mathbb{R}^N$. For both, we stress the measure-theoretical nature of the covering as opposed to the notion of covering a set by inclusion.

Coverings have made possible an understanding of the local properties of solutions of partial differential equations, chiefly the Harnack inequality for non-negative solutions of elliptic and parabolic equations. For this reason, in the Complements of this chapter, we have included various versions of the Vitali and Besicovitch covering theorems.

Chapter 4 introduces the Lebesgue integral. The theory is preceded by the notions of measurable functions, convergence in measure, Egorov's Theorem on selecting almost everywhere convergent subsequences from sequences convergent in measure, and Lusin's Theorem characterizing measurability in terms of quasi-continuity. This theorem is given relevance as it relates measurability and local behavior of measurable functions. It is also a concrete application of the necessary and sufficient criteria of measurability of the previous chapter.

The integral is constructed starting from non-negative simple functions by the Lebesgue procedure. Emphasis is placed on convergence theorems and the Vitali's Theorem on the absolute continuity of the integral. The Peano-Jordan and Riemann integrals are compared to the Lebesgue integral by pointing out differences and analogies.

The theory of product measures and the related integral is developed in the framework of the Charathéodory construction by starting from measurable rectangles. This construction provides a natural setting for the Fubini-Tonelli Theorem on multiple integrals.

Applications are provided ranging from the notion of convolution, the convergence of the Marcinkiewicz integral, to the interpretation of an integral in terms of the distribution function of its integrand.

The theory of measures is completed in this chapter by introducing the notion of signed measure and by proving Hahn's Decomposition Theorem. This leads to other natural notions of decompositions such as the Jordan and Lebesgue Decomposition Theorems. It also suggests naturally other notions of comparing two measures, such as the absolute continuity of a measure ν with respect to another measure μ. It also suggests representing ν, roughly speaking, as the integral of μ by the Radon-Nykodým Theorem.

Relating two measures finds application in the Besicovitch-Lebesgue Theorem, presented in the next chapter, and connecting integrability of a function to some of its local properties.

Chapter 5 is a collection of applications of measure theory to issues that are at the root of modern analysis. What does it mean for a function of one real variable to be differentiable? When can one compute an integral by the Fundamental Theorem of Calculus? What does it mean to take the derivative on an integral? These issues motivated a new way of measuring sets and the need for a new notion of integral.

The discussion starts from functions of bounded variation in an interval and their Jordan's characterization as the difference of two monotone functions. The notion of differentiability follows naturally from the definition of the four Dini's numbers. For a function of bounded variation, its Dini numbers, regarded as functions, are measurable. This is a remarkable fact due to Banach.

Functions of bounded variations are almost everywhere differentiable. This is a celebrated theorem of Lebesgue. It uses, in an essential way, Vitali's Covering Theorem of Chapter 3.

We introduce the notion of absolutely continuous functions and discuss similarities and differences with respect to functions of bounded variation. The Lebesgue theory of differentiating an integral is developed in this context. A natural related issue is that of the density of a Lebesgue measurable subset of an interval. Almost every point of a measurable set is a density point for that set. The proof uses a remarkable theorem of Fubini on differentiating term by term a series of monotone functions.

Similar issues for functions of N real variables are far more delicate. We present the theory of differentiating a measure ν with respect to another μ by identifying precisely such a derivative in terms of the singular part and the absolutely continuous part of μ with respect to ν. The various decompositions of measures of Chapter 4 find here their natural application, along with the Radon-Nykodým Theorem.

The pivotal point of the theory is the Besicovitch-Lebesgue Theorem asserting that the limit of the integral of a measurable function f when the domain of integration shrinks to a point x actually exists for almost all x and equals the value of f at x. The shrinking procedure is achieved by using balls centered at x, and the measure can be any Radon measure. This is the strength of the Besicovitch covering theorem. We discuss the possibility of replacing balls with domains that are, roughly speaking, comparable to a ball. As a consequence, almost every point of an N-dimensional Lebesgue-measurable set is a density point for that set.

The final part of the chapter contains an array of facts of common use in real analysis. These include basic facts on convex functions of one variable and their almost everywhere double differentiability. A similar fact for convex functions of several real variables (known as the Alexandrov Theorem) is beyond the scope of these notes. In the Complements, we introduce the Legendre transform and indicate the main properties and features.

We present the Ascoli-Arzelá Theorem, keeping in mind a description of compact subsets of spaces of continuous functions.

We also include a theorem of Kirzbraun and Pucci extending bounded, continuous functions in a domain into bounded, continuous functions in the whole $\mathbb{R}^N$ with the *same* upper bound and the *same* concave modulus of continuity. This theorem does not seem to be widely known.

The final part of the chapter contains a detailed discussion of the Stone-Weierstrass Theorem. We present first the Weierstrass Theorem (in N dimensions) as a pure fact of Approximation Theory. The polynomials approximating a continuous function f in the sup-norm over a compact set are constructed explicitly by means of the Bernstein polynomials. The Stone Theorem is then presented as a way of identifying the structure of a class of functions that can be approximated by polynomials.

Chapter 6 introduces the theory of L^p spaces for $1 \le p \le \infty$. The basic inequalities of Hölder and Minkowski are introduced and used to characterize the norm and the related topology of these spaces. A discussion is provided to identify elements of $L^p(E)$ as equivalence classes.

We introduce also the $L^p(E)$ spaces for $0 < p < 1$ and the related topology. We establish that there are not convex open sets except $L^p(E)$ itself and the empty set.

We then turn to questions of convergence in the sense of $L^p(E)$ and their completeness (Riesz-Fisher Theorem) as well as issues of separating such spaces by simple functions. The latter serves as a tool in the notion of weak convergence of sequences of functions in $L^p(E)$. Strong and weak convergence are compared and basic facts relating weak convergence and convergence of norms are stated and proved.

The Complements contain an extensive discussion comparing the various notions of convergence.

We introduce the notion of functional in $L^p(E)$ and its boundedness and continuity and prove the Riesz representation Theorem, characterizing the form of all the bounded linear functionals in $L^p(E)$ for $1 \le p \le \infty$. This proof is based on the Radon-Nykodým Theorem and as such is measure theoretical in nature.

We present a second proof of the same theorem based on the topology of L^p. The open balls that generate the topology of $L^p(E)$ are *strictly* convex for $1 < p < \infty$. This fact is proved by means of the Hanner and Clarkson's inequality, which while technical, are of interest in their own right.

The Riesz Representation Theorem permits one to prove that if E is a Lebesgue-measurable set in $\mathbb{R}^N$, then $L^p(E)$ for $1 \le p < \infty$, are separable. It also permits one to select weakly convergent subsequences from bounded ones. This fact holds in general, reflexive, separable Banach spaces (Chapter 7). We have chosen to present it independently as part of the L^p theory. It is our point of view that a good part of functional analysis draws some of its key facts from concrete spaces, such as spaces of continuous functions, the L^p, space and the spaces ℓ_p.

The remainder of the chapter presents some technical tools regarding $L^p(E)$ for E, a Lebesgue-measurable set in $\mathbb{R}^N$, to be used in various parts of the later chapters. These include the continuity of the translation in the topology of $L^p(E)$, the Friedrichs mollifyiers, and the approximation of functions in $L^p(E)$ with $C^\infty(E)$ functions. It includes also a characterization of the compact subsets in $L^p(E)$.

Chapter 7 is an introduction to those aspects of functional analysis closely related to the Euclidean spaces $\mathbb{R}^N$, the spaces of continuous functions defined on some open set $E \subset \mathbb{R}^N$, and the spaces $L^p(E)$. These naturally suggest the notion of finite dimensional and infinite dimensional normed spaces. The difference between the two is best characterized in terms of the compactness of their closed unit ball. This is a consequence of a beautiful counterexample of Riesz.

The notions of maps and functionals is rephrased in terms of the norm topology. In $\mathbb{R}^N$, one thinks of a linear functional as an affine functions whose level sets are hyperplanes through the origin. Much of this analogy holds in general normed spaces with the proper rephrasing.

Families of pointwise equi-bounded maps are proven to be uniformly equi-bounded as an application of Baire's Category Theorem.

We also briefly consider special maps such as those generated by Riesz potential (estimates of these potentials are provided in Chapter 9), and related Fredholm integral equations.

A proof of the classical Open Mapping Theorem and Closed Graph Theorem are presented as a way of inverting continuous maps to identify isomorphisms out of continuous linear maps.

The Hahn-Banach Theorem is viewed in its geometrical aspects of separating closed convex sets in a normed space and of "drawing" tangent planes to a convex set.

These facts all play a role in the notion of weak topology and its properties. Mazur's Theorem on weak and strong closure of convex sets in a normed space is related to the weak topology of the $L^p(E)$ spaces. These provide the main examples, as convexity is explicit through Clarkson's inequalities.

The last part of the chapter gives an introduction to Hilbert spaces and their geometrical aspects through the parallelogram identity. We present the Riesz Representation Theorem of functionals through the inner product. The notion of basis is introduced and its cardinality is related to the separability of a Hilbert space. We introduce orthonormal systems and indicate the main properties (Bessel's inequality) and some construction procedures (Gram-Schmidt). The existence of a complete system is a consequence of the Hausdorff maximum principle. We also discuss various equivalent notions of completeness.

Chapter 8 is about spaces of real-valued continuous functions, differentiable functions, infinitely differentiable functions with compact support in some open set $E \subset \mathbb{R}^N$, and weakly differentiable functions.

Together with the $L^p(E)$ spaces, these are among the backbone spaces of real analysis.

We prove the Riesz Representation Theorem for continuous functions of compact support in $\mathbb{R}^N$. The discussion starts from positive functionals and their representation. Radon measures are related to positive functionals and bounded, signed Radon measures are related to bounded linear functionals. Analogous facts hold for the space of continuous functions with compact support in some open set $E \subset \mathbb{R}^N$.

We then turn to making precise the notion of a topology for $C_o^\infty(E)$. Completeness and non-completeness are related to metric topologies in a constructive way. We introduce the Schwartz topology and the notion of continuous maps and functionals with respect to such a topology. This leads to the theory of distributions and its related calculus (derivatives, convolutions etc. of distributions).

Their relation to partial differential equations is indicated through the notion of fundamental solution. We compute the fundamental solution for the Laplace operator also in view of its applications to potential theory (Chapter 9) and to Sobolev inequalities (Chapter 10).

The notion of weak derivative in some open set $E \subset \mathbb{R}^N$ is introduced as an aspect of the theory distributions. We outline their main properties and state and prove the by now classical Meyers-Serrin Theorem. Extension theorems and approximation by smooth functions defined in domains larger than E are provided. This leads naturally to a discussion of the smoothness properties of ∂E for these approximations and/or extensions to take place (cone property, segment property, etc.).

We present some calculus aspects of weak derivatives (chair rule, approximations by difference quotients, etc.) and turn to a discussion of $W^{1,\infty}(E)$ and its relation to Lipschitz functions. For the latter, we conclude the chapter by stating and proving the Rademaker Theorem.

Chapter 9 is a collection of topics of common use in real analysis and its applications. First is the Wiener version of the Vitali Covering Theorem (commonly referred to as the "simple version" of Vitali's Theorem). This is applied to the notion of maximal function, its properties, and its related strong type L^p estimates for $1 < p < \infty$. Weak estimates are also proved and used in the Marcinkiewicz Interpolation Theorem. We prove the by now classical Calderón-Zygmund Decomposition Theorem and its applications to the space functions of bounded mean oscillation (BMO) and the Stein-Fefferman L^p estimate for the sharp maximal function.

The space of BMO is given some emphasis. We give the proof of the John-Nirenberg estimate and provide its counterexample. We have in mind here the limiting case of some potential estimates (later in the chapter) and the limiting Sobolev embedding estimates (Chapter 10).

We introduce the notion of rearranging the values of functions and provide their properties and the related notion of equi-measurable function. The discussion is for functions of one real variable. Extensions to functions of N real variables are indicated in the Complements.

The goal is to prove the Riesz convolution inequality by rearrangements. The several proofs existing (Riesz, Zygmund, Hardy-Littlewood-Polya) all use, one way or another, the symmetric rearrangement of an integrable function.

We have reproduced here the proof of Hardy-Littlewood-Polya as appearing in their monograph [70]. In the process, we need to establish Hardy's inequality, of interest in its own right.

The Riesz convolution inequality is presented in several of its variants, leading to an N-dimensional version of it through an application of the continuous version of the Minkowski inequality.

Besides its intrinsic interest of these inequalities, what we have in mind here is to recover some limiting cases of potential estimates an their related Sobolev embedding inequalities.

The final part of the chapter introduces the Riesz potentials and their related L^p estimates, including some limiting cases. These are on one hand based on the previous Riesz convolution inequality, and on the other hand to Trudinger's version of the BMO estimates for particular functions arising as potentials.

Chapter 10 provides an array of embedding theorems for functions in Sobolev spaces. Their importance to analysis and partial differential equations cannot be underscored. Although good monographs exist ([1, 104]), I have found it laborious to extract the main facts, listed in a clean manner and ready for applications.

We start from the classical Gagliardo-Nirenberg inequalitites and proceed to Sobolev inequalities. We have made an effort to trace, in the various embedding inequalities, how the smoothness of the boundary enters in the estimates. For example, whenever the cone condition is required, we trace back in the various constant the dependence on the height and the angle of the cone. We present the Poincaré inequalities for bounded, convex domains E, and trace the dependence of the various constants on the "modulus of convexity" of the domain through the ratio of the radius of the smallest ball containing E and the largest ball contained in E. The limiting case $p = N$ of the Sobolev inequality builds of the limiting inequalities for the Riesz potentials, and is preceded by an introduction to Morrey spaces and their connection to BMO.

The characterization of the compact subsets of $L^p(E)$ (Chapter 6) is used to prove Reillich's Theorem on compact Sobolev inequalities.

We introduce the notion of trace of function in $W^{1,p}(\mathbb{R}^N \times \mathbb{R}^+)$ on the hyperplane $x_{N+1} = 0$. Through a partition of unity and a local covering, this provides the notion of trace of functions in $W^{1,p}(E)$ on the boundary ∂E, provided such a boundary is sufficiently smooth. Sharp inequalities relating functions in $W^{1,p}(E)$ with the integrability and regularity of their traces on ∂E are established in terms of fractional Sobolev spaces. Such inequalities are first established for E being a half-space and ∂E an hyperplane, and then extended to general domains E with sufficiently smooth boundary ∂E. In the Complements we characterize functions f defined and integrable on ∂E as traces on ∂E of functions in some Sobolev spaces $W^{1,p}(E)$. The relation between p and the order of integrability of f on ∂E is shown to be sharp. For special geometries, such as a ball, the inequality relating the integral of the traces and the Sobolev norm can be made explicit. This is indicated in the Complements.

The last part of the chapter contains a newly established *multiplicative* Sobolev embedding for functions in $W^{1,p}(E)$ that do not necessarily vanish on ∂E. The open set E is required to be convex. Its value is in its applicability to the asymptotic behavior of solutions to Neumann problems related to parabolic partial differential equations.

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