Preface

What's special about this book? In writing it we had four goals in mind:

1. To guide students through the theory of ordinary differential equations (ODEs) from an introductory level up through fairly advanced material at the graduate level, including bifurcation theory but not chaos, in a one-semester course;

2. to support the theory with meaningful examples of ODEs drawn from diverse other fields;

3. to enable students to read it profitably on their own,[1] outside of a course, with no more frustration than is inherent in learning any mathematics;

4. to make the book *enjoyable* to read.[2]

★ Regarding our first goal, you do not need any previous exposure to ODEs to read this book. The only prerequisites are some experience in the analytical foundations of calculus and a course in linear algebra. Moreover, you need not be completely comfortable with either of these subjects—we believe that applying them to ODEs, a task that demonstrates their utility, will help you to master them.[3]

While we succeeded in making the transition from introductory theory to advanced topics, we failed spectacularly in writing a book that can be covered in a one-semester course—we couldn't shut off the spigot. (In the "Notes to the Professor" below, we offer guidance about what topics may be cut to fit a one-semester course.)

★ Regarding our second goal, we apply the theory of ODEs to help the student understand equations from the following fields.

[1] Professors, please take note: we believe that because of this feature the book can be comfortably used in an *inverted classroom*. (See "Notes for the Professor" below.)

[2] Hey, we had fun writing it . . . mostly.

[3] And to facilitate your task, in Appendices B and C we review the information from analysis and linear algebra that we actually use.

- Biology: interacting-species population models, activator–inhibitor systems including a toy model for the Turing instability, the chemostat, Sel'kov's model for glycolysis, simple models for biological switches and clocks, some neurological models including the Morris–Lecar equations.
- Physics: various mechanical systems including systems based on the pendulum, Duffing's equation, the Lorenz equations.
- Chemistry: the Michaelis–Menten reaction rate for enzymatic reactions.
- Chemical engineering: the continuous stirred tank reactor.
- Electrical engineering: van der Pol's equation.

A knowledge of high-school science is sufficient preparation for these applications. Indeed, we believe that math students can learn much of the science through ODEs, transferring intuition from one application to another.[4]

★ Regarding our third goal: We draw on our years of teaching courses on this material to try to anticipate as many confusions of the beginning student as possible.[5] We supplement many of the exercises with long discussions that explain their significance. We include in the written text the kind of parenthetical remarks that typically are part of the classroom experience. We distinguish between dull tasks that really are necessary (yes, learning the subject does require some of these) and those that can be omitted or at least postponed.[6] This readability comes at some cost: it makes the book more wordy than the usual mathematician's minimalist style, and this fact might put off some readers.

★ Regarding our fourth goal, we write in an informal style, as though we were speaking to a student in the room with us. If being understandable and being rigorous are in conflict, we lean toward the former. We strive to give counterexamples that

[4] Please forgive us a folksy story in support of this point. A graduate student in biology was explaining his research to one of us. The student began by asking what he thought was a rhetorical question: "Should I give you the equations, or should I explain the biology?" He had already launched into the latter and was astounded when we answered, "Just give us the equations, we'll make up our own story to go along with it." And after having seen and discussed the equations, we found that our story was pretty much on target.

[5] Dear Reader, please let us know at

`http://www.math.duke.edu/ode-book/contact-us`

where we have failed in this attempt. We'll put a better explanation on our web site and, so twentieth-century, fix it in the second edition.

[6] We use the euphemism "a task for the dedicated reader" to describe the latter.

are as dramatic as possible.[7] We make connections to interesting scientific problems related to ODEs, such as the surprising fact that a pendulum can be stabilized in an inverted, straight-up, position by rapidly vibrating its pivot point. We even try for the occasional joke.[8] For example, we call the more advanced problems “PHD exercises,” not because they relate in any way to the doctorate, but as an acronym for “piled higher and deeper.” (If you are not familiar with the mildly vulgar misinterpretation of the usual sequence of degrees, B.S., M.S., Ph.D., from which this phrase derives, ask an older colleague.) Similarly, we call the notes at the end of the chapters “Pearls of wisdom.” (Trust us, we don’t *really* take ourselves that seriously.)

Here are some other pedagogical features of the book:

- A dedicated website

  `http://www.math.duke.edu/ode-book`

  containing software templates for solving many ODEs in this book, some problem solutions, supplementary readings, a list of known errors, and a link to contact us.

- Very detailed references within this book and to other sources, including specific sections or even specific pages, which can be easily followed.

- Consistent color conventions in figures[9] for nullclines, periodic orbits, stable and unstable manifolds, which enhance their value greatly.

- Great attention to consistent notation, including a detailed overview of our conventions in Appendix A.[10]

- An extensive collection of exercises, some of which we are vain enough to regard as creative, along with explanations of why they matter.

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[7]For instance, the absolute value function is typically cited as an example of a Lipschitz continuous function on the line that is not differentiable (at the one point $x = 0$). However, in Section 3.6.1 we construct an example of a Lipschitz continuous function that is *not differentiable on any open interval in* $\mathbb{R}$. Besides being more interesting, the construction exercises the “analysis muscle.”

[8]Attempted humor nipped in the bud: the publishers vetoed the title “Come romp with Dave and John across ODE land.”

[9]If you are reading a black-and-white version of this book, color figures are available on the web site.

[10]We recommend that readers skim this appendix before starting the book and review it from time to time.

- An epilogue (Chapter 10) that gives an overview and references to ODE topics that continue the work begun in this book.

This book contains a great deal of information about properties of specific ODE models drawn from applications. Typically, after introducing an analytical technique, we illustrate it on one or more particular ODEs. The same equation may be used repeatedly in support of different techniques. For example, regarding activator–inhibitor systems, global existence is derived in Chapter 4 for such equations, scaling is used in Chapter 5 to reduce the equations to the simplest possible form, the stabilities of equilibria are determined in Chapter 6, the bifurcation of periodic solutions is analyzed in Chapter 8, and global bifurcation of solutions is considered in Chapter 9. This style of organization makes the theory less dry, but it also creates a problem: *how to learn what's known about any specific equation without paging through the whole book.* Well, here's a workaround: in the index under each specific ODE we list the pages where that equation is studied, and you can follow these references to focus on the behavior of a particular equation.

Notes for the Student

We are delighted that you are joining us for a journey through the study of ODEs, a subject that your authors love dearly. Part of the allure of ODEs is the "something for everyone" aspect of the field—whether you are theory-oriented or application-oriented, whether you prefer geometric reasoning or analytical reasoning, and whether you prefer computer-based or pen-and-paper-based calculations, your preferences will have a useful role in understanding ODEs.

At the same time, we urge you not to neglect the techniques you may find less congenial. You will achieve a deeper, more flexible, more portable understanding of the subject if you can synthesize theory, application, and numerics/asymptotics. Building on your current strengths, you can use this subject to enlarge your mathematical tool kit.

Let's talk about exercises, beginning with an inconvenient truth: *you don't understand the material in a chapter until you can do some of the exercises!* To help you derive maximal benefit from the exercises, we often include generous commentary on the significance of the problem. We tried to hold routine "apply the definition" exercises to a minimum. This makes the problems that remain more interesting, but it also means that lengthy hints may be needed (and are given).

In case you feel discouraged by the many pages of exercises following each chapter, let us point out that besides the obvious fact that you needn't do every problem, the problems are actually a lot shorter than they may appear—only a fraction of the exercise (in normal-sized type) actually asks you to do anything. Much of the physical space on the page is devoted to hints and commentary (in smaller type).

To help you focus your efforts, the exercises are divided into subsections as follows:

- core exercises,
- a group of exercises focused on a common theme,
- PHD exercises.

The core exercises give you a chance to practice using the basic ideas in the chapter; their specific purposes are listed at the start of the subsection. Focused subsections contain several related, not-to-be-neglected, exercises that may probe an area more thoroughly or may introduce ideas to be studied later. PHD exercises may be more difficult or develop some related topic not studied elsewhere in the text. (We won't refer to PHD exercises later in the text, except in other PHD exercises or in the Pearls, so these may be skipped without bad consequences.)

Besides the explicit exercises, every statement in the text is a sort of implied exercise—you should be able to prove these. (Sometimes we include a specific parenthetical remark like *"Prove this!"* or *"Why?"* at places where we think a reader might be tempted to move on without sufficient reflection, to the detriment of understanding.)

Notes for the Professor

This book is suited to a variety of courses. Because of its length, the most obvious fit is a two-semester graduate course covering the entire book, along with a supplemental foray into chaos.[11] However, to accommodate other options, large portions of the text could be cut without significant loss of continuity, such as:

→ Some longer proofs, like Sections 4.6.4–4.6.7 and 8.5.2.
→ The longer applications, like Sections 6.4, 8.6, and 9.8.
→ Perturbation-theory calculations, like Sections 7.5–7.7.
→ All appendices.
→ The discussion of scaling (Chapter 5).
→ The epilogue (Chapter 10).

[11] Appropriate chaos references are given in Section 10.6.

For additional savings, one could:

→ Assign Section 1.6 as independent reading.[12]

→ Cover fewer examples (of trapping regions in Sections 4.3 and 4.4 and of bifurcation in Chapters 8 and 9).

To make this quantitative: the book contains just over 300 pages of regular text.[13] What's left after the above cuts—a little over 150 pages of material[14]—can be covered in a reasonably paced one-semester course, perhaps with time left over for end-of-term projects or to reinstate some of the cut topics.

The book is also suited to a variety of students, not just the obvious audience of math graduate students. We believe that the level of exposition will be challenging, but still accessible, for strong upper-level undergrads in math,[15] all the more so if the primary focus is restricted to either theory or applications. The wealth of applications in the book should make it appealing to theoretically inclined graduate students in science or engineering.

In some semesters we used the book in an *inverted* classroom in which students were assigned reading before class and class time was freed up for more active learning. We liked the result.

To conclude, if you are weighing adoption of this text for a course at your institution, feel free to contact us (via the website) with questions regarding how, or whether, our book might serve the needs of your students.

Acknowledgments

We are greatly indebted to Tom Beale, Jim Nolen, and Steve Schecter, who tested early drafts of this book in their ODE courses. Please know that their careful reading and attention to detail has shielded you, dear reader, from *many* errors (not

[12]This section illustrates what the qualitative theory of ODEs, the focus of Chapters 6–9 of this book, can say about one specific ODE model. While students may benefit from a quick look at things to come, detailed understanding of the phenomena in this section will not be needed until Chapter 6.

[13]Plus roughly 65 pages of appendices, 35 pages of Pearls, and 110 pages of exercises.

[14]Moreover, the first four items on the cut list contain some of the densest writing in the book, so omitting them lightens the text more than a mere page count indicates.

[15]We are inordinately proud of our treatment of bifurcation theory, which leads us to make the following suggestion: a seminar, to follow a first course in ODEs, in bifurcation phenomena based on Chapters 8 and 9.

to mention patches of clunky exposition) that your authors may not have caught otherwise.

We are grateful to John Guckenheimer, whose feedback helped reshape our thinking about the book, and to Marty Golubitsky and Tom Witelski, who were most helpful consultants on multiple occasions. Thanks also to Michael Peper, the math librarian at Duke, for tracking down various obscure references we sent his way.

We are privileged to be acquainted with two talented artists who granted us permission to print/reprint several of their works. Jeff Poe sketched the frontispiece as well as three of the figures in Chapter 8. Fiona Ross allowed us to reprint her intricate example of a Jordan curve, appearing in Appendix B.

Bard Ermentrout facilitated creation of this book in ways he may not realize. His freely available XPPAUT software frequently saved the day when we needed to plot stable/unstable manifolds or bifurcation diagrams. Without XPPAUT, we shudder to contemplate how many hours would have been squandered writing our own (highly inferior) computer code to perform such computations.

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