Chapter 2
Basis of the Method of Polynomial Approximation

2.1 Extension Sets of Observations: The Heuristic Path for Nonlinear Estimation

The estimation algorithm outlined in Chap. 1 can be constructively implemented if some a priori data are known. However, the algorithm does not fully use information from observations, since its operations are linear over the results of observations. Is it possible to increase the accuracy of the estimate, to use more complex, nonlinear operations? A multidimensional version of K. Veyershtrassa's theorem answers this question affirmatively.

We believe that the real functions of many variables $\theta(Y_N)$, which are further defined as approximation representations, are continuous in the closed bounded domain Ω_{Y_N} of the multidimensional space. Fulfillment of this condition [1] allows the use of a multidimensional analog of Weierstrass's theorem (Stone's corollary theorems). The theorem states that for every ε, the following holds:

$$\sup_{Y_N \in \Omega_{Y_N}} |P(Y_N, \varepsilon) - E(\theta|Y_N)| \leq \varepsilon, \tag{1.1}$$

where Ω_N is compact, $E(\theta|Y_N)$ is a continuous function of N components of Y_N, and $P(Y_N, \varepsilon)$ is a polynomial from Y_N (linear combination of powers of the components of Y_N). If this condition is fulfilled, the approximation error ε tends to zero with increasing dimension of the vector Y_N.

Let's define the set $\Omega_{W_{Y_N}}$ polynomial observing $W(Y_N)$, which depends on the primary vector of observations Y_N. The input estimation algorithm is not supposed to be the vectors Y_N and $W(Y_N)$. Inequality (1.1) makes it natural to assign that the degree of the components of Y_N are components of the vectors $W(Y_N)$. The sum of the exponents of all the degrees does not exceed a given integer d. $\Omega_{W_{Y_N}}$ contains a set of Ω_{Y_N} original observations:

$$Y \in \Omega_Y, \Omega_Y \in R^N \in R^N, W \in \Omega_W \in R^{N_1}, N_1 > N. \tag{1.2}$$

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We will estimate the vector $\theta(W(Y_N))$ via a formula similar to (1.3) of Chap. 1:

$$\hat{\theta}(W(Y_N)) = E(\theta(W(Y_N))) + \Lambda^o(W(Y_N) - E(W(Y_N))), \tag{1.3}$$

where

$$\Lambda^o Q = L.$$

Algorithm (1.3) defines a vector of estimates, the optimal mean-square and linear on the set of degrees of the components of Y_N. This vector is a polynomial of the components of Y_N and the corresponding estimation of the errors' mean cannot have more errors; that delivers a suboptimal polynomial $P(Y_N, \varepsilon)$ of (1.1).

This statement is true because algorithm (1.3) defines the optimal mean-square evaluation on the set of linear combinations of the components of the degrees Y_N and $P(Y_N, 2_N)$ (linear combination of powers of the components of Y_N). However, the linear combinations $P(Y_N, \varepsilon)$ are not optimal in the mean-square and true matrix inequality C (1.3) $\leq$ C (1.1), where C (1.3) and C (1.1) are covariance matrices of the estimation errors, corresponding to the expressions (1.1) and (1.3) for the estimation methods.

The set of elements $W(Y_N)$ expands with increasing d; some of its elements are degrees of component Y_N in the polynomial $P(Y_N, \varepsilon)$. In this case, the estimation error, which corresponds to algorithm (1.3), is at least not greater than the ε in formula (1.1).

Next, Sect. 2.3 represents data on the construction of the vector $W(Y_N)$. The arrangement is such that with an increase in d, estimation errors are reduced and do not exceed ε in (1.1).

A priori data for formula (1.3) are the vector and matrix $E(\theta)$, $E(W(Y_N))$, C_0, Q, L numerically defined in Chap. 1, by replacing the symbol Y_N on the symbol $W(Y_N)$. The elements of the vectors $W(Y_N)$ linearly depend on degrees of the observations; therefore, formula (1.3) corresponds to a nonlinear algorithmic process. The estimation error we obtain when using an extended set of observations will always be less than many original observations Ω_Y.

In Chap. 1, the formula was determined by calculating the estimation error covariance matrix obtained with this nonlinear algorithm and finding the best of their reduction of the nonlinear terms in the function WY_N.

Membership of the vector $W(Y_N)$'s degrees and the plurality of a sufficiently large value of d in principle ensure the achievement of arbitrarily small mean-value estimation errors.

2.2 The Statistical Basis

We assume that parameter vector θ has components $\theta_1, \ldots, \theta_q$ and at fixed vector Y_N belongs to a region $\Omega_{\theta|Y_N} \in R^q$. This region can have a finite or infinite number of points. The latter will be the case, for example, if $\theta = F(Y_N, \xi)$, where ξ is an independent variable that varies in a region.

If the vector Y_N spans the region of Ω_{Y_N} points, then the vector θ spans points of some region Ω_θ.

We suppose that Y_N, θ are random vectors on region $\Omega_{Y_N} \asymp \Omega_{\theta_N|Y} \in R^{N+q}$ and that their joint stochastic measure is

$$p(\theta, Y_N) = p(Y)p(\theta|Y_N),$$

where function $p(\theta|Y_N)$ is the conditional density of probabilities of the random vector θ at the fixed vector Y_N. If the set $\Omega_{\theta|Y_N}$ is composed of points $\theta_1(Y_N), \ldots, \theta_r(Y_N)$, then

$$p(\theta|Y_N) = (\delta(\theta - \theta_1(Y_N)) + \cdots + \delta(\theta - \theta_r(Y_N)))/r,$$

where $\delta(\ldots)$ is a delta function of θ variables. The vector of conditional expectation $E(\theta|Y_N)$ is represented by

$$E(\theta|Y_N) = \int_{\theta \in \Omega_{\theta|Y_N}} \theta p(\theta|Y_N) d\theta. \tag{2.1}$$

Let the vector W be a function of components of the vector $Y_N : W = W(Y_N)$. If the random vector Y_N is fixed, then the algorithm outlined in Chap. 1 delivers an estimator of the vector θ (or a function of this vector) that is linear relative to vector W and optimal in the root-mean-square sense on a class of linear operators.

We will use this algorithm to construct an estimator of the vector $E(\theta|Y_N)$ that will be linear relative to a vector $W(Y_N)$ and optimal in the root-mean-square sense. We can construct an estimator because the joint density of probabilities $p(\theta, Y_N)$ permits us to find the first and second statistical moments of the random vectors $E(\theta|Y_N)$, $W(Y)$ that are necessary for using formulas of Chap. 1. From Eq. (2.1), we will find

$$E(E(\theta|Y_N)) = \int_{Y_N \in \Omega_{Y_N}, \theta \in \Omega_{\theta|Y_N}} \theta p(\theta, Y_N) d\theta dY_N, \tag{2.2}$$

$$E(W(Y_N)) = W(Y_N)p(\theta, Y_N) d\theta dY_n, \tag{2.3}$$

$$L = E((E(\theta|Y_N) - E(E\theta|Y_N)))(W(Y_N) - E(W(Y_N)))^T)$$

$$= \int_{Y_N \in \Omega_{Y_N}, \theta \in \Omega_{\theta|Y_N}} (E(\theta|Y_N) - E(E\theta|Y_N)))(W(Y_N) - E(W(Y_N)))^T p(\theta, Y_N) d\theta dY_N, \tag{2.4}$$

$$Q = E((W - E(W))((W - E(W))^T)$$

$$= \int_{Y \in \Omega_{Y_N}, \theta \in \Omega_{\theta|Y_N}} (W - E(W))(W - E(W))^T p(\theta, Y) d\theta dY. \tag{2.5}$$

We assume that $E(\theta|Y_N)$ is a continuous vector-function Y_N.

Let the components $w_1(Y_N), \ldots, w_m(Y_N)$ of the vector W be the first m components of W. Let's find the optimal in the root-mean-square-sense estimator of the vector $E(\theta|Y_N)$; we'll do it using a linear (relative to W) vector-function of orm (1.3) of Chap. 1. But under this condition, the vector W is a function of Y_N. Hence, additionally, the vector to estimate the conditional expectation—an estimator realized via an optimal in the root-mean-square-sense linear operator over the vector $W(Y_N)$—is denoted as $\hat{E}_{\theta|Y}(Y, m)^o$ and defined by

$$\hat{E}_{\theta|Y_N}(Y_N, m)^o = E(E(\theta|Y_N)) + \Lambda^o(W(Y_N) - E(W(Y_N))), \tag{2.6}$$

where

$$\Lambda^o Q = L. \tag{2.7}$$

2.3 Polynomial Approximation

Now, we believe that elements of the basic sequence are products of integer nonnegative power functions of components of the vector of primary observations Y_N:

$$w_{a_1,\ldots,a_N}(Y_N) = y_1^{a_1} \cdots y_N^{a_N}, \tag{3.1}$$

where the nonnegative integers $a_1, \ldots, a_N$ deliver all integer nonnegative solutions of the inequality $0 \le a_1 + \cdots + a_N \le d, d = 1, 2, \ldots$. For $d \to \infty$, we obtain a countable sequence of basic functions. We will notice that in this case, Stone algebra is a space of polynomial N variables, and Stone's theorem serves as a multidimensional analog of Weierstrass's theorem. For given integers d, N, we will denote the number of elements of the basic sequence as $m(d, N)$.

Lemma 4.1 *The value $m(d, N)$ is defined by the recurrent formula*

$$m(d, N) = m(d-1, N) + (1/d!)(N + d - 1) \cdots N, m(1, N) = N.$$

The formula is proved by induction.

With increasing d, the value $m(d, N)$ quickly increases. For example, if $N = 4$, then

d	1	2	3	4	5	6	7	8
$m(d, N)$	4	14	34	69	125	209	329	494.

The vectorial linear combination of basic functions that at fixed integer d delivers—onto regions $\Omega_Y \times \Omega_{\theta|Y}$—an optimal (in the root-mean-square-sense) estimator of the vector $\hat{E}(\theta|Y)$ of the form

$$\hat{E}(\theta|Y_N)(Y_N,d)^o = E(E(\theta|Y_N)) + \Lambda^o(W(Y_N) - E(W(Y_N))), \tag{3.2}$$

$$\Lambda^o Q = L, \tag{3.2*}$$

or

$$\hat{E}(\theta|Y)(Y,d)^o = \sum_{0\le(a_1+\cdots+a_N)\le d} \lambda(a_1,\ldots,a_N) y_1^{a_1}\cdots y_N^{a_N}, \tag{3.3}$$

where $\ldots, \lambda(a1,\ldots,aN),\ldots$ are the vectorial weight coefficients. We will find these coefficients if, to the right of Eq. (3.2), we substitute components of vectors and matrices from relationships (2.2)–(2.5) as well as components of the vector $W(Y)$ from Eq. (3.1), and set as equal the coefficients before identical products of power functions in Eqs. (3.2) and (3.3).

Vectors $\ldots, \lambda(a_1,\ldots,a_N),\ldots$ should be input to the computer. Then formula (3.2) or (3.3) solves the problem of polynomial approximation without solving matrix equation (3.2*) for any vectors $Y \in \Omega_{Y_N}$.

Equation (3.2) or (3.3) solves the problem of polynomial approximation of a vector of conditional expectation for the random vector of unknown parameters with an error, uniformly small on the given region Ω_Y:

$$\sup_{Y_N\in\Omega_{Y_N}} |E(\theta|Y_N) - \sum_{0\le a_1+\cdots+a_N\le d} \lambda(a_1,\ldots,a_N) y_1^{a_1}\cdots y_N^{a_N}| \to 0, d\to\infty.$$

We emphasize that with increasing number d, the polynomial, approximating vectorial series, contains a vector $E(\theta|Y)$ with arbitrary small root-mean-square error on region Ω_Y, despite having used the simple linear operator over polynomial functions of results of primary observations: Simplicity of the operator represented by its linearity, "is compensated" for by nonlinear (polynomial) functions of results of the primary observations, processed by the linear operator.

Then, by $W(d,k)$ we denote a vector whose components contain all possible products of the form $y_1^{a_1}\cdots y_k^{a_k},\ldots, 0 \le a_1+\cdots+a_k \le d$. We assume that all the components of the vector $W(d,N)$ are linearly independent. Then the numbering of these components can be arbitrary. However, to represent the recurrent form of an algorithm of polynomial approximation, the numbering, defined by recurrent relationships, is reasonable.

The recurrent form of writing the vector $W(d,k)$ is of the form

$$W(d,k)^T = \|W(d,k-1)^T \quad w(d-1,k-1,k-1,y_k)^T\|, \tag{3.5}$$

where

$$w(d-1,k-1,y_k)^T = \|W(d-1,k-1)^T y_k \cdots W(1,k-1)^T y_k^{d-1} \quad W(0,k-1)^T y_k^d\|, \tag{3.6}$$

$$W(0,i) = W(i,0) = 1, i = 0,1,2,\ldots$$

For a given integer d, successive application of relationships (3.5), (3.6) leads to the formulas

$$W(d,1)^T = \|1 y_1 \cdots y_1^{d-1} y_1^d\|,$$

$$W(d,2)^T = \|W(d,1)^T \quad W(d-1,1)^T y_2 \cdots W(1,1)^T y_2^{d-1} \quad y_2^d\|,$$

$$W(d,3)^T = \|W(d)^T \quad W(d-1,2)^T y_3 \cdots W(1,2)^T y_3^{d-1} \, y_3^d\|, \ldots$$

The formulas imply a way of successively numbering the components of vector $W(d,k)$. These components are then denoted as $w_1, w_2, \ldots, w_{m(d,k)}$.

2.4 Calculating Statistical Moments and Choice of Stochastic Measure

The preceding material points to the fact that at the exact calculation of integrals (2.2)–(2.5) and at a great value of an integer d, the presented method of approximation delivers an estimator vector of parameters in the form of a vector of conditional expectation. It is well known that the estimator is optimal in the root-mean-square sense for all vector-functions of a vector of observations. The efficiency of this estimator—as a carrier of information, contained in a vector of observations—depends on the a priori stochastic measure that was chosen.

It is likely that the problem of choosing an optimal stochastic measure can be formulated. However, a similar problem is not further considered here.

Besides the a priori region $\Omega_Y \times \Omega_{\theta|Y_N}$, there are commonly no a priori data for stochastic characteristics of the vectors Y_N, θ. Hence, it is natural to use the heuristic arguments when assigning a stochastic measure. However, the heuristics' role can be reduced if we connect choosing a stochastic measure with the problem of numerically determining integrals (2.2)–(2.5).

Let's consider a method of calculating multidimensional integrals, effectively used below for polynomial approximation in solving applied problems.

Let $\Omega \in R^m$; it is necessary to calculate the integral

$$J = \int_{x \in \Omega} F(x) dx, \tag{4.1}$$

where $F(x^1, \ldots, x^m)$ is a given integrand function, and Ω is a unity cube in R^m: $0 \le x^i \le 1$.

Every cube's edge is divided into r equal segments of length $1/r$, whose ends are vertices r^m of the smaller elementary cubes denoted as $E_1, \ldots, E_k, \ldots, E_{r^m}$. Then,

$$J = \sum_{k=1}^{k=r^m} J_k, \tag{4.2}$$

where

$$J_k = \int_{x \in E_k} F(x)dx. \tag{4.3}$$

To calculate J_k, one generalizes the method of trapezoids to a multidimensional case. Multidimensional linear interpolation is performed, at which the integrand function $F(x)$ is replaced with a multilinear function $F(x)'$. This function is a sum of products of functions, linear in variables $x^1, \ldots, x^m$, and coincides with $F(x)$ in 2^m cube vertices E_k.

We assume that $x_{k_1}, \ldots, x_{k_m}, 1 \le k_1, \ldots, k_m \le r-1$ are coordinates of that cube vertex E_k, whose coordinates have the smallest values of all 2^m cube vertices E_k. Then, coordinates of all 2^m cube vertices can be represented by the expressions

$$x_{k_1}(\alpha_1) = x_{k_1} + \alpha_1/r, \ldots, x_{k_m}(\alpha_m) = x_{k_m} + \alpha_m/r,$$

where quantities $\alpha_1, \ldots, \alpha_m$ assume—independently of one another—the value 0 or 1. Next, $x_{k_i}(1) - x_{k_i}(0) = 1/r, i = 1, \ldots, m$, are the coordinates $x_1, \ldots, x_m$ of a point, belonging to E_k, that satisfy the inequalities $x_{k_i}(0) \le x^i \le x_{k_i}(1)$, $i = 1, \ldots, m$.

Let's define linear functions of these point coordinates by

$$f_0(x^i) = r(x_{k_i}(1) - x^i),\ f_1(x^i) = r(x^i - xk_i(0)).$$

It is clear that $0 \le f_0(x^i) \le f_1(x^i)$, $f_0(x^i) + f_1(x^i) = 1$.

The interpolating function $F(x)'$ is defined by

$$F(x^1, \ldots, x^m)'$$

$$= \sum_{\alpha_1, \ldots, \alpha_m = 0,1} f_{\alpha_1}(x^1) \cdots f_{\alpha_n}(x^n) F(x_{k_1} + \alpha_1/r, \ldots, x_{k_m} + \alpha_m/r). \tag{4.4}$$

Summation in Eq. (4.4) is done over all binary numbers of the form $\alpha_1, \ldots, \alpha_m$, and the term's number is equal to 2^m.

Linear functions $f_0(x^i)$, $f_1(x^i)$ satisfy the identity

$$\sum_{\alpha_1, \ldots, \alpha_m = 0,1} f_{\alpha_1}(x^1) \cdots f_{\alpha_m}(x^m) = 1. \tag{4.5}$$

The identity is proved by induction.

Replacing in Eq. (4.3) the function $F(x)$ with $F(x)'$, after integrating over cube E_k, we will find an approximate expression for the integral J_k:

$$J_k \simeq (1/2r)^m \sum_{\alpha_1, \ldots, \alpha_m = 0,1} F(x_{k_1} + \alpha_1/r, \ldots, x_{k_m} + \alpha_m/r). \tag{4.6}$$

Hence, the approximate value of the integral over every elementary cube is proportional to an arithmetic average of values of the integrand function in vertices of this cube.

Let's consider a situation when every integral J_k is of the form

$$J_k = \int_{x \in E_k} F(x)dx, \tag{4.7}$$

where $F(x)$ is a yet-to-be-determined probability density of the random vector x. The value of J_k from (4.7) will become equal to the right of Eq. (4.6) if the probability density is assumed to be the proportional sum of products of delta-functions:

$$p(x^1, \ldots, x^m) = (1/2r)^m$$

$$\times \sum_{\alpha_1, \ldots, \alpha_m = 0,1} \delta(x_{k_1} + \alpha_1/r - x^1) \cdots \delta(x_{k_m} + \alpha_m/r - x^m). \tag{4.8}$$

For this function, there is a valid normalization condition on the unit cube.

Thus, assigning the probability density to be equal to a sum of products of delta functions delivers an exact value of the corresponding integral. But we get the same integral value if distribution of the random vector x on the unit cube is assumed uniform, and generalization of the method of trapezoids (presented above) is taken as an approximate method of calculating multidimensional integrals.

Hence, there are two possibilities.

1. Assign the probability density as a sum of products of delta functions on Ω and find an exact value of integral (4.7).
2. Assign the probability density as uniform on Ω and find an approximate value of this integral after using a generalized method of trapezoids.

In the latter case, approximate values should also be used; then some of the integrals being calculated can be determined analytically. This means that integrals being calculated are elements of a priori vectors and matrices needed to determine the vector of linear estimators in Chap. 1, optimal in the root-mean-square sense.

It should be emphasized that for a determinate connection of random vectors θ and Y, at the exact calculation of the integrals (2.2)–(2.5) and a sufficiently great value of d, the estimator $\hat{\theta}(Y, d)$ is close to the estimated vector θ and practically does not depend on the chosen probability density $p(Y)$. This statement follows from Eq. (4.8).

Let's assume that the a priori region Ω_Y is a cube in R_N, which is divided into r^N elementary cubes in realizing the generalized method of trapezoids (see above). Let's choose probability density $p(Y)$ as the corresponding sum of products of delta functions. Then Eq. (4.8) (representing, for the method of polynomial approximation, the key convergence of the calculations) will be true for any (including small) number r.

Of course, for small r, it will be necessary to use a greater number d. Hence, the "rough" (for small r) calculating weight coefficients $\lambda(a_1, \ldots, a_N)$ in Eq. (3.3) should be compensated for by a greater number $m(d, N)$ of terms in Eq. (3.3). The choice of rational—according to the criterion of minimum time—calculating numbers r and d is a subject of special consideration.

2.5 Fragment of Program of Modified Method of Trapezoids

An approximate value of a multidimensional integral over a unit cube is equal to a sum of approximate values, to be calculated under Eq. (4.2). However, practical use of this method is not rational, as it requires a huge volume of computing.

In fact, if a vertex of a small parallelepiped E_k is within a unit cube (it is surrounded by small parallelepipeds all around), then, for multiple uses of formulas like Eq. (4.6), one should calculate the value of the function $F(\ldots)$ in this vertex $2n$ times. However, if a vertex coincides with that of the unit cube, then the function value of $F(\ldots)$ is calculated only once.

It is reasonable to design an algorithm that would require calculating the function $F(\ldots)$ only once in every node of the grid covering the unit cube. In such a case, the explicit representation of an approximate integral as a linear combination of the function $F(\ldots)$'s values in nodes is rather difficult, because coefficients of this linear combination depend on the integer r.

We will present the offered algorithm as a fragment of a Pascal program for the case of $r = 5$:

```
J:=0;
for x1:=0 to r do for x2:=0 to r do
for x3:=0 to r do for x4:=0 to r do
for x5:=0 to r do
begin
x[1]:=x1;x[2]:=x22;x[3]:=x3; x[4]:=x4;x[5]:=x5;
nj:=0;
for i:=1 to 5 do if (x[i]=0) or (x[i]=r) then nj:=nj+1;
if nj=0 then mj:=32;
if nj=1 then mj:=16;
if nj=2 then mj:=8;
if nj=3 then mj:=4;
if nj=4 then mj:=2;
if nj=5 then mj:=1;
K:=mj/32;
J:=J+KF(x[1],x[2],x[3],x[4],x[5]);
end;
```

Table 2.1 Values of integers for different values of r

r	$k(1)$	$k(1/2)$	$k(1/4)$	$k(1/8)$	$k(1/16)$	$k(1/32)$
5	1,024	2,560	2,560	1,280	320	32
10	59,049	65,610	29,160	6,480	720	32
15	537,824	384,160	109,760	15,680	1,120	32
20	2,476,099	1,303,210	274,360	28,880	1,520	32

The fragment implies that the algorithm gives an approximate integral J as a linear combination of the function $F(\ldots)$'s values in vertices of small cubes. The coefficients K of this linear combination take the values 1, 1/2, 1/4, 1/8, 1/16, 1/32, multiplied by integers, automatically determined by the algorithm for the modified method of trapezoids.

Table 2.1 gives values of these integers for different values of r.

Reference

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