

## Chapter 2

# Kinetic Description of a Plasma

This chapter is devoted to the Boltzmann transport equation. The Boltzmann equation has been initially established for a dilute gas of neutral molecules and it provides the standard kinetic approach to describe the microscopic evolution of a gas to equilibrium. The Boltzmann equation can be applied to the description of a medium in which the dominant interactions are of short-range type so that it is suitable to describe the evolution of a plasma determined by electron-molecule and ion-molecule collisions. The Boltzmann equation fails however when applied to long-range Coulomb interactions. The collisional terms for both elastic and inelastic electron-molecule collisions are then consistently derived. The chapter ends with the establishment of the equations for the moments of the Boltzmann equation, i.e. the fluid equations, for electron-neutral interactions.

### 2.1 Boltzmann Transport Equation for Molecules

#### 2.1.1 Collisionless Boltzmann Equation

The system under consideration here is a dilute gas of molecules, such as usually considered in the classical kinetic theory of gases. The temperature is assumed sufficiently high and the density sufficiently low for each molecule may be considered a classical particle with a rather well defined position and momentum. Furthermore, the molecules interact with each other through collisions whose nature may be specified through the scattering cross section (Chapman and Cowling 1939; Huang 1963).

The analysis is based on the statistical role played by the particle distribution function of a system with one kind of molecules,  $f(\mathbf{r}, \mathbf{v}, t)$ , defined so that

$$f(\mathbf{r}, \mathbf{v}, t) \, d\mathbf{r} \, d\mathbf{v} \quad (2.1)$$

represents the number of particles which, at the instant  $t$ , have positions lying within a space-volume element  $d\mathbf{r}$  at the extremity of vector  $\mathbf{r}$ , and velocities lying within a velocity-volume element  $d\mathbf{v}$  at the extremity of vector  $\mathbf{v}$ . The volume elements  $d\mathbf{r}$  and  $d\mathbf{v}$  are large enough to contain a very large number of molecules but small as compared to the macroscopic dimensions of the whole gas. The distribution function changes with time, because the molecules continuously enter and leave a given double volume element  $d\mathbf{r} \, d\mathbf{v}$ .

Having a system with  $N$  molecules in a volume  $V$ , the normalization of the distribution function is as follows

$$\int_{\mathbf{r}} \int_{\mathbf{v}} f(\mathbf{r}, \mathbf{v}, t) \, d\mathbf{r} \, d\mathbf{v} = N. \quad (2.2)$$

If the molecules are uniformly distributed in space, so that  $f$  is independent of  $\mathbf{r}$ , we obtain

$$\int_{\mathbf{v}} f(\mathbf{v}, t) \, d\mathbf{v} = \frac{N}{V}. \quad (2.3)$$

On the contrary, if the distribution is not uniform in space, we may define the gas number density  $n(\mathbf{r}, t)$  by

$$\int_{\mathbf{v}} f(\mathbf{r}, \mathbf{v}, t) \, d\mathbf{v} = n(\mathbf{r}, t). \quad (2.4)$$

Let us start by considering first the gas in the absence of collisions. A given molecule with the coordinates  $(\mathbf{r}, \mathbf{v})$  at the instant  $t$  will have the coordinates  $(\mathbf{r} + \mathbf{v} \, dt, \mathbf{v} + \mathbf{a} \, dt)$  at the instant  $t + dt$ , with  $\mathbf{a} = \mathbf{F}/m$  denoting the particle acceleration,  $\mathbf{F}$  is the external force acting on the molecule, and  $m$  denotes its mass. Thus all the molecules that are in the double volume element  $d\mathbf{r} \, d\mathbf{v}$ , at the extremity of  $(\mathbf{r}, \mathbf{v})$  and at the instant  $t$ , will be in a volume element  $d\mathbf{r}' \, d\mathbf{v}'$ , at the extremity of  $(\mathbf{r} + \mathbf{v} \, dt, \mathbf{v} + \mathbf{F}/m \, dt)$  at the instant  $t + dt$ . Therefore, when the collisions do not exist we may write the following equality

$$f\left(\mathbf{r} + \mathbf{v} \, dt, \mathbf{v} + \frac{\mathbf{F}}{m} \, dt, t + dt\right) d\mathbf{r}' \, d\mathbf{v}' = f(\mathbf{r}, \mathbf{v}, t) \, d\mathbf{r} \, d\mathbf{v}. \quad (2.5)$$

Since  $d\mathbf{r} \, d\mathbf{v} = d\mathbf{r}' \, d\mathbf{v}'$  according to Liouville's theorem (see Appendix A.2.1), we still have

$$f\left(\mathbf{r} + \mathbf{v} \, dt, \mathbf{v} + \frac{\mathbf{F}}{m} \, dt, t + dt\right) = f(\mathbf{r}, \mathbf{v}, t). \quad (2.6)$$

Let us consider now the case in which the collisions exist in the gas. The equation (2.6) must be modified in order to include the introduction and removal of molecules from a given double volume element  $\mathbf{dr} \mathbf{dv}$  due to the collisions

$$f\left(\mathbf{r} + \mathbf{v} dt, \mathbf{v} + \frac{\mathbf{F}}{m} dt, t + dt\right) = f(\mathbf{r}, \mathbf{v}, t) + \left(\frac{\partial f}{\partial t}\right)_{coll} dt, \quad (2.7)$$

in which  $(\partial f / \partial t)_{coll}$  represents the time rate of change of the particle distribution function  $f(\mathbf{r}, \mathbf{v}, t)$  due to collisions. Expanding the left-hand side member to the first order in  $dt$ , when this time interval is vanishingly small, and subtracting it from the first right-hand side term, we obtain the following equation of motion for the particle distribution function

$$\frac{\partial f}{\partial t} + \left(\mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}}\right) + \left(\frac{\mathbf{F}}{m} \cdot \frac{\partial f}{\partial \mathbf{v}}\right) = \left(\frac{\partial f}{\partial t}\right)_{coll}. \quad (2.8)$$

Alternatively equation (2.8) may be written under the following form using the gradient operators with respect to  $\mathbf{r}$  and  $\mathbf{v}$

$$\frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla_{\mathbf{r}} f) + \left(\frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f\right) = \left(\frac{\partial f}{\partial t}\right)_{coll}. \quad (2.9)$$

When  $\mathbf{F}$  does not depend on  $\mathbf{v}$ , the equation (2.9) may still be written under the form

$$\frac{\partial f}{\partial t} + \nabla_{\mathbf{r}} \cdot (f \mathbf{v}) + \nabla_{\mathbf{v}} \cdot \left(f \frac{\mathbf{F}}{m}\right) = \left(\frac{\partial f}{\partial t}\right)_{coll}, \quad (2.10)$$

showing that the second and the third terms of the left-hand side member represent the divergence of the components of the current of molecules, respectively, in the space of positions and in the space of velocities.

The meaning of the equation (2.8) becomes clear if one notes that the left-hand side member,  $df/dt$ , represents the total (or the convective) derivate of  $f$  in the double, or phase space. The total derivative  $df/dt$  can be interpreted as the rate of change as seen in a frame moving with the molecule in  $(\mathbf{r}, \mathbf{v})$  space. The equation of motion simply says that  $df/dt$  is zero in the absence of collisions. The collisions have the effect of removing a particle from one element of phase space and introducing it in another. Obviously, this equation is not meaningful until the term  $(\partial f / \partial t)_{coll}$  may be explicitly specified. It is in specifying this term that the assumption that the system is a dilute gas of molecules becomes relevant. Here, we still note that the collision term may be expressed as follows

$$\left(\frac{\partial f}{\partial t}\right)_{coll} = R_{in} - R_{out}, \quad (2.11)$$

in which  $R_{in} \mathbf{dr} \mathbf{dv} dt$  represents the number of collisions occurring during the time interval  $dt$  in which one molecule enters into the double volume element  $\mathbf{dr} \mathbf{dv}$ , while  $R_{out} \mathbf{dr} \mathbf{dv} dt$  represents the number of collisions occurring during the same time interval in which one molecule leaves  $\mathbf{dr} \mathbf{dv}$ . Here, we are implicitly assuming that if a molecule enters or leaves the double volume element  $\mathbf{dr} \mathbf{dv}$  as a result of a collision, none of its partners enters or leaves this volume element in the same time interval  $dt$ . This error is negligible because of the smallness of the double volume element  $\mathbf{dr} \mathbf{dv}$ .

### 2.1.2 Binary Collisions

Let us consider here an elastic collision between two molecules of equal mass. The molecules have well defined positions and velocities so that the initial and final states of the collision may be described classically. If the velocities of the incoming molecules are  $\mathbf{v}_1$  and  $\mathbf{v}_2$ , and the velocities of the outgoing molecules are  $\mathbf{v}'_1$  and  $\mathbf{v}'_2$ , from the conservation of momentum and energy we can write

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}'_1 + \mathbf{v}'_2 \quad (2.12)$$

$$v_1^2 + v_2^2 = v_1'^2 + v_2'^2, \quad (2.13)$$

where  $v_k = |\mathbf{v}_k|$ , with  $k \equiv (1, 2)$ , denotes the absolute values of the velocities. As it is well known from any textbook on mechanics, when the new variables for the centre-of-mass velocity and relative velocity are introduced

$$\mathbf{V} = \frac{1}{2} (\mathbf{v}_1 + \mathbf{v}_2) \quad (2.14)$$

$$\mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 \quad (2.15)$$

and similar variables  $\mathbf{V}'$  and  $\mathbf{v}'$  are defined, the system of equations (2.12) and (2.13) can be rewritten as

$$\mathbf{V} = \mathbf{V}' \quad (2.16)$$

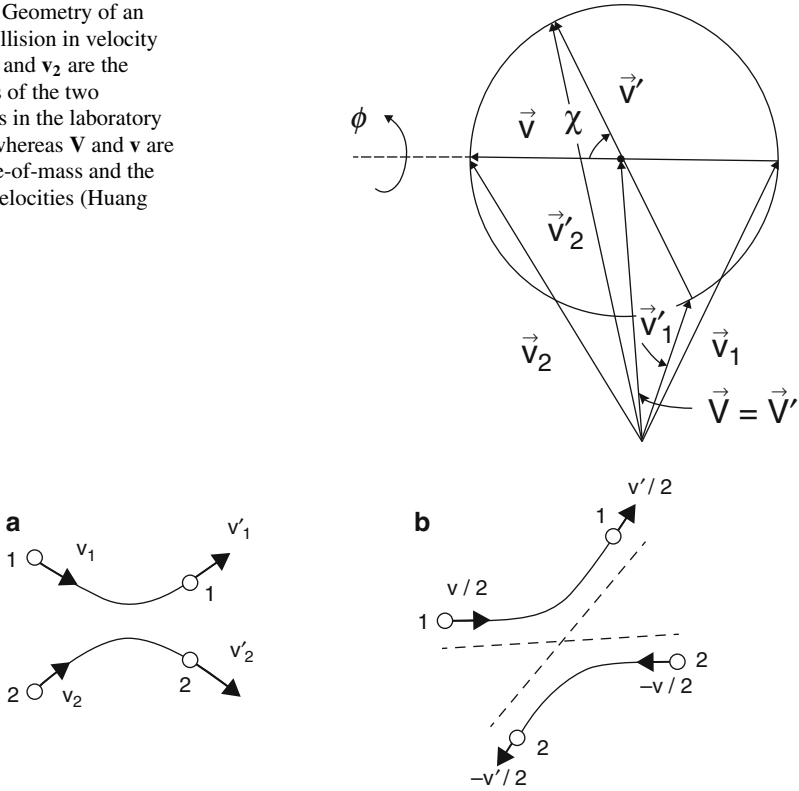
$$v = v', \quad (2.17)$$

where  $v = |\mathbf{v}|$  is the absolute value of the relative velocity. The collision is represented geometrically in Fig. 2.1. The relative velocity  $\mathbf{v}$  merely rotates to  $\mathbf{v}'$ , keeping constant its magnitude. The collision is completely determined by specifying  $\mathbf{V}$ ,  $\mathbf{v}$ , and the scattering angles  $\chi$  and  $\phi$  of  $\mathbf{v}'$  with respect to  $\mathbf{v}$ .

When  $\mathbf{V}$  and  $\mathbf{v}$  are slightly changed to  $\mathbf{V} + \mathbf{dV}$  and  $\mathbf{v} + \mathbf{dv}$ , respectively, with  $\chi$  and  $\phi$  kept constant, and  $\mathbf{V}'$  and  $\mathbf{v}'$  changed to  $\mathbf{V}' + \mathbf{dV}'$  and  $\mathbf{v}' + \mathbf{dv}'$  it is easy to verify that the following equality holds

$$d^3V d^3v = d^3V' d^3v'. \quad (2.18)$$

**Fig. 2.1** Geometry of an elastic collision in velocity space.  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are the velocities of the two molecules in the laboratory system, whereas  $\mathbf{V}$  and  $\mathbf{v}$  are the centre-of-mass and the relative velocities (Huang 1963)



**Fig. 2.2** (a) Scheme of a collision in the laboratory coordinate system between two molecules with initial  $\mathbf{v}_1$  and  $\mathbf{v}_2$  and final  $\mathbf{v}'_1$  and  $\mathbf{v}'_2$  velocities. (b) Collision view from the centre-of-mass system in which two molecules with the relative velocity  $\mathbf{v}$  interact each other outcoming with the relative velocity  $\mathbf{v}'$  (Huang 1963)

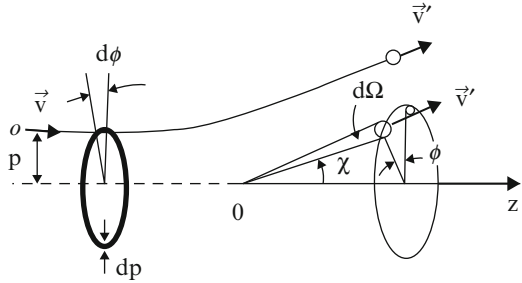
Here  $d^3V = dV_x dV_y dV_z$  and  $d^3v = dv_x dv_y dv_z$  denote equivalent representations for  $\mathbf{dV}$  and  $\mathbf{dv}$ . Using the velocities of molecules in the laboratory coordinate system, we may also write

$$d^3v_1 d^3v_2 = d^3v'_1 d^3v'_2. \quad (2.19)$$

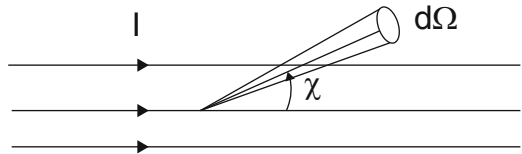
The velocity of the centre-of-mass system  $\mathbf{V}$  is not important here. In fact, if we translate the coordinate system with a uniform velocity  $\mathbf{V}$ , in the new coordinate system only the relative velocities  $\mathbf{v}$  and  $\mathbf{v}'$  need to be considered. Such a coordinate system is called the centre-of-mass system. The collision processes as viewed in the laboratory coordinate system and as viewed in the centre-of-mass system are shown in Fig. 2.2a, b.

In the centre-of-mass system it suffices to consider only one of the molecules, because its partner always moves oppositely. Thus the problem reduces to the

**Fig. 2.3** Scattering of a molecule by a fixed centre of forces specified by the impact parameter  $p$  (Huang 1963)



**Fig. 2.4** Scattering of an incident beam of molecules by a centre of forces within the solid-angle element  $d\Omega$



scattering of a molecule by a fictitious fixed centre of forces, represented by the point  $O$  in Fig. 2.3, in which  $p$  is the impact parameter.

The molecule approaches  $O$  with velocity  $\mathbf{v}$ , whose perpendicular distance to  $O$  is the impact parameter  $p$ . If the system of reference is chosen with  $O$  located at the origin of the coordinate system, with the  $z$  axis parallel to the velocity  $\mathbf{v}$ , and because  $|\mathbf{v}'| = |\mathbf{v}|$ , the final state of the molecule is specified by two scattering angles  $\chi$  and  $\phi$ , with  $\chi$  denoting the angle between  $\mathbf{v}'$  and the  $z$  axis, and  $\phi$  the azimuthal angle of  $\mathbf{v}'$  around the  $z$  axis. The two angles are collectively denoted by  $\Omega$ , with  $d\Omega = \sin \chi \, d\chi \, d\phi$  representing the solid-angle element. This totally completes the kinetic description of a binary collision.

The dynamical aspects of a binary collision are contained in the differential cross section  $\sigma(v, \Omega)$ . The initial velocities  $\mathbf{v}_1$  and  $\mathbf{v}_2$  of a collision do not uniquely determine the collision because they do not determine the impact parameter. Thus specifying  $\mathbf{v}_1$  and  $\mathbf{v}_2$  we specify a class of collisions with the same centre-of-mass system. They can be represented in Fig. 2.3 by trajectories corresponding to all possible impact parameters, and thus to all possible scattering angles. We may represent this class of collisions by imagining that a steady beam of particles of initial velocity  $\mathbf{v}$ , uniformly spread out in space, impinges on the centre of forces  $O$ .

Figure 2.4 represents the scattering of an incident beam with  $I$  representing the number of molecules crossing a unit area normal to the beam, per time unit.  $I$  is called the incident particle flux (expressed in molecules  $\text{m}^{-2} \text{s}^{-1}$ ). In the case of molecules with the absolute value of the relative velocity  $v$ , the differential cross section  $\sigma(v, \Omega)$  is defined so that

$$I \sigma(v, \Omega) \, d\Omega = I \sigma(v, \chi, \phi) \sin \chi \, d\chi \, d\phi \quad (2.20)$$

represents the number of molecules scattered, per time unit, in a direction lying within the solid-angle element  $d\Omega$ .

The differential cross section has hence the dimensions of an area. Using the concept of impact parameter Fig. 2.3 shows that the expression (2.20) can also be written under the form

$$I \sigma(v, \Omega) d\Omega = I p dp d\phi. \quad (2.21)$$

If the scattering is independent of the azimuthal angle  $\phi$ , we simply obtain

$$I \sigma(v, \chi) \sin \chi d\chi = I p dp. \quad (2.22)$$

Then the total cross section is obtained by integrating  $\sigma(v, \Omega)$  over all solid-angle elements

$$\sigma(v) = \int_{\Omega} \sigma(v, \Omega) d\Omega. \quad (2.23)$$

The differential cross section is a directly measured quantity or if the intermolecular potential is known it can also be calculated quantum mechanically. The detailed form of  $\sigma(v, \Omega)$  depends of the intermolecular potential describing the interactions in a given particular gas. For our purposes we consider the differential cross section as an input parameter specified when a given gas is chosen.

### 2.1.3 Collision Term of the Boltzmann Equation

The collision term  $(\partial f / \partial t)_{coll}$  is derived assuming the following approximations:

- (i) The collisions are strictly binary, so that the present derivation is valid only for a sufficiently dilute gas.
- (ii) The interactions between molecules are described by central forces, isotropic and of short-range, so that their effects are felt only in the interior of a small volume in the space of positions, i.e. in the interior of a small sphere of radius  $r_c$  called collision sphere. The interactions may be considered hence as true collisions.
- (iii) The external forces applied on the molecules are sufficiently weak in order they may be vanished in the interior of the collision sphere. The effect of the external forces on the collision cross section is thus neglected.
- (iv) For distances larger than the radius of the collision sphere there are no correlations. The velocity of a molecule is also uncorrelated with its position. This assumption is known as the hypothesis of *molecular chaos*.

The hypothesis (iv) is a crucial point in the derivation of the collision term of the Boltzmann equation, as we will discuss below, and it simply states that, in a spatial volume element  $d\mathbf{r}$  at the extremity of vector  $\mathbf{r}$ , the number of pairs of molecules

with velocities lying in the velocity volume elements  $\mathbf{dv}_1$  at the extremity of  $\mathbf{v}_1$  and  $\mathbf{dv}_2$  at the extremity of  $\mathbf{v}_2$  is

$$f(\mathbf{r}, \mathbf{v}_1, t) \, \mathbf{dr} \, \mathbf{dv}_1 \, f(\mathbf{r}, \mathbf{v}_2, t) \, \mathbf{dr} \, \mathbf{dv}_2. \quad (2.24)$$

This assumption is introduced here for simplification but it is not clear if this is a general condition for the description of the state of the gas under consideration. The collision term is hence derived from a model based on the classical kinetic theory of gases, in which the binary collisions occur in limited regions of space and time and where, between two collisions, the trajectory of a molecule is due to the action of external forces only.

Let us derive an explicit expression for the collision term  $(\partial f / \partial t)_{coll}$ . The rate of decrease of the velocity distribution of molecules whose velocity lies in  $\mathbf{dv}_1$  at the extremity of  $\mathbf{v}_1$  due to collisions, denoted by  $R_{out}$  in equation (2.11), can be obtained by considering that in the same spatial volume element  $\mathbf{dr}$  at the extremity of  $\mathbf{r}$ , there are molecules of any velocity  $\mathbf{v}_2$  impinging as a beam on the molecules of velocity  $\mathbf{v}_1$ . The flux of this incident beam (in molecules  $\text{m}^{-2} \text{s}^{-1}$ ) is hence

$$I_{21} = f(\mathbf{r}, \mathbf{v}_2, t) \, \mathbf{dv}_2 \, |\mathbf{v}_2 - \mathbf{v}_1|, \quad (2.25)$$

so that number of collisions of the type  $\{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\mathbf{v}'_1, \mathbf{v}'_2\}$  occurring in  $\mathbf{dr}$  during the time interval  $dt$ , leading to the exit of a molecule with velocity  $\mathbf{v}_1$  from  $\mathbf{dv}_1$ , is

$$I_{21} \, \sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega) \, d\Omega \, dt = f(\mathbf{r}, \mathbf{v}_2, t) \, \mathbf{dv}_2 \, |\mathbf{v}_2 - \mathbf{v}_1| \, \sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega) \, d\Omega \, dt, \quad (2.26)$$

where  $\sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega)$  is the differential cross section in the centre-of-mass system and  $\Omega \equiv (\chi, \phi)$  represents the deviation angle between  $\mathbf{v}'_2 - \mathbf{v}'_1$  and  $\mathbf{v}_2 - \mathbf{v}_1$ . The rate  $R_{out}$  is obtained by integrating (2.26) over all velocities  $\mathbf{v}_2$  and all angles  $\Omega$  and by multiplying the spatial density of molecules in  $\mathbf{dv}_1$ , so that we obtain

$$f(\mathbf{r}, \mathbf{v}_1, t) \, \mathbf{dv}_1 \int_{\Omega} \int_{\mathbf{v}_2} f(\mathbf{r}, \mathbf{v}_2, t) \, |\mathbf{v}_2 - \mathbf{v}_1| \, \sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega) \, d\Omega \, \mathbf{dv}_2 \, dt. \quad (2.27)$$

The expression (2.27) gives the number of collisions, per volume unit of the space of positions, that result in the removal of a molecule from the velocity element  $\mathbf{dv}_1$  in the time interval  $dt$ , that is it represents  $R_{out} \, \mathbf{dv}_1 \, dt$ , so that the time rate for the change of  $f$  is

$$R_{out} = f(\mathbf{r}, \mathbf{v}_1, t) \int_{\Omega} \int_{\mathbf{v}_2} f(\mathbf{r}, \mathbf{v}_2, t) \, |\mathbf{v}_2 - \mathbf{v}_1| \, \sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega) \, d\Omega \, \mathbf{dv}_2. \quad (2.28)$$

In a similar manner we can calculate now  $R_{in}$  defined in (2.11), in which a collision of the type  $\{\mathbf{v}'_1, \mathbf{v}'_2\} \rightarrow \{\mathbf{v}_1, \mathbf{v}_2\}$  produces an entrance of a molecule into the volume element  $\mathbf{dv}_1$ . In this case we start by considering the flux of incident molecules  $\mathbf{v}'_2$  impinging on a molecule  $\mathbf{v}'_1$



$$I'_{21} = f(\mathbf{r}, \mathbf{v}'_2, t) \, \mathbf{d}\mathbf{v}'_2 \, |\mathbf{v}'_2 - \mathbf{v}'_1|. \quad (2.29)$$

The number of collisions occurring in volume element  $\mathbf{dr}$  in the time interval  $dt$  is

$$f(\mathbf{r}, \mathbf{v}'_2, t) \, \mathbf{d}\mathbf{v}'_2 \, |\mathbf{v}'_2 - \mathbf{v}'_1| \, \sigma(|\mathbf{v}'_2 - \mathbf{v}'_1|, \Omega) \, d\Omega \, dt. \quad (2.30)$$

By integrating now over all velocities  $\mathbf{v}'_2$  and all angles and by multiplying the spatial density of molecules  $\mathbf{v}'_1$ , we obtain

$$f(\mathbf{r}, \mathbf{v}'_1, t) \, \mathbf{d}\mathbf{v}'_1 \int_{\Omega} \int_{\mathbf{v}'_2} f(\mathbf{r}, \mathbf{v}'_2, t) \, |\mathbf{v}'_2 - \mathbf{v}'_1| \, \sigma(|\mathbf{v}'_2 - \mathbf{v}'_1|, \Omega) \, d\Omega \, \mathbf{d}\mathbf{v}'_2 \, dt. \quad (2.31)$$

However, in accordance with equation (2.11) the expression (2.31) represents  $R_{in} \, \mathbf{d}\mathbf{v}_1 \, dt$ .

Since these collisions are reverse from  $\{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\mathbf{v}'_1, \mathbf{v}'_2\}$ , we still have

$$|\mathbf{v}'_2 - \mathbf{v}'_1| = |\mathbf{v}_2 - \mathbf{v}_1| \quad (2.32)$$

$$\mathbf{d}\mathbf{v}'_2 \, \mathbf{d}\mathbf{v}'_1 = \mathbf{d}\mathbf{v}_2 \, \mathbf{d}\mathbf{v}_1, \quad (2.33)$$

so that the rate for the entrance of a molecule in  $\mathbf{d}\mathbf{v}_1$  is

$$R_{in} = f(\mathbf{r}, \mathbf{v}'_1, t) \int_{\Omega} \int_{\mathbf{v}_2} f(\mathbf{r}, \mathbf{v}_2, t) \, |\mathbf{v}_2 - \mathbf{v}_1| \, \sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega) \, d\Omega \, \mathbf{d}\mathbf{v}_2. \quad (2.34)$$

Combining the results for  $R_{out}$  and  $R_{in}$  given by equations (2.28) and (2.34), we obtain the following collision term

$$\left( \frac{\partial f_1}{\partial t} \right)_{coll} = \int_{\Omega} \int_{\mathbf{v}_2} (f'_2 f'_1 - f_2 f_1) \, |\mathbf{v}_2 - \mathbf{v}_1| \, \sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega) \, d\Omega \, \mathbf{d}\mathbf{v}_2, \quad (2.35)$$

where  $\sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega)$  is the differential cross section for the collision  $\{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\mathbf{v}'_1, \mathbf{v}'_2\}$  and the following abbreviations have been used:  $f_1 \equiv f(\mathbf{r}, \mathbf{v}_1, t)$ , etc. Substituting (2.35) in (2.8), we obtain then the Boltzmann transport equation able to describe the evolution of the molecules with velocity  $\mathbf{v}_1$

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \left( \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{r}} \right) + \left( \frac{\mathbf{F}}{m} \cdot \frac{\partial f_1}{\partial \mathbf{v}_1} \right) = \\ \int_{\Omega} \int_{\mathbf{v}_2} (f'_2 f'_1 - f_2 f_1) \, |\mathbf{v}_2 - \mathbf{v}_1| \, \sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega) \, d\Omega \, \mathbf{d}\mathbf{v}_2, \end{aligned} \quad (2.36)$$

which is a nonlinear integro-differential equation for the unknown function  $f_1$ . This equation is irreversible with respect to the time, which is a paradoxical result, since it violates the reversibility of molecular dynamics and the Liouville equation from

which, in principle, the Boltzmann equation seems to be derived (Huang 1963; Liboff 1969; Sone 2007).

### 2.1.4 Validity of the Boltzmann Equation

At the end of nineteenth century a large controversy raised about the validity of Boltzmann equation through a famous paradox presented by Loschmidt. He objected that it should not be possible to deduce an irreversible equation from a time-symmetric dynamics using a time-symmetric formalism. The explanation is that the Boltzmann equation is based on the assumption of molecular chaos and this assumption breaks the time reversal symmetry. The apparent paradox is conveniently explained when we derive the collision integral of the Boltzmann equation from a more structured analysis. This derivation should be conducted with basis on the Liouville equation and it allows to obtain a system of  $N$  coupled equations for a system of  $N$  particles known as BBGKY (Born-Bogoliubov-Green-Kirkwood-Yvon) hierarchy (Delcroix 1963, 1966; Liboff 1969; Nicholson 1983; Delcroix and Bers 1994).

Let us define as  $F$  a function that contains the maximum of information about a given system. The probability of the states of a system in which the particle 1 has its position lying within a space-volume element  $d\mathbf{r}_1$  at the extremity of vector  $\mathbf{r}_1$  and its velocity lying within the velocity-volume  $d\mathbf{v}_1$  at the extremity of vector  $\mathbf{v}_1$ , being the states of particles  $2, \dots, N$  whatever, may be obtained by integrating over all space of positions  $\mathbf{r}_2, \dots, \mathbf{r}_N$  and over all space of velocities  $\mathbf{v}_2, \dots, \mathbf{v}_N$

$$d\mathbf{r}_1 d\mathbf{v}_1 \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} \dots \int_{\mathbf{r}_N} \int_{\mathbf{v}_N} F d\mathbf{r}_2 d\mathbf{v}_2 \dots d\mathbf{r}_N d\mathbf{v}_N. \quad (2.37)$$

The numbering of particles introduced to define  $F$  is artificial and arbitrary, because the particles are undistinguished.

The probable number  $\langle dN \rangle_1$  of particles that are in the interior of the double volume element  $d\mathbf{r}_1 d\mathbf{v}_1$  is equal to the above probability, valid for any particle, multiplied by the number of particles  $N$ , whose result may be written under the form

$$\langle dN_1 \rangle = f_1 d\mathbf{r}_1 d\mathbf{v}_1, \quad (2.38)$$

being  $f_1 \equiv f(\mathbf{r}_1, \mathbf{v}_1, t)$  the distribution function of the positions and velocities of one particle given by

$$f_1 = N \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} \dots \int_{\mathbf{r}_N} \int_{\mathbf{v}_N} F d\mathbf{r}_2 d\mathbf{v}_2 \dots d\mathbf{r}_N d\mathbf{v}_N. \quad (2.39)$$

On the other hand, the density of one particle  $n_1 \equiv n(\mathbf{r}_1, t)$  is obtained by integrating  $f_1$  over the space of velocities  $\mathbf{v}_1$

$$n_1 = \int_{\mathbf{v}_1} f_1 \, d\mathbf{v}_1 \quad (2.40)$$

and it allows to determine the probable number of particles placed in the volume element  $d\mathbf{r}_1$  whatever their velocities through the expression

$$\langle dN'_1 \rangle = n_1 \, d\mathbf{r}_1. \quad (2.41)$$

Let us consider now two volume elements  $d\mathbf{r}_1$  and  $d\mathbf{r}_2$  in the space of positions and two volume elements  $d\mathbf{v}_1$  and  $d\mathbf{v}_2$  in the space of velocities, and let us consider further the probable number of pairs of particles, in which the first particle is placed in the double volume element  $d\mathbf{r}_1 \, d\mathbf{v}_1$  and the second particle in the double volume element  $d\mathbf{r}_2 \, d\mathbf{v}_2$ . This probable number  $\langle dN_{12} \rangle$  is equal to the probability of the system to be in a state where two given particles, for instance the particles 1 and 2 satisfying imposed conditions, are placed in  $d\mathbf{r}_1 \, d\mathbf{v}_1 \, d\mathbf{r}_2 \, d\mathbf{v}_2$ , multiplied by the number of possible pairs, that is by  $N(N-1)$ . We may therefore write

$$\langle dN_{12} \rangle = f_{12} \, d\mathbf{r}_1 \, d\mathbf{v}_1 \, d\mathbf{r}_2 \, d\mathbf{v}_2, \quad (2.42)$$

being  $f_{12} \equiv f(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, t)$  the double distribution function defined by

$$f_{12} = N(N-1) \int_{\mathbf{r}_3} \int_{\mathbf{v}_3} \dots \int_{\mathbf{r}_N} \int_{\mathbf{v}_N} F \, d\mathbf{r}_3 \, d\mathbf{v}_3 \dots d\mathbf{r}_N \, d\mathbf{v}_N, \quad (2.43)$$

and where  $F$  is the function that contains all the information about the system. By integrating now  $f_{12}$  over all velocities we obtain the double density  $n_{12} \equiv n(\mathbf{r}_1, \mathbf{r}_2, t)$

$$n_{12} = \int_{\mathbf{v}_1} \int_{\mathbf{v}_2} f_{12} \, d\mathbf{v}_1 \, d\mathbf{v}_2. \quad (2.44)$$

From expressions (2.39) and (2.43) we obtain

$$\int_{\mathbf{r}_2} \int_{\mathbf{v}_2} f_{12} \, d\mathbf{r}_2 \, d\mathbf{v}_2 = (N-1) f_1 \quad (2.45)$$

and from (2.40) and (2.44)

$$\int_{\mathbf{r}_2} n_{12} \, d\mathbf{r}_2 = (N-1) n_1. \quad (2.46)$$

Since

$$N = \int_{\mathbf{r}_1} \int_{\mathbf{v}_1} f_1 \mathbf{dr}_1 \mathbf{dv}_1 = \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} f_2 \mathbf{dr}_2 \mathbf{dv}_2, \quad (2.47)$$

we still have from (2.45)

$$N \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} f_{12} \mathbf{dr}_2 \mathbf{dv}_2 = (N-1) f_1 \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} f_2 \mathbf{dr}_2 \mathbf{dv}_2. \quad (2.48)$$

If there are no correlations, we obtain

$$N f_{12} = (N-1) f_1 f_2 \quad (2.49)$$

and for high values of  $N$ , we have approximately

$$f_{12} \simeq f_1 f_2 \quad (2.50)$$

and

$$n_{12} \simeq n_1 n_2. \quad (2.51)$$

If the equalities (2.50) and (2.51) do not satisfy it is because there are binary correlations in result of interacting forces.

The above definitions may be still generalized to triple velocity distributions  $f_{123} \equiv f(\mathbf{r}_1, \mathbf{v}_1, \mathbf{r}_2, \mathbf{v}_2, \mathbf{r}_3, \mathbf{v}_3, t)$  and triple densities  $n_{123} \equiv n(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, t)$ , as follows

$$f_{123} = N(N-1)(N-2) \int_{\mathbf{r}_4} \int_{\mathbf{v}_4} \dots \int_{\mathbf{r}_N} \int_{\mathbf{v}_N} F \mathbf{dr}_4 \mathbf{dv}_4 \dots \mathbf{dr}_N \mathbf{dv}_N \quad (2.52)$$

$$n_{123} = \int_{\mathbf{v}_1} \int_{\mathbf{v}_2} \int_{\mathbf{v}_3} f_{123} \mathbf{dv}_1 \mathbf{dv}_2 \mathbf{dv}_3. \quad (2.53)$$

These triple functions are related with the double functions through the expressions

$$\int_{\mathbf{r}_3} \int_{\mathbf{v}_3} f_{123} \mathbf{dr}_3 \mathbf{dv}_3 = (N-2) f_{12} \quad (2.54)$$

$$\int_{\mathbf{r}_3} n_{123} \mathbf{dr}_3 = (N-2) n_{12}. \quad (2.55)$$

Let us establish now the kinetic equations governing the distribution functions  $f$ 's of different order. Starting from the Liouville equation (Delcroix 1963, 1966; Liboff 1969; Delcroix and Bers 1994)

$$\frac{\partial F}{\partial t} + \sum_{i=1}^N \left( \mathbf{v}_i \cdot \frac{\partial F}{\partial \mathbf{r}_i} \right) + \sum_{i=1}^N \left( \frac{\mathbf{F}_i}{m} \cdot \frac{\partial F}{\partial \mathbf{v}_i} \right) = - \sum_{i,j=1 (i \neq j)}^N \left( \frac{\mathbf{F}_{ij}}{m} \cdot \frac{\partial F}{\partial \mathbf{v}_i} \right), \quad (2.56)$$

where  $\mathbf{F}_i$  and  $\mathbf{F}_{ij}$  denote the external and the interacting forces acting upon the particle  $i$  belonging to a set of  $N$  particles. Multiplying this equation by  $N$  and  $d\mathbf{r}_2, d\mathbf{v}_2, \dots, d\mathbf{r}_N, d\mathbf{v}_N$ , and integrating over these variables, we obtain

$$\begin{aligned} \frac{\partial f_1}{\partial t} + \left( \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{r}_1} \right) + \left( \frac{\mathbf{F}_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{v}_1} \right) = \\ - N \sum_{j=2}^N \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} \dots \int_{\mathbf{r}_N} \int_{\mathbf{v}_N} \left( \frac{\mathbf{F}_{1j}}{m} \cdot \frac{\partial F}{\partial \mathbf{v}_1} \right) d\mathbf{r}_2 d\mathbf{v}_2 \dots d\mathbf{r}_N d\mathbf{v}_N, \end{aligned} \quad (2.57)$$

being  $f_1$  the distribution function of one particle (2.39). The right-hand side member of equation (2.57) can still be written as follows

$$- N(N-1) \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} \dots \int_{\mathbf{r}_N} \int_{\mathbf{v}_N} \left( \frac{\mathbf{F}_{12}}{m} \cdot \frac{\partial F}{\partial \mathbf{v}_1} \right) d\mathbf{r}_2 d\mathbf{v}_2 \dots d\mathbf{r}_N d\mathbf{v}_N \quad (2.58)$$

and making use of the definition of the double distribution function  $f_{12}$  (2.43), we still have

$$- \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} \left( \frac{\mathbf{F}_{12}}{m} \cdot \frac{\partial f_{12}}{\partial \mathbf{v}_1} \right) d\mathbf{r}_2 d\mathbf{v}_2. \quad (2.59)$$

The equation (2.57) for the evolution of the distribution function  $f_1$  writes then as follows

$$\frac{\partial f_1}{\partial t} + \left( \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{r}_1} \right) + \left( \frac{\mathbf{F}_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{v}_1} \right) = - \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} \left( \frac{\mathbf{F}_{12}}{m} \cdot \frac{\partial f_{12}}{\partial \mathbf{v}_1} \right) d\mathbf{r}_2 d\mathbf{v}_2. \quad (2.60)$$

The second term on the left-hand side member expresses the influence of diffusion phenomena, with  $\partial f_1 / \partial \mathbf{r}_1$  representing the gradient of  $f_1$  in the space of positions, while the third term expresses the action of external forces, with  $\mathbf{F}_1/m$  representing the particle acceleration, and  $\partial f_1 / \partial \mathbf{v}_1$  the gradient of  $f_1$  in the velocity space. In the right-hand side member,  $\mathbf{F}_{12}$  represents the interaction forces of particle 1 with any other particle.

By making the same calculation but with less one integration, we obtain the equation for the evolution of the double distribution function  $f_{12}$  (Delcroix 1963, 1966)

$$\begin{aligned} \frac{\partial f_{12}}{\partial t} + \left( \mathbf{v}_1 \cdot \frac{\partial f_{12}}{\partial \mathbf{r}_1} \right) + \left( \mathbf{v}_2 \cdot \frac{\partial f_{12}}{\partial \mathbf{r}_2} \right) + \left( \frac{\mathbf{F}_1 + \mathbf{F}_{12}}{m} \cdot \frac{\partial f_{12}}{\partial \mathbf{v}_1} \right) + \left( \frac{\mathbf{F}_2 + \mathbf{F}_{21}}{m} \cdot \frac{\partial f_{12}}{\partial \mathbf{v}_2} \right) \\ = - \int_{\mathbf{r}_3} \int_{\mathbf{v}_3} \left[ \left( \frac{\mathbf{F}_{13}}{m} \cdot \frac{\partial f_{123}}{\partial \mathbf{v}_1} \right) + \left( \frac{\mathbf{F}_{23}}{m} \cdot \frac{\partial f_{123}}{\partial \mathbf{v}_2} \right) \right] d\mathbf{r}_3 d\mathbf{v}_3, \end{aligned} \quad (2.61)$$

where the right-hand side term represents the effect of triple interactions.

The equations (2.60) and (2.61) are the first two equations of a system of infinity equations, as a matter of fact of  $(N - 1)$  equations, in which to determine  $f_1$  we need to know previously  $f_{12}$ , and to determine  $f_{12}$  we need to know  $f_{123}$ . We might write the equation for  $f_{123}$  but it still would include  $f_{1234}$ . Starting from the Liouville equation we obtain then a coupled system of equations known as BBGKY hierarchy. For practical purposes, we need to close the system making an hypothesis on the distribution functions. The simplest solution of BBGKY system consists in considering only double interactions, by making  $f_{123} = 0$  in the equation for  $f_{12}$ , which is surely valid only for a dilute gas.

By comparing the right-hand side member of equation (2.60) with the collision integral of the Boltzmann equation (2.36), we may verify at which point the time-symmetric formalism of mechanics, described by the Liouville equation, was broken and transformed to the irreversible description used in thermodynamics. The Boltzmann equation has been deduced assuming the hypothesis of molecular chaos, i.e. that the correlations between two or more particles are neglected everywhere. We can recall here that the assumption of molecular chaos states the following: If  $f(\mathbf{r}, \mathbf{v}, t)$  is the probability of finding a particle with velocity  $\mathbf{v}$ , at position  $\mathbf{r}$  and time  $t$ , the probability of simultaneously finding a molecule with velocity  $\mathbf{v}$  and another with velocity  $\mathbf{v}'$ , at position  $\mathbf{r}$  and time  $t$ , is  $f(\mathbf{r}, \mathbf{v}, t) f(\mathbf{r}, \mathbf{v}', t)$ . This assumption concerns with the absence of correlations and has nothing to say about the form of the actual distribution function of the real gas. Thus a true state of the gas possessing a given distribution function may or may not satisfy the assumption of molecular chaos. Furthermore, the molecular collisions which are responsible for the change of the velocity distribution can create the molecular chaos, when it does not exist, and destroy the molecular chaos, when it exists (Huang 1963).

Thus, the Boltzmann transport equation is not a rigorous consequence of molecular dynamics. The latter is invariant under thermal reversal, but the former is not. The Boltzmann equation is only valid for a gas at the instants when the gas is in a state of molecular chaos. But the collisions can destroy the molecular chaos once established. The Boltzmann equation is not rigorously valid hence for all times. The actual distribution function does not satisfy the Boltzmann equation at all the instants. It satisfies the Boltzmann equation only at the instants when the assumption of molecular chaos is valid. If the molecular chaos is a condition valid most of the time, the Boltzmann equation may be regarded as valid in a statistical sense.

In the derivation of the Boltzmann equation we have defined a collision sphere, of radius  $r_c$ , in the interior of which the interactions may only take place and the effects of external forces are not felt. On the contrary, in the exterior of this sphere it is assumed that there are no correlations and that the particles are governed by mechanical laws under the action of the external forces. Thus, the Boltzmann equation provides an average description of the approach to equilibrium. It gives the evolution within a time-scale  $\Delta t$  much larger than the time of collision  $t_c = r_c/v_1$ , with  $v_1 = |\mathbf{v}_1|$  denoting the absolute value of velocity, but it does not give the description in the interior of the collision sphere.

It can be shown that the equilibrium distribution function is the solution of the Boltzmann equation and that this function is independent of time or, which is the same, it is the limiting form of the distribution function as the time tends to

infinity (see e.g. Huang 1963). Let us assume then that there are no external forces in (2.36). In this case, it is consistent to assume further that the distribution function is independent of  $\mathbf{r}$ . The equilibrium distribution function, denoted by  $f_0(\mathbf{v}_1) \equiv f_{10}$ , is the solution of the equation when  $\partial f(\mathbf{v}_1, t)/\partial t = 0$ . According to the Boltzmann equation (2.36),  $f_{10}$  satisfies the integral relation

$$\int_{\Omega} \int_{\mathbf{v}_2} (f'_{20} f'_{10} - f_{20} f_{10}) |\mathbf{v}_2 - \mathbf{v}_1| \sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega) d\Omega d\mathbf{v}_2 = 0. \quad (2.62)$$

As long as the differential cross section  $\sigma(|\mathbf{v}_2 - \mathbf{v}_1|, \Omega)$  is non-vanishing, a sufficient condition for (2.62) is

$$f'_{20} f'_{10} - f_{20} f_{10} = 0, \quad (2.63)$$

for any possible collision  $\{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\mathbf{v}'_1, \mathbf{v}'_2\}$ . We thus arrive at the interesting conclusion that the equilibrium distribution  $f_0(\mathbf{v}_1)$  is independent of the cross section.

The equilibrium distribution function  $f_0(\mathbf{v}_1)$  for a gas of one type of molecules of mass  $M$ , labelled here  $f(\mathbf{v})$  for simplification, is a solution of (2.63) and this distribution is the so-called Maxwell-Boltzmann distribution (Huang 1963; Bittencourt 2004)

$$f(\mathbf{v}) = n \left( \frac{M}{2\pi k_B T} \right)^{3/2} \exp \left( - \frac{M |\mathbf{v} - \mathbf{v}_d|^2}{2k_B T} \right), \quad (2.64)$$

being  $n$  the particle number density

$$n = \frac{N}{V} = \int_{\mathbf{v}} f(\mathbf{v}) d\mathbf{v}, \quad (2.65)$$

$\mathbf{v}_d = \langle \mathbf{v} \rangle$  the drift or average vector velocity

$$\langle \mathbf{v} \rangle = \frac{1}{n} \int_{\mathbf{v}} \mathbf{v} f(\mathbf{v}) d\mathbf{v}, \quad (2.66)$$

and  $T$  the temperature of molecules

$$\frac{3}{2} k_B T = \langle E \rangle = \frac{1}{n} \int_{\mathbf{v}} E f(\mathbf{v}) d\mathbf{v}, \quad (2.67)$$

being  $E = (1/2)Mv^2$  the particle kinetic energy, and  $k_B$  the Boltzmann constant. The velocity distribution (2.64) gives the probability of finding a molecule with velocity  $v = |\mathbf{v}|$  in the gas, under equilibrium conditions. We have noted the interesting fact of the Maxwell-Boltzmann distribution to be independent of the detailed form of the molecular interactions, as long as they exist.

## 2.2 Boltzmann Transport Equation for Plasmas

### 2.2.1 Application to Charged Species

The Boltzmann transport equation has been initially established for a dilute gas of neutral molecules, but it may also be used in weakly ionized classical kinetic plasmas to describe the electron-molecule and ion-molecule collisions. A plasma is called classical kinetics when the electron number density  $n_e$  is low enough to exist a high number of electrons around each ion in the interior of the Debye sphere, i.e. when  $d_e \ll \lambda_D$ , with  $d_e \sim n_e^{-1/3}$  denoting the average distance between two electrons and  $\lambda_D$  the Debye length given by

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T_e}{e^2 n_e}}. \quad (2.68)$$

As seen in Sect. 1.3.1, the Debye length provides a measure of the distance over which the influence of the field of a charged particle is felt by the other particles. The Debye length defines the radius of a cloud around each charge particle outside which the plasma will tend to remain neutral. The condition for a classical kinetic plasma is hence

$$n_e \ll \left( \frac{\epsilon_0 k_B T_e}{e^2} \right)^3. \quad (2.69)$$

In the case of  $k_B T_e = 1$  eV (i.e.  $T_e = 11,604$  K), we have  $n_e \ll 1.68 \times 10^{23} \text{ m}^{-3}$ .

Under such conditions, the electrons produce a screen effect around each ion and the coulomb interactions are vanished for distances larger than the Debye length having the electric potential the form of the Yukawa potential

$$V(r) = \frac{Ze}{4\pi\epsilon_0 r} \exp\left(-\frac{r}{\lambda_D}\right). \quad (2.70)$$

The electron-ion interactions are of short distance due to this screening and they take place only as the electron trajectory with respect to an ion has an impact parameter with  $p < \lambda_D$ . Such plasmas are termed classical kinetic in parallel with the classical kinetic theory of gases, in which collective effects do not exist. In this category are included the plasmas created by conventional discharges. For instance, at  $k_B T_e = 1$  eV and  $n_e = 10^{16} \text{ m}^{-3}$ , we obtain  $\lambda_D = 7.43 \times 10^{-5} \text{ m}$  and  $d_e = 2.15 \times 10^{-6} \text{ m}$ . The number of electrons in the interior of the Debye sphere is  $N_e = \frac{4}{3}\pi\lambda_D^3 n_e = 1.72 \times 10^4$ . However, even in the case of classical kinetic plasmas, the Boltzmann equation cannot be used to describe the interactions between charged species, because these interactions do not occur through true collisions, but via a continuous addition of small deviations in the interior of the Debye sphere. The kinetic equation to be used for interactions of this type is the Fokker-Planck equation



(see e.g. Allis 1956). The Boltzmann equation may be used hence to describe non-Coulomb interactions, such as electron-molecule and ion-molecule collisions (Allis 1956; Nicholson 1983).

Furthermore, the ionization degree  $n_e/n_o$ , with  $n_e = n_i$ , and being  $n_e$ ,  $n_i$  and  $n_o$  the density of electrons, ions and molecules, must be low enough, in order the effects produced by the electron-electron, electron-ion and ion-ion interactions upon the trajectories of charged species are vanishingly small. The trajectories are then determined by collisions with neutral species only.

In a plasma there are different species present and we need to establish different Boltzmann equations. In the case of electrons, for instance, the kinetic equation yielding to determine the electron distribution function takes the form

$$\frac{\partial f_e}{\partial t} + \left( \mathbf{v}_e \cdot \frac{\partial f_e}{\partial \mathbf{r}} \right) + \left( \frac{\mathbf{F}_e}{m} \cdot \frac{\partial f_e}{\partial \mathbf{v}_e} \right) = \left( \frac{\partial f_e}{\partial t} \right)_{coll}, \quad (2.71)$$

in which the collision term includes, in principle, the contributions of electron-electron ( $e-e$ ), electron-ion ( $e-i$ ), and electron-molecule ( $e-o$ ) interactions, but where the third term is dominant

$$\left( \frac{\partial f_e}{\partial t} \right)_{coll} = \left( \frac{\partial f_e}{\partial t} \right)_{e-e} + \left( \frac{\partial f_e}{\partial t} \right)_{e-i} + \left( \frac{\partial f_e}{\partial t} \right)_{e-o} \simeq \left( \frac{\partial f_e}{\partial t} \right)_{e-o}. \quad (2.72)$$

Thus, in a weakly or medium ionized plasma, the collision terms for interactions between charged species are discarded in the kinetic equation for electrons, as compared with the term for electron-neutral collisions.

On the contrary, in the case of molecules the collisions between two molecules are the dominant interactions, which associated with the absence of spatial inhomogeneities, i.e. null gradients in the space of positions, and null external forces acting upon the molecules, allows to obtain an equation in which the interactions occurs among one type of particles only

$$\frac{\partial F_o}{\partial t} = \left( \frac{\partial F_o}{\partial t} \right)_{o-o}. \quad (2.73)$$

Assuming further the gas in equilibrium with the walls of the container, we obtain the Maxwell-Boltzmann distribution at temperature  $T_o$  as solution for this equation

$$F_o(\mathbf{v}_o) = n_o \left( \frac{M}{2\pi k_B T_o} \right)^{3/2} \exp \left( - \frac{M|\mathbf{v}_o|^2}{2k_B T_o} \right), \quad (2.74)$$

in which  $n_o$  is the gas number density and  $M$  the mass of molecules.

### 2.2.2 Vlasov Equation

In plasmas moderately ionized the electron density may not be so low that the long-distance interactions may be totally discarded, due to an incomplete screening produced by the electrons. In this case the interactions do not act as true collisions, since there are appreciable distances involved. Further, if the correlations at such distances are neglected, we may simply assume that the double distribution function is  $f_{12} = f_1 f_2$ . This condition introduced in (2.59) allows to write for the interaction term

$$\begin{aligned} \left( \frac{\partial f_1}{\partial t} \right)_{int} &= - \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} \left( \frac{\mathbf{F}_{12}}{m} \cdot \frac{\partial f_{12}}{\partial \mathbf{v}_1} \right) d\mathbf{r}_2 d\mathbf{v}_2 \\ &= - \left( \frac{\partial f_1}{\partial \mathbf{v}_1} \cdot \frac{1}{m} \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} \mathbf{F}_{12} f_2 d\mathbf{r}_2 d\mathbf{v}_2 \right). \end{aligned} \quad (2.75)$$

Introducing now the space-charge force defined by

$$\mathbf{F}'_1 = \int_{\mathbf{r}_2} \int_{\mathbf{v}_2} \mathbf{F}_{12} f_2 d\mathbf{r}_2 d\mathbf{v}_2, \quad (2.76)$$

or, in the case of a velocity independent force, by

$$\mathbf{F}'_1 = \int_{\mathbf{r}_2} \mathbf{F}_{12} n_2 d\mathbf{r}_2, \quad (2.77)$$

we may write then the interaction term (2.75) as

$$\left( \frac{\partial f_1}{\partial t} \right)_{int} = - \left( \frac{\partial f_1}{\partial \mathbf{v}_1} \cdot \frac{\mathbf{F}'_1}{m} \right). \quad (2.78)$$

This term has the same form as that due to the external forces in the left-hand side member of equation (2.60), so that as long-range interactions only exist the kinetic equation takes the form

$$\frac{\partial f_1}{\partial t} + \left( \mathbf{v}_1 \cdot \frac{\partial f_1}{\partial \mathbf{r}_1} \right) + \left( \frac{\mathbf{F}_1 + \mathbf{F}'_1}{m} \cdot \frac{\partial f_1}{\partial \mathbf{v}_1} \right) = 0. \quad (2.79)$$

The equation (2.79) known as Vlasov equation is a collisionless kinetic equation in which both the external forces applied to the charged particles and the space-charge forces resulting from Coulomb interactions are included. It allows to study, in a self-consistent manner, the collective movement of charged particles in moderately ionized plasmas, where the distributions are strongly non-Maxwellian, and therefore a fluid description is not accurate.

## 2.3 Collision Phenomena in Plasmas

### 2.3.1 Electron-Neutral Elastic Collisions

Let us consider a binary collision  $\{\mathbf{v}_e, \mathbf{v}_o\} \rightarrow \{\mathbf{v}'_e, \mathbf{v}'_o\}$  between an electron and a neutral molecule. In a collision, the positions of the particles do not change appreciably but their velocities may change a lot. Here, we do not consider the modification of velocities during the collision but only the velocities before and after the particles are within the range of their interaction forces. Introducing the position and the velocity of the centre-of-mass system, we get

$$\mathbf{R} = \frac{m \mathbf{r}_e + M \mathbf{r}_o}{m + M} \quad (2.80)$$

$$\mathbf{V} = \frac{d\mathbf{R}}{dt} = \frac{m}{m + M} \mathbf{v}_e + \frac{M}{m + M} \mathbf{v}_o. \quad (2.81)$$

where  $\mathbf{v}_e = d\mathbf{r}_e/dt$  and  $\mathbf{v}_o = d\mathbf{r}_o/dt$  are the electron and molecule velocities. Introducing now the relative velocity  $\mathbf{v} = \mathbf{v}_e - \mathbf{v}_o$ , the equation (2.81) can take one of the forms

$$\mathbf{v}_e - \mathbf{V} = \frac{M}{m + M} \mathbf{v} \quad (2.82)$$

$$\mathbf{v}_o - \mathbf{V} = -\frac{m}{m + M} \mathbf{v}. \quad (2.83)$$

In the same manner, we may write for the velocities after the collision

$$\mathbf{v}'_e - \mathbf{V}' = \frac{M}{m + M} \mathbf{v}' \quad (2.84)$$

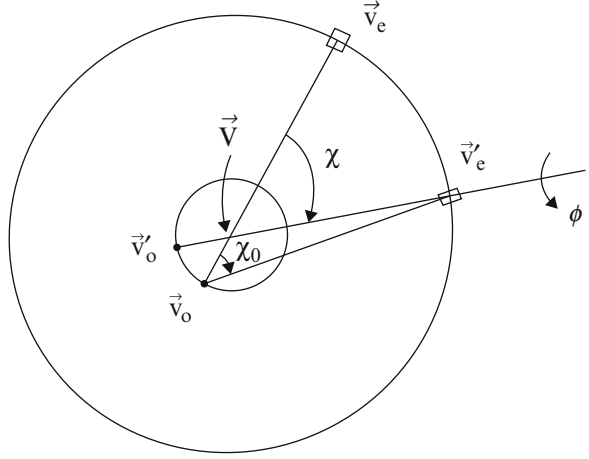
$$\mathbf{v}'_o - \mathbf{V}' = -\frac{m}{m + M} \mathbf{v}', \quad (2.85)$$

where  $\mathbf{v}' = \mathbf{v}'_e - \mathbf{v}'_o$  and  $\mathbf{V}' = \mathbf{V}$ . During the collision the velocity of the centre-of-mass,  $\mathbf{V}$ , remains fixed and if the collision is elastic, the relative velocity has the same magnitude before and after the collision,  $|\mathbf{v}'| = |\mathbf{v}|$ , but it changes in direction by an angle  $\chi$  called scattering angle in centre-of-mass system (see Fig. 2.5 and Appendix A.2.2). The centre-of-mass system divides the relative velocity inversely proportional to the masses, so that the velocities of electrons and molecules remain over two spheres in velocity space, centred in the centre-of-mass velocity, with the radii

$$\frac{M}{m + M} |\mathbf{v}| \quad \text{and} \quad \frac{m}{m + M} |\mathbf{v}|. \quad (2.86)$$

The radius of molecules' sphere is very small compared with that of electrons.

**Fig. 2.5** Geometry of an elastic collision between an electron and a molecule. If the molecule is at rest before the collision the origin of the system is at  $\mathbf{v}_0$ . The circles are spheres centred in  $\mathbf{V} = \mathbf{V}'$  (Allis 1956)



From equations (2.83) and (2.84), we still have

$$\mathbf{v}'_e - \mathbf{v}_0 = \frac{M}{m+M} \mathbf{v}' + \frac{m}{m+M} \mathbf{v} \quad (2.87)$$

and using the equality  $|\mathbf{v}| = |\mathbf{v}'|$ , we obtain

$$\begin{aligned} |\mathbf{v}'_e - \mathbf{v}_0|^2 &= (\mathbf{v}'_e - \mathbf{v}_0) \cdot (\mathbf{v}'_e - \mathbf{v}_0) \\ &= \frac{M^2}{(m+M)^2} |\mathbf{v}|^2 + \frac{m^2}{(m+M)^2} |\mathbf{v}|^2 + \frac{2mM}{(m+M)^2} (\mathbf{v} \cdot \mathbf{v}') \\ &= |\mathbf{v}|^2 - \frac{2mM}{(m+M)^2} |\mathbf{v}|^2 (1 - \cos \chi), \end{aligned} \quad (2.88)$$

with  $\chi$  denoting the scattering angle between the relative velocities  $\mathbf{v}$  and  $\mathbf{v}'$ .

If the molecule is at rest before the collision, the origin of velocity coordinates should be taken at  $\mathbf{v}_0$  and we have  $\mathbf{v} = \mathbf{v}_e$ . The electron velocities before and after the collision are then given by  $\overline{\mathbf{v}_0 \mathbf{v}_e}$  and  $\overline{\mathbf{v}_0 \mathbf{v}'_e}$  in Fig. 2.5. From equation (2.88) we obtain in this case

$$v_e'^2 = v_e^2 - \frac{2mM}{(m+M)^2} v_e^2 (1 - \cos \chi) \quad (2.89)$$

and being  $u = \frac{1}{2} m v_e^2$  the initial kinetic energy of the electron, the small fractional energy lost in striking a molecule at rest,  $\Delta u = u - u'$ , is

$$\frac{\Delta u}{u} = \frac{2mM}{(m+M)^2} (1 - \cos \chi). \quad (2.90)$$

This energy goes into the recoil energy of the molecule.

Another useful relation from the triangle  $\mathbf{v}_0$ ,  $\mathbf{V}$  and  $\mathbf{v}'_e$  gives the scattering angle in the laboratory system,  $\chi_0$ , in terms of the scattering angle in the centre-of-mass system,  $\chi$ , (see Appendix A.2.3)

$$\sin^2 \chi_0 = \frac{\sin^2 \chi}{1 + (m/M)^2 + (2m/M) \cos \chi}. \quad (2.91)$$

For practical purposes  $\chi_0 \simeq \chi$  because of  $m/M \ll 1$ .

The scattering cross section  $\sigma(v, \Omega) \equiv \sigma(v, \chi, \phi)$  is deduced from the force law as a function of  $v = |\mathbf{v}|$  and  $\chi$ , and it is measured experimentally in terms of  $|\mathbf{v}_e|$  and  $\chi_0$ . The dependence on the azimuthal angle  $\phi$  is never known so that it is assumed that  $\sigma$  does not depend on  $\phi$ .

### 2.3.2 Electron-Neutral Inelastic Collisions

Let us consider now an inelastic binary collision expressed through the reaction



in which a mono-energetic beam of particles (1) impinges on a sample of particles (2) at rest in a given system (see Fig. 2.6). From energy conservation, we have

$$E_1 + E_2 \rightarrow E_3 + E_4 + \Delta E, \quad (2.93)$$

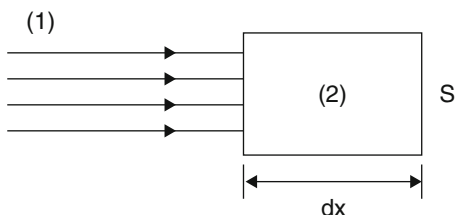
being  $\Delta E$  the energy threshold of the reaction.

The number of reactions of this type produced by one particle (1) striking on a sample of particles (2) of width  $dx$  is

$$n_2 \sigma(v) dx, \quad (2.94)$$

being  $n_2$  the density of particles (2) and  $\sigma(v)$  the collision cross section. Here,  $\sigma(v)$  is the total cross section, function of the absolute value of the relative velocity  $|\mathbf{v}| = |\mathbf{v}_1 - \mathbf{v}_2|$ , i.e. it does not specify the final scattering angles of the particles and, therefore, it does not specify as the final energy is distributed between the two product particles. If the particles (2) are not in rest but instead they have the velocity  $\mathbf{v}_2$ , the number of reactions produced by one particle (1) impinging the particles (2)

**Fig. 2.6** Mono-energetic beam of particles (1) impinging on a sample of particles (2) of width  $dx$  and area  $S$



with velocities  $\mathbf{dv}_2$  defined in the extremity of the velocity  $\mathbf{v}_2$  is given by

$$f(\mathbf{v}_2) \mathbf{dv}_2 \sigma(v) dx, \quad (2.95)$$

being  $f(\mathbf{v}_2)$  the velocity distribution of particles (2).

Let us consider now a beam of particles with velocity distribution  $f(\mathbf{v}_1)$ . The number of particles of velocities  $\mathbf{dv}_1$  defined in the extremity of velocity  $\mathbf{v}_1$  crossing a sample of particles (2) of area  $S$  in the time  $dt$  is

$$f(\mathbf{v}_1) \mathbf{dv}_1 v S dt, \quad (2.96)$$

where  $v$  is the absolute relative velocity, and hence the number of probable reactions produced in the volume of the sample  $\mathbf{dr} = S dx$ , in the time interval  $dt$ , by the crossing of the two beams is

$$dN = f(\mathbf{v}_1) \mathbf{dv}_1 v S dt f(\mathbf{v}_2) \mathbf{dv}_2 \sigma(v) dx. \quad (2.97)$$

The number of reactions produced in the sample, per volume and time units, is

$$\frac{dN}{\mathbf{dr} dt} = v \sigma(v) f(\mathbf{v}_1) f(\mathbf{v}_2) \mathbf{dv}_1 \mathbf{dv}_2. \quad (2.98)$$

If we assume now that the particles (1) and (2) may have different velocities, we obtain for the total rate of reaction (2.92), in  $\text{m}^{-3} \text{s}^{-1}$

$$\begin{aligned} R &= \int_{\mathbf{v}_1} \int_{\mathbf{v}_2} v \sigma(v) f(\mathbf{v}_1) f(\mathbf{v}_2) \mathbf{dv}_1 \mathbf{dv}_2 \\ &= n_1 n_2 C \end{aligned} \quad (2.99)$$

being  $C$  the collisional rate coefficient (in  $\text{m}^3 \text{s}^{-1}$ ) of the reaction defined as

$$C = \langle v \sigma(v) \rangle = \frac{1}{n_1 n_2} \int_{\mathbf{v}_1} \int_{\mathbf{v}_2} v \sigma(v) f(\mathbf{v}_1) f(\mathbf{v}_2) \mathbf{dv}_1 \mathbf{dv}_2. \quad (2.100)$$

In equation (2.98) the produced particles (3,4) may share the final energy  $E_3 + E_4$  through different forms. They depend of the scattering angles of the two particles. Let us define as  $\sigma(v, \chi, \phi)$  the differential cross section associated with the inelastic reaction (2.92), in which two particles (1,2) with the relative velocity  $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$  are transformed into two particles (3,4) with relative velocity  $\mathbf{v}' = \mathbf{v}'_3 - \mathbf{v}'_4$ , being  $|\mathbf{v}'| < |\mathbf{v}|$ , and where  $\chi$  and  $\phi$  are the scattering angles of velocity  $\mathbf{v}'$  with respect to  $\mathbf{v}$ . The probable number of reactions, per volume and time units, in which after the collision the two product particles are in the scattered solid-angle element  $d\Omega = \sin \chi d\chi d\phi$ , is

$$\frac{dN(\Omega)}{d\mathbf{r} dt} = v \sigma(v, \Omega) f(\mathbf{v}_1) f(\mathbf{v}_2) d\mathbf{v}_1 d\mathbf{v}_2 d\Omega. \quad (2.101)$$

In the case of an inelastic collision with velocities  $\{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\mathbf{v}'_3, \mathbf{v}'_4\}$ , we have from equations (2.82) and (2.83)

$$\mathbf{v}_1 - \mathbf{V} = \frac{m_2}{m_1 + m_2} \mathbf{v} \quad (2.102)$$

$$\mathbf{v}_2 - \mathbf{V} = -\frac{m_1}{m_1 + m_2} \mathbf{v}, \quad (2.103)$$

being  $\mathbf{V}$  the velocity of the centre-of-mass system. Before the collision the two particles have the momenta  $m_1(\mathbf{v}_1 - \mathbf{V})$  and  $m_2(\mathbf{v}_2 - \mathbf{V})$ , with equal absolute values and reverse directions, which may be expressed as

$$m_1(\mathbf{v}_1 - \mathbf{V}) = -m_2(\mathbf{v}_2 - \mathbf{V}) = \mu \mathbf{v}, \quad (2.104)$$

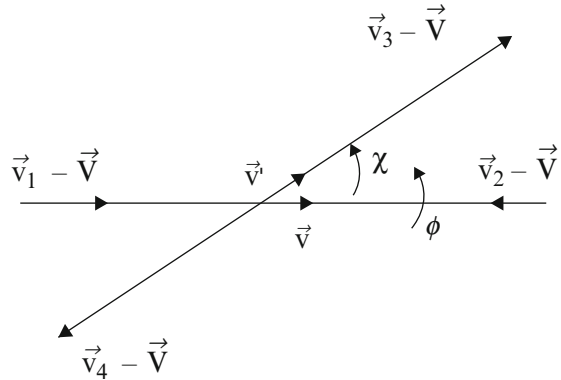
being  $\mu = m_1 m_2 / (m_1 + m_2)$  the reduced mass. After the collision the situation is likewise being now

$$m_3(\mathbf{v}'_3 - \mathbf{V}') = -m_4(\mathbf{v}'_4 - \mathbf{V}') = \mu' \mathbf{v}', \quad (2.105)$$

with  $\mu' = m_3 m_4 / (m_3 + m_4)$  (see Fig. 2.7). The relative velocity after the collision is determined by energy conservation

$$\frac{1}{2} \mu' |\mathbf{v}'|^2 = \frac{1}{2} \mu |\mathbf{v}|^2 - \Delta E. \quad (2.106)$$

**Fig. 2.7** Inelastic collision  $\{X_1, X_2\} \rightarrow \{X_3, X_4\}$  shown in the centre-of-mass system (Delcroix 1963, 1966)



### 2.3.3 Collision Term for Inelastic Collisions

Using a similar procedure used to derive the collision term of the Boltzmann equation for elastic collisions (2.35), we may derive now a similar term for the inelastic collisions as defined through reaction (2.92). The number of collisions, per volume unit in the space of positions and time unit, that result in the removal of a molecule from the velocities  $\mathbf{v}_1$  at the extremity of  $\mathbf{v}_1$  is given by

$$\left(\frac{\partial f_1}{\partial t}\right)_{coll} \mathbf{v}_1 = -f(\mathbf{r}, \mathbf{v}_1, t) \mathbf{v}_1 \int_{\Omega} \int_{\mathbf{v}_2} f(\mathbf{r}, \mathbf{v}_2, t) v \sigma_{12}^{34}(v, \Omega) d\Omega \mathbf{v}_2, \quad (2.107)$$

where  $\sigma_{12}^{34}(v, \Omega)$  is the differential cross section of reaction (2.92) and  $v = |\mathbf{v}| = |\mathbf{v}_1 - \mathbf{v}_2|$  is the absolute value of the relative velocity.

Let us consider now the reverse reaction



The number of collisions, per volume and time units, that result in the entrance of a molecule into the velocities  $\mathbf{v}_1$  is

$$\left(\frac{\partial f_1}{\partial t}\right)_{coll} \mathbf{v}_1 = f(\mathbf{r}, \mathbf{v}'_3, t) \mathbf{v}'_3 \int_{\Omega} \int_{\mathbf{v}'_4} f(\mathbf{r}, \mathbf{v}'_4, t) v' \sigma_{34}^{12}(v', -\Omega) (-d\Omega) \mathbf{v}'_4, \quad (2.109)$$

being now  $\sigma_{34}^{12}(v', -\Omega)$  the differential cross section of reaction (2.108) and  $v' = |\mathbf{v}'| = |\mathbf{v}'_3 - \mathbf{v}'_4|$  the absolute value of the relative velocity of particles (3) and (4). However, from micro-reversibility, or detailed balance, between the direct and reverse processes, we may write the following relation

$$v' \sigma_{34}^{12}(v', -\Omega) (-d\Omega) \mathbf{v}'_3 \mathbf{v}'_4 = v \sigma_{12}^{34}(v, \Omega) d\Omega \mathbf{v}_1 \mathbf{v}_2, \quad (2.110)$$

so that including this relation in equation (2.109), we obtain the following collision term gathering equations (2.107) and (2.109)

$$\left(\frac{\partial f_1}{\partial t}\right)_{coll} = \int_{\Omega} \int_{\mathbf{v}_2} (f'_3 f'_4 - f_1 f_2) v \sigma_{12}^{34}(v, \Omega) d\Omega \mathbf{v}_2, \quad (2.111)$$

in which the following abbreviations have been used:  $f_1 \equiv f(\mathbf{r}, \mathbf{v}_1, t)$ , etc.

In the case of a collision where the partners (1) and (3) are the same



the collision term (2.111) needs to be modified to take into account the indiscernibility of particles (1) and (3). In this case, besides the transitions  $\{\mathbf{v}_1, \mathbf{v}_2\} \rightleftharpoons \{\mathbf{v}'_1, \mathbf{v}'_4\}$ , we must also include the transitions  $\{\mathbf{v}_1, \mathbf{v}_4\} \rightleftharpoons \{\mathbf{v}''_1, \mathbf{v}''_2\}$ . The collision term takes hence the form



$$\begin{aligned} \left( \frac{\partial f_1}{\partial t} \right)_{coll} = & \int_{\Omega} \int_{\mathbf{v}_2} (f'_1 f'_4 - f_1 f_2) v_{12} \sigma_{12}^{14}(v_{12}, \Omega) d\Omega d\mathbf{v}_2 \\ & + \int_{\Omega} \int_{\mathbf{v}_4} (f'_1 f''_2 - f_1 f_4) v_{14} \sigma_{14}^{12}(v_{14}, \Omega) d\Omega d\mathbf{v}_4, \end{aligned} \quad (2.113)$$

being the relative velocities to be considered in the two terms  $\mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2$  and  $\mathbf{v}_{14} = \mathbf{v}_1 - \mathbf{v}_4$ , respectively.

Finally, let us consider now the direct and reverse electron-molecule inelastic collisions



with  $X$  and  $X^*$  denoting the ground-state and an excited-state of a given molecule. We may assume  $m_2 = m_4 \simeq \infty$  in reaction (2.113) and therefore  $\mathbf{v}_2 = \mathbf{v}_4 = 0$ . The molecule acts as a fixed centre of interactions and the centre-of-mass coincides with this fixed centre. We may replace then the velocity distributions  $f_2$  and  $f_4$  with delta functions  $n_2 \delta(\mathbf{v}_2)$  and  $n_4 \delta(\mathbf{v}_4)$  having, for instance, for particle (2)

$$\int_{\mathbf{v}_2} f_2 d\mathbf{v}_2 = \int_{\mathbf{v}_2} n_2 \delta(\mathbf{v}_2) d\mathbf{v}_2 = n_2, \quad (2.115)$$

being the relative velocities equal to  $\mathbf{v}_1$  in both terms of equation (2.113). The collision term (2.113) takes hence the form

$$\begin{aligned} \left( \frac{\partial f_1}{\partial t} \right)_{coll} = & \int_{\Omega} \int_{\mathbf{v}_2} f'_1 n_4 \delta(\mathbf{v}'_4) v_1 \sigma_{12}^{14}(v_1, \Omega) d\Omega d\mathbf{v}_2 \\ & - n_2 \int_{\Omega} f_1 v_1 \sigma_{12}^{14}(v_1, \Omega) d\Omega \\ & + \int_{\Omega} \int_{\mathbf{v}_4} f'_1 n_2 \delta(\mathbf{v}''_2) v_1 \sigma_{14}^{12}(v_1, \Omega) d\Omega d\mathbf{v}_4 \\ & - n_4 \int_{\Omega} f_1 v_1 \sigma_{14}^{12}(v_1, \Omega) d\Omega. \end{aligned} \quad (2.116)$$

Introducing now the total cross sections in the second and fourth terms

$$\int_{\Omega} \sigma_{12}^{14}(v_1, \Omega) d\Omega = \sigma_{12}^{14}(v_1) \quad (2.117)$$

$$\int_{\Omega} \sigma_{14}^{12}(v_1, \Omega) d\Omega = \sigma_{14}^{12}(v_1) \quad (2.118)$$

and the following micro-reversibility relations in the first and third terms

$$v_1 \sigma_{12}^{14}(v_1, \Omega) d\Omega d\mathbf{v}_1 d\mathbf{v}_2 = v'_1 \sigma_{14}^{12}(v'_1, -\Omega) (-d\Omega) d\mathbf{v}'_1 d\mathbf{v}'_4 \quad (2.119)$$

$$v_1 \sigma_{14}^{12}(v_1, \Omega) d\Omega d\mathbf{v}_1 d\mathbf{v}_4 = v''_1 \sigma_{12}^{14}(v''_1, -\Omega) (-d\Omega) d\mathbf{v}''_1 d\mathbf{v}''_2, \quad (2.120)$$

we may rewrite equation (2.116) as follows

$$\begin{aligned} \left( \frac{\partial f_1}{\partial t} \right)_{coll} = & \int_{\Omega} \int_{\mathbf{v}'_4} f'_1 n_4 \delta(\mathbf{v}'_4) v'_1 \sigma_{14}^{12}(v'_1, \Omega) d\Omega \frac{d\mathbf{v}'_1}{d\mathbf{v}_1} d\mathbf{v}'_4 - n_2 f_1 v_1 \sigma_{12}^{14}(v_1) \\ & + \int_{\Omega} \int_{\mathbf{v}''_2} f''_1 n_2 \delta(\mathbf{v}''_2) v''_1 \sigma_{12}^{14}(v''_1, \Omega) d\Omega \frac{d\mathbf{v}''_1}{d\mathbf{v}_1} d\mathbf{v}''_2 - n_4 f_1 v_1 \sigma_{14}^{12}(v_1). \end{aligned} \quad (2.121)$$

Taking into account that in the first and third terms of equation (2.121) we still have

$$\int_{\mathbf{v}'_4} n_4 \delta(\mathbf{v}'_4) d\mathbf{v}'_4 = n_4 \quad \text{and} \quad \int_{\mathbf{v}''_2} n_2 \delta(\mathbf{v}''_2) d\mathbf{v}''_2 = n_2 \quad (2.122)$$

and the following relationships between the velocity volume elements associated through an inelastic collision

$$d\mathbf{v}'_1 = \frac{v'_1}{v_1} d\mathbf{v}_1 \quad \text{and} \quad d\mathbf{v}''_1 = \frac{v''_1}{v_1} d\mathbf{v}_1, \quad (2.123)$$

the equation (2.121) may still be rewritten with the form

$$\begin{aligned} \left( \frac{\partial f_1}{\partial t} \right)_{coll} = & n_4 \frac{v_1'^2}{v_1} \int_{\Omega} f'_1 \sigma_{14}^{12}(v'_1, \Omega) d\Omega - n_2 v_1 f_1 \sigma_{12}^{14}(v_1) \\ & + n_2 \frac{v_1''^2}{v_1} \int_{\Omega} f''_1 \sigma_{12}^{14}(v''_1, \Omega) d\Omega - n_4 v_1 f_1 \sigma_{14}^{12}(v_1). \end{aligned} \quad (2.124)$$

The four terms in equation (2.124) refer by order to the following transitions

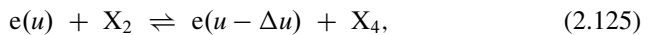
$$\{\mathbf{v}'_1, \mathbf{v}_4\} \rightarrow \{\mathbf{v}_1, \mathbf{v}_2\}$$

$$\{\mathbf{v}_1, \mathbf{v}_2\} \rightarrow \{\mathbf{v}'_1, \mathbf{v}_4\}$$

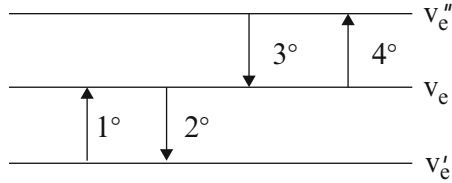
$$\{\mathbf{v}''_1, \mathbf{v}_2\} \rightarrow \{\mathbf{v}_1, \mathbf{v}_4\}$$

$$\{\mathbf{v}_1, \mathbf{v}_4\} \rightarrow \{\mathbf{v}''_1, \mathbf{v}_2\}.$$

To attribute significance we may assume that the reaction (1,2)  $\rightarrow$  (1,4) is endothermic, with the species (1) being an electron



in which a molecule in state  $X_2$  goes to a higher energy state  $X_4$ , suffering the electron an inelastic collision, whereas the reaction (1,4)  $\rightarrow$  (1,2) is exothermic, suffering the electron a superelastic collision, also called an inelastic collision of



**Fig. 2.8** Decrease and increase of the electron velocity as a result of an inelastic ( $v_e \rightarrow v_e'$ ;  $v_e'' \rightarrow v_e$ ) or of a superelastic ( $v_e' \rightarrow v_e$ ;  $v_e \rightarrow v_e''$ ) collision, respectively (Delcroix 1963, 1966)

second kind. In the two collisions the electron of energy  $u$  loses or gains the energy  $\Delta u$ , respectively.

Let us consider now a given molecule with different quantum states  $\dots, i, \dots, j, \dots$ . If among them we isolate the states  $i$  and  $j$ , being  $u_{ij} = (E_j - E_i) > 0$ , and  $v' < v < v''$  denoting three different values of electron velocity obeying to

$$v''^2 - v^2 = v^2 - v'^2 = \frac{2}{m} u_{ij}, \quad (2.126)$$

the collision term for inelastic collisions (2.124) may be written under the form, in the case of the electron distribution function  $f_e \equiv f(\mathbf{v}_e)$ ,

$$\begin{aligned} \left( \frac{\partial f_e}{\partial t} \right)_{coll} = & \sum_{i,j} \left\{ n_j \frac{v_e'^2}{v_e} \int_{\Omega} f_e' \sigma_{ji}(v_e', \Omega) d\Omega - n_i v_e f_e \sigma_{ij}(v_e) \right\} \\ & + \sum_{i,j} \left\{ n_i \frac{v_e''^2}{v_e} \int_{\Omega} f_e'' \sigma_{ij}(v_e'', \Omega) d\Omega - n_j v_e f_e \sigma_{ji}(v_e) \right\}, \quad (2.127) \end{aligned}$$

being  $f_e' \equiv f(\mathbf{v}_e')$  and  $f_e'' \equiv f(\mathbf{v}_e'')$  and where the sum indicates the possibility of different reactions to be considered.

The physical meaning of the four terms in equation (2.127) can be well understood through the diagram shown in Fig. 2.8.

The vertical rows from left to right represent by order the four terms of equation (2.127). The down rows represent the inelastic collisions in which an electron diminishes the absolute value of its velocity from  $v_e \rightarrow v_e'$  and from  $v_e'' \rightarrow v_e$ , whereas the up rows represent the reverse superelastic collisions, in which an electron increases the absolute value of its velocity from  $v_e' \rightarrow v_e$  and from  $v_e \rightarrow v_e''$ . The second and the fourth terms are loss terms for the electrons of velocity  $v_e$ , while the first and the third terms are gain terms. In this latter case, due to collisions with colder electrons and hotter electrons, respectively.

In the case of the terms taking into account the exit of an electron from velocity  $v_e$ , either due to an inelastic collision (second term) or a superelastic collision (fourth term), we do not need to specify the form as the electron enters into the new velocity element, so that the total cross section is enough. On the contrary, in

the case of the terms describing an entrance at velocity  $v_e$ , due to a superelastic collision (first term) or due to an inelastic collision (third term), we need to specify the orientation of the velocity vector before the collision, so that the differential cross section is needed. For instance,  $\sigma_{ij}(v_e'', \Omega)$  is the differential cross section of the inelastic collision for an electron with initial velocity  $\mathbf{v}_e''$  and final velocity  $\mathbf{v}_e$ , with  $v_e < v_e''$ , in which a rotation from  $\mathbf{v}_e''$  to  $\mathbf{v}_e$  occurs.

### 2.3.4 Rate Coefficients for Direct and Reverse Processes

In the case of inelastic and superelastic electron-molecule collisions of the type



with  $X_i$  denoting a molecule in a state with the level-energy  $E_i$  and  $X_j$  the same molecule in a upper state with the level-energy  $E_j > E_i$ , and since we have  $|\mathbf{v}_e| \gg |\mathbf{v}_0|$ , and therefore  $|\mathbf{v}| \simeq |\mathbf{v}_e|$ , we obtain from equation (2.100) the following expression for the rate coefficient of this inelastic process by electron impact

$$C_{ij} = \langle v_e \sigma_{ij}(v_e) \rangle = \frac{1}{n_e n_i} \int_{\mathbf{v}_e} \int_{\mathbf{v}_i} v_e \sigma_{ij}(v_e) f(\mathbf{v}_e) F(\mathbf{v}_i) d\mathbf{v}_e d\mathbf{v}_i. \quad (2.129)$$

Since for the heavy species we have

$$\int_{\mathbf{v}_i} F(\mathbf{v}_i) d\mathbf{v}_i = n_i, \quad (2.130)$$

we still obtain

$$C_{ij} = \langle v_e \sigma_{ij}(v_e) \rangle = \frac{1}{n_e} \int_{\mathbf{v}_e} v_e \sigma_{ij}(v_e) f(\mathbf{v}_e) d\mathbf{v}_e. \quad (2.131)$$

We will show in Chap. 3 that because of the integrand function  $v_e \sigma_{ij}(v_e)$  is an isotropic function of the electron velocity, depending hence on the absolute value of the velocity only, the electron rate coefficient  $C_{ij}$  is given by

$$C_{ij} = \frac{1}{n_e} \int_{v_{ij}}^{\infty} v_e \sigma_{ij}(v_e) f_e^0(v_e) 4\pi v_e^2 dv_e, \quad (2.132)$$

with  $f_e^0(v_e)$  representing the isotropic component of the electron velocity distribution, i.e. the component that depends on the absolute value of the velocity only, and  $v_{ij} = \sqrt{2u_{ij}/m} = \sqrt{2(E_j - E_i)/m}$  is the change of velocity in result of the inelastic collision.

Under thermodynamic equilibrium conditions it should be verified by micro-reversibility

$$n_e n_i C_{ij} = n_e n_j C_{ji}, \quad (2.133)$$

being  $n_i$  and  $n_j$  the populations of the states  $X_i$  and  $X_j$ , respectively, and  $C_{ij}$  and  $C_{ji}$  the electron rate coefficients of the direct and reverse processes (2.128). Since at equilibrium conditions the populations  $n_i$  and  $n_j$  follow a Maxwell-Boltzmann distribution at the common temperature  $T_e = T_o$ , we may write

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} \exp\left(-\frac{u_{ij}}{k_B T_e}\right), \quad (2.134)$$

with  $g_i$  and  $g_j$  denoting the statistical weights, or degenerescency degrees (number of distinguished quantum states), of the states of energy  $E_i$  and  $E_j$ , respectively. Matching equations (2.133) and (2.134) we obtain the following relation between the superelastic and the inelastic electron rate coefficients

$$\frac{C_{ji}}{C_{ij}} = \frac{g_i}{g_j} \exp\left(\frac{u_{ij}}{k_B T_e}\right). \quad (2.135)$$

The rate coefficients are hence larger for the superelastic processes.

At the first sight, we could think that equation (2.135) is valid at thermodynamic equilibrium only. However, the sole condition at which this equation is obliged to fulfill is the velocity distribution of electrons to be a Maxwellian at a given temperature  $T_e$ , being hence possible to have  $T_e \neq T_o$ . As a matter of fact, looking at the rate coefficient of the inelastic process given by equation (2.132), we observe that it only acts the collision cross section, which is a quantity that has nothing to do with the macroscopic properties of the medium. Thus, when the velocity distribution of electrons is Maxwellian, the equation (2.135) is verified although the electrons may not be in equilibrium with the molecules.

However, it is still possible to derive from equation (2.135) a relationship between the electron cross sections for the direct and reverse reactions, which obviously does not depend on the electron velocity distribution. Starting by expressing the equation (2.132) in terms of the electron energy  $u = \frac{1}{2} m v_e^2$ , we obtain

$$C_{ij} = \frac{1}{n_e} \frac{8\pi}{m^2} \int_{u_{ij}}^{\infty} u \sigma_{ij}(u) f_e^0(u) du \quad (2.136)$$

and therefore the expression for the ratio  $C_{ji}/C_{ij}$  between the reverse and direct rate coefficients can be written as follows

$$\begin{aligned} \frac{C_{ji}}{C_{ij}} &= \frac{\int_0^{\infty} u \sigma_{ji}(u) f_e^0(u) du}{\int_{u_{ij}}^{\infty} u \sigma_{ij}(u) f_e^0(u) du} \\ &= \frac{\int_0^{\infty} u \sigma_{ji}(u) f_e^0(u) du}{\int_0^{\infty} (u + u_{ij}) \sigma_{ij}(u + u_{ij}) f_e^0(u + u_{ij}) du}. \end{aligned} \quad (2.137)$$

In the case of a Maxwellian distribution the following dependence exists

$$f_e^0(u) \propto \exp\left(-\frac{u}{k_B T_e}\right), \quad (2.138)$$

so that the ratio (2.137) takes the form

$$\frac{C_{ji}}{C_{ij}} = \frac{\int_0^\infty u \sigma_{ji}(u) \exp(-u/(k_B T_e)) du}{\int_0^\infty (u + u_{ij}) \sigma_{ij}(u + u_{ij}) \exp(-u/(k_B T_e)) du} \exp\left(\frac{u_{ij}}{k_B T_e}\right). \quad (2.139)$$

Since the ratio  $C_{ji}/C_{ij}$  also fulfills equation (2.135) and the integrals of equation (2.139) are extended over the same domain, we obtain the following relationship from the integrand functions of this latter equation

$$g_j u \sigma_{ji}(u) = g_i (u + u_{ij}) \sigma_{ij}(u + u_{ij}). \quad (2.140)$$

Equation (2.140) expresses a relationship between the cross sections of direct and reverse processes known as Klein-Rosseland relation. It is a microscopic universal relation, independent hence of the macroscopic properties of the medium.

## 2.4 Moments of the Boltzmann Equation

### 2.4.1 Conservation Laws and Continuity Equation

As we have seen in the previous sections, the electron distribution function for a plasma out of equilibrium can be obtained by solving the electron Boltzmann transport equation. Then, once the electron distribution is known, we can obtain the macroscopic quantities of physical interest for the plasma, such as the electron number density  $n_e$ , the electron drift velocity  $\mathbf{v}_{ed} = \langle \mathbf{v}_e \rangle$ , or the electron mean energy  $\langle u \rangle = \frac{1}{2} m \langle v_e^2 \rangle$ . However, the solution of the Boltzmann equation is generally a matter of great difficulty, so that when a given problem only needs the macroscopic variables it is preferable to use an analysis based on the moments of the Boltzmann equation, i.e. the fluid equations, without having to solve the Boltzmann equation (Huang 1963; Delcroix 1963, 1966; Liboff 1969; Cherrington 1980; Bittencourt 2004). The resulting equations are significantly simpler to use, since they describe the plasma from a macroscopic point of view.

The moments of the Boltzmann equations are conservation laws for any quantity  $X(\mathbf{r}, \mathbf{v}_e, t)$  associated with an electron located at position  $\mathbf{r}$ , velocity  $\mathbf{v}_e$ , and time  $t$ . In order to derive a conservation law for this general quantity  $X$ , let us consider equation (2.71), with the collision term with electron-neutral collisions only such as indicated in (2.72). Multiplying both members of (2.71) by  $X$ , we can write

$$X \frac{\partial f_e}{\partial t} + \sum_i X v_{ei} \frac{\partial f_e}{\partial x_i} + \sum_i X \frac{F_{ei}}{m} \frac{\partial f_e}{\partial v_{ei}} = X \left( \frac{\partial f_e}{\partial t} \right)_{e-o}. \quad (2.141)$$

Integrating this equation in velocity space and defining the velocity averaged value of  $X$  by

$$\langle X \rangle(\mathbf{r}, t) = \frac{1}{n_e} \int_{\mathbf{v}_e} X(\mathbf{r}, \mathbf{v}_e, t) f_e(\mathbf{r}, \mathbf{v}_e, t) \mathbf{d}\mathbf{v}_e, \quad (2.142)$$

we obtain for each one the left-hand side terms of equation (2.141)

$$\begin{aligned} \int_{\mathbf{v}_e} X \frac{\partial f_e}{\partial t} \mathbf{d}\mathbf{v}_e &= \frac{\partial}{\partial t} \left( \int_{\mathbf{v}_e} X f_e \mathbf{d}\mathbf{v}_e \right) - \int_{\mathbf{v}_e} \frac{\partial X}{\partial t} f_e \mathbf{d}\mathbf{v}_e \\ &= \frac{\partial}{\partial t} (n_e \langle X \rangle) - n_e \left\langle \frac{\partial X}{\partial t} \right\rangle; \end{aligned} \quad (2.143)$$

$$\begin{aligned} \int_{\mathbf{v}_e} X v_{ei} \frac{\partial f_e}{\partial x_i} \mathbf{d}\mathbf{v}_e &= \frac{\partial}{\partial x_i} \left( \int_{\mathbf{v}_e} X v_{ei} f_e \mathbf{d}\mathbf{v}_e \right) - \int_{\mathbf{v}_e} \frac{\partial X}{\partial x_i} v_{ei} f_e \mathbf{d}\mathbf{v}_e \\ &= \frac{\partial}{\partial x_i} (n_e \langle X v_{ei} \rangle) - n_e \left\langle \frac{\partial X}{\partial x_i} v_{ei} \right\rangle; \end{aligned} \quad (2.144)$$

$$\begin{aligned} \int_{\mathbf{v}_e} X \frac{F_{ei}}{m} \frac{\partial f_e}{\partial v_{ei}} \mathbf{d}\mathbf{v}_e &= \int_{\mathbf{v}_e} \frac{\partial}{\partial v_{ei}} \left( X \frac{F_{ei}}{m} f_e \mathbf{d}\mathbf{v}_e \right) - \int_{\mathbf{v}_e} \frac{\partial}{\partial v_{ei}} \left( X \frac{F_{ei}}{m} \right) f_e \mathbf{d}\mathbf{v}_e \\ &= - n_e \left\langle \frac{\partial}{\partial v_{ei}} \left( X \frac{F_{ei}}{m} \right) \right\rangle. \end{aligned} \quad (2.145)$$

The first term of equation (2.145) vanishes if  $f_e(\mathbf{r}, \mathbf{v}_e, t)$  is assumed to vanish when  $v_{ei} \rightarrow \infty$ .

The integration in velocity space of equation (2.141) can then be written as follows assuming velocity independent external forces ( $\partial F_{ei}/\partial v_{ei} = 0$ , although  $F_{ei}$  may depend on the components of the drift velocity as in the Lorentz force)

$$\begin{aligned} \frac{\partial}{\partial t} (n_e \langle X \rangle) - n_e \left\langle \frac{\partial X}{\partial t} \right\rangle + \sum_i \frac{\partial}{\partial x_i} (n_e \langle X v_{ei} \rangle) - \sum_i n_e \left\langle \frac{\partial X}{\partial x_i} v_{ei} \right\rangle \\ - \sum_i n_e \left\langle \frac{\partial X}{\partial v_{ei}} \right\rangle \frac{F_{ei}}{m} = \int_{\mathbf{v}_e} X \left( \frac{\partial f_e}{\partial t} \right)_{e-o} \mathbf{d}\mathbf{v}_e. \end{aligned} \quad (2.146)$$

In the case of electrons the independent conserved properties are obtained by making  $X = 1$ ,  $X = mv_{ei}$ , and  $X = \frac{1}{2} mv_e^2$ . We obtain then the equation for

electron number density conservation, i.e. the continuity equation, the equation for momentum conservation, and the equation for energy conservation.

When  $X = 1$ , we simply obtain

$$\frac{\partial n_e}{\partial t} + \sum_i \frac{\partial}{\partial x_i} (n_e \langle v_{ei} \rangle) = \int_{\mathbf{v}_e} \left( \frac{\partial f_e}{\partial t} \right)_{e-o} \mathbf{d}\mathbf{v}_e, \quad (2.147)$$

which can be written as

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_{ed}) = \left( \frac{\partial n_e}{\partial t} \right)_{e-o}, \quad (2.148)$$

being  $\mathbf{v}_{ed}$  the electron drift velocity. If the collisions do not modify the number of electrons, we have  $(\partial n_e / \partial t)_{e-o} = 0$ , otherwise as the production of secondary electrons or the electron attachment are included, the right-hand side member of equation (2.148) becomes non-null. In the case of ionizing collisions, the right-hand side member of equation (2.146) takes the following form using equation (2.36)

$$\begin{aligned} \int_{\mathbf{v}_e} X \left( \frac{\partial f_e}{\partial t} \right)_{e-o} \mathbf{d}\mathbf{v}_e &= \int_{\mathbf{v}_e} \int_{\Omega} \int_{\mathbf{v}_o} (X' f'_e F'_o - X f_e F_o) v_e \sigma_{ion}(v_e, \Omega) \mathbf{d}\mathbf{v}_e d\Omega d\mathbf{v}_o \\ &= n_o \int_{\mathbf{v}_e} \int_{\Omega} (X' f'_e - X f_e) v_e \sigma_{ion}(v_e, \Omega) \mathbf{d}\mathbf{v}_e d\Omega. \end{aligned} \quad (2.149)$$

Furthermore, in an ionizing collision  $X = 1$  and  $X' = 2$ , so that we obtain

$$\begin{aligned} \left( \frac{\partial n_e}{\partial t} \right)_{e-o} &= \int_{\mathbf{v}_e} \left( \frac{\partial f_e}{\partial t} \right)_{e-o} \mathbf{d}\mathbf{v}_e = n_o \int_{\mathbf{v}_e} \int_{\Omega} v_e \sigma_{ion}(v_e, \Omega) f_e \mathbf{d}\mathbf{v}_e d\Omega \\ &= n_o \int_{v_e} v_e \sigma_{ion}(v_e) f_e^0(v_e) 4\pi v_e^2 dv_e, \end{aligned} \quad (2.150)$$

being  $\sigma_{ion}(v_e)$  the total electron ionization cross section and, as it was already referred in equation (2.132) and it will be shown later on in Chap. 3,  $f_e^0(v_e)$  is the isotropic component of the electron velocity distribution. Considering the ionization frequency  $\nu_{ion}(v_e) = n_o v_e \sigma_{ion}(v_e)$ , we obtain for equation (2.150)

$$\left( \frac{\partial n_e}{\partial t} \right)_{e-o} = n_e \langle \nu_{ion} \rangle. \quad (2.151)$$

Finally, introducing the electron particle current density  $\Gamma_e = n_e \mathbf{v}_{ed}$ , the continuity equation (2.148) takes the form

$$\frac{\partial n_e}{\partial t} + (\nabla \cdot \Gamma_e) = n_e \langle \nu_{ion} \rangle. \quad (2.152)$$



### 2.4.2 Equation for Momentum Conservation

The equation of the second moment is the equation for momentum conservation and it can be derived using  $\mathbf{X} = m\mathbf{v}_e$  in equation (2.146) and integrating in velocity space

$$\begin{aligned} & \frac{\partial}{\partial t} (n_e m \langle \mathbf{v}_e \rangle) + \sum_i \frac{\partial}{\partial x_i} (n_e m \langle \mathbf{v}_e v_{ei} \rangle) \\ & - \sum_i n_e m \langle \frac{\partial \mathbf{v}_e}{\partial v_{ei}} \rangle \frac{F_{ei}}{m} = \int_{\mathbf{v}_e} m \mathbf{v}_e \left( \frac{\partial f_e}{\partial t} \right)_{e-o} d\mathbf{v}_e. \end{aligned} \quad (2.153)$$

Since  $v_{ei} = (v_{ei} - v_{edi}) + v_{edi}$ , being  $v_{edi}$  the  $i$ -th component of the electron drift velocity  $\mathbf{v}_{ed} = \langle \mathbf{v}_e \rangle$ , we may write the second left-hand side term of equation (2.153) under the form

$$\begin{aligned} & \sum_{i,j} \frac{\partial}{\partial x_i} \{ n_e m \langle ((v_{ei} - v_{edi}) + v_{edi}) ((v_{ej} - v_{edj}) + v_{edj}) \rangle \} \\ & = \sum_{i,j} \frac{\partial}{\partial x_i} \{ n_e m \langle (v_{ei} - v_{edi}) (v_{ej} - v_{edj}) \rangle + v_{edi} v_{edj} \} \\ & = \sum_{i,j} \frac{\partial P_{ij}}{\partial x_i} + \sum_{i,j} m \frac{\partial (n_e v_{edi})}{\partial x_i} v_{edj} + \sum_{i,j} n_e m v_{edi} \frac{\partial v_{edj}}{\partial x_i}, \end{aligned} \quad (2.154)$$

where  $P_{ij}$  are the components of a dyadic called pressure tensor defined by

$$P_{ij} = n_e m \langle (v_{ei} - v_{edi}) (v_{ej} - v_{edj}) \rangle, \quad (2.155)$$

or under vectorial form as

$$\hat{P} = n_e m \langle (\mathbf{v}_e - \mathbf{v}_{ed}) (\mathbf{v}_e - \mathbf{v}_{ed}) \rangle = n_e m (\langle \mathbf{v}_e \mathbf{v}_e \rangle - \mathbf{v}_{ed} \mathbf{v}_{ed}). \quad (2.156)$$

Therefore, using vectorial notation, we may write (2.154) as follows

$$(\nabla \cdot \hat{P}) + m (\nabla \cdot (n_e \mathbf{v}_{ed})) \mathbf{v}_{ed} + n_e m (\mathbf{v}_{ed} \cdot \nabla) \mathbf{v}_{ed}. \quad (2.157)$$

On the other hand, the third left-hand side term of equation (2.153) takes the form

$$- \sum_{i,j} n_e m \langle \frac{\partial v_{ej}}{\partial v_{ei}} \rangle \frac{F_{ei}}{m} = - \sum_{i,j} n_e \delta_{ji} F_{ei} = - n_e \mathbf{F}_e. \quad (2.158)$$

Replacing equations (2.157) and (2.158) in equation (2.153), and making use of the continuity equation (2.152) in the first left-hand side term of equation (2.153), we obtain

$$n_e m \langle v_{ion} \rangle \mathbf{v}_{ed} + n_e m \frac{\partial \mathbf{v}_{ed}}{\partial t} + (\nabla \cdot \hat{P}) + n_e m (\mathbf{v}_{ed} \cdot \nabla) \mathbf{v}_{ed} - n_e \mathbf{F}_e = \mathbf{I}_1, \quad (2.159)$$

where the right-hand side member  $\mathbf{I}_1$  represents a source term in the equation for electron momentum transfer obtained from the collision integral

$$\mathbf{I}_1 = \int_{\mathbf{v}_e} m \mathbf{v}_e \left( \frac{\partial f_e}{\partial t} \right)_{e-o} d\mathbf{v}_e, \quad (2.160)$$

such as in the continuity equation we had

$$I_0 = \int_{\mathbf{v}_e} \left( \frac{\partial f_e}{\partial t} \right)_{e-o} d\mathbf{v}_e = n_e \langle v_{ion} \rangle. \quad (2.161)$$

The term  $\mathbf{I}_1$  may have a contribution from space-charge forces, as in the Vlasov equation (see Sect. 2.2.2), and a term due to the contribution of true collisions:

$$\mathbf{I}_1 = n_e \mathbf{F}_s + \mathbf{I}_{1(\text{coll})}. \quad (2.162)$$

In the case of weak anisotropies in the electron velocity, the off-diagonal elements of the pressure tensor  $\hat{P}$  can be neglected, so that we may write the divergence term in equation (2.159) under the form

$$(\nabla \cdot \hat{P}) = \sum_i \frac{\partial P_{ii}}{\partial x_i} \mathbf{e}_i, \quad (2.163)$$

with the diagonal elements of the pressure tensor (2.155) assumed equal

$$P_{ii} = n_e m \langle (v_{ei} - v_{edi})^2 \rangle = \frac{1}{3} n_e m \langle |\mathbf{v}_e - \mathbf{v}_{ed}|^2 \rangle. \quad (2.164)$$

If further the electron velocity distribution is Maxwellian at temperature  $T_e$ , we still have

$$\frac{1}{2} m \langle |\mathbf{v}_e - \mathbf{v}_{ed}|^2 \rangle = \frac{3}{2} k_B T_e, \quad (2.165)$$

so that we obtain for the divergence of the pressure tensor

$$(\nabla \cdot \hat{P}) = \nabla p_e, \quad (2.166)$$

being  $p_e = n_e k_B T_e$  the electron scalar pressure. With equations (2.162) and (2.166) the equation (2.159) takes the form

$$n_e m \left( \frac{\partial}{\partial t} + (\mathbf{v}_{ed} \cdot \nabla) + \langle v_{ion} \rangle \right) \mathbf{v}_{ed} + \nabla p_e - n_e (\mathbf{F}_e + \mathbf{F}_s) = \mathbf{I}_{I(coll)}. \quad (2.167)$$

### 2.4.3 Equation for Energy Conservation

Using the electron kinetic energy  $u = \frac{1}{2} m v_e^2$  as variable  $X$  in equation (2.146), we obtain the equation for energy conservation of electrons

$$\begin{aligned} & \frac{\partial}{\partial t} \left( n_e \frac{1}{2} m \langle v_e^2 \rangle \right) + \sum_i \frac{\partial}{\partial x_i} \left( n_e \frac{1}{2} m \langle v_e^2 v_{ei} \rangle \right) \\ & - \sum_i n_e \frac{1}{2} m \left\langle \frac{\partial v_e^2}{\partial v_{ei}} \right\rangle \frac{F_{ei}}{m} = \int_{\mathbf{v}_e} \frac{1}{2} m v_e^2 \left( \frac{\partial f_e}{\partial t} \right)_{e-o} d\mathbf{v}_e. \end{aligned} \quad (2.168)$$

The first left-hand side term is simply  $\partial(n_e \langle u \rangle) / \partial t$ , while for the second left-hand side term we may define a total heat flux vector by

$$\mathbf{q}_e = \frac{1}{2} n_e m \langle v_e^2 \mathbf{v}_e \rangle, \quad (2.169)$$

and writing this term under the form of a vector divergence,  $(\nabla \cdot \mathbf{q}_e)$ . In concerning now the third left-hand side term, we may write in case of velocity independent external forces

$$- \sum_{i,j} n_e \frac{1}{2} \left\langle \frac{\partial v_{ej}^2}{\partial v_{ei}} \right\rangle F_{ei} = - \sum_{i,j} n_e \delta_{ji} \langle v_{ej} \rangle F_{ei} = - n_e (\mathbf{v}_{ed} \cdot \mathbf{F}_e). \quad (2.170)$$

We may write then equation (2.168) under the form

$$\frac{\partial}{\partial t} (n_e \langle u \rangle) + (\nabla \cdot \mathbf{q}_e) - n_e (\mathbf{v}_{ed} \cdot \mathbf{F}_e) = I_2, \quad (2.171)$$

with

$$I_2 = \int_{\mathbf{v}_e} u \left( \frac{\partial f_e}{\partial t} \right)_{e-o} d\mathbf{v}_e. \quad (2.172)$$

For the analysis it is sometimes convenient to separate the heat flux into an agitation and a convective component. Thus taking into consideration that

$$v_e^2 = |\mathbf{v}_e - \mathbf{v}_{ed}|^2 + 2 ((\mathbf{v}_e - \mathbf{v}_{ed}) \cdot \mathbf{v}_{ed}) + v_{ed}^2 \quad (2.173)$$

and  $\langle (\mathbf{v}_e - \mathbf{v}_{ed}) \cdot \mathbf{v}_{ed} \rangle = 0$ , we may write

$$\begin{aligned} \langle u \rangle &= \frac{1}{2} m \langle v_e^2 \rangle = \frac{1}{2} m \langle |\mathbf{v}_e - \mathbf{v}_{ed}|^2 \rangle + \frac{1}{2} m v_{ed}^2 \\ &= \frac{3}{2} k_B T_e + \frac{1}{2} m v_{ed}^2. \end{aligned} \quad (2.174)$$

Defining the agitation heat flux (flux of the random or thermal energy across a surface element moving with the drift velocity  $\mathbf{v}_{ed}$ ) as

$$\begin{aligned} \mathbf{q}_e^* &= \frac{1}{2} n_e m \langle |\mathbf{v}_e - \mathbf{v}_{ed}|^2 (\mathbf{v}_e - \mathbf{v}_{ed}) \rangle \\ &= \frac{1}{2} n_e m \langle (v_e^2 - 2 (\mathbf{v}_e \cdot \mathbf{v}_{ed}) + v_{ed}^2) (\mathbf{v}_e - \mathbf{v}_{ed}) \rangle, \end{aligned} \quad (2.175)$$

we obtain

$$\begin{aligned} \mathbf{q}_e^* &= \mathbf{q}_e - \frac{1}{2} n_e m \langle v_e^2 \rangle \mathbf{v}_{ed} - n_e m ((\langle \mathbf{v}_e \mathbf{v}_e \rangle - \mathbf{v}_{ed} \mathbf{v}_{ed}) \cdot \mathbf{v}_{ed}) \\ &= \mathbf{q}_e - n_e \langle u \rangle \mathbf{v}_{ed} - (\hat{P} \cdot \mathbf{v}_{ed}). \end{aligned} \quad (2.176)$$

The heat current may be expressed in terms of the temperature gradient as  $\mathbf{q}_e^* = -k_T \nabla T_e$ , being  $k_T$  the thermal conductivity assumed independent of the temperature.

The equation (2.171) takes hence the following form using equation (2.176) and neglecting the dependence of the thermal conductivity on the temperature

$$\begin{aligned} \frac{\partial}{\partial t} \left( n_e \left( \frac{3}{2} k_B T_e + \frac{1}{2} m v_{ed}^2 \right) \right) + \nabla \cdot \left( n_e \left( \frac{3}{2} k_B T_e + \frac{1}{2} m v_{ed}^2 \right) \mathbf{v}_{ed} \right) \\ - k_T \nabla^2 T_e + \nabla \cdot (\hat{P} \cdot \mathbf{v}_{ed}) = n_e (\mathbf{v}_{ed} \cdot \mathbf{F}_e) + I_2. \end{aligned} \quad (2.177)$$

Finally, in case of weak velocity anisotropies we still have

$$\nabla \cdot (\hat{P} \cdot \mathbf{v}_{ed}) = \nabla \cdot (n_e k_B T_e \mathbf{v}_{ed}) \quad (2.178)$$

and therefore

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{3}{2} p_e + \frac{1}{2} n_e m v_{ed}^2 \right) + \nabla \cdot \left( \left( \frac{5}{2} p_e + \frac{1}{2} n_e m v_{ed}^2 \right) \mathbf{v}_{ed} \right) \\ - k_T \nabla^2 T_e = n_e (\mathbf{v}_{ed} \cdot \mathbf{F}_e) + I_2. \end{aligned} \quad (2.179)$$

At steady-state conditions and sufficiently high neutral gas pressure values, the left-hand side member of equation (2.179) vanishes and the energy conservation

is determined by the local balance between the work per time and volume units realized by the external forces (i.e. the electric field),  $n_e (\mathbf{v}_{ed} \cdot \mathbf{E}_e) = (\mathbf{J}_e \cdot \mathbf{E})$ , being  $\mathbf{J}_e = -en_e \mathbf{v}_{ed}$  the electron current density, and the power lost per volume unit in collisions,  $-I_2$ .

## Appendices

### A.2.1 Liouville Relation $dq dp = dq' dp'$

Let us consider a transformation to pass from the variables  $q(t)$  and  $p(t)$ , describing a system in a given instant  $t$ , to the variables  $q'(t + dt)$  and  $p'(t + dt)$ , describing the same system at the instant  $t + dt$ . The evolutions in time of  $q$  and  $p$  are given by Hamilton's equations

$$\dot{q} = \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q}, \quad (2.180)$$

so that we may write

$$q' = q + \dot{q} dt = q + \frac{\partial H}{\partial p} dt \quad (2.181)$$

$$p' = p + \dot{p} dt = p - \frac{\partial H}{\partial q} dt. \quad (2.182)$$

Differentiating  $q'$  and  $p'$  in terms of  $q$  and  $p$  we obtain

$$dq' = dq + \frac{\partial}{\partial q} \left( \frac{\partial H}{\partial p} dt \right) dq + \frac{\partial}{\partial p} \left( \frac{\partial H}{\partial p} dt \right) dp \quad (2.183)$$

$$dp' = dp + \frac{\partial}{\partial q} \left( -\frac{\partial H}{\partial q} dt \right) dq + \frac{\partial}{\partial p} \left( -\frac{\partial H}{\partial q} dt \right) dp, \quad (2.184)$$

so that we may write in accordance with Jacobi's theorem

$$dq' dp' = J dq dp, \quad (2.185)$$

where  $J$  is the determinant of the transformation

$$J = \begin{vmatrix} 1 + \frac{\partial^2 H}{\partial q \partial p} dt & \frac{\partial^2 H}{\partial p^2} dt \\ -\frac{\partial^2 H}{\partial q^2} dt & 1 - \frac{\partial^2 H}{\partial p \partial q} dt \end{vmatrix} \quad (2.186)$$

and therefore

$$\begin{aligned} J &= 1 + \left( \frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q} \right) dt + \mathcal{O}(dt^2) \\ &= 1 + \mathcal{O}(dt^2). \end{aligned} \quad (2.187)$$

Neglecting the terms of  $\mathcal{O}(dt^2)$  order, we have  $J = 1$ , so that

$$dq' dp' = dq dp. \quad (2.188)$$

This proof is equivalent to calculate the Poisson brackets of variables  $q'$  and  $p'$  with respect to  $q$  and  $p$

$$[q', p']_{qp} = \frac{\partial(q', p')}{\partial(q, p)} = \frac{\partial q'}{\partial q} \frac{\partial p'}{\partial p} - \frac{\partial q'}{\partial p} \frac{\partial p'}{\partial q} \quad (2.189)$$

and verifying thus that

$$[q', p']_{qp} = 1. \quad (2.190)$$

This shows that the transformation  $(q, p) \rightarrow (q', p')$  is canonical and the condition  $dq' dp' = dq dp$  is satisfied (see e.g. Goldstein 1980).

### A.2.2 Demonstration of $\mathbf{V} = \mathbf{V}'$ and $|\mathbf{v}| = |\mathbf{v}'|$ in Sect. 2.3.1

From equation (2.81) we may write before and after a given collision, respectively

$$(m + M) \mathbf{V} = m \mathbf{v}_e + M \mathbf{v}_o \quad (2.191)$$

$$(m + M) \mathbf{V}' = m \mathbf{v}'_e + M \mathbf{v}'_o. \quad (2.192)$$

The right-hand side members of these two expressions are equal due to momentum conservation, so that the left-hand side members are also equal, and hence  $\mathbf{V} = \mathbf{V}'$ .

On the other hand, from equations (2.82) and (2.83), we may write

$$v_e^2 = V^2 + \left( \frac{M}{m + M} \right)^2 v^2 + 2 \frac{M}{m + M} (\mathbf{V} \cdot \mathbf{v}) \quad (2.193)$$

$$v_o^2 = V^2 + \left( \frac{m}{m + M} \right)^2 v^2 - 2 \frac{m}{m + M} (\mathbf{V} \cdot \mathbf{v}) \quad (2.194)$$

and from equations (2.84) and (2.85), with  $\mathbf{V} = \mathbf{V}'$ , we get similar expressions for  $v_e'^2$  and  $v_o'^2$ . Substituting these expressions into the equation for energy conservation

$$m v_e'^2 + M v_o'^2 = m v_e'^2 + M v_o'^2, \quad (2.195)$$

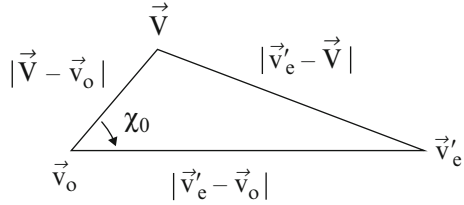
we rapidly obtain  $v = v'$ .

### A.2.3 Demonstration of Equation (2.91)

Let us consider the triangle  $\mathbf{v}_o$ ,  $\mathbf{V}$  and  $\mathbf{v}_e'$  shown in Fig. 2.9 obtained from Fig. 2.5. We easily obtain

$$\begin{aligned} |\mathbf{v}_e' - \mathbf{V}|^2 &= |\mathbf{V} - \mathbf{v}_o|^2 \sin^2 \chi_0 + (|\mathbf{v}_e' - \mathbf{v}_o| - |\mathbf{V} - \mathbf{v}_o| \cos \chi_0)^2 \\ &= |\mathbf{v}_e' - \mathbf{v}_o|^2 + |\mathbf{V} - \mathbf{v}_o|^2 - 2 |\mathbf{v}_e' - \mathbf{v}_o| |\mathbf{V} - \mathbf{v}_o| \cos \chi_0. \end{aligned} \quad (2.196)$$

**Fig. 2.9** Triangle obtained from Fig. 2.5, in which  $\chi_0$  is the scattering angle in the laboratory system



Substituting equations (2.84) and (2.85), with  $\mathbf{V}' = \mathbf{V}$  and  $|\mathbf{v}'| = |\mathbf{v}|$ , we may write

$$\left( \frac{M}{m+M} \right)^2 |\mathbf{v}|^2 = |\mathbf{v}_e' - \mathbf{v}_o|^2 + \left( \frac{m}{m+M} \right)^2 |\mathbf{v}|^2 - 2 |\mathbf{v}_e' - \mathbf{v}_o| \left( \frac{m}{m+M} \right) |\mathbf{v}| \cos \chi_0. \quad (2.197)$$

Inserting now (2.88), we still have

$$\cos \chi_0 = \frac{m + M \cos \chi}{\sqrt{m^2 + M^2 + 2 m M \cos \chi}}, \quad (2.198)$$

from which we obtain equation (2.91).

## Exercises

**Exercise 2.1.** Determine the electron rate coefficients of the inelastic electron-molecule processes for excitation of a given state  $j$  from state  $i$ , with  $u_{ij}$  denoting the threshold-energy:

- (a)  $\sigma_{ij}(u) = 0$ , for  $u < u_{ij}$ , and  $\sigma_{ij}(u) = a_{ij} = \text{const}$ , for  $u > u_{ij}$  ;  
 (b)  $\sigma_{ij}(u) = 0$ , for  $u \leq u_{ij}$ , and  $\sigma_{ij}(u) = a_{ij} (u - u_{ij})$ , for  $u \geq u_{ij}$ ,

in the case of electrons with a Maxwellian velocity distribution at temperature  $T_e$ .

**Resolution:** For electrons following a Maxwellian velocity distribution at temperature  $T_e$ , the isotropic component of the velocity distribution is

$$f_e^0(v_e) = n_e \left( \frac{m}{2\pi k_B T_e} \right)^{3/2} \exp \left( - \frac{mv_e^2}{2k_B T_e} \right),$$

obeying to the normalization condition

$$\int_0^\infty f_e^0(v_e) 4\pi v_e^2 dv_e = n_e,$$

whereas the electron rate coefficient (2.132) is given by

$$C_{ij} = \langle v_e \sigma_{ij}(v_e) \rangle = \frac{1}{n_e} \int_{v_{ij}}^\infty v_e \sigma_{ij}(v_e) f_e^0(v_e) 4\pi v_e^2 dv_e,$$

being  $v_{ij} = \sqrt{2u_{ij}/m}$ .

Rewriting these expressions in terms of the electron energy  $u = \frac{1}{2}mv_e^2$ , we have

$$f_e^0(u) = n_e \left( \frac{m}{2\pi k_B T_e} \right)^{3/2} \exp \left( - \frac{u}{k_B T_e} \right)$$

and

$$C_{ij} = \frac{1}{n_e} \frac{8\pi}{m^2} \int_{u_{ij}}^\infty u \sigma_{ij}(u) f_e^0(u) du,$$

from which we obtain the following expressions for the two cases proposed:

(a)

$$C_{ij} = \sqrt{\frac{8k_B T_e}{\pi m}} a_{ij} \left( 1 + \frac{u_{ij}}{k_B T_e} \right) \exp \left( - \frac{u_{ij}}{k_B T_e} \right) ;$$



(b)

$$C_{0j} = \sqrt{\frac{8k_B T_e}{\pi m}} 2k_B T_e a_{ij} \left(1 + \frac{u_{ij}}{2k_B T_e}\right) \exp\left(-\frac{u_{ij}}{k_B T_e}\right).$$

**Exercise 2.2.** Using the Klein-Rosseland relation (2.140) on the cross sections of Exercise 2.1 show that the electron rate coefficients of direct and reverse processes obey the equation (2.135).

**Resolution:** Using the relation (2.140) we obtain the reverse cross sections of Exercise 2.1: (a)  $\sigma_{ji}(u) = a_{ij} ((u + u_{ij})/u)$ , for  $u > 0$ ; (b)  $\sigma_{ji}(u) = a_{ij} (u + u_{ij})$ , for  $u \geq 0$ . Calculating then the electron rate coefficient of the superelastic process

$$C_{ji} = \frac{1}{n_e} \frac{8\pi}{m^2} \int_0^\infty u \sigma_{ji}(u) f_e^0(u) du,$$

we obtain

$$\frac{C_{ji}}{C_{ij}} = \exp\left(\frac{u_{ij}}{k_B T_e}\right)$$

in both situations.

**Exercise 2.3.** The cross sections for electron impact excitation of a dipole-allowed transition at high energies is of the form (Massey and Burhop 1969)

$$\sigma_{ij}(u) = \frac{a_{ij}}{u} \ln\left(b_{ij} \frac{u}{u_{ij}}\right),$$

being  $u = \frac{1}{2}mv_e^2$  the electron energy,  $u_{ij}$  the threshold-energy, and  $a_{ij}$  and  $b_{ij}$  two constants dependent of the gas. Determine the expression for the corresponding electron rate coefficient in the case of a Maxwellian velocity distribution at temperature  $T_e$ .

**Resolution:** In the case of a Maxwellian velocity distribution, the expression for the rate coefficient is as follows

$$C_{ij} = \sqrt{\frac{8}{\pi m}} \frac{1}{(k_B T_e)^{3/2}} \int_{u_{ij}}^\infty u \sigma_{ij}(u) \exp\left(-\frac{u}{k_B T_e}\right) du,$$

so that assuming, as in Exercise 2.1-(ii), a linear dependence  $\sigma_{ij}(u) = c_{ij} (u - u_{ij})$  at low energies up to the energy  $u^* > u_{ij}$ , we may write

$$C_{ij} = \sqrt{\frac{8k_B T_e}{\pi m}} 2k_B T_e c_{ij} \left\{ \left(1 + \frac{u_{ij}}{2k_B T_e}\right) \exp\left(-\frac{u_{ij}}{k_B T_e}\right) - \left(1 + \frac{u^*}{2k_B T_e}\right) \right\}$$

$$\exp\left(-\frac{u^*}{k_B T_e}\right)\left\} + \sqrt{\frac{8}{\pi m}} \frac{a_{ij}}{(k_B T_e)^{3/2}} \int_{u^*}^{\infty} \ln\left(b_{ij} \frac{u}{u_{ij}}\right) \exp\left(-\frac{u}{k_B T_e}\right) du.$$

Making the substitution  $\eta = b_{ij} u/u_{ij}$ , we still obtain for the high-energy term

$$C_{ij} = \dots\dots\dots + \sqrt{\frac{8}{\pi m}} \frac{a_{ij}}{(k_B T_e)^{3/2}} \frac{u_{ij}}{b_{ij}} \int_{\eta^*}^{\infty} \ln \eta \exp\left(-\frac{u_{ij}}{b_{ij} k_B T_e} \eta\right) d\eta,$$

which presents the following primitive for the integrand function with  $a = u_{ij}/(b_{ij} k_B T_e)$

$$\begin{aligned} \int \ln \eta \exp(-a\eta) d\eta &= -\frac{1}{a} \ln \eta \exp(-a\eta) + \frac{1}{a} \int \frac{1}{\eta} \exp(-a\eta) d\eta \\ &= -\frac{1}{a} \ln \eta \exp(-a\eta) + \frac{1}{a} \left( \ln \eta - a\eta + \frac{(a\eta)^2}{2 \cdot 2!} - \frac{(a\eta)^3}{3 \cdot 3!} + \dots\dots\dots \right). \end{aligned}$$

**Exercise 2.4.** In a plasma column of radius  $R$  and infinite length the gas temperature presents a parabolic radial profile

$$T_o(r) = (T_o(0) - T_w) \left(1 - \frac{r^2}{R^2}\right) + T_w,$$

being  $T_o(0)$  and  $T_w = T_o(R)$  the gas temperatures at the axis and at the wall, respectively. Neglecting the radial dependence of the thermal conductivity coefficient through the gas temperature, determine the power loss by thermal conduction, per volume unity.

**Resolution:** Using an analogous expression for molecules as indicated in Sect. 2.4.3, the power loss by thermal conduction by volume unity is  $P_{cond} = -k_T \nabla^2 T_o$ , being  $k_T$  the thermal conductivity coefficient. We obtain therefore

$$P_{cond} = -k_T \frac{1}{r} \frac{d}{dr} \left( r \frac{dT_o}{dr} \right) = \frac{4}{R^2} k_T (T_o(0) - T_w).$$

On the other hand, the radially averaged temperature is

$$\bar{T} = \frac{1}{R^2} \int_0^R T_o(r) 2r dr = \frac{1}{2} (T_o(0) + T_w),$$

so that the gas temperature at the axis of the column is  $T_o(0) = 2\bar{T} - T_w$ . Using this latter relation in the expression above, we obtain at the end

$$P_{cond} = \frac{8}{R^2} k_T (\bar{T} - T_w).$$

## References

- W.P. Allis, Motions of ions and electrons, in *Handbuch der Physik*, vol. 21. ed. by S. Flügge (Springer, Berlin, 1956), pp. 383–444
- J.A. Bittencourt, *Fundamentals of Plasma Physics* (Springer, New York, 2004)
- S. Chapman, T.G. Cowling, *The Mathematical Theory of Non-uniform Gases* (Cambridge University Press, Cambridge, 1939)
- B.E. Cherrington, *Gaseous Electronics and Gas Lasers* (Pergamon Press, Oxford, 1980)
- J.-L. Delcroix, *Physique des Plasmas: Volumes 1 and 2*. Monographies Dunod (Dunod, Paris, 1963, 1966) (in French)
- J.-L. Delcroix, A. Bers, *Physique des Plasmas: Volumes 1 and 2*. Savoirs Actuels (InterÉditions/CNRS Éditions, Paris, 1994) (in French)
- H. Goldstein, *Classical Mechanics*, 2nd edn. (Addison-Wesley, Reading, 1980)
- K. Huang, *Statistical Mechanics* (Wiley, New York, 1963)
- R.L. Liboff, *Introduction to the Theory of Kinetic Equations* (Wiley, New York, 1969)
- H.S.W. Massey, E.H.S. Burhop, *Electronic and Ionic Impact Phenomena: Volume 1 – Collisions of Electrons with Atoms* (Oxford at the Clarendon Press, Oxford, 1969)
- D.R. Nicholson, *Introduction to Plasma Theory* (Wiley, New York, 1983)
- Y. Sone, *Molecular Gas Dynamics: Theory, Techniques and Applications* (Birkhäuser, Boston, 2007)

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