# Chapter 2
# Separably Injective Banach Spaces

It is no exaggeration to say that the theory of separably injective spaces is quite different from that of injective spaces. In this chapter we will explain why. Indeed, we will enter now in the main topic of the monograph, namely, separably injective spaces and their "universal" version. After giving the main definitions and taking a look at the first natural examples one encounters, we present the basic characterizations and a number of structural properties of (universally) separable injective Banach spaces. We will show, among other things, that 1-separably injective spaces are not necessarily isometric to $C$-spaces, that (universally) separably injective spaces are not necessarily complemented in any $C$-space—the separably injective part of the assertion will be shown here while the "universal" part can be found in the next chapter—and that there exist essential differences between 1-separably injective and 2-separably injective spaces.

Moreover, in contrast with the scarcity of examples and general results concerning the class of injective Banach spaces, there exist many different types of separably injective spaces and a rich theory around them. In fact, most of the chapter is devoted to examples: Some of them are rather natural, while others are Banach spaces introduced elsewhere for different purposes and that, at the end of the day, turn out to be separable injective.

**Definition 2.1** A Banach space $E$ is *separably injective* if for every separable Banach space $X$ and each subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : X \to E$. If some extension $T$ exists with $\|T\| \le \lambda\|t\|$ we say that $E$ is *$\lambda$-separably injective*.

We are especially interested in the following subclass of separably injective spaces.

**Definition 2.2** A Banach space $E$ is said to be *universally separably injective* if for every Banach space $X$ and each separable subspace $Y \subset X$, every operator $t : Y \to E$ extends to an operator $T : X \to E$. If some extension $T$ exists with $\|T\| \le \lambda\|t\|$ we say that $E$ is *universally $\lambda$-separably injective*.



A. Avilés et al., *Separably Injective Banach Spaces*, Lecture Notes in Mathematics 2132, DOI 10.1007/978-3-319-14741-3_2

Before going any further we will present a couple of examples to give the flavor of the chapter. Recall that $c_0(I)$ denotes the space of all functions $f : I \to \mathbb{R}$ such that, for every $\varepsilon > 0$, the set $\{i \in I : |x(i)| > \varepsilon\}$ is finite. We present first Sobczyk's theorem [236], with Veech's proof [244]. See also [56] for an account of different proofs for this result.

**Theorem 2.3 (Sobczyk's Theorem)** *The space $c_0(I)$ is 2-separably injective in the sup norm for every index set $I$.*

*Proof* Since the elements of $c_0(I)$ have countable support, every $c_0(I)$-valued operator from a separable space has its range contained in a copy of $c_0$. So, it suffices to prove the result when $I$ is countable; i.e., when $c_0(I)$ is $c_0$, the space of null sequences. So, let $X$ be a separable Banach space and $t : Y \to c_0$ a norm one operator, where $Y$ is a subspace of $X$. Write $t$ as a sequence of functionals $t_n \in Y^*$, so that $t(y) = (t_n(y))$ for every $y \in Y^*$ and $\|t_n\| \leq 1$ for every $n \in \mathbb{N}$. The sequence $(t_n)$ is weakly* null in $Y^*$ and one has to find a sequence of extensions $(T_n)$ which is again weakly* null in $X^*$, with $\|T_n\| \leq 2$. For each $n$, let $\tau_n : X \to \mathbb{R}$ be a Hahn-Banach extension of $t_n : Y \to \mathbb{R}$. Recall that the weak* topology is metrizable on every bounded subset of $X^*$ by a translation-invariant metric $d$.

If $\Lambda$ is the set of weak* accumulation points of the sequence $(\tau_n)$, then $d(\tau_n, \Lambda) \to 0$ as $n \to \infty$ (a sequence such that every subsequence contains a further subsequence converging to zero is itself convergent to zero). Choose $\lambda_n \in \Lambda$ such that $d(\tau_n, \lambda_n) \leq d(\tau_n, \Lambda) + 1/n$. Then $\tau_n - \lambda_n$ is an extension of $t_n$ (since any functional in $\Lambda$ vanishes on $Y$) and $\|\tau_n - \lambda_n\| \leq \|\tau_n\| + \|\lambda_n\| \leq 2$. Clearly, the sequence $(\tau_n - \lambda_n)_n$ is weakly*-null in $X^*$. The operator $T : X \to c_0$ defined by $T(x) = ((\tau_n - \lambda_n)(x))$ is an extension of $t$ and $\|T\| \leq 2$. □

The space $c_0(I)$ is not universally separably injective (unless $I$ is finite) since $c_0$ is not complemented in $\ell_\infty$ (Example 1.25). By the same token, and Corollary 1.17, no separable space can be universally separably injective. A deep result of Zippin [252] puts an end to the story for separable spaces: every infinite dimensional separable separably injective space is isomorphic to $c_0$. Zippin's theorem has a long and delicate proof; we refer to [253] for what is perhaps the simplest proof due to Benyamini [33].

Thus, the results in this monograph belong naturally to the theory of non-separable Banach spaces. The "basic case" of Pełczyński-Sudakov spaces (see Proposition 1.28) provides a typical universally separably injective space. Although simple, this natural example shows that the theory of universally separably injective spaces does not run parallel with that of injective spaces: contrary to what happens in the injective case, 1-universally separably injective spaces need not be isometric to any $C(K)$ space.

*Example 2.4* Let $\Gamma$ be an uncountable set and let $\ell_\infty^c(\Gamma)$ denote the space of countably supported bounded functions $f : \Gamma \to \mathbb{R}$. Then $\ell_\infty^c(\Gamma)$ is:

1. 1-universally separably injective,
2. not isometric to any $C$-space,

3. isomorphic to a $C$-space,
4. not injective.

*Proof*

1. Every separable subspace of $\ell_\infty^c(\Gamma)$ is contained in another subspace isometric to $\ell_\infty$.
2. The unit ball of every $C(K)$ has extreme points. In fact $f$ is an extreme point if and only if $|f(x)| = 1$ for every $x \in K$. Quite clearly, the ball of $\ell_\infty^c(\Gamma)$ has no extreme points.
3. Consider the unitization of $\ell_\infty^c(\Gamma)$ inside $\ell_\infty(\Gamma)$, that is,

$$\ell_\infty^c(\Gamma)_+ = \{f \in \ell_\infty(\Gamma) : f = \lambda 1_\Gamma + g : \lambda \in \mathbb{R}, g \in \ell_\infty^c(\Gamma)\}.$$

   It is clear that $\ell_\infty^c(\Gamma)_+$ is 2-isomorphic to $\ell_\infty^c(\Gamma) \oplus \mathbb{R}$, endowed with the sup-norm, and this is in turn isomorphic to $\ell_\infty^c(\Gamma)$; and, as every unital subalgebra of $\ell_\infty(\Gamma)$, it is isometrically isomorphic to the algebra of all continuous real-valued functions on certain compact space $K$ (much more general results are available, see Sect. 2.2.1). In fact, if $A$ is a unital subalgebra of $\ell_\infty(\Gamma) = C(\beta\Gamma)$, we can identify $A$ with $C(K)$, where $K$ is the quotient space of $\beta\Gamma$ by the equivalence $x \sim y$ if $f(x) = f(y)$ for every $f \in A$.
4. The space $\ell_\infty^c(\Gamma)$ contains a complemented subspace isometric to $\ell_\infty^c(\aleph_1) = \ell_\infty^<(\aleph_1)$, which is not injective by the result of Pełczyński and Sudakov quoted in Theorem 1.26. □

## 2.1 Basic Properties

### 2.1.1 Characterizations

Separably injective spaces can be characterized as follows.

**Proposition 2.5** *For a Banach space $E$ the following properties are equivalent.*

1. *$E$ is separably injective.*
2. *Every operator from a subspace of $\ell_1$ into $E$ extends to $\ell_1$.*
3. *For every Banach space $X$ and each subspace $Y$ such that $X/Y$ is separable, every operator $t : Y \to E$ extends to $X$.*
4. *If $Z$ is a Banach space containing $E$ and $Z/E$ is separable, then $E$ is complemented in $Z$.*
5. *For every separable space $S$ one has $\mathrm{Ext}(S, E) = 0$.*

*Proof* It is clear that (3) $\Rightarrow$ (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5). We prove now that (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3). Since every separable space $X/Y$ can be set as a quotient $q : \ell_1 \to X/Y$ of $\ell_1$, the lifting property of $\ell_1$ provides an operator $Q : \ell_1 \to X$

yielding a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker q & \xrightarrow{j} & \ell_1 & \xrightarrow{q} & X/Y & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\big\downarrow & & {\scriptstyle Q}\big\downarrow & & \big\| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow[p]{} & X/Y & \longrightarrow & 0 \end{array} \tag{2.1}$$

Let $t : Y \to E$ be an operator, for which there must be an extension $\tau : \ell_1 \to E$ of $t\phi$ provided by (2). When $X$ is separable (case (2) $\Rightarrow$ (1)) then $Q$ can be chosen surjective and then an extension $T : X \to E$ of $t$ can be defined as follows: if $x = Q(\xi)$ then

$$T(x) = \tau\xi.$$

The map is well defined because if $0 = Q\xi$ then $q\xi = 0$ and thus also $\phi\xi = 0$ from where it follows $\tau\xi = t\phi\xi = 0$. The map $T$ is continuous since the open mapping theorem yields the existence of some $\mu$ so that norm one elements $x \in X$ are images of $x = Q\xi$ of some $\xi$ with $\|\xi\| \le \mu$. Thus $\|Tx\| = \|Q\xi\| \le \|Q\|\mu$. It extends $t$ because $T(y) = t(y)$ choosing the representation $y = \phi(\xi)$.

But even if $X$ is not separable (case (2) $\Rightarrow$ (3)), diagram (2.1) implies that $X$ is a quotient of $Y \oplus_1 \ell_1$ via the operator $\overline{Q}(y,\xi) = y + Q\xi$. Thus, yields an extension $T : X \to E$ defined as

$$T(x) = ty + \tau\xi.$$

Indeed, the map is well defined because if $0 = y + Q\xi$ then $Q\xi = -y$ and thus $q\xi = pQ\xi = p(-y) = 0$ from where $\xi \in \ker q$ and moreover $\phi\xi = Q\xi = -y$; therefore $ty + \tau\xi = ty + t\phi\xi = ty - ty = 0$. The map $T$ is continuous since the open mapping theorem yields the existence of some $\mu$ so that norm one elements $x \in X$ admit a representation as $x = y + Q\xi$ with $\|y\| + \|\xi\| \le \mu$. Thus $\|Tx\| = \|ty + Q\xi\| \le \max(\|t\|, \|Q\|\mu)$. It extends $t$ because $T(y) = ty$ choosing the representation $y + Q(0)$.

That (4) $\Rightarrow$ (3) follows from the existence of the push-out diagram: given an operator $t : Y \to E$ one gets

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\imath} & X & \xrightarrow{\pi} & X/Y & \longrightarrow & 0 \\ & & {\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle t'} & & \big\| & & \\ 0 & \longrightarrow & E & \xrightarrow{\imath'} & \mathrm{PO} & \longrightarrow & \mathrm{PO}/E & \longrightarrow & 0 \end{array}$$

and thus the existence of a projection $p'$ through $\imath'$ yields the existence of an extension $p't' : X \to E$ of $t$. $\square$

Analogous characterizations can be given for universal separable injectivity.

**Proposition 2.6** *For a Banach space $E$ the following properties are equivalent.*

1. *$E$ is universally separably injective.*
2. *Every operator $t : S \to E$ from a separable Banach space $S$ can be extended to an operator $T : \ell_\infty \to E$ through any embedding $S \to \ell_\infty$.*
3. *For every Banach space $X$ and each subspace $Y$, every operator $t : Y \to E$ with separable range extends to $X$.*

*Proof* The equivalence of (1) and (2) is clear: since $\ell_\infty$ is injective, once an operator can be extended from $S$ to $\ell_\infty$ it can be extended anywhere. That (1) implies (3) only requires to draw a push-out diagram:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "t"'] & X \arrow[d, "t'"] \\
\overline{t[Y]} \arrow[r] \arrow[d, "\imath"'] & \mathrm{PO} \\
E &
\end{tikzcd}$$

where $\imath$ denotes the canonical inclusion. Since $\imath$ can be extended to an operator $I : \mathrm{PO} \to E$, the composition $It'$ yields an extension of $t$. □

**Proposition 2.7** *Every (universally) separably injective Banach space is (universally) $\lambda$-separably injective for some $\lambda \geq 1$.*

*Proof* One only has to modify the proof of Proposition 1.6 assuming $X_n$ separable. For the part concerning universally separably injective spaces just shift the separability assumption from $X_n$ to $Y_n$. □

### *2.1.2 First Structural Properties*

Recall that a Banach space $X$ has *Pełczyński's property* ($V$) if each operator defined on $X$ is either weakly compact or it is an isomorphism on a subspace isomorphic to $c_0$. The indulgent reader (and Rosenthal, we hope) will forgive us for saying that $X$ has *Rosenthal's property* ($V$) if it satisfies the preceding condition with $\ell_\infty$ replacing $c_0$.

All $C$-spaces as well as their complemented subspaces have Pełczyński's property ($V$) [212]. Lindenstrauss spaces (i.e., $\mathscr{L}_{\infty,1+}$-spaces) also have this property [147], although there are $\mathscr{L}_\infty$-spaces that do not have it. For example, the ones constructed by Bourgain and Delbaen [46] that contain no copies of $c_0$, or the space $\Omega$ constructed in [57] as a twisted sum

$$0 \longrightarrow C[0,1] \longrightarrow \Omega \longrightarrow c_0 \longrightarrow 0$$

with strictly singular quotient map. Of course Argyros-Haydon's hereditarily indecomposable $\mathscr{L}_\infty$ space is also a counter-example, although this is a clear case of using a sledgehammer to crack an almond.

We say that $X$ is a *Grothendieck space* if every operator from $X$ to a separable Banach space (equivalently, to $c_0$) is weakly compact; equivalently, weak* and weak convergent sequences in $X^*$ coincide. Clearly, a Banach space with property $(V)$ is a Grothendieck space if and only if it has no complemented subspace isomorphic to $c_0$. It is well-known that $\ell_\infty$ is a Grothendieck space. In fact, it has Rosenthal's property $(V)$ (see Proposition 1.15), which is clearly stronger.

**Proposition 2.8**

1. *A separably injective space is of type $\mathscr{L}_\infty$, has Pełczyński's property (V) and, when it is infinite dimensional, contains copies of $c_0$.*
2. *A universally separably injective space is a Grothendieck space of type $\mathscr{L}_\infty$, it has Rosenthal's property $(V)$ and, when it is infinite dimensional, contains $\ell_\infty$.*

*Proof*

1. A separably injective space is obviously locally injective and thus (see Proposition 1.4) an $\mathscr{L}_\infty$-space.

   To show that $E$ contains $c_0$ and has property $(V)$, let $T : E \to X$ be a non-weakly compact operator ($E$ being an infinite dimensional $\mathscr{L}_\infty$ space cannot be reflexive). Choose a bounded sequence $(x_n)$ in $E$ such that $(Tx_n)$ has no weakly convergent subsequences and let $Y$ be the subspace spanned by $(x_n)$ in $E$. As $Y$ is separable we can regard it as a subspace of $C[0,1]$. Let $J : C[0,1] \to E$ be any operator extending the inclusion of $Y$ into $E$. We already mentioned that $C$-spaces have property $(V)$, so since $TJ : C[0,1] \to E$ is not weakly compact, $TJ$ is an isomorphism on some subspace isomorphic to $c_0$ ; and the same occurs to $T$.
2. To show that an universally separably injective space $E$ has Rosenthal's property $(V)$ we may take $T : E \to Z$ and $Y \subset E$ as in the previous argument, but this time we consider $Y$ as a subspace of $\ell_\infty$. If $J : \ell_\infty \to E$ is any extension of the inclusion of $Y$ into $E$, then $TJ : \ell_\infty \to Z$ is not weakly compact. Hence it is an isomorphism on some subspace isomorphic to $\ell_\infty$ and so is $T$. □

The list of spaces with Pełczyński's property $(V)$ includes Lindenstrauss spaces (see [147]) and, by Proposition 2.8(1), separably injective spaces. Consequently:

**Corollary 2.9** *A separably injective space is a Grothendieck space if and only if it does not contain complemented copies of $c_0$.*

Let us mention another similarity between separably injective spaces and complemented subspaces of $C[0,1]$.

**Proposition 2.10** *If a separably injective space contains a subspace with nonseparable dual then it also contains* $C[0,1]$.

*Proof* Assume that a separably injective space $X$ contains a subspace $Z$ with nonseparable dual through some embedding $j$. Consider an embedding $i : Z \to C[0,1]$ and get an extension $J : C[0,1] \to X$ of $j$ through $i$, that is $Ji = j$. The operator $J^*$ must have nonseparable range; hence, a result of Rosenthal [224] yields that $J$ fixes a copy of $C[0,1]$. $\square$

### *2.1.3 Stability Properties*

In this section we study the stability properties of (universally) separably injective spaces under some natural "operations" such as taking subspaces and quotients, forming direct products and twisted sums. This will allow us to present many natural examples of (universally) separably injective spaces as soon as we have the basic ingredients to start.

**Proposition 2.11** *Let* $0 \longrightarrow A \xrightarrow{i} B \xrightarrow{q} C \longrightarrow 0$ *be an exact sequence of Banach spaces.*

1. *If A and C are separably injective, then so is B.*
2. *If A and B are separably injective, then so is C.*
3. *If A is separably injective and B is universally separably injective then C is universally separably injective.*

*In particular, products and complemented subspaces of (universally) separably injective spaces are (universally) separably injective. Moreover,* $1$*-complemented subspaces of (universally)* $\lambda$*-separably injective spaces are (universally)* $\lambda$*-separably injective.*

*Proof* The simplest proof for (1) follows from characterization (2) in Proposition 2.5. Let $j : K \to \ell_1$ be an isomorphic embedding and let $\phi : K \to B$ be an operator. Then $q\phi$ can be extended to an operator $\Phi : \ell_1 \to C$, which can in turn be lifted to an operator $\Psi : \ell_1 \to B$. The difference $\phi - \Psi j$ takes values in $A$ and can thus be extended to an operator $\nu : \ell_1 \to A$. The desired operator is $\Psi + i\nu$.

To prove (2) and (3) suppose $A$ is separably injective and $B$ is (resp. universally) separably injective. Let $Y$ be a subspace of a separable (resp. arbitrary) space $X$ and let $\phi : Y \to C$ be an operator. Consider the pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] & B \arrow[r, "q"] & C \arrow[r] & 0 \\
0 \arrow[r] & C \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r, "'q"] \arrow[u, "'\phi"'] & Y \arrow[r] \arrow[u, "\phi"'] & 0
\end{tikzcd}$$

Since $C$ is separably injective, the lower exact sequence splits, so $'q$ admits a linear continuous selection $s : Y \to \mathrm{PB}$. By the assumption about $B$, the operator $'\phi s$ can be extended to an operator $T : X \to E$. Thus, $qT : X \to C$ is the desired extension of $\phi$. □

Thus, if $0 \to A \to B \to C \to 0$ is an exact sequence of Banach spaces, we know that $B$ is separably injective if the other two relevant spaces are; and the same happens with $C$. What about $A$? Bourgain showed in [44] that $\ell_1$ contains an uncomplemented subspace isomorphic to $\ell_1$ which yields an exact sequence $0 \to \ell_1 \to \ell_1 \to B \to 0$ that does not split (see Sect. 6.3). By Lindenstrauss' lifting (Proposition A.18) $B$ is not an $\mathscr{L}_1$ space. Its dual sequence $0 \to B^* \to \ell_\infty \to \ell_\infty \to 0$ shows that the kernel of a quotient mapping between two injective spaces may fail to be even an $\mathscr{L}_\infty$-space.

In [20, Proposition 5.3] it was claimed that universal separable injectivity is a 3-space property; but the proof contains a gap we have been unable to fill. Consequently, other claims also remain without proper justification, namely Propositions 5.4 and 5.6 and Theorem 5.5 in [20] and Example 4.5(a) and the second part of Proposition 5.1 in [21]. See Sect. 6.2 for a more detailed account of the situation.

Several variations of these results can be seen in [70]. Regarding infinite products, it is obvious that if $(E_i)_{i\in I}$ is a family of $\lambda$-separably injective Banach spaces, then $\ell_\infty(I, E_i)$ is $\lambda$-separably injective. The non-obvious fact that also $c_0(I, E_i)$ is separably injective can be considered as a vector valued version of Sobczyk's theorem. Proofs for this result were obtained by Johnson-Oikhberg [146], Rosenthal [225], Cabello Sánchez [52] and Castillo-Moreno [65], each with its own estimate for the constant: respectively, $2\lambda^2$ (implicitly), $\lambda(1+\lambda)^+$, $(3\lambda^2)^+$ and $6\lambda^+$. Here we present a proof like that of Castillo and Moreno [65] based on an idea of Sánchez et al. [57] and giving the same bound as [225].

**Proposition 2.12** *If $(E_i)_{i\in I}$ is a family of $\lambda$-separably injective spaces, then $c_0(I, E_i)$ is $\lambda(1+\lambda)^+$-separably injective.*

*Proof* Since the elements of $c_0(I, E_i)$ have countable "supports" it suffices to prove the result for countable families. So, let $(E_n)$ be a sequence of $\lambda$-separably injective spaces, $X$ a separable Banach space, $Y$ a subspace of $X$, and $t : Y \to c_0(\mathbb{N}, E_n)$ a norm one operator that we can write as $t = (t_n)$, where each $t_n : Y \to E_n$ has norm at most 1.

Fix $\varepsilon \in (0, 1)$. Set $Z = X/Y$ and let $\pi : X \to Z$ denote the natural quotient map. Let $(Z_k)$ be an increasing sequence of finite dimensional subspaces of $Z$ whose union is dense in $Z$. For each $k$, let $X_k$ be a finite dimensional subspace of $X$ so that $\pi[X_k] = Z_k$. We may assume that $(X_k)$ is an increasing sequence whose union is dense in $X$. We require, moreover, that for every $z \in Z_k$ there is $x \in X_k$ such that $\pi(x) = z$, with $\|x\| \le (1+\varepsilon)\|z\|$. This implies that $Z_k$ is $(1+\varepsilon)$-isomorphic to the quotient of $X_k$ by $Y_k = Y \cap X_k$ through the obvious map. It is clear that, for every $k \in \mathbb{N}$, the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y_k & \longrightarrow & X_k & \xrightarrow{\pi_k} & Z_k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \xrightarrow{\pi} & Z & \longrightarrow & 0 \end{array}$$

in which $\pi_k$ is the restriction of $\pi$ to $X_k$ and the vertical arrows are the canonical embeddings is commutative.

For each $n$, let $\tau_n : X \to E_n$ be an extension of $t_n$ with $\|\tau_n\| \leq \lambda\|t_n\|$, which exists by hypothesis. Let $t_{n,k}$ denote the restriction of $t_n$ to $Y_k$. Let $\tau_{n,k} : X \to E_n$ be an extension of $t_{n,k}$ such that $\|\tau_{n,k}\| \leq \lambda\|t_{n,k}\|$ which, once again, exists by hypothesis. Since $\tau_n - \tau_{n,k}$ vanishes on $Y_k$ there is an operator $\phi_{n,k} : Z_k \to E_n$ such that $\tau_n - \tau_{n,k} = \phi_{n,k} \circ \pi_k$. Besides, the norm of $\phi_{n,k}$ on $X_k/Y_k$ is $\|\tau_n - \tau_{n,k}\|$ and we have

$$\|\phi_{n,k} : Z_k \longrightarrow E_n\| \leq (1+\varepsilon)\|\tau_n - \tau_{n,k}\|.$$

Let $\Phi_{n,k} : Z \to E_n$ be an extension of $\phi_{n,k}$ with $\|\Phi_{n,k}\| \leq \lambda\|\phi_{n,k}\|$.

Since for every $y \in Y$ one has $\lim \|t_n(y)\| = 0$ and $Y_k$ is finite dimensional, for fixed $k$, one has $\lim_n \|t_{n,k}\| = 0$. Put $N(k) = \max\{n : \|t_{n,k}\| > \varepsilon^k\}$. Then $N(k)$ is increasing, and $N(k) \to \infty$ as $k \to \infty$.

We define a sequence of operators $T_n : X \to E_n$ as follows:

$$T_n(x) = \begin{cases} \tau_n(x) - \Phi_{n,k}(\pi(x)) & \text{for } N(k) < n \leq N(k+1), \\ \tau_n(x) & \text{if } n \leq N(1). \end{cases}$$

These $T_n$ are uniformly bounded and thus define an operator $T : X \longrightarrow \ell_\infty(\mathbb{N}, E_n)$ given by $T(x) = (T_n(x))$. Let us see that $T$ is the desired extension of $t$:

1. To check that $T$ takes values in $c_0(\mathbb{N}, E_n)$ it is sufficient to work on $\bigcup_k X_k$. So, take $x \in X_k$, with $\|x\| = 1$. Then for $n > N(k)$ one has

$$T_n(x) = \tau_n(x) - \Phi_{n,i}(\pi(x)) = \tau_n(x) - \Phi_{n,k}(\pi_k(x)) = \tau_n(x) - \tau_n(x) + \tau_{n,k}(x) = \tau_{n,k}(x).$$

   Thus, for $n > N(k)$, one has

$$\|T_n(x)\| \leq \|\tau_{n,k}\| \leq \lambda\|t_{n,k}\| \leq \lambda\varepsilon^k.$$

2. The operator $T$ is an extension of $t$. Indeed, if $y \in Y$, then for every $n$ one has $T_n(y) = \tau_n(y) = t_n(y)$, by the very definitions.
3. To estimate $\|T\|$ it is enough to bound each coordinate. If $n \leq N(1)$, then $T_n = \tau_n$, so $\|T_n\| \leq \lambda\|t_n\| \leq \lambda$. Otherwise $N(k) < n \leq N(k+1)$ for some $k \geq 1$ and we have $T_n = \tau_n - \Phi_{n,k} \circ \pi$ and so $\|T_n\| \leq \|\tau_n\| + \|\Phi_{n,k}\|$. But $\|\tau_n\| \leq \lambda\|t_n\| \leq \lambda$;

as for the other chunk, we have

$$\begin{aligned}\|\Phi_{n,k}\| &\leq \lambda\|\phi_{n,k}\| \leq \lambda(1+\varepsilon)\|\tau_n - \tau_{n,k}\| \\ &\leq \lambda(1+\varepsilon)(\|\tau_n\| + \|\tau_{n,k}\|) \leq \lambda^2(1+\varepsilon)(\|t_n\| + \|t_{n,k}\|) \\ &\leq \lambda^2(1+\varepsilon)(\|t_n\| + \varepsilon^k) \leq \lambda^2(1+\varepsilon)^2,\end{aligned}$$

from where it follows that $c_0(\mathbb{N}, E_n)$ is $(\lambda+\lambda^2)^+$-separably injective. □

## 2.2 Examples of Separably Injective Spaces

In this section we will present a number of separably injective spaces appearing in nature. The first obvious example, since $\ell_\infty$ is injective and $c_0$ is separably injective, follows from Proposition 2.11: $\ell_\infty/c_0$ is universally separably injective. In fact, it will be shown later that $\ell_\infty/c_0$ is 1-universally separably injective (Theorem 2.40 and Corollary 2.41) and non injective (Theorem 1.25 and also Proposition 2.43).

The non isomorphic [74] spaces $c_0(\ell_\infty)$ and $\ell_\infty(c_0)$ are also separably injective and not universally separably injective; the quotients $\ell_\infty/c_0(\ell_\infty)$ and $\ell_\infty/\ell_\infty(c_0)$ are universally separably injective as well. It also follows from Proposition 2.11 that for $\Gamma$ an uncountable set, $\ell_\infty^c(\Gamma)/c_0(\Gamma)$ is universally separably injective non-injective. It is worth noticing that it is possible to identify such spaces with $C(K)$ spaces (perhaps after unitization, see Sect. 2.2.1 below).

$C(K)$-spaces (and their ideals) in which $K$ is either of finite height or an $F$-space, twisted sums of separably injective spaces and quotients of separably injective spaces will be our next examples. We will also show the first examples of separably injective spaces that are not isomorphic to a complemented subspace of any $C$-space (which is clearly impossible for an injective space). Further examples will be exhibited in Chaps. 4 and 5, when other important classes of separably injective spaces will be presented.

### *2.2.1 C(K)-Spaces When K Is an F-Space*

There are close connections between the 1-separable injectivity of $C(K)$, the topological properties of $K$ and the lattice structure of $C(K)$. Let us recall some separation conditions that compacta may or may not have.

**Definition 2.13** A compact Hausdorff space is said to be:

- An $F$-space if disjoint open $F_\sigma$ sets (equivalently, cozeroes) have disjoint closures.
- Basically disconnected (or $\sigma$-Stonian) if the closure of every open $F_\sigma$ set is open.

- Extremely disconnected (or Stonian) if the closure of every open set is open.
- Zero-dimensional if the topology has a base of clopen sets.

Recall that a cozero set of $K$ is one of the form $\{x \in K : f(x) \neq 0\}$, for some continuous function $f$. Cozeroes and open $F_\sigma$ sets agree on a normal space. Indeed, for any $f \in C(K)$ one has

$$\{x \in K : f(x) \neq 0\} = \bigcup_{n=1}^{\infty} |f|^{-1}\big([1/n, \infty)\big).$$

Thus each cozero is $F_\sigma$. Conversely, if $V = \bigcup_n V_n$ is open with all $V_n$ closed, according to Tietze, we may take for each $n$ a continuous $0 \leq f_n \leq 2^{-n}$ vanishing off $V$ and such that $f_n = 2^{-n}$ on $V_n$. Clearly $V$ is the cozero set of $\sum_n f_n$.

Of course Stonian implies $\sigma$-Stonian and this implies $F$-space. $\beta\mathbb{N}$ is perhaps the most natural example of extremely disconnected compactum. It is obvious that closed sets of $F$-spaces are $F$-spaces, so $\mathbb{N}^* = \beta\mathbb{N}\backslash\mathbb{N}$ is an $F$-space.

**Theorem 2.14** *Let $K$ be a compact space. The following conditions are equivalent:*

1. *$C(K)$ is 1-separably injective.*
2. *Given sequences $(f_i)$ and $(g_j)$ in $C(K)$ such that $f_i \leq g_j$ for each $i, j \in \mathbb{N}$, there exists $h \in C(K)$ such that $f_i \leq h \leq g_j$ for each $i, j \in \mathbb{N}$.*
3. *Every sequence of mutually intersecting balls in $C(K)$ has nonempty intersection.*
4. *$K$ is a $F$-space.*
5. *For every $f \in C(K)$ there is $u \in C(K)$ such that $f = u|f|$.*
6. *Every operator from a two-dimensional space into $C(K)$ has a norm preserving extension to any three-dimensional space.*

*Proof* The equivalence of (1), (2), (3) and (4) is a special case of the equivalence of the corresponding conditions in Theorem 5.16, where details and accurate references are provided. The proof that (4) and (5) are equivalent is based on the fact that open $F_\sigma$ sets and cozeroes agree on a normal space:

That (5) holds when $K$ is an $F$-space is clear: take $f \in C(K)$ and consider the sets $P = f^{-1}(0, \infty)$ and $N = f^{-1}(-\infty, 0)$. These are disjoint cozeroes and so they have disjoint closures. Therefore there is $u \in C(K)$ such that $u = 1$ on $P$ and $u = -1$ on $N$. Clearly, $f = u|f|$. The converse is also easy: let $P$ and $N$ be disjoint cozero sets and take $f, g \in C(K)$ such that $P = \{x \in K : f(x) \neq 0\}$ and $N = \{x \in K : g(x) \neq 0\}$. Define $h(x) = |f(x)| - |g(x)|$. Now, if $h = u|h|$ for some continuous $u$, then since $u = 1$ on $P$ and $u = -1$ on $N$ we see that $P$ and $N$ have disjoint closures.

That (6) implies the separable injectivity of $C(K)$ is proved in [177], and the converse implication is trivial. □

The correspondence between "$K$ is an $F$-space" and "$C(K)$ is 1-separably injective" does not extend to $C_0(L)$, the space of continuous functions vanishing at infinity on a locally compact space $L$: indeed, $\mathbb{N}$ is an $F$-space while $c_0$ is not 1-separably injective. However, one has the following:

**Proposition 2.15** *Let $L$ be a locally compact space. Then $C_0(L)$ is 1-separably injective if and only if every compact subset of $L$ is an $F$-space and the infinity point is a $P$-point in $\alpha L$.*

*Proof* Assume first that $C_0(L)$ is 1-separably injective. If $K$ is a compact subset of $L$ we have an exact sequence

$$0 \longrightarrow \ker r \longrightarrow C_0(L) \xrightarrow{\ r\ } C(K) \longrightarrow 0, \tag{2.2}$$

where $r$ is the restriction map and since $\ker r = \{f \in C_0(L) : f|_K = 0\}$ is an $M$-ideal in $C_0(L)$ we have that $C(K)$ is 1-separably injective (Theorem 2.21) and so $K$ is an $F$-space (Theorem 2.14).

To prove that the infinity point is a $P$-point in $\alpha L$ let us assume on the contrary that there is a sequence $(x_n)$ in $L$ such that $x_n \to \infty$ in $\alpha L$. We may and do assume that $x_n \neq x_m$ for $n \neq m$. Then the evaluation map $\pi : C_0(L) \to c_0$ given by $\pi f = (f(x_n))_n$ is an "isometric quotient" whose kernel is an $M$-ideal in $C_0(L)$ and reasoning as before the space $c_0$ would be 1-separably injective, a contradiction.

As for the other implication, let $t : Y \to C_0(L)$ be an operator, where $Y$ is a closed subspace of a separable space $X$. As the infinity point is a $P$-point in $\alpha L$ and $Y$ is separable it is clear that there is a compact $K \subset L$ such that $\operatorname{supp} t(y) \subset K$ for every $y \in Y$. Let us define $\tau : Y \to C(K)$ by $\tau(y) = t(y)|_K$. Since $K$ is an $F$-space $\tau$ has an extension $\hat{\tau} : X \to C(K)$ with $\|\hat{\tau}\| = \|\tau\| = \|t\|$. But $\hat{\tau}[X]$ is a separable subspace of $C(K)$ and Proposition 2.20 applied to the sequence (2.2) provides a "lifting" of $\hat{\tau}$ to $C_0(L)$ which is the required extension of $t$. □

It is not true that $K$ is an $F$-space when $C(K)$ is only *isomorphic to* a 1-separably injective space. To see this we proceed as follows: identify two points $u, v \in \mathbb{N}^*$ that we may consider as two free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ and let us call $\beta(u, v)$ to the corresponding quotient space of $\beta\mathbb{N}$. The space $C(\beta(u, v)) = \{f \in C(\beta\mathbb{N}) : f(u) = f(v)\}$ is a closed hyperplane of $C(\beta\mathbb{N})$ and thus it is 2-isomorphic to $\ell_\infty$. However, $\beta(u, v)$ is not an $F$-space: pick $U \in \mathcal{U}\backslash\mathcal{V}$, so that $V = \mathbb{N}\backslash U$ belongs to $\mathcal{V}$. Set the function $f : \mathbb{N} \to \mathbb{R}$ given by

$$f(n) = \frac{1_U(n) - 1_V(n)}{n}$$

and extend it to a continuous function on $\beta\mathbb{N}$ denoted again $f$. As $f(u) = f(v) = 0$ we have $f \in C(\beta(u, v))$. However there is no factorization $f = g|f|$ with $g \in C(\beta(u, v))$ since it this case we would have $g(u) = 1$ and $g(v) = -1$.

It is important to realize that many Banach algebras are $C(K)$ spaces though given in a disguised form. The most convenient characterization of the algebras of all continuous functions on compact spaces in our real-valued setting is the one due to Albiac and Kalton [1, 2]: if $A$ is a (real, unital) Banach algebra whose norm satisfies the inequality

$$2\|fg\| \leq \|f^2 + g^2\| \qquad (f, g \in A),$$

then, as a Banach algebra, $A$ is isometrically isomorphic to $C(K)$, for some compact space $K$. See [1, 2] for the remarkably simple proof. The next example is just one application.

**Proposition 2.16** *The space of all bounded Borel (respectively, Lebesgue) measurable functions on the line is 1-separably injective in the sup norm.*

*Proof* Clearly, the given spaces are in fact Banach algebras satisfying the inequality required by Albiac-Kalton characterization. Thus they can be represented as $C(K)$ spaces. On the other hand, each measurable function can be decomposed as $f = u|f|$, with $u$ (and $|f|$, of course) measurable. This clearly implies that the corresponding compacta satisfy the fifth condition in Theorem 2.14. □

### *2.2.2 M-ideals of Separably Injective Spaces*

Let $M$ be a closed subset of the compact space $K$. By Tietze's extension theorem each continuous function on $M$ is the restriction of some continuous function on $K$ having the same norm. The space $L = K \setminus M$ is locally compact and one has the exact sequence

$$0 \longrightarrow C_0(L) \longrightarrow C(K) \stackrel{r}{\longrightarrow} C(M) \longrightarrow 0, \tag{2.3}$$

where the map $r$ is plain restriction. Even if this sequence does not split (as a rule), one has the following result, which can be regarded as a linear version of Tietze's extension theorem.

**Proposition 2.17 (Borsuk-Dugundji Theorem)** *Let $M$ be a closed set in the compact space $K$. For every separable subspace $S \subset C(M)$ there is a norm-one operator $s : S \longrightarrow C(K)$ such that $rs = \mathbf{1}_S$.*

Borsuk proved this result in [42] for $K$ a metrizable and separable space (not necessarily compact), setting as $C(K)$ the space of continuous bounded functions and $S = C(M)$. Separability was removed by Dugundji in [91, Theorem 5.1], see [230, Sect. 21]. The version of the theorem as it is stated in Proposition 2.17 is a corollary of the more general Proposition 2.20 that we shall discuss later. We can rephrase Borsuk-Dugundji Theorem by saying that, with the same notations as before, if $t : X \to C(M)$ has separable range, then the lower sequence in the pull back diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_0(L) & \longrightarrow & C(K) & \xrightarrow{r} & C(M) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle t} & & \\
0 & \longrightarrow & C_0(L) & \longrightarrow & \mathrm{PB} & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

splits.

**Theorem 2.18** *Let $K$ be a compact space, $M$ a closed subset of $K$ and $L = K\backslash M$.*

1. *If $C(K)$ is (universally) $\lambda$-separably injective, then so is $C(M)$.*
2. *If $C(K)$ is $\lambda$-separably injective, then $C_0(L)$ is $2\lambda$-separably injective.*

*Proof*

1. Let $Y$ be a separable subspace of $X$ and $t : Y \longrightarrow C(M)$ an operator. Let $S \subset C(M)$ any separable subspace containing the image of $t$ and $s : S \longrightarrow C(K)$ the lifting provided by the Borsuk-Dugundji theorem. If $T : X \longrightarrow C(K)$ is an extension of $st$, then $rT : X \longrightarrow C(M)$ is an extension of $t$, and $\|rT\| = \|T\|$.
2. Let us remark that if $S$ is a subspace of $C(K)$ containing $C_0(L)$ and $S/C_0(L)$ is separable, then there is a projection $p : S \longrightarrow C_0(L)$ of norm at most 2. Indeed, $S/C_0(L)$ is a separable subspace of $C(M)$ and there is a lifting $s : S/C_0(L) \longrightarrow C(K)$, with $\|s\| = 1$, and $p = \mathbf{1}_S - sr$ is the required projection. Now, let $t : Y \longrightarrow C_0(L)$ be an operator, where $Y$ is a subspace of a separable Banach space $X$. Considering $t$ as taking values in $C(K)$, there is an extension $T : X \longrightarrow C(K)$ with $\|T\| \le \lambda\|t\|$. Let $S$ denote the least closed subspace of $C(K)$ containing the range of $T$ and $C_0(L)$ and $p : S \longrightarrow C_0(L)$ a projection with $\|p\| \le 2$. The composition $pT : X \longrightarrow C_0(L)$ is an extension of $t$ and thus $\|pT\| \le 2\lambda\|t\|$. □

It is easy to see that every closed ideal of $C(K)$ has the form $\{f \in C(K) : f|_S = 0\}$ for some closed subset $S \subset K$ (see [249, III.D.1]). Thus, part (1) of the theorem above can be reformulated as:

**Corollary 2.19** *Let $K$ be a compact space and let $J$ be an ideal of $C(K)$. If $C(K)$ is (universally) $\lambda$-separably injective, then so is $C(K)/J$.*

Let us consider the following construction introduced by Dashiell and Lindenstrauss [80] with the declared purpose of exhibiting spaces admitting a strictly convex renorming but no injective operator into any $c_0(\Gamma)$. Take $\mathbb{I} = [0, 1]$ in its natural topology. For every $A \subset \mathbb{I}$ and every countable ordinal $\alpha$, let $A^{(\alpha)}$ be the $\alpha^{\text{th}}$-derived set of $A$. Given $\varepsilon > 0$ and $f \in \ell_\infty^c(\mathbb{I})$, let $\sigma_\varepsilon(f) = \{t \in \mathbb{I} : |f(t)| \ge \varepsilon\}$. For each countable ordinal $\alpha$ we set

$$X_\alpha = \{f \in \ell_\infty(\mathbb{I}) : \sigma_\varepsilon(f)^{(\alpha)} = \varnothing \;\; \forall\, \varepsilon > 0\}.$$

If $X = \bigcup_{\alpha<\omega_1} X_\alpha$ one has the chain

$$c_0(\mathbb{I}) = X_1 \subset X_2 \subset \cdots \subset X_\alpha \subset X_{\alpha+1} \subset \cdots \subset X \subset \ell_\infty^c(\mathbb{I}) \subset \ell_\infty(\mathbb{I}).$$

The function spaces in the preceding chain are all ideals in $\ell_\infty(\mathbb{I})$. Let $Y$ denote any of them. After representing $\ell_\infty(\mathbb{I})$ as a suitable $C(K)$ space (notice that $K$ is just the Stone-Čech compactification of $\mathbb{I}$ viewed as a discrete set) we have $Y = C_0(L)$, where $L = \{k \in K : f(k) \neq 0 \text{ for some } f \in Y\}$. As $\ell_\infty(\mathbb{I})$ is 1-injective, we get from Theorem 2.18 that $Y$ is 2-separably injective.

These spaces are all different—in fact, none is complemented in the next—since [80, Theorem 2]: for $\alpha < \beta$ there is no linear continuous operator $T : X_\beta \to X_\alpha$ whose restriction to $c_0(\mathbb{I})$ is injective; the same is true for any operator $\ell_\infty^c(\mathbb{I}) \to X$. Moreover, Dashiell and Lindenstrauss show that $X$ is the space of Baire 1 class functions having countable support, namely

$$X = B_1 \cap \ell_\infty^c(\mathbb{I}).$$

This should be compared with Proposition 6.10 where we show that $B_1$ is not 1-separably injective.

A remarkable generalization of Borsuk-Dugundji theorem for $M$-ideals was provided by Ando [7] and, independently, Choi and Effros [76]. In order to state it let us recall that a closed subspace $J \subset X$ is called an $M$-ideal (see [121, Definition 1.1]) if its annihilator $J^\perp = \{x^* \in X^* : \langle x^*, x\rangle = 0 \text{ for every } x \in J\}$ is an $L$-summand in $X^*$. This just means that there is a linear projection $P$ on $X^*$ whose range is $J^\perp$ and such that $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. The easier examples of $M$-ideals are just ideals in $C(K)$-spaces, which arise as in (2.3). The fact that such a $C_0(L)$ is an $M$-ideal in $C(K)$ is straightforward from the Riesz representation of $C(K)^*$.

**Proposition 2.20** *Let $J$ be an $M$-ideal in the Banach space $E$ and $\pi : E \to E/J$ the natural quotient map. Let $Y$ be a separable Banach space and $t : Y \to E/J$ be an operator. Assume further that one of the following conditions is satisfied:*

1. *$Y$ has the $\lambda$-AP.*
2. *$J$ is a Lindenstrauss space.*

*Then $t$ can be lifted to $E$, that is, there is an operator $L : Y \to E$ such that $\pi L = t$. Moreover one can get $\|L\| \leq \lambda\|t\|$ under the assumption* (1) *and $\|L\| = \|t\|$ under* (2).

We refer the reader to [121, Theorem 2.1] for a proof. In a similar way as Theorem 2.18 was deduced from the Borsuk-Dugundji Theorem (Proposition 2.17 above), one gets from Proposition 2.20:

**Theorem 2.21** *Let $J$ be an $M$-ideal in a Banach space $E$.*

1. *If $E$ is (universally) $\lambda$-separably injective, then $E/J$ is (universally) $\lambda^2$-separably injective.*
2. *If $E$ is $\lambda$-separably injective, then $J$ is $2\lambda^2$-separably injective.*

*Proof*

1. By Proposition 1.5 $E^{**}$ is $\lambda$-injective and so it has the $\lambda$-AP. As $E^{**} = J^{**} \oplus_\infty (E/J)^{**}$ we see that also $J^{**}$ and $(E/J)^{**}$ have the $\lambda$-AP. Hence both $J$ and $(E/J)$ have the $\lambda$-AP. Let $Y$ be a separable subspace of $X$ and $t : Y \longrightarrow E/J$ an operator. Let $S$ be a separable subspace of $E/J$ containing the image of $t$. By [60, Theorem 9.7] we may assume $S$ has the $\lambda$-AP. Let $s : S \longrightarrow E$ be the lifting provided by Proposition 2.20, so that $\|s\| \leq \lambda$. Now, if $T : X \longrightarrow E$ is an extension of $st$, then $\pi T : X \longrightarrow E/J$ is an extension of $t$, and this can be achieved with $\|\pi T\| = \|T\| \leq \lambda^2 \|t\|$.
2. The proof is similar to that of Theorem 2.18(2) and is left to the reader. □

Observe that when $E$ is a Lindenstrauss space then $J$ is also a Lindenstrauss space and then the exponent 2 can be eliminated everywhere in Theorem 2.21. This result also provides a different proof for Proposition 2.12. Indeed, suppose $E_i$ are $\lambda$-separably injective for every $i \in I$. Then so is $E = \ell_\infty(I, E_i)$ and therefore its $M$-ideal $J = c_0(I, E_i)$ is $2\lambda^2$-separably injective. This argument, taken from [146], gives the best constant when each $E_i$ is 1-separably injective; otherwise the value $\lambda(1+\lambda)^+$ we got in the proof of Proposition 2.12 is smaller than $2\lambda^2$.

As we mentioned in Proposition 1.21, Rosenthal constructed in [222] the first injective Banach space not isomorphic to a dual space. The example appears as a space $C(G)$ where $G$ is a closed part of $\beta\mathbb{N}$. One therefore has an exact sequence

$$0 \longrightarrow J_G \longrightarrow \ell_\infty \longrightarrow C(G) \longrightarrow 0$$

in which $J_G$ is an $M$-ideal, hence separably injective. In the remarks after the proof of Proposition 2.11 it was already noticed that the kernel of a quotient map $\ell_\infty \to \ell_\infty$ need not to be an $\mathscr{L}_\infty$ space.

### 2.2.3 *Compact Spaces of Finite Height*

Given a compact space $K$, recall that we write $K'$ for its derived set, that is, the set of non-isolated points of $K$. This process can be iterated to define $K^{(n+1)}$ as $(K^{(n)})'$. We say that $K$ has finite height if $K^{(n)} = \varnothing$ for some $n \in \mathbb{N}$, the least of which is called the (Cantor-Bendixson) height of $K$.

**Proposition 2.22** *Let $K$ be an infinite compact space of finite height. Then $C(K)$ is separably injective but not universally separably injective.*

*Proof* Let us show that $C(K)$ is separably injective if and only if $C(K')$ is separably injective; which yields the result since after finite number of derivations one necessarily arrives to a finite compact set. Let $I$ be the set of isolated points of $K$. The restriction operator $C(K) \longrightarrow C(K')$ induces a short exact sequence

$$0 \longrightarrow c_0(I) \longrightarrow C(K) \longrightarrow C(K') \longrightarrow 0.$$

Since separable injectivity is a 3-space property (Proposition 2.11(1)) and $c_0(I)$ is separably injective, if $C(K')$ is separably injective then also $C(K)$ is separably injective. When $K$ is scattered (in particular, of finite height) then the dual of every separable subspace is separable [92], hence $C(K)$ does not contain $\ell_\infty$ and thus it follows from Proposition 2.8(2) that it cannot be universally separably injective. The only if follows from Proposition 2.11(2). □

Of course that spaces of continuous functions on countable height compacta, such as $C(\omega^\omega)$, need not be separably injective. An alternative proof for the result above provides more information about the constants involved:

**Proposition 2.23** *If $K$ is a compact space of height $n$, then $C(K)$ is $(2n-1)$-separably injective.*

*Proof* Let $Y \subset X$ with $X$ separable and let $t : Y \to C(K)$ be a norm one operator. The range of $t$ is separable and every separable subspace of a $C(K)$ is contained in an isometric copy of $C(L)$, where $L$ is the quotient of $K$ after identifying $k$ and $k'$ when $y(k) = y(k')$ for all $y \in Y$. This $L$ is metrizable because $Y$ is separable. Moreover, if $K$ has height $n$, then $L$ has height at most $n$ and so it is homeomorphic to $[0, \omega^r \cdot k]$ with $r < n$, $k < \omega$ (see [36]; or else [120, Theorem 2.56]). Since $C[0, \omega^r \cdot k]$ is $(2r+1)$-separably injective [25], our operator can be extended to an operator $T : X \to C(K)$ with norm

$$\|T\| \le (2r+1)\|t\| \le (2n-1)\|t\|,$$

concluding the proof. □

When $K$ is a metrizable compact of finite height $n$, Baker [25] showed that $2n-1$ is the best constant for separable injectivity, using arguments from Amir [5]. There are some difficulties in generalizing those arguments for nonmetrizable compact spaces, so we do not know if it could exist a nonmetrizable compact space $K$ of height $n$ such that $C(K)$ is $\lambda$-separably injective for some $\lambda < 2n-1$.

### 2.2.4 Twisted Sums of $c_0(I)$

By Proposition 2.11, twisted sums of separably injective spaces are separably injective, so making twisted sums is an effective method to obtain new separably injective spaces. The simplest examples will be provided by twisted sums of two $c_0(\aleph)$. There exist in the literature several examples of nontrivial twisted sums of the type

$$0 \longrightarrow c_0(I) \longrightarrow E \longrightarrow c_0(J) \longrightarrow 0 \tag{2.4}$$

with different properties. The twisted sum space $E$ is separably injective but not universally separably injective, just because $E$ cannot be a Grothendieck space unless both $I$ and $J$ are finite [see Proposition 2.8(2)]. All the examples of such twisted sums $E$ that exist in the literature are of the form $C(K)$ with $K$ a compact space of finite height as in Sect. 2.2.3. It is an open problem whether a twisted sum $E$ of $c_0(I)$ and $c_0(J)$ exists that is not a $C(K)$-space. It is shown in [68] that every twisted sum of $c_0(\Gamma)$ and a space with property $(V)$ has property $(V)$.

When $J$ is countable the sequence splits since $c_0(I)$ is separably injective. For $I = \mathbb{N}$ and $\aleph_0 < |J| \leq \mathfrak{c}$ a nontrivial extension can be obtained (see [144, Example 2]; and also [61]) from an almost-disjoint family $\mathcal{M}$ of size $|J|$ of infinite subsets of $\mathbb{N}$; which means that $M \cap N$ is finite for different $M, N \in \mathcal{M}$. The existence of such a family was first observed by Sierpiński; see the proof of Theorem 1.25. Let $E(\mathcal{M})$ be the closure of the linear span in $\ell_\infty$ of the characteristic functions $\{1_n : n \in \mathbb{N}\}$ and $\{1_M : M \in \mathcal{M}\}$. Since the images of $\{1_M : M \in \mathcal{M}\}$ in $\ell_\infty/c_0$ generate a copy of $c_0(J)$ we have the pull-back diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & \ell_\infty/c_0 & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & c_0 & \longrightarrow & E(\mathcal{M}) & \longrightarrow & c_0(J) & \longrightarrow & 0
\end{array}$$

Recall that weakly compactly generated (in short, WCG) subspaces of $\ell_\infty$ are separable: if $K$ is weakly compact in $\ell_\infty$, then the coordinates of $\ell_\infty$ provide countably many real-valued continuous functions on the compact $K$ that separate the points, hence $K$ is metrizable and separable. From this, we get that $E(\mathcal{M})$ is not WCG. Hence the lower sequence in the preceding diagram does not split because $c_0$ and $c_0(J)$ are WCG and the product of two WCG spaces is WCG.

It is easily seen that $E(\mathcal{M})$ is a subring of $\ell_\infty$ and so it can be represented as (that is, it is isometric through a ring isomorphism to) certain $C_0(L)$, where $L$ is a locally compact space. It is actually simpler to consider the unitization of $E(\mathcal{M})$ in $\ell_\infty$, that is,

$$E(\mathcal{M})_+ = \{\lambda 1_{\mathbb{N}} + f : \lambda \in \mathbb{R}, f \in E(\mathcal{M})\}.$$

In this way $E(\mathcal{M})_+$ is a (closed, unital) subalgebra of $\ell_\infty$ that can be identified with a $C(K)$ for a compact space $K = K_{\mathcal{M}}$ (the one-point compactification of the just mentioned $L$). The description of $K_{\mathcal{M}}$ is an amusing exercise. It has three levels: isolated points, that correspond to natural numbers; points in the second level correspond to elements of the family $\mathcal{M}$, and a neighborhood of $M$ contains $M$ together with almost all elements of $M$. The point in the third level is the infinity point in the one-point compactification of the first two levels. One has a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & C(\beta\mathbb{N}) & \longrightarrow & C(\mathbb{N}^*) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & c_0 & \longrightarrow & C(K_{\mathcal{M}}) & \longrightarrow & C(K'_{\mathcal{M}}) & \longrightarrow & 0 \end{array}$$

where, moreover, $K'_{\mathcal{M}}$ is the one-point compactification of $J$.

Other twisted sums of $c_0(I)$ and $c_0(J)$ spaces were obtained by Ciesielski and Pol (see [81, Definition 8.8.2]). They are $C$-spaces $C(\mathcal{CP})$, where the Ciesielski-Pol compacta $\mathcal{CP}$ have both the derived set $\mathcal{CP}'$ and its complement $\mathcal{CP} \setminus \mathcal{CP}'$ uncountable, and the second derived set $\mathcal{CP}''$ is a singleton. Moreover, $C(\mathcal{CP})$ has a subspace $Y$ isometric to $c_0(I)$ with $C(\mathcal{CP})/Y$ isomorphic to $c_0(J)$, for some uncountable sets $I$ and $J$. They have the property that there is no injective operator from $C(\mathcal{CP})$ into $c_0(\Gamma)$, for any $\Gamma$, so they are not WCG.

Nontrival WCG twisted sums of $c_0(\Gamma)$ also exist. In [13] it is obtained an exact sequence

$$0 \longrightarrow c_0(\aleph) \longrightarrow C(K) \longrightarrow c_0(\aleph) \longrightarrow 0$$

in which $K$ is an Eberlein compact. Under GCH one can choose $\aleph = \aleph_\omega$ (and this is the smallest cardinal allowing a WCG nontrivial twisted sum of $c_0(\Gamma)$. Bell and Marciszewski construct in [29] an Eberlein compact $\mathcal{BM}$ of weight $\mathfrak{c}$ and height 3 that cannot be embedded into the space of all characteristic functions of subsets of cardinality lesser than or equal to $n$ of a given set; Marciszewski shows in [192] that $C(\mathcal{BM})$ is actually a nontrivial twisted sum of two $c_0(\Gamma)$. On the other hand, given a compact space $K$ of weight smaller than $\aleph_\omega$, the space $C(K)$ is isomorphic to $c_0(I)$ if and only if $K$ is an Eberlein compact of finite height [109, 192].

### 2.2.5 Twisted Sums of $c_0$ and $\ell_\infty$

The next simplest twisted sum of separably injective spaces are those of $c_0$ and $\ell_\infty$. Nontrivial twisted sums $0 \to c_0 \to X \to \ell_\infty \to 0$ exist and explicit examples can be seen in [54], obtained as the lower sequence in a certain pull-back diagram

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r, "\imath"] & \ell_\infty \arrow[r] & C(\mathbb{N}^*) \arrow[r] & 0 \\
0 \arrow[r] & c_0 \arrow[r] \arrow[u, equal] & E(\mathcal{M}) \arrow[r] \arrow[u] & c_0(J) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & c_0 \arrow[r] \arrow[u, equal] & \mathrm{PB} \arrow[r] \arrow[u] & \ell_2(J) \arrow[r] \arrow[u, "\varkappa"'] & 0 \\
0 \arrow[r] & c_0 \arrow[r, "j"] \arrow[u, equal] & CC \arrow[r] \arrow[u] & \ell_\infty \arrow[r] \arrow[u, "q"'] & 0
\end{tikzcd}$$

Here $E(\mathcal{M})$ is obtained using an almost disjoint family of size $\mathfrak{c} = |J|$, $\varkappa$ is any operator providing a non WCG pull-back space PB (such as the canonical inclusion, in which case PB is the Johnson-Lindenstrauss space [144, Example 1]) and $q$ is a quotient map. The twisted sum space in the lower sequence was baptized $CC$ in [157].

The lower sequence cannot split since otherwise there would be a quotient map $Q : c_0 \oplus \ell_\infty \to \mathrm{PB}$. The restriction of $Q$ to $\ell_\infty$ cannot be weakly compact, since otherwise PB would be WCG; therefore, $Q$ must be an isomorphism on a copy of $\ell_\infty$; but PB does not contain $\ell_\infty$ because "not containing $\ell_\infty$" is a 3-space property [61, Theorem 3.2.f]. The space $CC$ cannot be universally separably injective: since $\imath$ admits the obvious extension through $j$, if $j$ would also extend through $\imath$ then the diagonal principles (Proposition A.22) would yield an isomorphism $\ell_\infty \oplus \ell_\infty = CC \oplus C(\mathbb{N}^*)$, which makes $C(\mathbb{N}^*)$ complemented in $\ell_\infty$ which is not.

### 2.2.6 A Separably Injective Space Not Isomorphic to a Complemented Subspace of Any $C(K)$

This counterexample depends on Benyamini's construction appearing in [32] of an $M$-space not isomorphic to any complemented subspace of a $C$-space. The basic element in that construction can be described as follows. Let $\tilde{\mathbb{N}}$ denote a copy of the set of the integers. Given $x \in \beta\mathbb{N}$, we denote by $\tilde{x}$ the corresponding element in $\beta\tilde{\mathbb{N}}$. Set $\tilde{\mathbb{N}}^* = \beta\tilde{\mathbb{N}} \setminus \tilde{\mathbb{N}}$ and put $B = \beta\mathbb{N} \oplus \tilde{\mathbb{N}}^*$. Now, for $0 < \tau < 1$, consider

$$\mathscr{B}_\tau = \{f \in C(B) : f(x) = \tau f(\tilde{x}) \text{ for all } x \in \mathbb{N}^*\},$$

equipped with the restriction of the sup norm in $C(B)$. Quite clearly, $\mathscr{B}_\tau$ is a renorming of $\ell_\infty$. However, and this is the crux, $\mathscr{B}_\tau$ is far away from the complemented subspaces of any $C(K)$ space in the following precise sense: if $K$ is a compact space, $u : \mathscr{B}_\tau \to C(K)$ is an isomorphic embedding and $p$ is a projection of $C(K)$ onto the range of $u$, then $\|u\|\|u^{-1}\|\|p\| \geq 1/\tau$.

*Example 2.24* Suppose $\tau(n) \to 0$. Then the spaces $c_0(\mathbb{N}, \mathscr{B}_{\tau(n)})$ and $\ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)})$ are separably injective yet they are isomorphic to no direct factor of a $C$-space. They are not universally separably injective and $\ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)})$ is a Grothendieck space.

*Proof* It suffices to see that $\mathscr{B}_\tau$ is 5-separably injective for $0 < \tau \leq 1$. Notice that the characteristic functions of the points of $\mathbb{N}$ generate an ideal in $C(B)$ which is fact an isometric copy of $c_0$ in $B_\tau$ that we will denote $c_0(\mathbb{N})$. Clearly, $c_0(\mathbb{N})$ is an $M$-ideal in $B_\tau$. After a moment's reflection one realizes that the quotient $B_\tau/c_0(\mathbb{N})$ is isometric to $\ell_\infty/c_0 = C(\mathbb{N}^*)$. Thus, even if $\mathscr{B}_\tau$ is badly isomorphic to $\ell_\infty$ we have an isometric exact sequence

$$0 \longrightarrow c_0(\mathbb{N}) \xrightarrow{\ \imath\ } \mathscr{B}_\tau \xrightarrow{\ \pi\ } C(\mathbb{N}^*) \longrightarrow 0$$

whose kernel is an $M$-ideal.

Let now $X$ be a separable Banach space and $t : Y \to \mathscr{B}_\tau$ be a norm one operator, where $Y$ is a subspace of $X$. As $C(\mathbb{N}^*)$ is 1-separably injective one can find a norm one $T : X \to C(\mathbb{N}^*)$ extending the composition $\pi t$. As $T$ has separable range, by Proposition 2.20, $T$ can be lifted to an operator $L : X \to \mathscr{B}_\tau$, again with $\|L\| = 1$. Clearly, $t - L|_Y$ takes values in $c_0(\mathbb{N})$ and it can be extended to an operator $S : X \to c_0(\mathbb{N})$, with $\|S\| \leq 2\|t - L|_Y\| \leq 4$. Hence $S + L$ is an extension of $t$ to $X$, and has norm at most 5. Thus we see that $\ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)})$ is 5-separably injective and $c_0(\mathbb{N}, \mathscr{B}_{\tau(n)})$ is 10-separably injective.

As for the last statement, $c_0(\mathbb{N}, \mathscr{B}_{\tau(n)})$ cannot be universally separably injective since it contains a complemented copy of $c_0$, which is not. To see that $\ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)})$ is not universally separably injective, observe that $\mathscr{B}_\tau$ is (isometric to) the pull-back space in the diagram

$$\begin{tikzcd}
0 \arrow[r] & c_0(\mathbb{N}) \arrow[r] & C(\beta\mathbb{N}) \arrow[r, "r"] & C(\mathbb{N}^*) \arrow[r] & 0 \\
0 \arrow[r] & c_0(\mathbb{N}) \arrow[r] \arrow[u, equal] & \mathrm{PB}(r, \tau) \arrow[r] \arrow[u] & C(\tilde{\mathbb{N}}^*) \arrow[r] \arrow[u, "\tau"'] & 0
\end{tikzcd} \tag{2.5}$$

where $r$ is plain restriction and $\tau$ denotes multiplication (by $\tau$). Indeed, by the very definition we have

$$\begin{aligned}
\mathrm{PB}(r, \tau) &= \{(f, g) \in C(\beta\mathbb{N}) \oplus_\infty C(\tilde{\mathbb{N}}^*) : rf = \tau g\} \\
&= \{(f, g) \in C(\beta\mathbb{N}) \oplus_\infty C(\tilde{\mathbb{N}}^*) : f(x) = \tau g(\tilde{x}) \text{ for every } x \in \mathbb{N}^*\} \\
&= \mathscr{B}_\tau.
\end{aligned}$$

Therefore, for each $n$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & c_0 & \longrightarrow & \ell_\infty & \longrightarrow & C(\mathbb{N}^*) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & c_0 & \xrightarrow{j_n} & \mathscr{B}_{\tau(n)} & \longrightarrow & C(\mathbb{N}^*) & \longrightarrow & 0 \end{array} \tag{2.6}$$

All these can be amalgamated into a unique diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ell_\infty(c_0) & \longrightarrow & \ell_\infty(\ell_\infty) & \longrightarrow & \ell_\infty(C(\mathbb{N}^*)) & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & \ell_\infty(c_0) & \xrightarrow{j} & \ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)}) & \longrightarrow & \ell_\infty(C(\mathbb{N}^*)) & \longrightarrow & 0 \end{array} \tag{2.7}$$

If $\ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)})$ were universally separably injective, then it should be $\lambda$-universally separably injective, for some $\lambda$. This would imply that every $\mathscr{B}_{\tau(n)}$ is $\lambda$-universally separably injective and so the operator $j_n$ in (2.6) admits an extension $J_n : \ell_\infty \to \mathscr{B}_{\tau(n)}$, with $\|J_n\| \leq \lambda$. The "diagonal" operator $J : \ell_\infty(\ell_\infty) \to \ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)})$ given by $J((f_n)) = (J_n(f_n))$ is then an extension of the operator $j$ in diagram (2.7). Applying Proposition A.22 we would obtain an isomorphism

$$\ell_\infty(\ell_\infty) \oplus \ell_\infty(C(\mathbb{N}^*)) = E \oplus \ell_\infty(C(\mathbb{N}^*)).$$

This is impossible, since $\ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)})$ is not complemented in any $C$-space.

It follows from results of Leung and Räbiger in [174] that $\ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)})$ is a Grothendieck space: A set $I$ is said to have real-valued measurable cardinal if there exists a countably additive measure $\mu : \mathbb{P}(I) \to [0, 1]$ vanishing on the singletons of $I$. The existence of real-valued measurable cardinals cannot be proved in ZFC and the fact that $\aleph_0$ is not real-valued measurable is obvious. Leung and Räbiger proved in [174, Theorem] that if $(E_i)$ is a family of Banach spaces indexed by a set $I$ whose cardinal is not real-valued measurable, then the Banach space product $\ell_\infty(I, E_i)$ contains a complemented copy of $c_0$ (if and) only if some $E_i$ does. As each $\mathscr{B}_{\tau(n)}$ is a renorming of $\ell_\infty$ we see that $\ell_\infty(\mathbb{N}, \mathscr{B}_{\tau(n)})$ has no complemented subspace isomorphic to $c_0$. Since it is separably injective, has Pełczyński's property $(V)$ and, consequently, is a Grothendieck space. □

## 2.3 Universally Separably Injective Spaces

It was proved in Proposition 2.8(2) that universally separably injective spaces contain $\ell_\infty$. In this section we will show that they are in fact $\ell_\infty$-upper-saturated, according to the next definition.

**Definition 2.25** Let $X$ and $Y$ be Banach spaces. We say that $X$ is *$Y$-upper-saturated* if every separable subspace of $X$ is contained in some (isomorphic) copy of $Y$ inside $X$.

It is clear that $c_0$-upper-saturated spaces are separably injective and $\ell_\infty$-upper-saturated spaces are universally separably injective. One moreover has:

**Theorem 2.26** *An infinite-dimensional Banach space is universally separably injective if and only if it is $\ell_\infty$-upper-saturated.*

*Proof* The sufficiency is a clear consequence of the injectivity of $\ell_\infty$. In order to show the necessity, let $Y$ be a separable subspace of a universally separably injective space $X$. We consider a subspace $Y_0$ of $\ell_\infty$ isomorphic to $Y$ and an isomorphism $t : Y_0 \to Y$. We can find projections $p$ on $X$ and $q$ on $\ell_\infty$ such that $Y \subset \ker p$, $Y_0 \subset \ker q$, and both $p$ and $q$ have range isomorphic to $\ell_\infty$. Indeed, let $\pi : X \to X/Y$ be the quotient map. Since $X$ contains $\ell_\infty$ and $Y$ is separable, $\pi$ is not weakly compact so, by Proposition 2.8(2), there exists a subspace $M$ of $X$ isomorphic to $\ell_\infty$ where $\pi$ is an isomorphism. Now $X/Y = \pi[M] \oplus N$, with $N$ a closed subspace. Hence $X = M \oplus \pi^{-1}[N]$, and it is enough to take $p$ as the projection with range $M$ and kernel $\pi^{-1}[N]$.

Since $\ker p$ and $\ker q$ are universally separably injective spaces, we can take operators $u : X \to \ker q$ and $v : \ell_\infty \to \ker p$ such that $v = t$ on $Y_0$ and $u = t^{-1}$ on $Y$. Let $w : \ell_\infty \to \operatorname{ran} p$ be an operator satisfying $\|w(x)\| \geq \|x\|$ for all $x \in \ell_\infty$. We will show that the operator $T = v + w(\mathbf{1}_{\ell_\infty} - uv)$ is an into isomorphism $\ell_\infty \to X$. This suffices to end the proof since $\operatorname{ran} T$ is isomorphic to $\ell_\infty$ and both $T$ and $v$ agree with $t$ on $Y_0$, so $Y \subset \operatorname{ran} T \subset X$. Since $\operatorname{ran} v \subset \ker p$ and $\operatorname{ran} w \subset \operatorname{ran} p$, there exists $C > 0$ such that for all $x \in \ell_\infty$ one has

$$\|Tx\| \geq C \max\{\|v(x)\|, \|w(\mathbf{1}_{\ell_\infty} - uv)x\|\}.$$

Now, if $\|vx\| < (2\|u\|)^{-1}\|x\|$, then $\|uvx\| < \frac{1}{2}\|x\|$; hence

$$\|w(\mathbf{1}_{\ell_\infty} - uv)x\| \geq \|(\mathbf{1}_{\ell_\infty} - uv)x\| > \tfrac{1}{2}\|x\|.$$

Thus $\|Tx\| \geq C(2\|u\|)^{-1}\|x\|$ for every $x \in X$. □

Another similarity between $\ell_\infty$ and universally separably injective spaces is given in the next Proposition 2.27, which extends [182, Proposition 2.f.12(iii)]. Recall that an operator is Fredholm if its kernel and its cokernel are finite dimensional. Here, the cokernel of an operator $T : X \to Y$ is defined as $\operatorname{coker} T = Y/\operatorname{ran} T$. The index of a Fredholm operator $T$ is defined by

$$\operatorname{ind}(T) = \dim \ker T - \dim \operatorname{coker} T.$$

Note that if $Y/\operatorname{ran}(T)$ is finite dimensional, then $T$ has closed range [242, Theorem IV.5.10].

**Proposition 2.27** *Let $X$ be universally separably injective and let $\imath : Y \to X$ and $j : Y \to X$ be two into isomorphisms. Suppose that $X/j[Y]$ and $X/\imath[Y]$ are separable. Then every extension $I : X \to X$ of $\imath$ through $j$ (i.e., $Ij = i$) is a Fredholm operator and all these extensions have the same index.*

*Proof* Since $X$ is separably injective, we can find $u : X \to X$ and $v : X \to X$ operators such that $uj = \imath$ and $v\imath = j$. Let us denote $w = \mathbf{1}_X - vu$. Since $j(Y)$ is contained in the kernel of $w$, the operator $w$ factors through $X/j[Y]$. Recall that $\mathscr{L}_\infty$ spaces have the Dunford-Pettis property (every weakly compact operator defined on those spaces takes weakly convergent sequences into convergent ones; see Proposition A.2). Thus, $X$ has the Dunford-Pettis property and its separable quotients must be reflexive by Proposition 2.8(2). Therefore, the operator $w$ is weakly compact and completely continuous; hence $w^2$ is compact. From this fact it follows that $vu = \mathbf{1}_X - w$ is a Fredholm operator with $\mathrm{ind}(vu) = 0$. Similarly we can show that $uv$ is a Fredholm operator with $\mathrm{ind}(uv) = 0$. Thus $u$ and $v$ are Fredholm operators with $\mathrm{ind}(u) + \mathrm{ind}(v) = 0$, and the proof is done. □

Proposition 2.27 remains valid for $X$ separably injective provided one asks the quotients to be separable and reflexive (e.g., when $X$ is Grothendieck). Recall that two Banach spaces $X$ and $Y$ are said to be essentially incomparable (see [110]) if for each pair of operators $t : X \to Y$ and $s : Y \to X$, $\mathbf{1}_X - st$ is a Fredholm operator. Since it follows from Proposition 2.8(2) that a quotient of a universally separably injective space is either reflexive or it contains copies of $\ell_\infty$, the proof of Proposition 2.27 shows that universally separable injective spaces and spaces containing no copies of $\ell_\infty$ are essentially incomparable.

## 2.4 1-Separably Injective Spaces

While regarding injectivity it is unknown whether the parameter $\lambda$ in "$\lambda$-injective" has real content (after all, it could still be true that every $\lambda$-injective space can be renormed to become 1-injective) in this section we shall see that the parameter $\lambda$ in "$\lambda$-separably injective" has *some* meaning (but we do not know which). For instance, 1-separably injective spaces enjoy several properties that, say, 2-separably injective spaces lack; and spaces such as $c_0(\Gamma)$ are 2-separably injective but not $\lambda$-separably injective for $\lambda < 2$; at the same time, $C(K)$-spaces $\lambda$-separably injective for $\lambda < 2$ are automatically 1-separably injective (Proposition 2.34).

Keeping in mind that separably injective spaces are Grothendieck if and only if they do not contain $c_0$ complemented, it is possible to establish a major difference between 1-separably injective and general separably injective spaces: 1-separably injective spaces are Grothendieck (hence they cannot be separable or WCG)—see Proposition 2.31 below—while a 2-separably injective space, such as $c_0$, can be even separable.

To prove that 1-separably injective spaces cannot contain $c_0$ complemented, the following lemma due to Lindenstrauss [177, p. 221, proof of (i) ⇒ (v)] provides a useful technique.

**Lemma 2.28** *Let $E$ be a 1-separably injective space, $X$ a Banach space of density $\aleph_1$, and $Y$ a separable subspace of $X$. Then every operator $t : Y \to E$ can be extended to an operator $T : X \to E$ with the same norm.*

*Proof* We write $X$ as the union of a continuous $\omega_1$-sequence of separable spaces $(X_\alpha)_{\alpha<\omega_1}$ beginning with $X_0 = Y$. This just means (see Appendix A.6)

- $X_\alpha \subset X_\beta$ if $\alpha \leq \beta$.
- $X = \bigcup_{\alpha<\omega_1} X_\alpha$.
- For every limit ordinal $\beta < \omega_1$ one has $X_\beta = \overline{\bigcup_{\alpha<\beta} X_\alpha}$.

Then we define inductively a coherent family of operators $T_\alpha : X_\alpha \longrightarrow E$, all of them with the same norm as $T_0 = t$. We can do this using the 1-separable-injectivity of $E$ and, in the limit ordinals, using that given $T_{\alpha_n} : X_{\alpha_n} \longrightarrow E$, a coherent sequence of operators of norm $\|t\|$, they determine a unique operator $\bigcup_n X_{\alpha_n} \longrightarrow E$ of norm $\|t\|$. □

**Proposition 2.29 (CH)** *Every 1-separably injective Banach space is universally 1-separably injective and therefore a Grothendieck space.*

*Proof* Let $E$ be 1-separably injective, $X$ an arbitrary Banach space and $t : Y \to E$ an operator, where $Y$ is a separable subspace of $X$. Then $\overline{t[Y]}$, the closure of the image of $t$, is a separable subspace of $E$ and so there is an isometric embedding $u : \overline{t[Y]} \to \ell_\infty$. As $\ell_\infty$ is 1-injective there is an operator $T : X \to \ell_\infty$ whose restriction to $Y$ agrees with $ut$. Thus it suffices to extend the inclusion of $\overline{t[Y]}$ into $E$ to $\ell_\infty$. But, under CH, the density character of $\ell_\infty$ is $\aleph_1$ and Lemma 2.28 applies. The "therefore" part is now a consequence of Proposition 2.8(2). □

We will prove later (Theorem 2.39) that CH cannot be dropped in general from Proposition 2.29. However the "therefore" part survives in ZFC. The following characterization of 1-separable injectivity, apart from its intrinsic interest, will help with the proof. Its general version will be stated and proved in Proposition 5.12.

**Proposition 2.30** *A Banach space $E$ is 1-separably injective if and only if every countable family of mutually intersecting balls has nonempty intersection.*

*Proof* SUFFICIENCY. Take an operator $t : Y \to E$, where $Y$ is a closed subspace of a separable space $X$. We may and do assume $\|t\| = 1$. Let $z \in X\backslash Y$ and let $Y_0$ be a dense countable subset of $Y$ and, for each $y \in Y_0$, consider the ball $B(ty, \|y - z\|)$ in $E$. Any two of these balls intersect, since for $y_1, y_2 \in Y_0$ we have

$$\|ty_2 - ty_1\| \leq \|t\|\|y_2 - y_1\| \leq \|y_2 - z\| + \|y_1 - z\|.$$

The hypothesis is that there is

$$f \in \bigcap_{y \in Y_0} B(ty, \|y - z\|) = \bigcap_{y \in Y} B(ty, \|y - z\|).$$

It is clear that the map $T : Y + [z] \to E$ defined by $T(y + cz) = ty + cf$ is an extension of $t$ with $\|T\| = 1$. The rest is clear: use induction.

NECESSITY. We begin with the observation that if two closed balls of any Banach (or metric) space have a common point, then the distance between the centers is at most the sum of the radii. In $\ell_\infty$ that necessary condition suffices and every family of mutually intersecting balls has nonempty intersection.

Let $E$ be 1-separably injective and let $B(e_n, r_n)$ be a sequence of mutually intersecting balls in $E$. Let $Y$ be the closed separable subspace of $E$ spanned by the centers. Let $\kappa : Y \to \ell_\infty$ be any isometric embedding. Notice that even if $B_Y(e_n, r_n) = B(e_n, r_n) \cap Y$ need not be mutually intersecting in $Y$, any two balls of the sequence $B(\kappa(e_n), r_n)$ meet in $\ell_\infty$ because the distance between the centers does not exceed the sum of the radii. Therefore the intersection

$$\bigcap_n B(\kappa(e_n), r_n)$$

contains some point, say $x \in \ell_\infty$. Let $X$ be the subspace spanned by $x$ and $\kappa(Y)$ in $\ell_\infty$ so that $\dim X/Y \leq 1$. The hypothesis on $E$ allows one to extend the inclusion of $Y$ into $E$ to $X$ through $\kappa : Y \to X$ without increasing the norms. The image of $x$ in $E$ under any such extension belongs to the intersection of all the $B(e_n, r_n)$. □

**Proposition 2.31** *Every* 1*-separably injective space is a Grothendieck and a Lindenstrauss space.*

*Proof* To prove that a 1-separably injective space is Lindenstrauss we recall that a Banach space is a Lindenstrauss space if and only if every finite set of mutually intersecting balls has nonempty intersection [175]. Proposition 2.30 now concludes. A different argument can be derived from Proposition 1.5 that yields the bidual of a 1-separably injective space $X$ is 1-injective, hence $X^{**}$ and so $X$ is a Lindenstrauss space.

It remains to prove that a 1-separably injective space $X$ must be Grothendieck. Since $X$ has property $(V)$ by Proposition 2.8, it suffices to show that $c_0$ is not complemented in $X$, so let $j : c_0 \longrightarrow X$ be an embedding. Consider an almost-disjoint family $\mathcal{M}$ of size $\aleph_1$ formed by infinite subsets of $\mathbb{N}$. Proceeding as in Sect. 2.2.4 we get a nontrivial exact sequence

$$0 \longrightarrow c_0 \longrightarrow E(\mathcal{M}) \longrightarrow c_0(\aleph_1) \longrightarrow 0$$

where the space $E(\mathcal{M})$ has density character $\aleph_1$. The embedding $j$ can be extended to all of $E(\mathcal{M})$ by Lemma 2.28, which yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & c_0 \arrow[r] \arrow[d, equal] & E(\mathcal{M}) \arrow[r] \arrow[d] & c_0(\aleph_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & c_0 \arrow[r, "j"] & X \arrow[r] & X/j[c_0] \arrow[r] & 0
\end{tikzcd}$$

Thus, were $c_0$ complemented in $X$ it would be complemented in $E(\mathcal{M})$ as well, which is not. $\square$

### *2.4.1 On λ-Separably Injective Spaces When λ < 2*

As we have already mentioned, it is an open problem whether a $\lambda$-injective space is isomorphic to a 1-injective space. From Proposition 2.31 it is clear that 2-separably injective spaces cannot be, in general, be renormed to become 1-separably injective. We do not know whether a $\lambda$-separably injective space, $\lambda < 2$ must be (isomorphic to a) 1-separably injective or, at least, a Grothendieck space. We have, however, the following result, based on an idea of Ostrovskii [208]:

**Proposition 2.32** *A $\lambda$-separably injective space with $\lambda < 2$ is either finite dimensional or has density character at least $\mathfrak{c}$.*

*Proof* Let $X$ be an infinite dimensional $\lambda$-separably injective space for $\lambda < 2$. In Proposition 2.8 it is shown that $X$ contains $c_0$, and thus by a result of James [139] it contains, for each $\varepsilon > 0$, an $(1 + \varepsilon)$-isomorphic copy of $c_0$. With a standard renorming [211, Proposition 1] we may assume $X$ contains $c_0$ isometrically and it is $\lambda'$-separably injective, still with $\lambda' < 2$. So, let $u : c_0 \to X$ be an isometric embedding and let $u_n = u(e_n)$, where $(e_n)$ is the unit basis of $c_0$. For each element $f \in \ell_\infty$ with all coordinates $\pm 1$, let $T_f : c_0 + [f] \to X$ be an extension of $u$ with norm at most $\lambda'$. For two different $f, g$ pick $n$ so that $f(n) = 1$ and $g(n) = -1$. One has $\|u_n - T_f(f/2)\| = \|u_n + T_g(g/2)\| \le \lambda'/2$, and thus

$$\begin{aligned}
\|T_f(f/2) - T_g(g/2)\| &= \|T_f(f/2) - u_n - u_n - T_g(g/2) + 2u_n\| \\
&\ge 2 - \|T_f(f/2) - u_n - u_n - T_g(g/2)\| \\
&\ge 2 - \lambda'/2 - \lambda'/2 \\
&= 2 - \lambda'.
\end{aligned}$$

So $\operatorname{dens} X \ge \mathfrak{c}$. $\square$

In any case, it seems that some break occurs at $\lambda = 2$. As a preparation for the following result, let us record the following observation:

**Lemma 2.33** *If $E$ is $\lambda$-separably injective, then given a countable family of mutually intersecting balls $B(e_n, r_n)$ one has $\bigcap_n B(e_n, \lambda r_n) \neq \varnothing$.*

*Proof* Just read the "necessity part" of the proof of Proposition 2.30. □

According to Lindenstrausss (see [177, Remarks 3]) the following result "is similar to a result due to Amir [4] and Isbell and Semadeni [138] that if a $C$-space has projection constant $\lambda < 2$ then it has projection constant 1" (i.e., it is 1-injective).

**Proposition 2.34** *If a $C$-space is $\lambda$-separably injective for some $\lambda < 2$, then it is 1-separably injective.*

*Proof* What one actually proves is that if a $C(K)$-space is $\lambda$-separably injective for some $\lambda < 2$ then $K$ actually is an $F$-space, in the formulation: for every $f \in C(K)$ there is $g \in C(K)$ and $\delta > 0$ such that $f(k) > 0$ implies $g(k) \geq \delta$ and $f(k) < 0$ implies $g(k) \leq -\delta$. Now, if $C(K)$ is $\lambda$-separably injective then it has property $(c_\lambda)$, and therefore any family $B(x_\alpha, r_\alpha)$ of mutually intersecting balls whose centers lie on a separable subspace is such that $\bigcap_\alpha B(x_\alpha, \lambda r_\alpha) \neq \varnothing$.

Pick now $f \in C(K)$ and set $r_n(t) = 1$ for $t \geq 1/n$ and $r_n(t) = -1$ for $t \leq -1/n$ and linear in $[-1/n, 1/n]$. The balls in the sequence $B(r_n \circ f, 1/2)$ are mutually intersecting. By the preceding Lemma there exists $g \in \bigcap_n B(r_n \circ f, \lambda/2)$. Since $\lambda < 2$, set $\delta = 1 - \lambda/2 > 0$ and observe that $g(k) \geq \delta$ when $f(k) > 0$ and $g(k) \leq -\delta$ when $f(k) < 0$. □

### 2.4.2 A *C*-space 1-Separably Injective But Not Universally 1-Separably Injective

We show now that without CH, 1-separably injectivity does not longer imply universal 1-separable injectivity. To this end we will produce, assuming that $\mathfrak{c} = \aleph_2$ and also Martin's axiom, a 1-separably injective space $C(K)$ and an operator $c \to C(K)$ that does not admit norm-preserving extensions to $\ell_\infty$. In order to state Martin's axiom we need a few definitions. Suppose that we have a partially ordered set $P$. Two elements $p, q \in P$ are compatible if there exists $r \in P$ such that $r < p$ and $r < q$. A filter is a subset $\mathscr{F} \subset P$ of pairwise compatible elements such that if $p \in \mathscr{F}$ and $p < q$, then $q \in \mathscr{F}$. A subset $D \subset P$ is called dense if for every $p \in P$ there exists $q \in D$ such that $q < p$. We say that $P$ has the countable chain condition (or ccc) if every uncountable subset of $P$ contains a pair of compatible elements.

**Martin's Axiom** [MA] If $P$ is a ccc partially ordered set and $\{D_i : i \in I\}$ is a family of dense subsets of $P$ with $|I| < \mathfrak{c}$, then there exists a filter $\mathscr{F} \subset P$ such that $\mathscr{F} \cap D_i \neq \varnothing$ for every $i \in I$.

This axiom has become a standard tool with a number of applications in analysis. It is compatible with the ZFC system of axioms of set theory, and it is also compatible with different values of the continuum, in particular with $\mathfrak{c} = \aleph_2$; that is what we shall use.

**Definition 2.35** Let $L$ be a zero-dimensional compact space. An $\aleph_2$-Lusin family on $L$ is a family $\mathcal{F}$ of pairwise disjoint nonempty clopen subsets of $L$ with $|\mathcal{F}| = \aleph_2$, such that whenever $\mathcal{G}$ and $\mathcal{H}$ are subfamilies of $\mathcal{F}$ with $|\mathcal{G}| = |\mathcal{H}| = \aleph_2$, then

$$\overline{\bigcup\{G \in \mathcal{G}\}} \cap \overline{\bigcup\{G \in \mathcal{H}\}} \neq \varnothing.$$

**Lemma 2.36 (MA, $\mathfrak{c} = \aleph_2$)** *There exists an $\aleph_2$-Lusin family on $\mathbb{N}^*$.*

*Proof* We are going to construct a family $\{A_\alpha\}_{\alpha<\omega_2}$ of infinite subsets of $\mathbb{N}$ such that

1. $A_\alpha \cap A_\beta$ is finite for $\alpha < \beta < \omega_2$,
2. for every $B \subset \mathbb{N}$ either $\{\alpha : |A_\alpha \setminus B| \text{ is finite}\}$ or $\{\alpha : |A_\alpha \cap B| \text{ is finite}\}$ has cardinality strictly lesser than $\aleph_2$.

Once we obtain this family, we can consider the family the clopens $C_\alpha = \{\mathcal{U} \in \mathbb{N}^* : A_\alpha \in \mathcal{U}\}$ of $\mathbb{N}^*$. The family $\mathcal{C} = \{C_\alpha : \alpha < \omega_2\}$ is an $\aleph_2$-Lusin family on $\mathbb{N}^*$, because they are disjoint by (1), and if we have $\mathcal{G}$ and $\mathcal{H}$ subfamilies of $\mathcal{C}$ whose unions have disjoint closures, then these unions can be separated by a clopen set of $\mathbb{N}^*$, which is of the form $\{\mathcal{U} \in \mathbb{N}^* : B \in \mathcal{U}\}$. Property (2) of our family prevents that both $\mathcal{G}$ and $\mathcal{H}$ have cardinality $\aleph_2$.

So let us proceed now to the construction of the sets $A_\alpha$. Let $\{B_\alpha : \alpha < \omega_2\}$ be an enumeration of all infinite subsets of $\mathbb{N}$. We construct the $A_\alpha$'s inductively on $\alpha$. Suppose $A_\gamma$ has been constructed for $\gamma < \alpha$. We define a partially ordered set $\mathbb{P}_\alpha$ whose elements are pairs $p = (f_p, F_p)$ where $f_p$ is a $\{0, 1\}$-valued function on a finite subset $\mathrm{dom}(f_p)$ of $\mathbb{N}$ and $F_p$ is a finite subset of $\alpha$. The order relation is that $p < q$ if

- $\mathrm{dom}(f_p) \supset \mathrm{dom}(f_q)$ and $f_p|_{\mathrm{dom}(q)} = f_q$,
- $F_p \supset F_q$,
- $f_p$ vanishes in $A_\gamma \cap \mathrm{dom}(f_p) \setminus \mathrm{dom}(f_q)$ for $\gamma \in F_q$.

First, notice that this partially ordered set is ccc. This is simply because if $Q \subset \mathbb{P}_\alpha$ is an uncountable set, we can find $p, q \in Q$ with $f_p = f_q$, and any two such functions are compatible, since $r = (f_p, F_p \cup F_q) = (f_q, F_p \cup F_q)$ satisfies $r < p$ and $r < q$. Thus for any family of $\aleph_1$ many dense subsets we can find a filter $\mathscr{F} \subset \mathbb{P}_\alpha$ that intersects all of them. The family of dense subsets is the following:

- $D_n = \{p \in \mathbb{P}_\alpha : n \in \mathrm{dom}(p)\}$, for $n \in \mathbb{N}$,
- $D'_\beta = \{p \in \mathbb{P}_\alpha : \beta \in F_p\}$, for $\beta < \alpha$,
- $D''_{\gamma,m} = \{p \in \mathbb{P}_\alpha : \text{there is } n > m \text{ such that } n \in B_\gamma \cap \mathrm{dom} f_p \text{ and } f_p(n) = 1\}$, where $m \in \mathbb{N}$ and $\gamma < \alpha$ are such that $B_\gamma \setminus \{0, \ldots, m\}$ is not contained in any finite union of $A_\beta$'s with $\beta < \alpha$.

It is easily seen that all these sets are dense in $\mathbb{P}_\alpha$. Let $\mathscr{F} \subset \mathbb{P}_\alpha$ be the filter provided by MA and take $A_\alpha = \{n \in \mathbb{N} : \text{ there is } p \in \mathfrak{F} \text{ such that } f_p(n) = 1\}$. We check:

1. $A_\alpha \cap A_\beta$ is finite for every $\beta < \alpha$. To check this, pick $q \in \mathscr{F} \cap D'_\beta$. We claim that $A_\alpha \cap A_\beta \subset \mathrm{dom}(f_q)$. Suppose on the contrary that we have $n \in A_\alpha \cap A_\beta \setminus \mathrm{dom}(f_q)$. Since $n \in A_\alpha$ we can find $p \in \mathscr{F}$ such that $f_p(n) = 1$. Since $\mathscr{F}$ is a filter, $p$ and $q$ must be compatible, so pick $r \in \mathbb{P}_\alpha$ such that $r < p$ and $r < q$. We can now apply the third condition of the definition of the order relation, because $\beta \in F_q$, and $n \in A_\beta \cap \mathrm{dom}(f_r) \setminus \mathrm{dom}(f_q)$. So we conclude that $f_r(n) = 0$. But we supposed that $f_p(n) = 1$ and $r < p$, a contradiction.
2. For every $\gamma < \alpha$, if $B_\gamma$ is not contained in any finite union of $A_\delta$'s and a finite set then $A_\alpha \cap B_\gamma$ is infinite. To prove this, it is enough to check that $A_\alpha \cap B_\gamma$ contains some $n > m$ for every $m \in \mathbb{N}$. For this, just use $p \in \mathscr{F} \cap D''_{\beta,m}$.

This finishes the inductive construction of the $A_\alpha$'s. They form indeed an almost disjoint family by property 1 above. It remains to check the second property that we claimed about the family $\{A_\alpha\}_{\alpha<\omega_2}$ at the beginning of the proof. So pick $B \subset \mathbb{N}$. If $B$ is contained in a finite union of $A_\delta$'s and a finite set $F$, $B \subset F \cup \left(\bigcup_{\delta\in\Delta} A_\delta\right)$, then just using the almost disjointness, we check that $\{\alpha < \omega_2 : |A_\alpha \setminus B| \text{ is finite }\} \subset \Delta$ so we are done. Similarly, if $\mathbb{N} \setminus B$ is contained in a union $F \cup \bigcup_{\delta\in\Delta} A_\delta$, then $\{\alpha < \omega_2 : |A_\alpha \cap B| \text{ is finite }\} \subset \Delta$. So we assume that neither $B$ nor $\mathbb{N} \setminus B$ is contained in a finite union of $A_\delta$'s and a finite set. Then pick $\alpha_0$ such that $B = B_\beta$ and $\mathbb{N} \setminus B = B_\gamma$ with $\beta, \gamma < \alpha_0$. Then, using condition 2 above, we get that for every $\alpha > \alpha_0$, both $A_\alpha \cap B = A_\alpha \cap B_\beta$ and $A_\alpha \cap (\mathbb{N} \setminus B) = A_\alpha \cap B_\gamma$ are infinite, and we are done. □

In the next theorem we provide the compact space $\mathscr{A}$ whose space of continuous functions will provide the desired example. This compactum is constructed in ZFC, though we will focus on the case when $\mathfrak{c} = \aleph_2$. This value of the continuum is taken mainly for convenience. The construction is the same as the one performed in [16, 87], that we present in purely topological language. It is an inductive construction of length $\mathfrak{c}$ in which at each successor step we split a couple of disjoint $F_\sigma$ open sets, and we do this exhaustively. Those successive steps can be interpreted as pull-backs with respect to metrizable quotients, cf. Sect. 3.4.5 for further information about this compactum.

**Theorem 2.37** ($\mathfrak{c} = \aleph_2$) *There exists an infinite zero-dimensional compact $F$-space $\mathscr{A}$ such that no closed $G_\delta$ subset of $\mathscr{A}$ contains any $\aleph_2$-Lusin family.*

*Proof* We construct this compact space as an inverse limit of length $\mathfrak{c}$. So, we shall produce compact spaces $\{K_\alpha : \alpha < \mathfrak{c}\}$ and continuous onto maps $\{\pi_{\beta\alpha} : K_\beta \longrightarrow K_\alpha : \alpha \leq \beta\}$ such that $\pi_{\alpha\alpha}$ is the identity in $K_\alpha$ and $\pi_{\beta\alpha} \circ \pi_{\gamma\beta} = \pi_{\gamma\alpha}$ for all $\alpha < \beta < \gamma$, and then $\mathscr{A}$ will be the limit of the system in the sense that

$$\mathscr{A} = \left\{ (x_\alpha)_{\alpha<\mathfrak{c}} \in \prod_{\alpha<\mathfrak{c}} K_\alpha : \pi_{\beta\alpha}(x_\beta) = x_\alpha \text{ for all } \alpha < \beta < \mathfrak{c} \right\}$$

We fix a partition $\mathfrak{c} = \bigcup_{\alpha<\mathfrak{c}} S_i$ into $\mathfrak{c}$ many subsets such that $|S_\alpha| = \mathfrak{c}$ and $\alpha \leq \min(S_\alpha)$ for all $\alpha$. The inductive construction is as follows. Let $K_0$ be the Cantor set. Once the compact space $K_\alpha$ is constructed, we produce an enumeration

$$\{(V_\beta, W_\beta) : \beta \in S_\alpha\}$$

of all pairs of disjoint open $F_\sigma$ subsets of $K_\alpha$. This will be possible because the weight of each $K_\alpha$ will be less than $\mathfrak{c}$. If the system has been defined for all ordinals below a given $\gamma$, we distinguish two cases. If $\gamma$ is a limit ordinal, then we take $K_\gamma$ to be the inverse limit of the preceding system:

$$K_\gamma = \left\{ (x_\alpha)_{\alpha<\gamma} \in \prod_{\alpha<\gamma} K_\alpha : \pi_{\beta\alpha}(x_\beta) = x_\alpha \text{ for all } \alpha < \beta < \gamma \right\}.$$

If $\gamma = \beta + 1$ is a successor, we pick the $\alpha$ such that $\beta \in S_\alpha$, and then define

$$K_\gamma = (K_\beta \setminus \pi_{\beta\alpha}^{-1}(V_\beta)) \times \{0\} \ \cup \ (K_\beta \setminus \pi_{\beta\alpha}^{-1}(W_\beta)) \times \{1\}$$

and then $\pi_{\gamma\delta}(x, i) = \pi_{\beta\delta}(x)$ for $\delta < \gamma$. This finishes the inductive construction. We will denote by $\pi_\alpha : \mathscr{A} \longrightarrow K_\alpha$ the canonical projection.

We start now exploring the properties of $\mathscr{A}$. First, it is clear that $\mathscr{A}$ is zero-dimensional since each $K_\alpha$ along the construction is such. Second, we check that $\mathscr{A}$ is an $F$-space. If we take $V$ and $W$ two disjoint open $F_\sigma$-sets, then they must be of the form $V = \pi_\alpha^{-1}(V')$ and $W = \pi_\alpha^1(W')$ for some disjoint open $F_\sigma$ sets in $K_\alpha$ for some $\alpha < \mathfrak{c}$. But then, $(V, W')' = (V_\beta, W_\beta)$ for some $\beta > \alpha$ and

$$\pi_\beta^{-1}\left((K_\beta \setminus \pi_{\beta\alpha}^{-1}(V_\beta)) \times \{0\}\right) \text{ and } \pi_\beta^{-1}\left((K_\beta \setminus \pi_{\beta\alpha}^{-1}(W_\beta)) \times \{1\}\right)$$

are disjoint clopens that separate $W$ and $V$. Third, we prove that if $c$ is a clopen subset of $\mathscr{A}$, then $\pi_\alpha(c)$ is a closed $G_\delta$ set for all $\alpha < \mathfrak{c}$. Indeed, the set $c$ must be of the form $\pi_\beta^{-1}(b)$ for some clopen subset $b \subset K_\beta$ and some $\beta < \mathfrak{c}$. So it is enough to show that $\pi_{\beta\alpha}(b)$ is a $G_\delta$ for every clopen $b$ of $K_\beta$ and every $\alpha < \beta < \mathfrak{c}$. We prove it by induction on $\beta$. If $\beta = \gamma + 1$ is a successor, it is easily checked that the one-step map $\pi_{\beta\gamma}$ takes clopen sets onto $G_\delta$-sets. If $\beta$ is a limit ordinal, there exists indeed $\alpha < \gamma < \beta$ such that $\pi_{\beta\gamma}(b)$ is a clopen set.

Finally, we fix a closed $G_\delta$ set $F$ and we prove that $F$ does not contain any $\aleph_2$-Lusin family of clopen subsets of $F$. So suppose that we have such a family $\mathfrak{F}$ , and we construct by induction subfamilies $\mathfrak{F}_i \subset \mathfrak{F}$ and ordinals $\alpha(i) < \mathfrak{c}$ for $i < \omega_1$ with the following properties:

1. $\mathfrak{F}_i \subsetneq \mathfrak{F}_j$ and $\alpha(i) < \alpha(j)$ if $i < j$.
2. Each family $\mathfrak{F}_i$ has cardinality $\aleph_1$.

3. Each $a \in \mathcal{F}_i$ is determined up to $\alpha(i+1)$, in the sense that $a$ is of the form $a = \pi^{-1}_{\alpha(i+1)}(a')$ for some clopen set $a'$ of $K_{\alpha(i+1)}$.
4. If $b$ is a clopen subset of $K_{\alpha(i)}$ such that $|\{a \in \mathcal{F} : a \subset \pi^{-1}_{\alpha(i)}(b)\}| \leq \aleph_1$, then $\{a \in \mathcal{F} : a \subset \pi^{-1}_{\alpha(i)}(b)\} \subset \mathcal{F}_{i+1}$

The construction is possible because each $K_\alpha$ has weight less than $\mathfrak{c} = \aleph_2$ so it has at most $\aleph_1$ many clopens. Now, consider $\alpha(\infty) = \sup_{i<\omega_1} \alpha(i)$. We pick $a_\infty \in \mathcal{F} \setminus \bigcup_{i<\omega_1} \mathcal{F}_i$. Write $a_\infty = c_\infty \cap F$ where $c_\infty$ is a clopen subset of $\mathscr{A}$. Now, $\pi_{\alpha(\infty)}(c_\infty)$ is a closed $G_\delta$ subset of $K_{\alpha(\infty)}$ which is disjoint from $\pi_{\alpha(\infty)}(a)$ for all $a \in \bigcup_{i<\omega_1} \mathcal{F}_i$, because $c_\infty$ is disjoint from every such $a$, which is determined up to $\alpha(i) < \alpha(\infty)$. The complement of $\pi_{\alpha(\infty)}(c_\infty)$ is a countable union of clopen sets, so we can conclude that there exists a clopen subset $b$ of $K$ which depends up to $\alpha(\infty)$ such that

$$|\{i < \omega_1 : \text{there is } a \in \mathcal{F}_i \text{ such that } a_\infty \subset b \text{ and } a \cap b = \varnothing\}| = \aleph_1$$

The clopen $b$ must in fact depend up to $\alpha_k$ for some $k < \omega_1$. On the one hand,

$$|\{a \in \mathcal{F} : a \subset b\}| = \aleph_2$$

because of property (4) of the families $\mathcal{F}_i$ since $a_\infty \notin \mathcal{F}_{k+1}$. On the other hand,

$$|\{a \in \mathcal{F} : a \cap b = \varnothing\}| = \aleph_2$$

again by property (4) of the families $\mathcal{F}_i$ because there exist $a \in \mathcal{F}_i$ with $a \subset \mathcal{A} \setminus b$ for many $i > k+1$. □

**Lemma 2.38** *Let $K, L, M$ be compact spaces and let $f : K \longrightarrow M$ be a continuous map. We denote by $j = f^\circ : C(M) \longrightarrow C(K)$ the composition operator induced by $f$. Let $\imath : C(M) \longrightarrow C(L)$ be a positive operator of norm one and suppose that $S : C(L) \longrightarrow C(K)$ is an operator with $\|S\| = 1$ and $S\imath = j$. Then $S$ is a positive operator.*

*Proof* Obviously $S \geq 0$ if and only if $S^*\delta_x \geq 0$ for all $x \in K$, where $\delta_x$ is the unit mass at $x$ and $S^* : C(K)^* \to C(L)^*$ is the adjoint operator. Fix $x \in K$. By Riesz theorem we have that $S^*\delta_x = \mu$ is a measure of total variation $\|\mu\| \leq 1$. Let $\mu = \mu^+ - \mu^-$ be the Hahn-Jordan decomposition of $\mu$ as the difference of two disjointly supported positive measures, so that $\|\mu\| = \|\mu^+\| + \|\mu^-\|$, with $\mu^+, \mu^- \geq 0$. We have that $\delta_{f(x)} = j^*\delta_x = \imath^* S^* \delta_x = \imath^*\mu$, thus

$$\delta_{f(x)} = \imath^*\mu^+ - \imath^*\mu^- \quad \text{and} \quad \|\delta_{f(x)}\| = 1 \geq \|\imath^*\mu^+\| + \|\imath^*\mu^-\|.$$

Since $\imath$ is a positive operator, $\imath^*\mu^+$ and $\imath^*\mu^-$ are positive measures, so all this implies that the above is the Hahn-Jordan decomposition of $\delta_{f(x)}$, and in particular $\imath^*\mu^- = 0$, hence $\mu^- = 0$. □

**Theorem 2.39 (MA, $\mathfrak{c} = \aleph_2$)** *The Banach space $C(\mathscr{A})$ is 1-separably injective but not universally 1-separably injective.*

*Proof* Since $\mathscr{A}$ is an $F$-space, $C(\mathscr{A})$ is 1-separably injective by Theorem 2.14. We suppose that $C(\mathscr{A})$ is universally 1-separably injective, and we will derive a contradiction. We pick $\{U_n : n \in \mathbb{N}\}$ a sequence of pairwise disjoint clopen subsets of $\mathscr{A}$, and let $U = \bigcup_n U_n$.

Let $c \subset \ell_\infty$ be the Banach space of convergent sequences, and let $t : c \longrightarrow C(\mathscr{A})$ be the operator given by $t(z_1, z_2, \ldots)(x) = z_n$ if $x \in U_n$ and $t(z_1, z_2, \ldots)(x) = \lim z_n$ if $x \notin U$. If $C(\mathscr{A})$ were universally 1-separably injective, we should have an extension $T : \ell_\infty \longrightarrow C(\mathscr{A})$ of $t$ with $\|T\| = 1$. We shall derive a contradiction from the existence of such operator.

The first observation is that $T$ must be a positive operator because we are in a position to apply Lemma 2.38. It might not be obvious at first glance how we apply the lemma. Let $\alpha\mathbb{N} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of the natural numbers. The space $c$ of convergent sequences is naturally identified with $C(\alpha\mathbb{N})$ and $\ell_\infty$ with $C(\beta\mathbb{N})$. The operator $t : c \longrightarrow C(\mathscr{A})$ is thus identified with $f^\circ : C(\alpha\mathbb{N}) \longrightarrow C(\mathscr{A})$ where $f : \mathscr{A} \longrightarrow \alpha\mathbb{N}$ is given by $f(x) = n$ if $x \in U_n$ and $f(x) = \infty$ if $x \notin U$. After this translation, it is clear that we can apply Lemma 2.38, and thus $T$ is positive.

For every $A \subset \mathbb{N}$ we will denote $[A] = \overline{A}^{\beta\mathbb{N}} \setminus \mathbb{N}$. The clopen subsets of $\mathbb{N}^*$ are exactly the sets of the form $[A]$, and we have that $[A] = [B]$ if and only if $(A \setminus B) \cup (B \setminus A)$ is finite.

Let $\mathfrak{F}$ be an $\aleph_2$-Lusin family in $\mathbb{N}^*$, which exists by Lemma 2.36. For $F \in \mathfrak{F}$ and $0 < \varepsilon < \frac{1}{2}$, let

$$F_\varepsilon = \{x \in \mathscr{A} \setminus U : T(1_A)(x) > 1 - \varepsilon\},$$

where $F = [A]$. This $F_\varepsilon$ depends only on $F$ and not on the choice of $A$ because if $[A] = [B]$, then $1_A - 1_B \in c_0$, hence $T(1_A - 1_B) = t(1_A - 1_B)$ vanishes out of $U$, so $T(1_A)|_{\mathscr{A}\setminus U} = T(1_B)|_{\mathscr{A}\setminus U}$.

Claim 1 If $\delta < \varepsilon$ and $F \in \mathfrak{F}$ then $\overline{F_\delta} \subset F_\varepsilon$.

Claim 2 $F_\varepsilon \cap G_\varepsilon = \varnothing$ for every $F \neq G$.

*Proof of Claim 2* Since $F \cap G = \varnothing$ we can choose $A, B \subset \mathbb{N}$ such that $F = [A]$, $G = [B]$ and $A \cap B = \varnothing$. If $x \in F_\varepsilon \cap G_\varepsilon$, $T(1_A + 1_B)(x) > 2 - 2\varepsilon > 1$ which is a contradiction because $1_A + 1_B = 1_{A\cup B}$ and $\|T(1_{A\cup B})\| \leq \|T\|\|1_{A\cup B}\| = 1$. End of the Proof of Claim 2.

For every $F \in \mathfrak{F}$, let $\tilde{F}$ be a clopen subset of $\mathscr{A} \setminus U$ such that $\overline{F_{0.2}} \subset \tilde{F} \subset F_{0.3}$. By the preceding claims, this is a disjoint family of clopen sets. As we mentioned above, the key property of $\mathscr{A}$ is that $\mathscr{A} \setminus U$ does not contain any $\aleph_2$-Lusin family.

Therefore we can find $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ with $|\mathcal{G}| = |\mathcal{H}| = \aleph_2$ such that

$$\overline{\bigcup\{\tilde{G} : G \in \mathcal{G}\}} \cap \overline{\bigcup\{\tilde{H} : H \in \mathcal{H}\}} = \varnothing.$$

Now, for every $n \in \mathbb{N}$ choose a point $p_n \in U_n$. Let $g : \beta\mathbb{N} \longrightarrow \mathscr{A}$ be a continuous function such that $g(n) = p_n$.

CLAIM 3 For $u \in \beta\mathbb{N}$ and $A \subset \mathbb{N}$ one has $T(1_A)(g(u)) = \begin{cases} 1, & \text{if } u \in \overline{A}^{\beta\mathbb{N}}; \\ 0, & \text{if } u \notin \overline{A}^{\beta\mathbb{N}}. \end{cases}$

*Proof of Claim 3* It is enough to check it for $u \in \mathbb{N}$. This is a consequence of the fact that $T$ is positive, because if $m \in A$, $n \notin A$, then $0 \le t(1_m) \le T(1_A) \le t(1_{\mathbb{N}\setminus\{n\}}) \le 1$. END OF THE PROOF OF CLAIM 3.

The function $g$ is one-to-one because

$$\overline{\{p_n : n \in A\}} \cap \overline{\{p_n : n \notin A\}} = \varnothing$$

for every $A \subset \mathbb{N}$, as the function $T(1_A)$ separates these sets. On the other hand, as a consequence of Claim 3 above, for every $F \in \mathcal{F}$ and every $\varepsilon$, $g^{-1}(F_\varepsilon) \cap \mathbb{N}^* = F$, and also $g^{-1}(\tilde{F}) \cap \mathbb{N}^* = F$. But then, for the families $\mathcal{H}$ and $\mathcal{G}$ that we found before, we have

$$\overline{\bigcup \mathcal{G}} \cap \overline{\bigcup \mathcal{H}} \subset g^{-1}\left(\overline{\bigcup\{\tilde{G} : G \in \mathcal{G}\}} \cap \overline{\bigcup\{\tilde{H} : H \in \mathcal{H}\}}\right) \cap \mathbb{N}^* = \varnothing.$$

And this contradicts that $\mathcal{F}$ is an $\aleph_2$-Lusin family in $\mathbb{N}^*$. □

We do not know whether the space $C(\mathscr{A})$ is universally separably injective, or whether it contains copies of $\ell_\infty$.

## 2.5 Injectivity Properties of $C(\mathbb{N}^*)$

In this section we take a closer look at $C(\mathbb{N}^*)$. As usual, $\mathbb{N}^* = \beta\mathbb{N}\setminus\mathbb{N}$ is the growth of the integers in its Stone-Čech compactification. Since $\ell_\infty$ can be identified with $C(\beta\mathbb{N})$ we also have $C(\mathbb{N}^*) = \ell_\infty/c_0$ in the obvious way and therefore the exact sequence

$$0 \longrightarrow c_0 \longrightarrow \ell_\infty \xrightarrow{\pi} \ell_\infty/c_0 \longrightarrow 0 \tag{2.8}$$

can be thought as

$$0 \longrightarrow c_0(\mathbb{N}) \longrightarrow C(\beta\mathbb{N}) \xrightarrow{r} C(\mathbb{N}^*) \longrightarrow 0 \tag{2.9}$$

where $r$ is plain restriction. Most of the injectivity properties of $C(\mathbb{N}^*)$ can be deduced from these representations. Indeed, in view of (2.8) and Proposition 2.11(3), the injectivity of $\ell_\infty$ and Sobczyk theorem already imply that $C(\mathbb{N}^*)$ is universally separably injective, hence $\ell_\infty$-upper-saturated.

What about the constants? That $C(\mathbb{N}^*)$ is 1-separably injective follows from the fact that being $\mathbb{N}^*$ a closed subset of the $F$-space $\beta\mathbb{N}$, it is itself an $F$-space. See Theorem 2.14, especially the equivalences between (1), (4) and (5). Also, it is clear from (2.9) that $C(\mathbb{N}^*)$ is universally 1-separably injective, according to Borsuk-Dugundji. All these properties of $C(\mathbb{N}^*)$ are implied by the conclusion of the following result we shall prove from scratch.

**Theorem 2.40** *Every separable subspace of $\ell_\infty/c_0$ is contained in a subalgebra of $\ell_\infty/c_0$ isometrically isomorphic to $\ell_\infty$. That algebra can be lifted through the quotient homomorphism $\pi : \ell_\infty \longrightarrow \ell_\infty/c_0$ by means of an isometric homomorphism.*

*Proof* The assertion is a consequence of an analogue result in the category of Boolean algebras: every countable Boolean subalgebra of $\mathbb{P}(\mathbb{N})/\operatorname{fin}$ is contained in a subalgebra isomorphic to $\mathbb{P}(\mathbb{N})$. Those readers that are familiar with the relations of a Boolean algebra with its Stone compact and the corresponding space of continuous functions will have little difficulties in deriving the functional analytic result from the Boolean algebraic one. Anyway, we present a detailed account of the proof.

Let $\mathcal{P}$ be a partition of $\mathbb{N}$ into infinite sets. Associated with $\mathcal{P}$ we define:

$$Y_\mathcal{P} = \{x \in \ell_\infty : x \text{ is constant on every } P \in \mathcal{P}\}$$
$$\mathfrak{Y}_\mathcal{P} = \pi(Y_\mathcal{P}) \subset \ell_\infty/c_0$$

Notice that $Y_\mathcal{P}$ is a subspace of $\ell_\infty$ (actually a subalgebra: it is closed under products and contains constant functions) isometric to $\ell_\infty$, and that $\pi : Y_\mathcal{P} \longrightarrow \mathfrak{Y}_\mathcal{P}$ is an isometry and hence $\mathfrak{Y}_\mathcal{P}$ is a subalgebra of $\ell_\infty/c_0$ isometric to $\ell_\infty$. We will show that for every separable subspace $S \subset \ell_\infty/c_0$ there exists a partition $\mathcal{P}$ of $\mathbb{N}$ into infinite sets such that $S \subset \mathfrak{Y}_\mathcal{P}$.

Let $\mathbb{P}(\mathbb{N})$ be the Boolean algebra of all subsets of $\mathbb{N}$. Each $A \in \mathbb{P}(\mathbb{N})$ can be identified with its characteristic function $1_A \in \ell_\infty$. Let $\mathbb{P}(\mathbb{N})/\operatorname{fin}$ be the quotient Boolean algebra obtained from $\mathbb{P}(\mathbb{N})$ by the equivalence relation $A \sim B$ if $(A \setminus B) \cup (B \setminus A)$ is finite. The operations of union, intersection and complement and the inclusion relation are defined in $\mathbb{P}(\mathbb{N})/\operatorname{fin}$ as inherited from $\mathbb{P}(\mathbb{N})$ modulo finite sets. Let $\pi' : \mathbb{P}(\mathbb{N}) \longrightarrow \mathbb{P}(\mathbb{N})/\operatorname{fin}$ be the canonical projection. Elements of $\mathbb{P}(\mathbb{N})/\operatorname{fin}$ can be viewed as elements of $\ell_\infty/c_0$ by identifying each $a = \pi'(A)$ with $x_a = \pi(1_A)$. For a subset $\mathcal{A} \subset \mathbb{P}(\mathbb{N})/\operatorname{fin}$, we define $X_\mathcal{A}$ to be the Banach subalgebra of $\ell_\infty/c_0$ generated by $\{x_a : a \in \mathcal{A}\}$.

CLAIM 1 For every separable subspace $S \subset \ell_\infty/c_0$ there exists a countable Boolean subalgebra $\mathcal{A} \subset \mathbb{P}(\mathbb{N})/\operatorname{fin}$ such that $S \subset X_\mathcal{A} \subset \ell_\infty/c_0$.

*Proof of the claim* It is clear that the algebra generated by $\{1_A : A \in \mathbb{P}(\mathbb{N})\}$ is the whole space $\ell_\infty$. Thus, each vector of $\ell_\infty$ is in the algebra generated by a countable subset of $\{1_A : A \in \mathbb{P}(\mathbb{N})\}$. Hence, every separable subspace of $\ell_\infty$ is contained in the algebra generated by a countable subset of $\{1_A : A \in \mathbb{P}(\mathbb{N})\}$. It follows that every separable subspace of $\ell_\infty/c_0$ is contained in the subalgebra generated by $\{x_a : a \in \mathscr{A}\}$ with $\mathscr{A}$ a countable set. But the Boolean algebra generated by a countable set is countable, so we can assume that $\mathcal{A} = \mathscr{A}$ is a Boolean subalgebra, and the claim follows.

We will also make use of the following standard fact about the Boolean algebra $\mathbb{P}(\mathbb{N})/\operatorname{fin}$:

CLAIM 2 Let $\mathcal{U}$ be a countable set of nonzero elements of $\mathbb{P}(\mathbb{N})/\operatorname{fin}$ that is closed under finite intersections. Then there exists a nonzero $b$ in $\mathbb{P}(\mathbb{N})/\operatorname{fin}$ such that $b \subset a$ for all $a \in \mathcal{U}$.

*Proof of the claim* Let us enumerate $\mathcal{U} = \{u_1, u_2, \ldots\}$ and set $v_n = \bigcap_{i \le n} u_i$. We have $v_1 \supset v_2 \supset \cdots$ and all $v_n$'s are nonzero. Choose sets $V_n \subset \mathbb{N}$ with $v_n = \pi'(V_n)$. All $V_n$'s are infinite and $V_n \setminus V_m$ is finite whenever $n < m$. Inductively, construct a sequence of natural numbers $k_1 < k_2 < \cdots$ such that $k_n \in \bigcap_{i \le n} V_i$. Set $A = \{k_1, k_2, \ldots\}$ and $a = \pi'(A)$. This is the desired element $a$. It is nonzero since $A$ is infinite. And $a \subset u$ for $u \in \mathcal{U}$ because $V_n \setminus A$ is finite for all $n$. This proves the claim.

By Claim 1 it is enough to prove that for every countable subalgebra $\mathcal{A} \subset \mathbb{P}(\mathbb{N})/\operatorname{fin}$ there exists a partition $\mathcal{P}$ such that $X_{\mathcal{A}} \subset \mathfrak{Y}_{\mathcal{P}}$. So we fix such a subalgebra. We take $\{\mathcal{U}_n : n \in \mathbb{N}\}$ a sequence of ultrafilters of the Boolean algebra $\mathcal{A}$ such that for every nonzero $a \in \mathcal{A}$ there exists $n$ with $a \in \mathcal{U}_n$. In other words, $\{\mathcal{U}_n\}$ is a dense sequence in the Stone space of $\mathcal{A}$. For every $n$, either

1. $\mathcal{U}_n$ is principal, in which case we define $a_n = \min \mathcal{U}_n$, or
2. $\mathcal{U}_n$ is nonprincipal. In this case, by the previous Claim 2 we can pick a nonzero $a_n \in \mathbb{P}(\mathbb{N})/\operatorname{fin}$ such that $a_n \subset a$ for all $a \in \mathcal{U}_n$.

The elements $a_n$ defined above are pairwise disjoint in $\mathbb{P}(\mathbb{N})/\operatorname{fin}$: If $n \neq m$, then $\mathcal{U}_n \neq \mathcal{U}_m$, there exists $b \in \mathcal{U}_n$, with $\pi'(\mathbb{N}) \setminus b \in \mathcal{U}_m$, so $a_n \cap a_m \subset b \cap \pi'(\mathbb{N}) \setminus b = \varnothing$. The partition $\mathcal{P} = \{P_n\}$ we are looking for will be such that $a_n = \pi'(P_n)$ for all $n$. It remains to carefully choose each $P_n$ in the equivalence class $a_n$, so that they constitute a partition and $X_{\mathcal{A}} \subset \mathfrak{Y}_{\mathcal{P}}$. Since in any case, $\mathfrak{Y}_{\mathcal{P}}$ will be a Banach subalgebra, we just have to take care that $x_a \in \mathfrak{Y}_{\mathcal{P}}$ for all $a \in \mathcal{A}$. We will actually show that

$$a = \pi' \left( \bigcup_{a \in \mathcal{U}_n} P_n \right).$$

Since every finitely generated Boolean algebra is finite, we can write $\mathcal{A}$ as an increasing union of finite subalgebras $\mathcal{A} = \bigcup_{m=1}^{\infty} \mathcal{A}_m$. For fixed $m$, it is easy to choose a partition $\mathcal{P}^m = \{P_n^m\}_{n \in \mathbb{N}}$ of $\mathbb{N}$ with $\pi'(P_n^m) = a_n$ and $a = \pi' \left( \bigcup_{a \in \mathcal{U}_n} P_n^m \right)$

for all $a \in \mathcal{A}_m$: one only has to take care of each of the finitely many atoms (minimal nonzero elements) of $\mathcal{A}_m$. Moreover, if we choose the partitions $\mathcal{P}^m$ inductively one after another, it is possible to do it in such a way that the following conditions hold:

1. $P_k^m = P_k^{m-1}$ for all $k \leq m$,
2. $P_k^m \cap \{0, \dots, m\} = P_k^{m-1} \cap \{0, \dots, m\}$ for all $k$.
3. $\bigcup\{P_k^m : a \in U_k\} = \bigcup\{P_k^{m-1} : a \in U_k\}$ for all $a \in \mathcal{A}_{m-1}$.

The sets $\{P_n = P_n^n : n \in \mathbb{N}\}$ constitute the partition $\mathcal{P}$ that we are looking for. □

**Corollary 2.41** *$\ell_\infty/c_0$ is universally 1-separably injective.*

*Proof* It follows from Theorem 2.40, taking into account that $\ell_\infty$ is 1-injective. □

The subtleties in the proof of Theorem 2.40 are necessary only to construct the "enveloping" subspace isometric to $\ell_\infty$ in the right position since the lifting of a separable subspace of $\ell_\infty/c_0$ to $\ell_\infty$ is nearly trivial. The following result is, formally, a Corollary of Theorem 2.40:

**Proposition 2.42** *If $S$ is a separable subspace of $\ell_\infty$ containing $c_0$ then there is a contractive projection $p$ on $S$ whose kernel is $c_0$. When, additionally, $S$ is a subalgebra of $\ell_\infty$ then $p$ is a unital homomorphism. In any case, $\mathbf{1}_S - p$ is a projection onto $c_0$ of norm at most 2.*

*Proof* Let us show that if $S$ is a separable subalgebra of $\ell_\infty/c_0$ then there is a continuous homomorphism $\varphi : S \to \ell_\infty$ such that $\pi \circ \varphi = \mathbf{1}_S$, where $\pi : \ell_\infty \to \ell_\infty/c_0$ is the natural quotient map. From here, the proposition follows.

It is clear that for every $f \in \ell_\infty$ and $\varepsilon > 0$ there is a partition $\mathbb{N} = A_1 \cup \dots \cup A_k$ and numbers $t_i$ such that

$$\|f - \sum_{i=1}^{k} t_i 1_{A_i}\| \leq \varepsilon.$$

It follows that $S$ is contained in the closure of the union of a (increasing) sequence of algebras of the form $S_n = \pi(R_n)$, where $R_n$ is the algebra associated to a certain (finite) partition of $\mathbb{N}$. From the viewpoint of $\ell_\infty/c_0$ we see that each $S_n$ has a basis of idempotents whose sum is 1. Adding some "intermediate" subalgebras if necessary we may an do assume $\dim S_n = n$. Let us construct the required homomorphism

$$\varphi : \bigcup_{n=1}^{\infty} S_n \longrightarrow \ell_\infty$$

by showing that every lifting $\varphi : S_n \to \ell_\infty$ (in the category of unital algebras) extends to a lifting of $S_{n+1}$. Write $S_n = \operatorname{span}\{u_1, \dots, u_n\}$ where $u_k$ are idempotents such that $u_1 + \dots + u_n = 1$. Clearly, $\varphi(u_k) = 1_{A_k}$, where $\mathbb{N} = A_1 \cup \dots \cup A_n$ (this is the "induction" hypothesis). We may assume $S_{n+1} = \operatorname{span}\{u_1, \dots, u_{n-1}, v, w\}$

where $v$ and $w$ are idempotents such that $v + w = u_n$. Since $v$ is idempotent there is $V \subset \mathbb{N}$ such that $v = \pi 1_V$. We extend $\varphi$ to $S_{n+1}$ taking $\varphi(v) = 1_{V \cap A_n}$ (which forces $\varphi(w) = 1_{A_n \setminus V}$). The definition is correct since $V \setminus A_n$ is at most finite (otherwise the decomposition $\pi 1_{A_n} = \pi 1_V + w$ with $w \geq 0$ is impossible). □

We have already shown (several times) that $\ell_\infty/c_0$ is not injective. The simplest argument was to observe that the (images of the) characteristic functions of the elements of an almost disjoint family $\mathcal{M}$ of infinite subsets of $\mathbb{N}$ having size $\mathfrak{c}$ generate a subspace isometric to $c_0(\mathfrak{c})$; that $\ell_\infty/c_0$ has density character $\mathfrak{c}$ and that therefore it cannot contain any copy of $\ell_\infty(\mathfrak{c})$, which has density character $2^{\mathfrak{c}}$. The above argument is quite rough in a sense: it says that $\ell_\infty/c_0$ is uncomplemented in its bidual, a huge superspace. Not being injective, $\ell_\infty/c_0$ cannot be complemented in its bidual and therefore it cannot be complemented in any dual space (see [196]). In any case, Amir had shown in [5] that $C(\mathbb{N}^*)$ is not complemented in $\ell_\infty(\mathbb{P}(\mathbb{N}^*)) \approx \ell_\infty(2^{\mathfrak{c}})$, which provides another proof that $\ell_\infty/c_0$ is not injective. Amir's proof can be refined in order to get $C(\mathbb{N}^*)$ uncomplemented in a much smaller space. We are indebted to Anatolij Plichko for calling our attention to Amir's paper.

**Proposition 2.43** *There exists a Banach space of density character* $\mathfrak{c}$ *that contains an uncomplemented copy of* $C(\mathbb{N}^*)$.

*Proof* Following Amir's paper [5], let $\Sigma$ be a family of subsets of $\mathbb{N}^*$ that contains a basis of open sets of the topology of $\mathbb{N}^*$, and which is closed under complementation, finite union and the closure operation. We can consider the Banach space $B(\Sigma)$, sitting as $C(\mathbb{N}^*) \subset B(\Sigma) \subset \ell_\infty(\mathbb{N}^*)$ defined as the subspace of $\ell_\infty(\mathbb{N}^*)$ generated by the characteristic functions of the elements of $\Sigma$. Let also $D_\Sigma$ be the union of the boundaries of all open sets living in $\Sigma$. By [5, Corollary 1], if $C(\mathbb{N}^*)$ is complemented in $B(\Sigma)$, then $D_\Sigma$ is nowhere dense in $\mathbb{N}^*$. We indicate now how to construct such a family $\Sigma$ of cardinality $\mathfrak{c}$ and with $D_\Sigma$ dense in $\mathbb{N}^*$, so that the space $X = B(\Sigma)$ is as stated in the Proposition. For every clopen subset $A$ of $\mathbb{N}^*$, choose $U_A \subset A$ to be an open not closed set. Consider then $\Sigma$ the least family of subsets of $\mathbb{N}^*$ that contains all clopens $A$ and all open sets $U_A$ and that is closed under complementation, finite union and the closure operation. □

A different proof of Proposition 2.43 can be found in [18]. We do not know whether the space $X$ in the preceding result can be obtained so that $\operatorname{dens}(X/C(\mathbb{N}^*)) = \aleph_1$. By Parovičenko's theorem [40], [245, p. 81], $\mathbb{N}^*$ can be mapped onto any compact space having weight at most $\aleph_1$. Consequently:

**Lemma 2.44** *Every Banach space of density character* $\aleph_1$ *or less is isometric to a subspace of* $C(\mathbb{N}^*)$.

*Proof* Let $X$ denote a Banach space with $\operatorname{dens} X \leq \aleph_1$. Its dual unit ball $B_{X^*}$ in the weak* topology has weight at most $\aleph_1$. Let $\varphi : \mathbb{N}^* \to B_{X^*}$ be the surjective mapping given by Parovičenko theorem. The operator $\varphi^\circ : C(B_{X^*}) \longrightarrow C(\mathbb{N}^*)$ given by

$$(\varphi^\circ f)(u) = f(\varphi(u))$$

is an into isometry. The space $X$ is isometric to a subspace of $C(B_{X^*})$ and this concludes the proof. □

The following immediate application can be found in [67, Proposition 5.3]:

**Corollary 2.45 (CH)** *$C(\mathbb{N}^*)$ contains an uncomplemented subspace isometric to $C(\mathbb{N}^*)$.*

*Proof* By Lemma 2.44, the space in Proposition 2.43 is a subspace of $C(\mathbb{N}^*)$. □

The argument of Lemma 2.44, together with some interesting applications to the existence of nontrivial twisted sums, can be found in [251]. Lemma 2.44 is actually related to the topic of universal disposition discussed in Chap. 3. Although $C(\mathbb{N}^*)$ cannot be of almost universal disposition (since no $C$-space can be of almost universal disposition—see the discussion before Theorem 3.34), the compact space $\mathbb{N}^*$ is of "universal co-disposition in the category of compact spaces"—see Definition 5.23 and Corollary 5.24; and the Boolean algebra $\mathbb{P}(\mathbb{N})/\text{fin}$ is of "universal disposition in the category of Boolean algebras". All this can be understood as the real content of Parovičenko theorem.

## 2.6 Automorphisms of Separably Injective Spaces

Lindenstrauss and Rosenthal proved in [180] that every isomorphism between two infinite codimensional subspaces of $c_0$ can be extended to an automorphism of $c_0$.

**Definition 2.46** A Banach space is said to be *automorphic* if every isomorphism between two subspaces whose corresponding quotients have the same density character can be extended to an automorphism of the whole space.

Observe that the extension trivially exists when the subspaces are finite dimensional or, in the hypothesis above, finite codimensional. It is clear that Hilbert spaces are automorphic, and in [199] it was proved that also $c_0(\Gamma)$ is automorphic. Lindenstrauss and Rosenthal formulated what has been called *the automorphic space problem*: Does there exist an automorphic space different from $c_0(\Gamma)$ and $\ell_2(\Gamma)$? Different approaches and partial positive answers to the automorphic space problem have been considered and obtained in [17, 19, 63, 67, 199]. There emerged the notion of partially automorphic space, of which we isolate now the following:

**Definition 2.47** Let $X, Y$ be Banach spaces.

- We say that $X$ is $Y$-automorphic if every isomorphism $\tau : A \to B$ between two subspaces of $X$ isomorphic to $Y$ with $\text{dens}(X/A) = \text{dens}(X/B)$ can be extended to an automorphism of $X$.

- A Banach space $X$ is said to be separably automorphic if it is $Y$-automorphic for every separable $Y$.

Observe that $X$ is $Y$-automorphic if and only if given two embeddings $i, j : Y \to X$ with $\operatorname{dens}(X/i[Y]) = \operatorname{dens}(X/j[Y])$ there is an automorphism $\tau$ of $X$ such that $j = \tau \circ i$.

Lindenstrauss and Rosenthal also prove in [180] that $\ell_\infty$ is separably automorphic (see also [182, Theorem 2.f.12]). The proof can be easily adapted to the general case to obtain that, for every set $\Gamma$, the space $\ell_\infty(\Gamma)$ is separably automorphic. We shall see that indeed every universally separably injective space is separably automorphic.

**Lemma 2.48** *Let $Y$ be a Banach space isomorphic to its square. Assume that every copy of $Y$ is complemented in $X$. Then $X$ is $Y$-automorphic if and only if every complement of $Y$ with the same density character as $X$ contains $Y$.*

*Proof* As every copy of $Y$ is complemented in $X$ it is clear that $X$ is $Y$-automorphic if and only if the complements of copies of $Y$ in $X$ with the same density character are all isomorphic. In fact, using that $Y$ is isomorphic to its square, all the complements with density character equal to $\operatorname{dens} X$ must be isomorphic to $X$. Now, the "if" part is as follows. Let $Y_1$ be a subspace of $X$ isomorphic to $Y$. We have $X = Y_1 \oplus Z$ and $Z = Y_2 \oplus A$, with $Y_2 \sim Y$. Hence $X = Y_1 \oplus Y_2 \oplus A \sim Y_2 \oplus A = Z$. The converse is also easy: if $X \sim Y \oplus Z$, with $\operatorname{dens} Z = \operatorname{dens} X$, then $X \sim Y \oplus Y \oplus Z$ and if $X$ is $Y$-automorphic, then $Z \sim Y \oplus Z$. □

**Corollary 2.49** *Universally separably injective spaces are $\ell_\infty$-automorphic.*

*Proof* We apply Lemma 2.48 for $Y = \ell_\infty$. Observe that every copy of $\ell_\infty$ is complemented as $\ell_\infty$ is an injective space, and that every complemented subspace of a universally separably injective space is universally separably injective, so it contains $\ell_\infty$ by Theorem 2.26. □

Now we want to jump from "$X$ is $Y$-automorphic" to "$X$ is $H$-automorphic for every subspace $H$ of $Y$". The obvious result is:

**Lemma 2.50** *Let $X$ be $Y$-automorphic and let $H_1 \subset Y_1 \subset X$ and $H_2 \subset Y_2 \subset X$ be spaces where $H_1, H_2$ and $Y_1, Y_2$ are isomorphic to $H$ and $Y$, respectively. If there is an automorphism of $Y$ transforming $H_1$ into $H_2$ then there is an automorphism of $X$ transforming $H_1$ into $H_2$ every time* $\operatorname{dens}(X/Y_1) = \operatorname{dens}(X/Y_2)$.

There is an alternative approach to obtain the partially automorphic character of a space: to combine the $Y$-upper-saturaturation and the fact that every copy of $Y$ is complemented instead of relying on the $Y$-automorphic character of the space:

**Lemma 2.51** *Let $E, Y$ and $X$ be Banach spaces. Suppose that $Y$ is $E$-automorphic, and that every two copies $E_1, E_2$ of $E$ inside $X$ are contained in a single complemented copy $Y_0$ of $Y$ inside $X$ such that* $\operatorname{dens}(Y_0/E_1) = \operatorname{dens}(Y_0/E_2)$. *Then $X$ is $E$-automorphic.*

*Proof* Let $i, j : E \to X$ be two embeddings of a space $E$ into $X$, with $\mathrm{dens}(X/i[E]) = \mathrm{dens}(X/j[E])$. Obviously, $i$ and $j$ factorize through the inclusion $\omega : \overline{i[E] + j[E]} \longrightarrow X$. By hypothesis, there is a complemented subspace $Y_0$, which is isomorphic to $Y$ and contains both $i[E]$ and $j[E]$. If $\tau_0$ is an automorphism of $Y_0$ such that $j = \tau_0 \circ i$ and $A$ is a complement of $Y_0$ in $X$, then the automorphism of $X$ is $\tau = \tau_0 \oplus \mathbf{1}_A$. $\square$

Thus, using Theorem 2.26, and the result of Lindenstrauss and Rosenthal asserting that $\ell_\infty$ is separably automorphic, we can apply Lemma 2.51 for $Y = \ell_\infty$ and $E$ any separable space, to obtain:

**Proposition 2.52** *Universally separably injective spaces are separably automorphic.*

**Corollary 2.53** *The space $\ell_\infty/c_0$ is separably automorphic.*

The separably automorphic character of $C(\mathbb{N}^*)$ seems to be connected with the fact that the underlying Boolean algebra has analogous properties; namely, it is "countably automorphic" (every isomorphism between countable Boolean algebras is extended to an automorphism of $\mathbb{P}(\mathbb{N})/\,\mathrm{fin}$) and every countable Boolean algebra is contained in a copy of $\mathbb{P}(\mathbb{N})$, cf. [77]. The proof of this fact is just the last step (after Claim 2) of the proof of Theorem 2.40. If we are given $\mathcal{A}$ and $\mathcal{A}'$ two isomorphic countable subalgebras of $\mathbb{P}(\mathbb{N})/\,\mathrm{fin}$, that proof provides two copies of $\mathbb{P}(\mathbb{N})$ of the form $\mathfrak{Y}_{\mathcal{P}}$ and $\mathfrak{Y}_{\mathcal{P}'}$, and the isomorphism between $\mathcal{A}$ and $\mathcal{A}'$ induces naturally a bijection between the partitions $\mathcal{P}$ and $\mathcal{P}'$, that gives an extended isomorphism between $\mathfrak{Y}_{\mathcal{P}}$ and $\mathfrak{Y}_{\mathcal{P}'}$. It is not however so clear how to pass from "Boolean-automorphic" to "Banach-automorphic".

Other quotients of $\ell_\infty$ also have a partially automorphic character. We require a lemma.

**Lemma 2.54** *Assume that for $k = 1, 2$ one has pull-back diagrams*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i"] & X \arrow[r, "q"] & X/A \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, equal] & \mathrm{PB}_k \arrow[r, "q_k"'] \arrow[u, "\Delta_k"'] & Y \arrow[r] \arrow[u, "\delta_k"'] & 0
\end{tikzcd}$$

*where $\delta_k$ are isomorphic embeddings. If there exists isomorphisms $\Theta : X \to X$ and $\theta : \mathrm{PB}_1 \to \mathrm{PB}_2$ such that $\Theta\Delta_1 = \Delta_2\theta$ and $q_2\theta = q_1$ then there is an automorphism $\tau : X/A \to X/A$ such that $\tau\delta_1 = \delta_2$.*

*Proof* Observe first that $\theta(A) \subset A$ since $q_2\theta(a) = q_1(a) = 0$; and then that $\theta(A) = A$ since if $p \in \mathrm{PB}_1$ is such that $\theta(p) \in A$ then $0 = q_2\theta(p) = q_1(p)$, so $p \in A$. Since $\Theta$ is an automorphism of $X$ that extends $\theta$, then also $\Theta(A) = A$. One can then define an automorphism of $X/A$ by $\tau(x + A) = \Theta(x) + A$. It verifies

$$\tau\delta_1 q_1(p) = \Theta(p) + A = q\Delta_2\theta(p) + A = \delta_2 q_2\theta(p) = \delta_2 q_1(p)$$

and thus $\tau\delta_1 = \delta_2$. $\square$

The lemma says, in particular, that if the pull-back sequences are isomorphically equivalent and $X$ is $\mathrm{PB}_k$-automorphic then $X/A$ is $Y$-automorphic. So, it provides relevant information about the automorphic character of quotient spaces. When applied to quotients of $\ell_\infty$ one gets:

**Proposition 2.55**

1. *If $E$ is a separably injective subspace of $\ell_\infty$, then $\ell_\infty/E$ is separably automorphic.*
2. *Let $A$ be any subspace of $\ell_\infty$ complemented in its bidual and so that $\ell_\infty/A$ is not reflexive. Then $\ell_\infty/A$ is automorphic for all $\mathscr{L}_1$-spaces.*
3. *For every subspace $H$ of $c_0$ the space $\ell_\infty/H$ is automorphic for all separable $\mathscr{L}_1$-spaces.*

*Proof* Part (1) follows from a general fact: $\ell_\infty/E$ is universally separably injective, hence separably automorphic. An independent proof is however as follows: Let $\delta_k : Y \to \ell_\infty/E$ be two embeddings of a separable space $Y$ into $\ell_\infty/E$. Since $\mathrm{Ext}(Y, E) = 0$ then $\mathrm{PB}_k \sim E \oplus Y$. Since $\ell_\infty/\,\mathrm{PB}_k$ is isomorphic to $(\ell_\infty/E)/\delta_k[Y]$ one gets that $\ell_\infty/\,\mathrm{PB}_k$ contains $\ell_\infty$ and this implies that $\ell_\infty$ is $\mathrm{PB}_k$-automorphic, because Lindenstrauss and Rosenthal [180] proved in fact that $\ell_\infty$ is $Z$-automorphic whenever $\ell_\infty/Z$ is not reflexive. So, Lemma 2.54 applies, which proves (1). Assertions (2) and (3) follow the same schema: (2) using Lindenstrauss' lifting (i.e., $\mathrm{Ext}(\mathscr{L}_1, A) = 0$ for every Banach space $A$ complemented in its bidual; see Proposition A.18) and (3) using the identity $\mathrm{Ext}(\mathscr{L}_1, H) = 0$ obtained in [65] (see also [62]). □

Separably injective spaces are not necessarily separably automorphic as the example of $c_0 \oplus \ell_\infty$ shows: no automorphism can send a complemented copy of $c_0$ such as $c_0 \oplus 0$ onto an uncomplemented copy such as $0 \oplus c_0$. And automorphic spaces, such as $\ell_2$, are not necessarily separably injective. One however has:

**Proposition 2.56**

1. *Every separably automorphic space containing $\ell_1$ is separably injective.*
2. *Every separably automorphic space containing $\ell_\infty$ is universally separably injective.*

*Proof* Let $i : \ell_1 \to X$ be an into isomorphism. By Proposition 2.5 it is enough to prove that for every closed subspace $K$ of $\ell_1$, every operator $K \to X$ extends to $\ell_1$. Assume otherwise; let $\delta : K \to \ell_1$ be an into isomorphism and let $t : K \to X$ be an operator that cannot be extended through $\delta$, therefore neither it can be extended to $X$ through $i\delta$. Then, for some $\varepsilon > 0$, the operator $i\delta + \varepsilon t$ is an into isomorphism. If $X$ is separably automorphic, we could find an isomorphism $F : X \longrightarrow X$ such that $Fi\delta = i\delta + \varepsilon t$. But then $\varepsilon^{-1}(T - \mathbf{1}_X)i$ would be an extension of $t$ through $\delta$, and this contradicts our hypothesis. The proof of the second assertion is simpler: every separable subspace of $X$ must be contained in a copy of $\ell_\infty$ and thus the space is universally separably injective. □

Many other Banach spaces have been shown to have a partially automorphic character; for instance, Lindenstrauss and Pełczyński prove in [178] that $C[0,1]$ is $H$-automorphic for all subspaces $H$ of $c_0$ while Kalton shows in [155] that $C[0,1]$ is also $\ell_1$-automorphic and in [156] that it is not $\ell_2$-automorphic. Another example of separably injective spaces that are also separably automorphic are the $C(K)$ spaces with $K$ Eberlein compact of finite height.

**Proposition 2.57**

1. *If $K$ is an Eberlein compact, then $C(K)$ is $c_0$-automorphic.*
2. *Every $c_0$-upper-saturated WCG-space is separably automorphic.*
3. *A $C(K)$ space that is $c_0$-automorphic is also $H$-automorphic for every subspace $H$ of $c_0$.*
4. *Let $E$ be an Eberlein compact. Then $C(E)$ is $H$-automorphic for every subspace $H$ of $c_0$.*

*Proof* Assertion (1) follows from Lemma 2.48 for $Y = c_0$. On the one hand, every infinite dimensional complemented subspace of a $C$-space contains a copy of $c_0$, cf. (Proposition A.5). On the other hand, if $C(K)$ is WCG, then it has the separable complementation property (every separable subspace is contained in a separable complemented subspace), hence by Sobczyk's theorem, every copy of $c_0$ is complemented. For assertion (2), we have again that every copy of $c_0$ is complemented, and since $c_0$ is automorphic by the Lindenstrauss-Rosenthal theorem, Lemma 2.51 applies. To get (3), assume first that neither $C(K)$ nor $H$ are isomorphic to $c_0$, otherwise the result is trivial. Now, since $c_0$ is automorphic, we only need to prove that every subspace $H$ of $c_0$ contained in $C(K)$ is actually contained in a copy of $c_0$ contained in $C(K)$. But every separable subspace $S$ of $C(K)$ is contained in a separable subspace $C(T)$ of $C(K)$. This subspace $C(T)$ is $H$-automorphic [178], and the result follows. Assertion (4) is an immediate consequence of (1) and (2). □

Assertion (4) is actually a non-separable extension of the main result in [71] asserting that separable Lindenstrauss-Pełczyński spaces are characterized as those which are $H$-automorphic for all subspaces $H$ of $c_0$. Concerning the automorphic character of $C(K)$-spaces we obtain from Lemma 6.2 and the proof of Lemma 2.22 that every $C(K)$-space with $K$ a compact of finite height is $c_0$-upper-saturated. As a consequence, applying Lemma 2.57(2):

**Corollary 2.58** *Every $C(K)$-space with $K$ an Eberlein compact of finite height is separably automorphic.*

A generalization of the preceding result was obtained in [17]:

**Proposition 2.59** *If $K$ is an Eberlein compact of finite height, the (separably injective) space $C(K)$ is automorphic for all possible subspaces of density character less than $\aleph_\omega$.*

These results are somewhat optimal: there exist separably injective $C(K)$-spaces such as $c_0 \oplus \ell_\infty$ which are not $c_0$-automorphic; there also exist non-Eberlein compacta of height 3 which are not $c_0$-automorphic (since they contain complemented and uncomplemented copies of $c_0$); while Eberlein compacta of infinite height, such as $C[0,1]$, are not separably automorphic.

See Theorem 5.30 and Sect. 6.4.3 for further information and open problems on partially automorphic spaces.

## 2.7 Notes and Remarks

### 2.7.1 Extensions vs. Projections

In this section we take a closer look at the constants implicit in the characterizations given in Proposition 2.5 and we consider the corresponding "quantified" properties and the relationships between the involved constants.

**Proposition 2.60**

1. *If $E$ is $\lambda$-separably injective, for every Banach space $X$ and each subspace $Y$ such that $X/Y$ is separable, every operator $t : Y \to E$ admits an extension $T : X \to E$ with $\|T\| \leq 3\lambda\|t\|$.*
2. *A space $E$ is $\lambda$-complemented in every $Z$ such that $Z/E$ is separable if and only if whenever $Y$ is a subspace of $X$ with $X/Y$ separable every operator $t : Y \to E$ admits an extension $T : X \to E$ with $\|T\| \leq \lambda\|t\|$.*

*Proof*

1. We have to follow the trace of $\lambda$ through the proof of (2) $\Rightarrow$ (3) in Proposition 2.5. With the same notation, consider thus the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker q & \xrightarrow{j} & \ell_1 & \xrightarrow{q} & X/Y & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\downarrow & & {\scriptstyle Q}\downarrow & & \| & & \\ 0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & X/Y & \longrightarrow & 0 \end{array}$$

Let us construct the true push-out of the couple $(\phi, j)$ and the corresponding complete diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker q & \xrightarrow{j} & \ell_1 & \xrightarrow{q} & X/Y & \longrightarrow & 0 \\ & & {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi'} & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{j'} & \mathrm{PO} & \longrightarrow & X/Y & \longrightarrow & 0 \end{array}$$

We can consider without loss of generality that $\|\phi\| = 1$. Let $S : \ell_1 \to E$ be an extension of $t\phi$ with $\|S\| \leq \lambda\|t\phi\| \leq \lambda\|t\|$. By the universal property of the push-out, there exists an operator $L : \mathrm{PO} \to E$ such that $L\phi' = S$ and

$$\|L\| \le \max\{\|t\|, \|S\|\} \le \lambda \|t\|.$$

Again by the universal property of the push-out, there is a diagram of equivalent exact sequences

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, "j'"] \arrow[d, equal] & \mathrm{PO} \arrow[r] \arrow[d, "\gamma"] & X/Y \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & Y \arrow[r, "\imath"] & X \arrow[r, "p"] & X/Y \arrow[r] & 0 \end{tikzcd}$$

where the isomorphism $\gamma$ is defined as $\gamma((y,u) + \Delta) = j(y) + Q(u)$ is such that $\|\gamma\| \le \max\{\|j\|, \|Q\|\} \le 1$. Remember that there exists a selection $s : X/Y \to \ell_1$ for the quotient map $q : \ell_1 \to X/Y$ which is not necessarily linear but is homogeneous and has $\|s\| = 1$ (that is, $s(\mu u) = \mu s(u)$ and $\|s(u)\| \le \|u\|$ for all $u \in X/Y$ and scalar $\mu$). The desired extension of $t$ to $X$ is $T = L\gamma^{-1}$, where $\gamma^{-1}$ comes defined by

$$\gamma^{-1}(x) = (x - Qspx, spx) + \Delta.$$

Notice that $\gamma^{-1}$ is well defined because $x - Qspx \in Y$ since $p(x - Qspx) = px - pQspx = px - qspx = 0$. Notice also that $\gamma^{-1}$ is linear because for $x, z \in X$ we can write $\gamma^{-1}(x+z) - \gamma^{-1}(x) - \gamma^{-1}(z)$ as

$$Q(spx + spz - sp(x+z)), sp(x+z) - spx - spz) + \Delta$$

and this is zero because $spx + spz - sp(x+z) \in \ker q$ as $s$ is a selection for $q$. Finally, one clearly has $\|\gamma^{-1}\| \le 3$, and therefore $\|T\| \le 3\lambda$.

2. The complementation of $E$ is achieved simply considering $t$ as the identity on $E$. The other implication is contained in the proof of the implication (4) $\Rightarrow$ (3) in Proposition 2.5: if $p' : \mathrm{PO} \to E$ is a projection with norm at most $\lambda$, since $\|t'\| \le 1$, the composition $p't' : X \to E$ yields an extension of $t$ with norm at most $\lambda$. □

We do not know if the bound 3 appearing in Proposition 2.60(1) is sharp. Observe that if every operator $t : Y \to E$ can be extended preserving the norm to any superspace $X$ such that $\dim X/Y = 1$, then $E$ is 1-injective, as can be seen by transfinite induction. Thus one cannot replace 3 by 1. The following example shows that at least 2 is required. See Example 2.4:

*Example 2.61* The projection constant of $\ell_\infty^c(\aleph_1)$ in $\ell_\infty^c(\aleph_1)_+$ is 2.

*Proof* Each element of $\ell_\infty^c(\aleph_1)_+$ can be written as $\lambda + f$, with $f \in \ell_\infty^c(\aleph_1)$. The map $\lambda + f \mapsto f$ is a projection of norm 2, so the projection constant is at most 2. To see the reversed inequality, let $p : \ell_\infty^c(\aleph_1)_+ \longrightarrow \ell_\infty^c(\aleph_1)$ be any linear projection and take $\phi = p(1)$, so that $p(\lambda + f) = \lambda\phi + f$. Take any $i$ such that $\phi(i) = 0$. Then $\|1 - 21_i\| = 1$ but $\|p(1 - 21_i)\| = 2$ since $p(1 - 21_i)(i) = -2$. $\square$

Moreover, a $\lambda$-separably injective space $E$ is not necessarily $\lambda$-complemented in every superspace $Z$ such that $Z/E$ is separable. Let us therefore consider the following "one dimensional" version of Proposition 2.5: A Banach space $E$ is said to enjoy property $(c_\lambda)$ when for every Banach space $X$ and each subspace $Y$ such that $\dim X/Y = 1$, every operator $t : Y \to E$ extends to an operator $X \to E$ with norm at most $\lambda\|t\|$. Lemma 2.33 says:

**Proposition 2.62** *A Banach space E has property* $(c_\lambda)$ *if and only if given a family* $B(x_\alpha, r_\alpha)$ *of mutually intersecting balls whose centers lie on a separable subspace there exists a point p such that* $\|x_\alpha - p\| \le \lambda r_\alpha$.

Every $\lambda$-separably injective space has property $(c_\lambda)$, although it is not clear if there is a function $f$ so that a space with property $(c_\lambda)$ is $f(\lambda)$-separably injective. Kalton is able to show in [155, Theorem 5.2] that such is the case of $c_0$:

**Lemma 2.63** *Let Y be a closed subspace of a Banach space X such that X/Y is separable. Let* $\tau : Y \to c_0$ *be a norm one operator. If for every* $x \in X$ *there is an extension* $\tau_x : Y + [x] \to c_0$ *with norm at most* $\lambda$ *then there is an extension* $T : X \to c_0$ *with norm at most* $\lambda$.

Of course the result contains extra information only for $\lambda < 2$. As it is clear from Propositions 2.32 or 2.34, some break occurs at $\lambda = 2$. Moreover, all $C$-spaces have property $(c_2)$ since they actually have the following property: for every family $B(x_\alpha, r_\alpha)$ of mutually intersecting balls $\bigcap_\alpha B(x_\alpha, 2r_\alpha) \neq \varnothing$. Indeed, every Banach space has the property that $\bigcap_\alpha B(x_\alpha, 2r_\alpha + \varepsilon) \neq \varnothing$ for every family $B(x_\alpha, r_\alpha)$ of mutually intersecting balls [116, p.198]. To deduce from here that a 2 is enough in $C$-spaces, Lindenstrauss [177] reasons as follows: in a $C(K)$ space $\bigcap_\alpha B(f_\alpha, t_\alpha) \neq \varnothing$ if and only if for every $k_0 \in K$

$$\limsup_{k\to k_0} \sup_\alpha (f_\alpha(k) - t_\alpha) \le \liminf_{k\to k_0} \inf_\alpha (f_\alpha(k) + t_\alpha).$$

From here it is clear that if the inequality holds for all $t_\alpha + \varepsilon$ then it also holds for $t_\alpha$. Observe that Proposition 2.32 actually shows that a space with property $(c_\lambda)$ for $\lambda < 2$ and containing almost isometric copies of $c_0$ has density character $\mathfrak{c}$.

### 2.7.2 *Complex Separably Injective Spaces*

Although these notes deal with real Banach spaces we will make a few remarks on injective-like complex Banach spaces. First of all one can consider (universally) separably injective complex spaces just assuming that the underlying field in the definitions is $\mathbb{C}$. Then Sects. 2.1, 2.3 and 2.4 apply verbatim to complex spaces, with the sole exception that the characterization of 1-separable injectivity by intersection properties of balls has to be reformulated. The new property required here is the *weak intersection property*, introduced by Hustad [137] as follows: a family of balls $\{B(x_\alpha, r_\alpha)\}_\alpha$ in a Banach space $X$ said to be weakly intersecting if for every norm one $f \in X^*$ the balls $\{B(f(x_\alpha), r_\alpha)\}_\alpha$ in the scalar field have nonempty intersection. All what remains is to replace "mutually intersecting balls" (real case) by "weakly intersecting balls" (complex case); see Proposition 2.30.

In general, given a real vector space $X$ one can "change" the scalar field just taking $X \otimes_{\mathbb{R}} \mathbb{C}$, which is a complex vector space by the very definition. Observe that there is a natural embedding of $X$ into $X \otimes_{\mathbb{R}} \mathbb{C}$ given by $x \mapsto x \otimes 1$ and that every $z \in X \otimes_{\mathbb{R}} \mathbb{C}$ has a unique decomposition $z = x + iy$, where $x, y \in X$. If, besides, $X$ is a real Banach space, then $X \otimes_{\mathbb{R}} \mathbb{C}$ can be equipped with a variety of norms, making it a complex Banach space, which is called a *complexification* of $X$ when its norm is reasonable in the sense that $\|x \otimes \zeta\| = \|x\| \cdot |\zeta|$ for every $x \in X, \zeta \in \mathbb{C}$.

Examples of such can be found in [3, 41, 182, 241]. Unfortunately, those complexifications that are suitable for some purposes may be not for others, and the interested reader may peruse the paper [200] to get an idea of the situation. Those readers familiar with tensor products of Banach spaces will guess that for what these notes are concerned (namely, the extension of operators) the most convenient norm on $X \otimes_{\mathbb{R}} \mathbb{C}$ is that arising from the *injective tensor product* $X \check{\otimes}_{\mathbb{R}} \mathbb{C}$. Without entering into any details, the injective norm in $X \otimes_{\mathbb{R}} Y$ is given by

$$\|u\|_\varepsilon = \sup\{|u(x^* \otimes y^*)| : \|x^*\|, \|y^*\| \le 1\},$$

where $x^*$ and $y^*$ are real-linear functionals on $X$ and $Y$, respectively (see [83]). Needless to say, every real-linear functional on $\mathbb{C}$ has the form $\zeta = \alpha + i\beta \mapsto s\alpha + t\beta$ for some fixed $s, t \in \mathbb{R}$ and the norm of such functionals is just $\|(s, t)\|_2 = \sqrt{s^2 + t^2}$. Thus, the injective norm of $z = x + iy = x \otimes 1 + y \otimes i$ in $X \otimes_{\mathbb{R}} \mathbb{C}$ is

$$\begin{aligned} \|z\|_\varepsilon &= \sup \{|sx^*(x) + tx^*(y)| : \ \|x^*\|, \ s^2 + t^2 \le 1\} \\ &= \sup \{|x^*(sx + ty)| : \ \|x^*\| \le 1, \ s^2 + t^2 \le 1\} \\ &= \sup \{\|sx + ty\|_X : \ s^2 + t^2 \le 1\}. \end{aligned} \tag{2.10}$$

Let us denote by $X_{\mathbb{C}}$ the complexification of $X$ associated to the just defined norm. Observe that this is not the same complexification as in, say, [181, p. 81]. The basic property of this construction is the following: if $u : Y \to X$ is a linear isometry between two real Banach spaces, then $u \otimes \mathbf{1}_{\mathbb{C}} : Y_{\mathbb{C}} \longrightarrow X_{\mathbb{C}}$ is again an isometry. Another pleasant feature of this norm is that if $X = C(K)$, then $X_{\mathbb{C}} = C(K, \mathbb{C})$, with the sup norm. It follows that if the real space $X$ is a Lindenstrauss space, then $X_{\mathbb{C}}$ is a (complex) Lindentrauss space: if $X^* = L_1(\mu, \mathbb{R})$, then $X^*_{\mathbb{C}} = L_1(\mu, \mathbb{C})$. One has

**Proposition 2.64** *Let $E$ be a real Banach space $E$ and $\lambda \ge 1$. Then $E$ is (universally) $\lambda$-separably injective, as a real Banach space if and only if $E_{\mathbb{C}}$ is (universally) $\lambda$-separably injective, as a complex Banach space.*

*Proof* Observe that the inclusion map $E \to E_{\mathbb{C}}$ given by $x \mapsto x \otimes 1$ has a contractive real-linear left-inverse $\Re : E_{\mathbb{C}} \longrightarrow E$ given by $\Re(x + iy) = x$. Actually this map is nothing different from the tensorization of the "real part" map $\mathbb{C} \to \mathbb{R}$ with the identity on $E$. Anyway is trivial to check that $\|x\|_E \leq \|x + iy\|_{E_{\mathbb{C}}}$ in view of (2.10).

Suppose $E$ is (universally) $\lambda$-separably injective, as a real Banach space. Let $X$ be a complex Banach space and $t : Y \to E_{\mathbb{C}}$ a complex-linear operator, where $Y$ is a closed subspace of $X$. Then $\Re(t) : Y \to E$ is a real-linear operator with $\|\Re(t)\| \leq \|t\|$. If $\tau : X \to E$ is a real-linear extension of $\Re(t)$, then the map $T : X \to E_{\mathbb{C}}$ defined by

$$T(x) = \frac{\tau(x) - i\tau(ix)}{2}$$

is a complex-linear extension of $t$. This establishes the "only if" part.

To prove the converse, let us assume that $E_{\mathbb{C}}$ is (universally) $\lambda$-separably injective, as a complex Banach space. Let $Y$ be a subspace of a real Banach space $X$, and let $t : Y \to E$ be a real-linear operator. Consider $Y_{\mathbb{C}}$ as a complex subspace of $X_{\mathbb{C}}$ and the complex operator $t_{\mathbb{C}} : Y_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}$ defined as

$$t_{\mathbb{C}}(x + iy) = t(x) + it(y)$$

If this operator extends to a complex operator $T : X_{\mathbb{C}} \longrightarrow E_{\mathbb{C}}$ then the "restriction" of $\Re(T)$ to $X$ is a real-linear extension of $t$. □

It follows, for instance, that a compact space $K$ is an $F$-space if and only if the complex space $C(K, \mathbb{C})$ is 1-separably injective.

In the opposite direction, every complex space is *also* a real space. One has:

**Lemma 2.65** *A complex Banach space is (universally) separably injective if and only if its underlying real space is (universally) separably injective.*

*Proof* First, we suppose that $E$ is a complex (universally) separably injective Banach space. Let $Y$ be a subspace of a real Banach space $X$, and let $t : Y \to E$ be a real-linear operator.

Consider $Y_{\mathbb{C}}$ as a complex subspace of $X_{\mathbb{C}}$ and the complex operator $\tau : Y_{\mathbb{C}} \longrightarrow E$ defined as

$$\tau(x + iy) = t(x) + it(y)$$

If this operator extends to a complex operator $T : X_{\mathbb{C}} \longrightarrow E$ then the "restriction" of $T$ to $X$ is a real-linear extension of $t$. For the converse implication, assume now that the underlying real space of $E$ is (universally) separably injective Let $X$ be complex Banach space, $Y$ a complex subspace of $X$ and $t : Y \longrightarrow E$ be a complex operator.

If $\tau : X \longrightarrow E$ a real-linear extension of $t$, it is easy to check that the formula

$$T(x) = \frac{\tau(x) - i\tau(ix)}{2}$$

defines a complex operator $T : X \longrightarrow E$ that extends $t$. $\square$

The preceding proof shows that if $E$ is (universally) $\lambda$-separably injective as a real space, then so is as a complex space. However, when $E$ is a complex (universally) $\lambda$-separably injective, the proof gives only that $E$ is (universally) $\lambda\sqrt{2}$-separably injective as a real Banach space. No more can be expected: $\mathbb{C}$ is 1-injective in the complex domain, while, being isometric to $\ell_2^2$, it is only $\sqrt{2}$-separably injective as a real space.

Springer
http://www.springer.com/978-3-319-14740-6

Separably Injective Banach Spaces
Avilés, A.; Cabello Sánchez, F.; Castillo, J.M.F.; González, M.; Moreno, Y.
2016, XXII, 217 p., Softcover
ISBN: 978-3-319-14740-6