Preface

The Plot

Injective Banach spaces are those spaces that allow the extension of any operator with values in them to any superspace. Finite dimensional and ℓ_∞ are the simplest examples of injective spaces. When a Banach space is injective, there automatically appears a constant that controls the norms of the extensions. At the other end of the line, we encounter the Banach spaces allowing the "controlled" extension of finite-rank operators, which form the well-known class of $\mathscr{L}_\infty$-spaces.

The main topic of this monograph lies between these two extremes: Banach spaces that allow a controlled extension of operators under certain restrictions on the size of their range or the size of the spaces to where they can be extended. The basic case is that of Banach spaces allowing the extension of operators from subspaces of separable spaces, called separably injective, for which c_0 is the simplest example. The second more important case is that of Banach spaces allowing the extension of operators from separable spaces elsewhere, called universally separably injective, for which the space $\ell_\infty^c(\Gamma)$ is the perfect example.

This monograph contains most of what is currently known about (universally) separably injective spaces; and certainly it contains all we know plus a good part of all we don't. Chapters 1–5, whose content we describe below, give a rather detailed account of the current theory of separable injectivity, with its many connections and applications. At the same time, the "Notes and remarks" sections at the end of each chapter and the entire "Open problems" section in Chap. 6 provide a large-scale map of the land beyond the sea. We have decided not to reproduce here several items already covered in books. The best example of this would be Zippin's theorem asserting that c_0 is the only separable separably injective Banach space, for which an exceptionally limpid exposition can be found in [253]; see also [33]. Other seemingly relevant pieces of information can be found in the literature, but as we were unable to establish precise connections with our topic, we chose to omit them.

Injectivity vs. (Universal) Separable Injectivity

The reader is referred to either the Preliminaries or the Appendix for all unexplained notation.

A Banach space E is said to be injective if for every Banach space X and every subspace Y of X, each operator $t: Y \to E$ admits an extension $T: X \to E$. The space is said to be λ-injective if, besides, T can be chosen so that $\|T\| \leq \lambda \|t\|$.

The space ℓ_∞ is the perfect example of 1-injective space, and a key point is to determine to what extent other injective spaces share the properties of ℓ_∞. On the positive side, one should score the results of Nachbin and Kelley who characterized the 1-injective spaces as those Banach spaces linearly isometric to a $C(K)$ space, with K an extremely disconnected compact space. On the other hand, the research of Argyros, Haydon, Rosenthal, and other authors provided several general structure theorems, good examples of exotic injective spaces, and the feeling that the complete classification of injective Banach spaces is an unmanageable problem.

Most of the research on injective spaces has revolved around the following problems:

(1) Is every injective space isomorphic to a 1-injective space?
(2) Is every injective space isomorphic to some $C(K)$-space?
(3) What is the structure of an injective space?

Admittedly, (3) is rather vague and (2) is a particular case of the more general problem of finding out whether or not a complemented subspace of a $C(K)$-space is again isomorphic to a $C(K)$-space (the compact space may vary). These problems have remained open for 50 years, (1) and (3) even in the case in which the injective space is a $C(K)$-space. We refer the reader to Chap. 1 for a summary of what is known about injective spaces.

In this monograph, we deal with several weak forms of injectivity, mainly *separable injectivity* and *universal separable injectivity*. A Banach space E is said to be separably injective if it satisfies the extension property in the definition of injective spaces under the restriction that X is separable; it is said to be a universally separably injective if it satisfies the extension property when Y is separable. Obviously, injective spaces are universally separably injective, and these, in turn, are separably injectives; the converse implications fail. The corresponding definitions of λ-separably injective and universally λ-separably injective should be clear.

The study of separably injective spaces was initiated by Phillips [216] and Sobczyk [235], who showed that c (resp. c_0) is 2-separably injective. Later Zippin [252] proved that c_0 is the only (infinite dimensional) separable space that is separably injective. Moreover, Ostrovskii [208] proved that a λ-separably injective space with $\lambda < 2$ cannot be separable, Baker [25] and Seever [228] studied separably injective $C(K)$ spaces, and several results in the literature on the extension of operators can also be formulated in terms of separably injective spaces. Separably injective spaces have been studied more recently by several authors such as Rosenthal [225], Zippin [253], Johnson and Oikhberg [146], and also by the present authors in [20], where the notion of universal separable injectivity was formally introduced.

The theory of separably injective and universally separably injective spaces is quite different from that of injective spaces, is much richer in examples, and contains interesting structure results and homological characterizations. For instance, problems (1) and (2) above have a negative answer for separably injective spaces, and quite interesting information about the structure of (universally) separably injective spaces can be offered. Moreover this theory is far from being complete: many open problems can be formulated (see Chap. 6) that we expect can be attractive for Banach spacers and could foster the interest in studying injectivity-like properties of Banach spaces.

A Brief Description of the Contents of This Monograph

After this Preface, "Preliminaries" section contains all due preliminaries about notation and basic definitions. An Appendix at the end of the book describes the basic facts about $\mathscr{L}_\infty$ and $\mathscr{L}_1$-spaces, homological techniques, and transfinite chains that will appear and be used throughout the monograph. Other definitions, properties, or required constructions will be given when needed.

In Chap. 1, we gather together properties of injective Banach spaces, basic examples and counterexamples, and criteria that are useful to prove that certain spaces are not injective. These facts will be used later and will allow the reader to compare the stability properties and the variety of examples of injective spaces with those of (universally) separably injective spaces.

In Chap. 2, we introduce the separably injective and the universally separably injective Banach spaces, as well as their quantitative versions, and obtain their basic properties and characterizations. We establish that infinite-dimensional separably injective spaces are $\mathscr{L}_\infty$-spaces, contain c_0, and have Pełczyński's property (V). Universally separably injective spaces, moreover, are Grothendieck spaces, contain ℓ_∞, and enjoy Rosenthal's property (V). We also prove a number of stability results that allow us to present many natural examples of separably injective spaces, such as $C(K)$-space when K is either an F-space or has finite height, twisted sums and c_0-vector sums of separably injective spaces, etc., including an example of a separably injective space that is not isomorphic to any complemented subspace of a $C(K)$-space, which solves problem (2) above for separable injectivity. In passing, these facts provide a major structural difference between λ-separably injective spaces for various values of λ, something that currently does not exist for injective spaces: 1-separably injective spaces are Grothendieck and Lindenstrauss spaces; hence they must be nonseparable when they are infinite dimensional. However, 2-separably injective spaces can be even separable.

The fundamental structure theorem for universally separably injective spaces is that a Banach space E is universally separably injective if and only if every separable subspace is contained in a copy of ℓ_∞ inside E. This establishes a bridge toward the study of the partially automorphic character of (universally) separably injective spaces, namely, toward determining in which cases an isomorphism between two subspaces extends to an automorphism of the whole space.

Homological characterizations are possible: recall that $\operatorname{Ext}(Z, Y) = 0$ means that whenever a Banach space isomorphic to Y is contained in a Banach space X in such a way that X/Y is isomorphic to Z, it must be complemented. In this language, a space E is separably injective if and only if $\operatorname{Ext}(S, E) = 0$ for every separable space S. For universally separably injective spaces, we have less: if $\operatorname{Ext}(\ell_\infty/S, U) = 0$ for every separable space S, then U is universally separably injective. A problem that is crying out to be solved is whether also the converse holds: Does the identity $\operatorname{Ext}(\ell_\infty, U) = 0$ characterize universally separably injective spaces U? A rather detailed discussion can be found in Sect. 6.2.

Section 2.4 is specifically devoted to 1-separably injective spaces. At this point, set theory axioms enter the game. Indeed, Lindenstrauss obtained in the mid-1960s what can be understood as a proof that, under the continuum hypothesis CH, 1-separably injective spaces are 1-universally separably injective; he left open the question in general. We show how to construct (in a way consistent with ZFC) an example of a Banach space of type $C(K)$ that is 1-separably injective but not 1-universally separably injective. The chapter closes with a detailed study of the separable injectivity and related homological properties of $C(\mathbb{N}^*)$.

Chapter 3 focuses on the study of spaces of universal disposition because they will provide new (and, in the case of p-Banach spaces, the only currently known) examples of (universally) separably injective spaces. We present a basic device to generate such spaces. The device is rather flexible and thus, when performed with the appropriate input data, is able to produce a great variety of examples: the Gurariy space $\mathscr{G}$ [118], the p-Gurariy spaces [58], the Kubiś space [169], new spaces such as $\mathscr{F}^{\omega_1}$ which is of universal disposition for all finite-dimensional spaces but not for separable spaces, or the $\mathscr{L}_\infty$-envelopes obtained in [69]. Remarkable outputs are the examples of a 1-separably injective spaces $\mathscr{S}^{\omega_1}$ and a 1-universally separably injective space $\mathscr{U}^{\omega_1}$ that are not isomorphic to complemented subspaces of any C-space (or an M-space), which solves problem (2) for the classes of (universally) 1-separably injective space in a somewhat surprising way. These spaces turn out to be of universal disposition for separable spaces, which confirms a conjecture made by Gurariy in the 1960s. Moreover, under CH, the space $\mathscr{S}^{\omega_1}$ coincides with the Fraïssé limit in the category of separable Banach spaces considered by Kubiś [169] and with the countable ultrapowers of the Gurariy space.

In Chap. 4, we study injectivity properties of ultraproducts. We show that ultraproducts built over countably incomplete ultrafilters are universally separably injective as long as they are $\mathscr{L}_\infty$-spaces, in spite of the fact that they are never injective. Then, we focus our attention on ultraproducts of Lindenstrauss spaces, with special emphasis in ultrapowers of $C(K)$-spaces and the Gurariy space. With the aid of M-ideal theory, the results can be applied to the lifting of operators and to the study of the behavior of the functor Ext regarding duality. One section is devoted to the Henson-Moore classification problem of isomorphic types of ultrapowers of $\mathscr{L}_\infty$-spaces. A particular concern of the authors has been to make clear which results can be proved without invoking "model theory" and which ones still belong to that domain.

In Chap. 5, we consider a natural generalization of separable injectivity to higher cardinals. Namely, given an infinite cardinal $\aleph$, we say that E is $\aleph$-injective (respectively, universally $\aleph$-injective) if it satisfies the extension property stated at the beginning for spaces X (respectively, for subspaces Y) having density character strictly smaller than $\aleph$. Stopping at the first uncountable cardinal $\aleph_1$, one reencounters the classes of (universally) separably injective spaces. Some facts generalize to the higher cardinal context (say, there exist $(1, \aleph)$-injective spaces that are complemented in no M-space), but some others offer difficulties. For example, we need to restrict to what we call $c_0(\aleph)$-supplemented subspaces in order to extend the results on universally separably injective spaces; or, concerning ultraproducts, we have to deal with $\aleph$-good ultrafilters. The chapter includes a study of spaces $C(K)$ which are $(1, \aleph)$-injective and a study of the interplay with topological properties of the compact space K in the form of a rather satisfactory duality between extension properties of operators into $C(K)$ and lifting properties of continuous maps from K.

At the end of each chapter, we have included a "Notes and Remarks" section containing additional information, mainly considered from the positive side. A final Chap. 6 describes problems related to the topics of this monograph that remain open. We have chosen to give the available information about them—including partial solutions, connections with other results, etc.—in order that it could be helpful for further research.

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