Chapter 2
Invertible Mappings

2.1 Injective, Surjective and Bijective Mappings

Given the map $f : A \to B$, and $I \subset A$, the set

$$f(I) = \{f(x) : x \in I\}$$

is called the image of I under f. If $I = A$, then $f(A)$ is called the *image* of f, or the *range* of f, and denoted $\mathrm{Im}(f)$. Observe that $f(A) \subset B$ but that, in general, $f(A) \neq B$.

Definition 2.1 The map $f : A \to B$ is called *surjective* if $f(A) = B$, that is, if for every $b \in B$ there exists $a \in A$ such that $f(a) = b$, and it is called *injective* if it never sends distinct points into the same point, that is, if $f(a_1) \neq f(a_2)$ for any $a_1, a_2 \in A$ with $a_1 \neq a_2$. Finally, f is called *bijective* if it is both injective and surjective.

From the definition of injective map it follows at once that f is not injective whenever there exist $a_1, a_2 \in A$ with $a_1 \neq a_2$ such that $f(a_1) = f(a_2)$. Hence, even functions, defined on symmetric subsets of $\mathbb{R}$, are never injective. Similarly, f is not surjective if there exists $b \in B$ that is not in the image of f.

Definition 2.2 Let $f : A \to B$ be a map and take a subset $J \subseteq B$. The *inverse image*, or *preimage*, of J under f is the set of points of A that are sent by f into J, that is

$$f^{-1}(J) = \{a \in A : f(a) \in J\} \subseteq A.$$

Evidently, f is injective exactly when the inverse image of any singleton (that is, a set of the form $J = \{b\}$, for some $b \in B$) is either a singleton (a set of the form $\{a\}$, for some $a \in A$) or empty, and is surjective when the inverse image of any singleton is not empty.

It is possible and useful to interpret the notions of injectivity and surjectivity for maps that are defined on subsets of $\mathbb{R}$ into $\mathbb{R}$ in terms of their graphs. Take such a

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map $f : I \subseteq \mathbb{R} \to \mathbb{R}$ and denote as usual by $\Gamma(f)$ its graph. For any $y_0 \in \mathbb{R}$,

$$f^{-1}(\{y_0\}) = \{x \in I : (x, y_0) \in \Gamma(f)\}.$$

Therefore, the preimage of a point is found by considering the horizontal line $y = y_0$ and then by collecting all the abscissae of the points that lie on the intersection between the horizontal line and $\Gamma(f)$. It follows in particular that f is injective if and only if every horizontal line intersects $\Gamma(f)$ in at most one point and it is surjective if and only if every horizontal line intersects $\Gamma(f)$ in at least one point. Therefore, f is bijective if and only if every horizontal line intersects $\Gamma(f)$ in exactly one point.

If one considers a function $f : I \subset \mathbb{R} \to J$, where J is a prescribed subset of $\mathbb{R}$, then the previous graphical interpretations must be modified by taking horizontal lines of the form $y = y_0$ only for the values y_0 that belong to J. For example, the function $f : [0, 1] \to [0, 1]$ defined by $x \mapsto x$ is bijective, as well as $x \mapsto x^n$ for any positive integer n. Indeed, for any positive integer n, and every horizontal line with equation $y = y_0$ with $y_0 \in [0, 1]$ intersects the graph of the function $x \mapsto x^n$ in the single point $x_0 = \sqrt[n]{y_0} \in [0, 1]$. Further, the mapping $\varphi : [-\pi/2, \pi/2] \to [-1, 1]$ defined by $\varphi(x) = \sin x$ is also bijective, as the reader is urged to check with a simple drawing and then appealing to elementary trigonometry, whereas the map $\psi : \mathbb{R} \to \mathbb{R}$ defined by $\psi(x) = \sin x$ is neither injective (because $\sin x = \sin(x + 2k\pi)$ for any $k \in \mathbb{Z}$) nor surjective (because $2 \in \mathbb{R}$ is not in the image of ψ).

The property of being injective is somehow intrinsic to a map, whereas the property of being surjective can always be achieved by suitably changing the codomain. Indeed, given $f : A \to B$, the new map

$$\tilde{f} : A \to f(A), \qquad \tilde{f}(a) = f(a) \tag{2.1}$$

is defined by the same law and on the same set as f, and is automatically surjective. Occasionally, $\tilde{f}$ will be referred to as the *surjective map naturally associated* with f.

2.2 Inversion of a Map

Definition 2.3 Take any set A. The map $\mathrm{id}_A : A \to A$ defined by $\mathrm{id}_A(a) = a$ for every $a \in A$ is called the *identity mapping* of A.

Definition 2.4 The map $f : A \to B$ is called *invertible* if there is a map $g : B \to A$ such that:

(i) $g \circ f = \mathrm{id}_A$;
(ii) $f \circ g = \mathrm{id}_B$.

In this case, the map g, necessarily unique, is called the *inverse map* of f and is denoted $g = f^{-1}$.

Proposition 2.1 *The map $f : A \to B$ is invertible if and only if it is bijective.*

Although Proposition 2.1 clarifies that only the bijective maps are invertible, it is customary to relax the notion of invertibility in view of the fact that surjectivity can always be achieved, as discussed at the end of the previous section. In what follows, the notion of invertible map is used to mean that f is injective. If this is the case, the surjective map $\tilde{f}$ naturally associated with f by (2.1) is actually bijective and hence invertible in the strict sense. This slight ambiguity is best circumvented by requiring to explicitly determine the image of f, which coincides with the domain of the inverse (whenever f is injective), and then, with slight abuse of notation, to identify f with $\tilde{f}$.

Another issue that often occurs naturally is local invertibility. By this it is meant that a function f might fail to be injective on its domain, for example $f(x) = x^2$ is not injective on $\mathbb{R}$, but perhaps its restriction to a proper subset of its domain is injective and thus, in the broader sense just discussed, invertible. For example the restriction of $f(x) = x^2$ to $[0, +\infty)$ is injective. This justifies the following definition.

Definition 2.5 If $f : A \to B$ is a map and $I \subset A$ is a subset of A such that the restriction $f|_I$ is injective, then f is said to be invertible on I (onto its image).

A map $f : A \to B$ is invertible on I onto its image if and only if for any b in $f(I)$ the equation $f(a) = b$ has a unique solution $a \in I$.

2.3 Monotone Functions

Definition 2.6 The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called:

(i) *increasing* if, whenever $x_1, x_2 \in I$ are such that $x_1 < x_2$, then $f(x_1) < f(x_2)$;
(ii) *nondecreasing* if, whenever $x_1, x_2 \in I$ are such that $x_1 < x_2$, then $f(x_1) \leq f(x_2)$;
(iii) *decreasing* if, whenever $x_1, x_2 \in I$ are such that $x_1 < x_2$, then $f(x_1) > f(x_2)$;
(iv) *nonincreasing* if, whenever $x_1, x_2 \in I$ are such that $x_1 < x_2$, then $f(x_1) \geq f(x_2)$.

If f satisfies either of the above conditions, then it is called *monotone* or *monotonic*. If it satisfies either (a) or (c), then it is called *strictly monotone*, or *strictly monotone*. The strictly monotone functions are those that either preserve or invert the order relations. Sometimes the increasing functions are called *strictly increasing* and the decreasing functions are called *strictly decreasing*.

Observe that the composition of monotone maps is always monotone. The point is that, loosely speaking, a monotone map either preserves or inverts the order, either in the strong sense (strictly monotone maps) or in the weak sense (kinds (ii) and (iv) in the definition), so that in the end the order is either preserved or reversed (strongly or weakly), according to which kind of monotoneities were involved. The reader is urged to check which compositions lead to which monotone maps.

Table 2.1 monotoneity of $f+g$ with $f, g : I \to \mathbb{R}$

f	g	$f+g$
Increasing	Increasing	Increasing
Decreasing	Decreasing	Decreasing

Table 2.2 monotoneity of fg with $f, g : I \to \mathbb{R}$

f	Sign of f	g	Sign of g	fg
Increasing	Positive	Increasing	Positive	Increasing
Increasing	Positive	Decreasing	Negative	Decreasing
Increasing	Negative	Decreasing	Positive	Increasing
Decreasing	Positive	Increasing	Negative	Increasing
Decreasing	Negative	Increasing	Positive	Decreasing
Decreasing	Negative	Decreasing	Negative	Increasing

Table 2.3 monotoneity of $f \circ g$, with $\mathrm{Im}(g) \subset \mathrm{Dom}(f)$

f	g	$f \circ g$
Increasing	Increasing	Increasing
Increasing	Decreasing	Decreasing
Decreasing	Decreasing	Increasing
Decreasing	Increasing	Decreasing

Proposition 2.2 *If $f : I \subset \mathbb{R} \to \mathbb{R}$ is strictly monotone, then f is injective.*

Notice that the reverse implication is false, for example the function $f(x) = 1/x$ is injective but not monotone on its natural domain $\mathbb{R} \setminus \{0\}$. Observe also that the inverse of a strictly monotone map is again a strictly monotone map, with the same type of monotoneity.

Monotoneity of sums, products and compositions of functions can be inferred, but not always. The results are summarized in Tables 2.1, 2.2 and 2.3 in the cases in which a conclusion can be drawn. In each of the remaining cases it is possible to produce examples with different behaviours.

2.4 Guided Exercises on Invertible Mappings

2.1 Consider the function $f : [0, +\infty) \to [0, \pi/4)$ defined by $f(x) = \arctan \dfrac{x}{x+1}$. Prove that f is invertible and write the explicit expression of its inverse.

Answer. Consider the auxiliary maps $h : [0, +\infty) \to [0, 1)$ and $g : [0, 1) \to [0, \pi/4)$ defined by

$$h(x) = \frac{x}{x+1}, \qquad g(x) = \arctan x.$$

Then f is the composition $f(x) = g(h(x)) = g \circ h(x)$. Now, both h and g are strictly increasing maps, so that f is also strictly increasing, hence invertible. Indeed, the arctangent map is monotone, being the inverse map of a strictly increasing map (the restriction of the tangent to $(-\pi/2, \pi/2)$), whereas to see that h is monotone just observe that

$$h(x) = \frac{x+1-1}{x+1} = 1 - \frac{1}{x+1}.$$

In order to find the expression of the inverse, for any given $y \in [0, \pi/4)$ one must find $x \in [0, +\infty)$ such that $f(x) = y$. Now

$$\begin{aligned} \arctan \frac{x}{x+1} = y \quad &\Longrightarrow \quad \tan y = \frac{x}{x+1} \\ &\Longrightarrow \quad x(1 - \tan y) = \tan y \\ &\Longrightarrow \quad x = \frac{\tan y}{1 - \tan y} \end{aligned}$$

Therefore $f^{-1} : [0, \pi/4) \to [0, +\infty)$ is defined by $f^{-1}(y) = \tan y/(1 - \tan y)$.

This exercise is a direct application of two properties: the composition of strictly monotone maps is strictly monotone and a strictly monotone map is invertible. In the case at hand, the search of the inverse map leads to an answer by "undoing" each operation, in the correct order, or, more technically, observing that

$$f = g \circ h \quad \Longrightarrow \quad f^{-1} = h^{-1} \circ g^{-1}.$$

Evidently, here $g^{-1}(x) = \tan x$ and $h^{-1}(x) = x/(1 - x)$.

2.2 Determine if the function

$$f(x) = \begin{cases} -x^2 - 1 & x < 0 \\ 0 & x = 0 \\ x + 1 & x > 0 \end{cases}$$

is invertible and, if yes, find an explicit expression of its inverse.

Answer. The restriction $f_1 = f\big|_{(-\infty,0]} \to (-\infty, -1) \cup \{0\}$ is a bijection. Indeed, if $x_1, x_2 \in (-\infty, 0)$ are such that $x_1 < x_2$, then

$$f(x_1) - f(x_2) = -x_1^2 + x_2^2 = (x_2 - x_1)(x_1 + x_2) \neq 0,$$

and, more precisely, $f_1(x_1) - f_1(x_2) < 0$, so that f_1 is strictly increasing. Take now $y \in (-\infty, -1)$. Then $y \in \mathrm{Im}(f_1)$ because the negative number $x = -\sqrt{-1-y}$ satisfies

$$-x^2 - 1 = -\left(-\sqrt{-1-y}\right)^2 - 1 = y.$$

Observe further that $f_1(0) = 0$, so that f_1 is a bijection.

Next, $f_2 = f\big|_{(0,+\infty)} \to (1, +\infty)$, which is defined by $f_2(x) = x + 1$, is clearly also a bijection and hence $f : \mathbb{R} \to (-\infty, 1) \cup \{0\} \cup (1, +\infty)$ is a bijection, with inverse

$$f^{-1}(x) = \begin{cases} -\sqrt{-1-x} & x < -1 \\ 0 & x = 0 \\ x - 1 & x > 1. \end{cases}$$

In this exercise the given function has different expressions in different intervals and therefore needs to be analyzed in each of them separately. Now, 0 goes to 0, and it is rather clear that in fact the negative real numbers are sent to negative real numbers and likewise for the positive real numbers, so that in the end the map is a bijection. All remains to be done is to write the explicit inverse mappings.

2.3 Prove that the map $f : [1, +\infty) \to [1, +\infty)$ defined by $f(x) = \mathrm{e}^{\log^2 x}$ is invertible, and write the explicit expression of f^{-1}.

Answer. Take x_1 and x_2 with $1 \le x_1 < x_2$. Then

$$0 \le \log x_1 < \log x_2 \implies \mathrm{e}^{\log^2 x_1} < \mathrm{e}^{\log^2 x_2} \implies f(x_1) < f(x_2).$$

Therefore f is strictly increasing, hence invertible. Take now $y \ge 1$. The point $x \ge 1$ satisfies $f(x) = y$ provided that

$$\mathrm{e}^{\log^2 x} = y = \mathrm{e}^{\log y} \implies \log^2 x = \log y \implies \log x = \sqrt{\log y} \implies x = \mathrm{e}^{\sqrt{\log y}}.$$

Evidently $\sqrt{\log y} \ge 0$ and hence its exponential is in $[1, +\infty)$. Therefore the image of f is $[1, +\infty)$ and $f^{-1} : [1, +\infty) \to [1, +\infty)$ is given by $f^{-1}(y) = \mathrm{e}^{\sqrt{\log y}}$.

This exercise is standard and simply requires to see that the given function actually maps the set $[1, +\infty)$ bijectively onto itself. For injectivity, it is immediately seen that f is increasing. For surjectivity, it is easy to find the solution of $f(x) = y$, that is, to find the inverse map.

2.4 Consider the function $g(x) = \dfrac{1}{4 \arcsin x - \pi}$.

(a) Find the domain of g.
(b) Determine the image of g.
(c) Prove that the restriction of g to $(-1, 1/\sqrt{2})$ is invertible and write its inverse.

Answer. (a) Put $f(x) = \arcsin x$ and $h(y) = 1/(4y - \pi)$, so that $g = h \circ f$. The domain of g is therefore:

$$\begin{aligned}\mathrm{Dom}(g) &= \{x \in \mathbb{R} : x \in \mathrm{Dom}(f),\ f(x) \in \mathrm{Dom}(h)\} \\ &= \{x \in \mathbb{R} : x \in [-1, 1],\ \arcsin x \neq \frac{\pi}{4}\} \\ &= \{x \in \mathbb{R} : x \in [-1, 1],\ x \neq \frac{\sqrt{2}}{2}\} \\ &= [-1, \frac{\sqrt{2}}{2}) \cup (\frac{\sqrt{2}}{2}, 1].\end{aligned}$$

(b) The image of g consists of those $y \in \mathbb{R}$ for which there exists $x \in \mathrm{Dom}(g)$ such that $y = g(x)$. Thus

$$\begin{aligned} y \in \mathrm{Im}(g) &\iff \text{there exists } x \in \mathrm{Dom}(g) \text{ such that: } y = (4\arcsin x - \pi)^{-1} \\ &\iff \text{there exists } x \in \mathrm{Dom}(g) \text{ such that: } 4y \arcsin x - \pi y = 1.\end{aligned}$$

From the latter it follows that

$$\begin{aligned} y \neq 0 \text{ and } y \in \mathrm{Im}(g) &\iff \text{there exists } x \in \mathrm{Dom}(g) \text{ such that: } \arcsin x = \frac{\pi y + 1}{4y} \\ &\iff \frac{\pi y + 1}{4y} \in \mathrm{Im}(f) \setminus \{f(\frac{\sqrt{2}}{2})\} = [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{\frac{\pi}{4}\},\end{aligned}$$

where again $f(x) = \arcsin x$. Hence, if $y \neq 0$ and $y \in \mathrm{Im}(g)$, then

$$-\frac{\pi}{2} \leq \frac{\pi y + 1}{4y} \leq \frac{\pi}{2} \quad \text{and} \quad \frac{\pi y + 1}{4y} \neq \frac{\pi}{4}.$$

It follows that $y \in (-\infty, -1/(3\pi)] \cup [1/\pi, +\infty)$ and hence

$$\mathrm{Im}(g) = (-\infty, -\frac{1}{3\pi}] \cup [\frac{1}{\pi}, +\infty).$$

(c) The function $f(x) = \arcsin x$ is increasing. Hence

$$-\frac{\pi}{2} = f(-1) < f(x) < f\Big(\frac{1}{\sqrt{2}}\Big) = \frac{\pi}{4},$$

namely $f(x) \in (-\pi/2, \pi/4)$, for every $x \in (-1, 1/\sqrt{2})$. Further, $h(y)$ is decreasing in $(-\pi/2, \pi/4)$. Since g is the composition of f and h, which are both strictly monotone but with opposite monotoneity, g is decreasing on $(-1, 1/\sqrt{2})$ and hence invertible on this interval. Finally, from (b) it follows that

$$y = g(x) \iff \arcsin x = \frac{\pi y + 1}{4y}, \qquad \frac{\pi y + 1}{4y} \in [-\frac{\pi}{2}, \frac{\pi}{2}].$$

From this it follows that $x = \sin((\pi y + 1)/(4y))$, and finally

$$g^{-1}(y) = \sin\Big(\frac{\pi y + 1}{4y}\Big).$$

In this exercise, the basic idea is again to view g as a composite function. Once this is done, then finding the domain and the image is achieved by carefully following what each map (f and h) does. Most of the effort actually goes into finding the image of g. Inversion is done by inverting each of f and h.

2.5 Consider the function $f(x) = |x| + x^2 - 1$.

(a) Establish if $-5/4 \in \mathrm{Im}(f)$.
(b) Find the largest neighborhood of $x_0 = 1$ on which the function is invertible, and write the explicit analytic expression of the inverse.

Answer. (a) Observe that $-5/4 \in \mathrm{Im}(f)$ if and only if the equation

$$-\frac{5}{4} = |x| + x^2 - 1$$

has a solution in $\mathbb{R}$, the domain of f. However, for every $x \in \mathbb{R}$

$$f(x) = |x| + x^2 - 1 \geq -1 > -\frac{5}{4},$$

so that $-5/4 \notin \mathrm{Im}(f)$.

(b) Since f is an even function, it is not injective, hence not invertible. Consider the restriction of f to the interval $[0, +\infty)$ and denote it by g, explicitly $g(x) = x + x^2 - 1$. It is easy to see that g is increasing in its domain, for if $0 \leq x_1 \leq x_2$, then $x_1^2 < x_2^2$ and hence

$$g(x_1) = x_1^2 + x_1 - 1 < x_2^2 + x_2 - 1 = g(x_2).$$

Therefore g is invertible. In order to find the explicit analytic expression of the inverse of (the surjective map naturally associated with) g, the second order equation $y = x + x^2 - 1$ must be solved for x as a function of y. The roots of the polynomial $x^2 + x - 1 - y$ are

$$x_1(y) = \frac{-1 - \sqrt{5 + 4y}}{2}, \qquad x_2(y) = \frac{-1 + \sqrt{5 + 4y}}{2}.$$

Clearly, $x_1(y) < 0$, whereas $x_2(y) > 0$ because $y > -1$. Therefore

$$g^{-1}(y) = x_2(y) = \frac{-1+\sqrt{5+4y}}{2}$$

and it follows that $\mathrm{Dom}(g^{-1}) = \mathrm{Im}(g) = [-1, +\infty)$.

This is a basic exercise on invertible mappings, where it is required to show that globally the function is not invertible but it is so if properly restricted. The presence of both an absolute value and a quadratic term imply that f is actually even, hence non invertible. But if one looks at one of the "branches" of f, namely $[0, +\infty)$, then f is monotone hence invertible. The explicit expression comes from taking the appropriate square root.

2.6 Consider the function $f(x) = \dfrac{1}{\log_{1/3}(x-2)} - 1$.

(a) Find the domain of f.
(b) Establish if f is invertible for $x > 3$ and, if yes, find the explicit analytic expression of the inverse g^{-1}, where $g = f|_{(3,+\infty)}$, specifying its domain.

Answer. (a) In order for $\log_{1/3}(x-2)$ to be well defined, it must be $x - 2 > 0$, that is $x > 2$. Further, $\log_{1/3}(x-2) \neq 0$ if $x - 2 \neq 1$. Therefore $\mathrm{Dom}(f) = (2, 3) \cup (3, +\infty)$.

(b) For $x > 3$, f is increasing. Indeed, $x \mapsto \log_{1/3}(x-2)$ is decreasing and positive. Hence $x \mapsto 1/\log_{1/3}(x-2)$ is increasing and such is also the function $x \mapsto (1/\log_{1/3}(x-2)) - 1$. As f is increasing on $(3, +\infty)$, the restriction g of f to this interval is invertible. Since $x > 3$, one has $\log_{1/3}(x-2) < 0$ and $f(x) < -1$. Therefore, $f((3, +\infty)) \subset (-\infty, -1)$. Furthermore, if $y < -1$, then the equation $y = f(x)$ has a solution if and only if

$$\begin{aligned} \frac{1}{\log_{1/3}(x-2)} - 1 = y \iff & \frac{1}{\log_{1/3}(x-2)} = y + 1 \\ \iff & \log_{1/3}(x-2) = \frac{1}{y+1} \\ \iff & x - 2 = \left(\frac{1}{3}\right)^{\frac{1}{y+1}} \\ \iff & x = 2 + \left(\frac{1}{3}\right)^{\frac{1}{y+1}}. \end{aligned}$$

Since $y < -1$, it follows that $(1/3)^{\frac{1}{x+1}} > 1$ and $x > 3$, that is $(-\infty, -1) \subset f((3, +\infty))$. Hence $g^{-1}(x) = 2 + (1/3)^{\frac{1}{x+1}}$.

Here the proof that the appropriate restriction of f is increasing can be carried out by viewing f as a composition of functions. The final formula is obtained undoing each of the several functions in the correct order.

2.5 Problems on Invertible Mapings

2.7 Consider the function $f(x) = 1/(1-3^x)$.

(a) Find the domain and the image of f.
(b) Study the monotoneity of f and establish if f is invertible.
(c) Write f^{-1}, if it exists, and specify both its domain and its image.

2.8 Consider the function $f(x) = -1/(\log_2 |x| + 1)$.

(a) Find the domain and the image of f.
(b) Study the monotoneity of f and establish if f is invertible.
(c) Denote by g the restriction of f to $(1/2, +\infty)$. Determine whether g is invertible and, if yes, write g^{-1} explicitly, specifying its domain.

2.9 Consider the function $f(x) = \log\left(x^2 - 3x + 1\right)$.

(a) Find the domain of f.
(b) Find the largest interval I containing $x_0 = 3$ on which f is injective.
(c) Determine $J = f(I)$ and compute the inverse $(f|_I)^{-1} : J \to I$.

2.10 Consider the function

$$f(x) = \frac{1}{x^2 - x + 3} - \frac{3}{11}$$

and denote by g the restriction of f to $[1/2, +\infty)$. Determine g^{-1}, if it exists, and specify its domain.

2.11 Consider the function $f : \mathbb{R} \to [-1, 1]$ defined by

$$f(x) = \begin{cases} \dfrac{1}{x} & x \in (-\infty, -1] \\ x & x \in (-1, 0) \\ 1 - x & x \in [0, 1) \\ -\dfrac{1}{x} & x \in [1, +\infty). \end{cases}$$

(a) Draw the graph of f.
(b) Establish whether f is injective and/or surjective.
(c) Establish in which intervals f is increasing.
(d) Draw the graph of $f(|x|)$.

2.12 Consider the function $f(x) = \dfrac{1}{1 + \sqrt{|x-2|}}$.

(a) Find the domain of f.
(b) Check if f is invertible for $x > 2$ and, if yes, find the inverse function g^{-1} of the restriction $g = f|_{(2,+\infty)}$.
(c) Find the even and odd parts of $F(x) = f(x + 2)$.

2.13 On which maximal intervals, if any, is $f(x) = \dfrac{1}{\sqrt{1-\log x}}$ injective?

2.14 Show that the restriction of $f(x) = x - \dfrac{1}{x}$ to $(0, +\infty)$ is invertible and find an explicit expression of the inverse.

2.15 Consider the function $f(x) = \dfrac{e^x}{\sqrt{e^x - 1}}$. Find an interval I on which f is invertible and write the explicit expression of $(f|_I)^{-1}$.

2.16 Consider the function $f : (-\infty, +\infty) \to (-\pi/2, \pi/2]$ defined by

$$f(x) = \begin{cases} \arctan x & x \in (-\infty, -\frac{\pi}{4}] \cup [\frac{\pi}{4}, +\infty) \\ 2x + \dfrac{\pi}{2} & x \in (-\frac{\pi}{4}, 0] \\ -2x & x \in (0, \frac{\pi}{4}). \end{cases}$$

(a) Draw the graph of f.
(b) Establish whether f is injective and/or surjective.
(c) Establish in which intervals f is increasing.

2.17 On which intervals, if any, is the function $f(x) = \dfrac{e^{-x}}{\sqrt{\log x - 5}}$ injective?

2.18 Consider the function $f(x) = \dfrac{e^x}{e^{2x} + e^x + k}$.

(a) For which values of the real parameter k the function f is defined on $\mathbb{R}$?
(b) Put $k = 1$. Find a neighborhood of $x_0 = -1$ in which f is invertible and write an explicit analytic expression of the inverse.

2.19 Is the function

$$f(x) = \begin{cases} \dfrac{1}{2}\left[\left(\dfrac{2}{3}\right)^x + \dfrac{1}{x+1}\right] & x \geq 0 \\[2ex] \dfrac{2x+3}{x+1} & x < -1 \end{cases}$$

invertible in its domain?

2.20 Consider the function $g(x) = \sqrt{\log_2 x - \log_4 (x-1)^2}$.

(a) Find the domain of g.
(b) Establish if g is invertible in its domain.

2.21 Consider the function

$$f(x) = \begin{cases} 2^{\frac{x^2+1}{x}} & x < 0 \\ -\frac{1}{5}\cos(2\arctan x) & x \geq 0. \end{cases}$$

(a) Find the image of f.
(b) Put $I = (-1, 1)$. Establish if $g = f|_I$ is invertible and, if yes, find g^{-1}.
(c) Determine a neighborhood of $x_0 = -2$ in which f is invertible.

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