

Chapter 2

Inherited Properties of Descendants

Abstract Given the results of Chap. 1 that every descendant may be constructed from a complete family of ancestor genes, we now investigate how the properties of the descendant correspond to the properties of those genes. In particular we consider the properties of Hamiltonicity, bipartiteness and planarity. In all three cases we prove that a descendant may only possess the property if all of its ancestor genes do. In the case of bipartiteness and planarity we also establish sufficient conditions. These results allow us to analyse the properties of a graph by considering its ancestor genes, or alternatively, to construct a graph with desired properties by choosing smaller genes with those properties. We follow each section with a discussion of famous results and conjectures relating to the graph properties, and how the results of this chapter relate to them.

2.1 Introduction

The results of the previous chapter indicate that every cubic graph is either a gene, or can be constructed from a unique complete family of ancestor genes via some sequence of the six breeding operations. It is therefore obvious that any such descendant has obtained the entirety of its structure either from its complete family of ancestor genes, or via the breeding operations used to construct it. A reasonable assumption is that some of the graph theoretic properties present in the descendant may have been inherited from its ancestor genes. In this chapter we investigate three such properties – Hamiltonicity, bipartiteness and planarity – and demonstrate that this is indeed the case. Specifically, we will prove that, in order for a descendant to possess any of the three properties, it is necessary for all of its ancestor genes to possess the property as well. In the cases of bipartiteness and planarity, this condition is both necessary and sufficient. This chapter not intended to represent an exhaustive list of inherited properties in descendants, but merely a first exploration into such inheritance. The discovery of other such inherited properties is a ripe subject for future research.

2.2 Hamiltonicity

The *Hamiltonian cycle problem* is a famously difficult graph theory problem that is known to be NP-complete, even when restricted to the class of cubic graphs [7].¹ The problem can be stated simply: given a graph Γ containing N vertices, determine whether or not any simple cycles of length N exist in the graph. Such a simple cycle of length N is called a *Hamiltonian cycle*. If a graph contains at least one Hamiltonian cycle, it is said to be a *Hamiltonian graph*, whereas a graph with no Hamiltonian cycles is said to be a *non-Hamiltonian graph*. A Hamiltonian graph is said to possess the property of *Hamiltonicity*.

Rather than focusing on Hamiltonian descendants, in this section we primarily investigate the different methods in which a non-Hamiltonian descendant may be constructed, as non-Hamiltonian cubic graphs are known to be relatively rare (e.g. see Robinson and Wormald [15]) and are therefore simpler to characterise. Recall from Chap. 1 that non-Hamiltonian genes are called *mutants*. We separate non-Hamiltonian descendants into two categories – mutant descendants, which possess at least one mutant in their complete family of ancestor genes; and non-mutant non-Hamiltonian (NMNH) descendants. In order to aid the flow of proofs, we consider the latter category first.

2.2.1 Non-mutant Non-Hamiltonian Descendants

Consider a descendant graph Γ_D . From Theorem 1.2, we know that Γ_D has a unique complete family of ancestor genes. Suppose that none of these genes are mutants. Then in general Γ_D could be Hamiltonian or non-Hamiltonian. We now investigate, exhaustively, all methods by which non-Hamiltonian descendants can be constructed from Hamiltonian parents for each of the six breeding operations.

2.2.1.1 Type 1 Breeding

Lemma 2.1. *If a descendant graph Γ_D is output by the type 1 breeding operation $\mathcal{B}_1(\Gamma_1, \Gamma_2, e_1, e_2)$ for some Γ_1, Γ_2, e_1 and e_2 , then Γ_D is non-Hamiltonian (Fig. 2.1).*

Proof. Type 1 breeding always introduces a 1-cracker into the graph, and so Γ_D is a bridge graph and therefore non-Hamiltonian. $\square$

Corollary 2.1. *Any descendants of the descendant Γ_D described in Lemma 2.1 are also non-Hamiltonian.*

¹ In fact, the Hamiltonian cycle problem is still NP-complete even for cubic planar graphs [8]. Cubic planar graphs are investigated in Sect. 2.4 of the present manuscript.

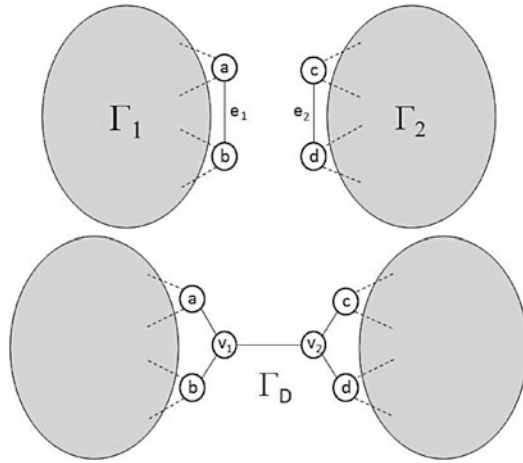


Fig. 2.1 A descendant obtained from type 1 breeding, and its two parents

Proof. Γ_D was produced by a type 1 breeding operation, and therefore contains a 1-cracker. This 1-cracker is preserved regardless of future operations, because type 2 and 3 breeding operations are not permitted where they would break the 1-cracker, type 2 parthenogenesis is only permitted on 2-crackers, and types 1 and 3 parthenogenesis preserve (by the introduction of equivalent) 1-crackers. Therefore, any descendants of Γ_D also contain at least one 1-cracker and are themselves bridge graphs. $\square$

Type 1 breeding is the easiest method of producing NMNH descendants, as it can be performed on any two parents, and is guaranteed to output a NMNH descendant, specifically a bridge graph.

2.2.1.2 Type 2 Breeding

Recall that when type 2 breeding is performed, an edge is broken in each of two parent graphs, and the broken edges are joined together to form the descendant. This process always outputs a 2-connected graph, and so any Hamiltonian cycle in the descendant must traverse both edges in the 2-cracker introduced by the type 2 breeding operation.

Definition 2.1. If a graph Γ contains an edge e that is never traversed in any Hamiltonian cycles in Γ , we say that e is a *non-Hamiltonian edge* (NH-edge).

Note that, by definition, every edge in a non-Hamiltonian graph is an NH-edge.

Lemma 2.2. Consider a descendant graph Γ_D formed by the type 2 breeding operation $\mathcal{B}_2(\Gamma_1, \Gamma_2, a, b, c, d)$, for some $\Gamma_1 = \langle V_1, E_1 \rangle$, $\Gamma_2 = \langle V_2, E_2 \rangle$, $e_1 := (a, b) \in E_1$ and $e_2 := (c, d) \in E_2$. If either e_1 or e_2 is a non-Hamiltonian edge, then Γ_D is non-Hamiltonian.

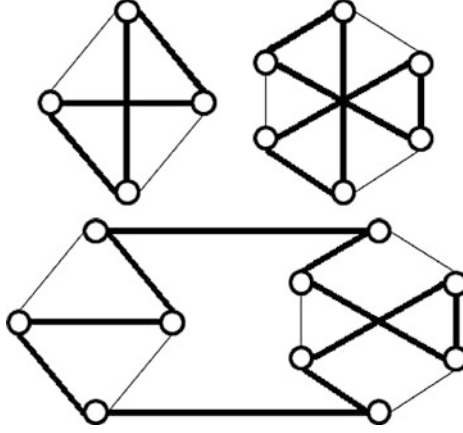


Fig. 2.2 Hamiltonian cycles in two parents, and the corresponding Hamiltonian cycle in one of their descendants

Proof. Any Hamiltonian cycle in Γ_D must be composed from a Hamiltonian cycle in Γ_1 that traverses e_1 , and a Hamiltonian cycle in Γ_2 that traverses e_2 (e.g. see Fig. 2.2). However, at least one of e_1 and e_2 is a non-Hamiltonian edge, and therefore these two Hamiltonian cycles cannot both exist. Therefore, Γ_D has no Hamiltonian cycles. $\square$

Corollary 2.2. *Without loss of generality, if Γ_1 is non-Hamiltonian, then Γ_D is non-Hamiltonian as well.*

Proof. Since Γ_1 is non-Hamiltonian, it is clear that e_1 is an NH-edge. Then the result follows immediately from Lemma 2.2. $\square$

NH-edges can occur naturally in Hamiltonian genes. The smallest such example, on 14 vertices, is displayed in Fig. 2.3. However, NH-edges are more commonly induced by selective breeding, such as in the following example.

Lemma 2.3. *Consider a descendant Γ_D formed by the type 2 parthenogenic operation $\mathcal{P}_2(\Gamma_1, e_1, e_2)$. Then, the parthenogenic bridge (v_1, v_2) in Γ_D constitutes an NH-edge.*

Proof. Consider a Hamiltonian cycle in Γ_D that traverses the parthenogenic bridge. For convenience, assume the Hamiltonian cycle begins on vertex v_2 . Then, once the parthenogenic bridge is crossed, there are two choices of which vertex to visit next, say vertex a or vertex b . If we visit vertex a , we can no longer visit vertex b without revisiting a vertex, and vice versa. This is because it is now impossible to get to the other side of the parthenogenic bridge without closing the cycle. This situation can be seen in Fig. 2.4. Therefore, no Hamiltonian cycles can use the parthenogenic bridge, and it constitutes an NH-edge. $\square$

Another method of inducing an NH-edge is to manually select a set of edges that do not exist simultaneously in any Hamiltonian cycle in a graph.

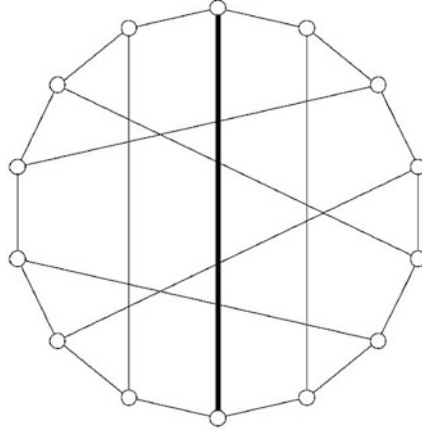


Fig. 2.3 The smallest Hamiltonian gene containing an NH-edge, which is shown here in *bold*

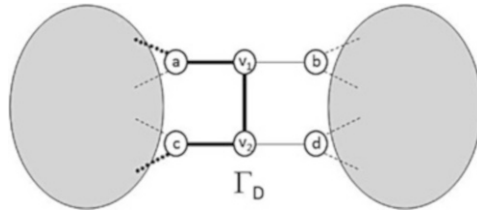


Fig. 2.4 A (short) cycle using a parthenogenic bridge

Definition 2.2. An *NH-edge-set* for a graph is a set of edges that are not all traversed in any single Hamiltonian cycle in that graph.

For example, any three edges adjacent to a single vertex in a cubic graph form an NH-edge-set of cardinality three. It is clear that after performing type 2 breeding on all but one edge in an NH-edge-set, the remaining edge constitutes an NH-edge. An example of a NHNM descendant produced using this method is shown in Fig. 2.5. Of course, an NH-edge is merely a special case of an NH-edge-set, with cardinality 1.

2.2.1.3 Type 3 Breeding

Recall that when type 3 breeding is performed, a vertex is removed from each parent graph, and the three vertices from each parent which were adjacent to the removed vertices are joined together by a 3-cracker. Since this process provides three passages between the two sides of the 3-cracker, any Hamiltonian cycle in the descendant must traverse exactly 2 of the 3 edges. It is important to note that any Hamiltonian cycle must travel through all vertices on one side of the 3-cracker first before crossing the 3-cracker.

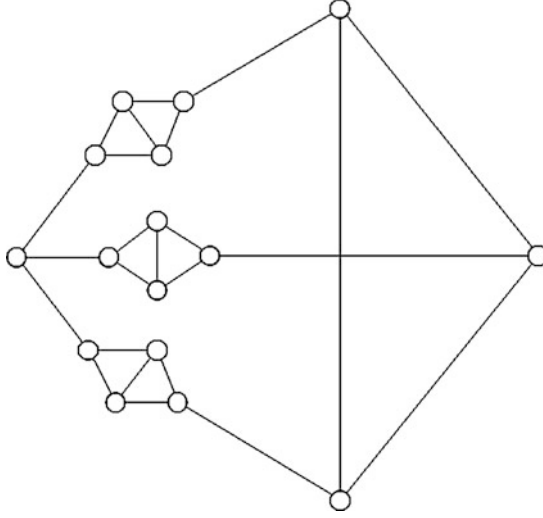


Fig. 2.5 A non-Hamiltonian graph formed by performing type 2 breeding on all edges in a NH-edge-set consisting of three edges adjacent to a vertex in one of the parents

Consider a descendant Γ_D formed by the type 3 breeding operation $\mathcal{B}_3(\Gamma_1, \Gamma_2, v_1, v_2, a, b, c, d, e, f)$, for some $\Gamma_1 = \langle V_1, E_1 \rangle$, $\Gamma_2 = \langle V_2, E_2 \rangle$, $v_1, a, b, c \in V_1$ and $v_2, d, e, f \in V_2$. Then if a Hamiltonian cycle in Γ_D traverses edges (a, d) and (b, e) say, it can be composed from two smaller Hamiltonian cycles, one in each parent. The corresponding Hamiltonian cycle in Γ_1 must contain the path $a - v_1 - b$, and the corresponding Hamiltonian cycle in Γ_2 must contain the path $d - v_2 - e$. This situation is displayed in Fig. 2.6.

Note that the presence of an NH-edge in a parent is not sufficient on its own to permit the construction of a non-Hamiltonian descendant via type 3 breeding, because the other two edges adjacent to the same vertex can be used instead. Also, it is clear that in a Hamiltonian graph, no two NH-edges can be adjacent. Therefore, producing a non-Hamiltonian descendant from Hamiltonian parents with type 3 breeding requires the consideration of *forced edges*, which are edges that exist in every Hamiltonian cycle in a given graph.

Lemma 2.4. *If a cubic Hamiltonian graph contains an NH-edge e , then the four adjacent edges to e are all forced edges.*

Proof. Hamiltonian cycles traverse two edges adjacent to each vertex. Edge e is adjacent to two vertices, and so the two remaining adjacent edges for both vertices must be used in every Hamiltonian cycle. $\square$

Lemma 2.5. *Consider a descendant graph Γ_D formed by the type 3 breeding operation $\mathcal{B}_3(\Gamma_1, \Gamma_2, v_1, v_2, a, b, c, d, e, f)$, for some $\Gamma_1 = \langle V_1, E_1 \rangle$, $\Gamma_2 = \langle V_2, E_2 \rangle$, $v_1, a, b, c \in V_1$ and $v_2, d, e, f \in V_2$. Without loss of generality, if $(a, v_1) \in E_1$ is a forced edge, and $(d, v_2) \in E_2$ is an NH-edge, then Γ_D is non-Hamiltonian.*

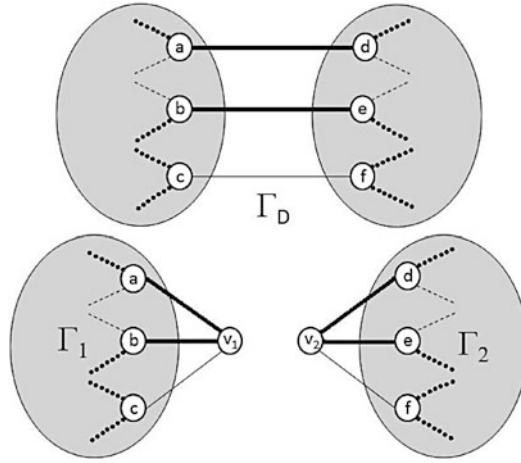


Fig. 2.6 A Hamiltonian cycle in a descendant obtained from type 3 breeding, and the corresponding Hamiltonian cycles in the parents

Proof. Any Hamiltonian cycle in Γ_D must be composed of a Hamiltonian cycles in each of Γ_1 and Γ_2 . Since (a, v_1) is a forced edge, it must be traversed in all Hamiltonian cycles in Γ_1 , and therefore 3-cracker edge (a, d) must be traversed in any corresponding Hamiltonian cycle in Γ_D . However, this implies that edge (d, v_2) must be traversed in any corresponding Hamiltonian cycle in Γ_2 . Edge (d, v_2) is an NH-edge, and so no Hamiltonian cycle traversing this edge can be found in Γ_2 . Therefore, there are no Hamiltonian cycles in Γ_D . $\square$

Corollary 2.3. *Without loss of generality, if Γ_1 is non-Hamiltonian, then Γ_D is also non-Hamiltonian.*

Proof. In Lemma 2.5, non-Hamiltonicity results because it is impossible to avoid traversing an NH-edge. If Γ_1 is non-Hamiltonian, all edges $e \in E_1$ are NH-edges, and it is again impossible to avoid traversing one. Then, by the same arguments used in the proof of Lemma 2.5, Γ_D is non-Hamiltonian. $\square$

Lemma 2.6. *If Γ_1 and Γ_2 are Hamiltonian, and it is not the case that one graph contains an NH-edge and the other contains a forced edge, then any type 3 breeding operation involving Γ_1 and Γ_2 outputs a Hamiltonian graph.*

Proof. If neither graph contains an NH-edge then the result is trivial. Consider the situation where a graph contains NH-edges. As stated earlier, no two NH-edges can be adjacent in a Hamiltonian cubic graph, and therefore any 3-cracker involving an NH-edge is still traversable by a Hamiltonian cycle by using the other two edges. Only if the NH-edge is demanded does non-Hamiltonicity result, which requires the presence of a forced edge in the second parent. $\square$

While forced edges do occur naturally in some genes, inducing them in descendants is a simple task; any type 2 breeding operation produces a descendant containing a 2-cracker, and if the descendant is Hamiltonian, this 2-cracker comprises two

forced edges. Inducing NH-edges also results in the production of forced edges, as demonstrated by Lemma 2.4. It should be noted that performing a type 3 breeding operation cannot introduce an NH-edge or a forced edge if the two parents did not contain any to begin with.

2.2.1.4 Parthenogenic Operations

It is easy to see that parthenogenic operations alone cannot alter the Hamiltonicity of a graph.

Lemma 2.7. *If Γ_D is a descendant produced by any parthenogenic operation, and Γ_1 is its parent graph, then Γ_D and Γ_1 have the same Hamiltonicity.*

Proof. If types 1 or 3 parthenogenesis are used to create Γ_D , then by definition, Γ_1 must be a bridge graph, and the result follows immediately from Corollary 2.1. So we only need to consider type 2 parthenogenesis. Then, Γ_1 contains a 2-cracker, comprising edges, say, (a, b) and (c, d) .

Consider first the case where Γ_1 is Hamiltonian. Clearly, both edges in the 2-cracker are forced edges. Consider the effect of placing a degree-2 vertex inside each of these edges, say vertices v_1 and v_2 . That is, edge (a, b) is replaced with edges (a, v_1) , (v_1, b) , and edge (c, d) is replaced with edges (c, v_2) and (v_2, d) . It is clear that the resulting graph is still Hamiltonian, and that every Hamiltonian cycle in Γ_1 corresponds to a Hamiltonian in the new graph with the 2-cracker edges replaced accordingly. Then, the addition of the edge (v_1, v_2) , which gives Γ_D , cannot destroy Hamiltonicity, and therefore Γ_D is Hamiltonian as well (Fig. 2.7).

Next, consider the case where Γ_1 is non-Hamiltonian. Similar to the argument in the previous case, we can introduce vertices v_1 and v_2 inside the 2-cracker edges without altering the Hamiltonicity. Then, the addition of the edge (v_1, v_2) , which gives Γ_D , also retains non-Hamiltonicity, because from Lemma 2.3, edge (v_1, v_2) is an NH-edge. Therefore Γ_D is non-Hamiltonian. $\square$

2.2.2 Main Theorem and Discussion

The results so far allow us to propose the main theorem of this section.

Theorem 2.1. *If a descendant has at least one non-Hamiltonian parent, then the descendant is also non-Hamiltonian.*

Proof. The descendant can only be obtained from one of the three breeding operations, or one of the three parthenogenic operations. Then, the result follows immediately from Corollaries 2.1–2.3 and Lemma 2.7. $\square$

The strength of Theorem 2.1 is that it ensures that once a non-Hamiltonian descendant has been constructed, it is impossible to “reintroduce” Hamiltonicity in future descendants.

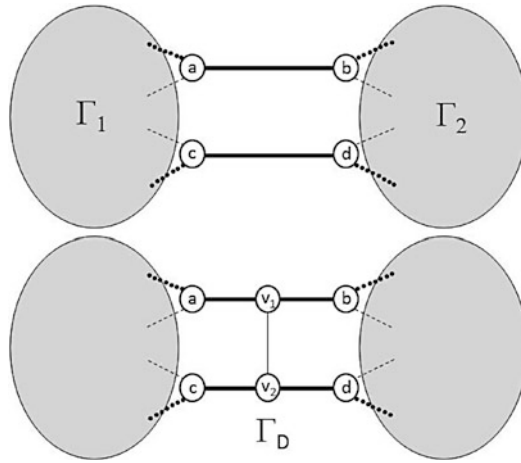


Fig. 2.7 A Hamiltonian cycle in a descendant involving a 2-cracker, and the corresponding cycle in its child containing a parthenogenic bridge

2.2.2.1 Mutant Descendants

Consider a descendant graph Γ_D . From Theorem 1.2, we know that Γ_D has a unique complete family of ancestor genes. Suppose that one of these genes is a mutant. We then refer to Γ_D as a *mutant descendant*.

Lemma 2.8. *If a descendant Γ_D is a mutant descendant, then it is non-Hamiltonian.*

Proof. The proof follows immediately from Theorems 2.1 and 1.2. $\square$

Figure 2.8 shows Tietze's graph, which is the smallest mutant descendant.

2.2.2.2 Relative Cardinalities of Non-Hamiltonian Graphs

We could now partition non-Hamiltonian cubic graphs into three disjoint sets: mutant descendants, NMNH descendants and mutants. However, note that bridge graphs inhabit both of the former two sets. For the sake of simplicity, we will treat bridge graphs as their own category of non-Hamiltonian graphs, and thus we partition non-Hamiltonian cubic graphs into four disjoint sets: cubic bridge graphs, (non-bridge) mutant descendants, (non-bridge) NMNH descendants, and mutants. By convention, when we refer to mutant descendants or NMNH descendants, it will be assumed that such descendants are not bridge graphs. We refer to the set of graphs of size N in each of these four categories as $\mathcal{NH}_N^b$, $\mathcal{NH}_N^{md}$, $\mathcal{NH}_N^{nm}$ and $\mathcal{H}_N^m$ respectively.

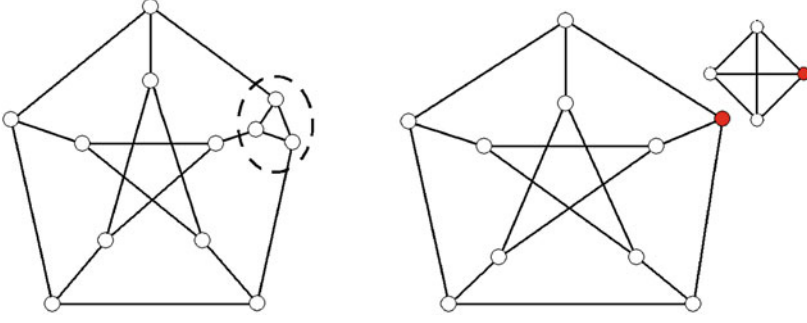


Fig. 2.8 Tietze's graph, and its parents, which include the (mutant) Petersen graph

In [6] it was conjectured that the relative cardinality of cubic bridge graphs of a given size, as a ratio of all non-Hamiltonian graphs of the same size, converges to 1 as the size increases. We display in Table 2.1 the relative cardinalities of the four different types of non-Hamiltonian cubic graphs for different fixed sizes.

N	$\mathcal{NH}_N^b$ (%)	$\mathcal{NH}_N^{md}$ (%)	$\mathcal{NH}_N^{nm}$ (%)	$\mathcal{NH}_N^m$ (%)	# of NH graphs
10	50.00	0	0	50.00	2
12	80.00	20.00	0	0	5
14	82.86	14.29	2.86	0	35
16	84.93	11.87	3.20	0	219
18	86.13	11.28	2.46	0.12	1666
20	87.40	9.53	3.02	0.05	14498
22	88.59	8.65	2.74	0.02	148790

Table 2.1 Relative cardinalities of the four different types of non-Hamiltonian cubic graphs containing up to 20 vertices

Conjecture 2.1. Consider the set $\mathcal{NH}_N^{nb}$ of all non-bridge non-Hamiltonian cubic graphs of size N . In terms of the categories given in Table 2.1, we have $\mathcal{NH}_N^{nb} = \mathcal{NH}_N^{md} \cup \mathcal{NH}_N^{nm} \cup \mathcal{NH}_N^m$. Then,

$$\lim_{N \rightarrow \infty} \frac{\mathcal{NH}_N^{nm}}{\mathcal{NH}_N^{nb}} = 1.$$

That is, NMNH descendants become the dominant set of non-bridge non-Hamiltonian graphs as the size of the graph increases. This conjecture is natural extension of the conjecture given in [6]. Bridge graphs are relatively easy to produce, as they can be constructed from any two parents through the use of type 1 breeding. Similarly, NMNH descendants can be constructed from any three parents,

through the use of type 2 breeding, type 2 parthenogenesis to create a parthenogenic bridge (which constitutes an NH-edge by Lemma 2.3), and then type 2 breeding on the parthenogenic bridge. Mutant descendants, on the other hand, can only be constructed from the extremely rare mutants. It is this line of thought that motivates the conjecture. It should be noted that in Table 2.1 the proportion of mutant descendants is still larger than the proportion of NHNB descendants for $N = 22$. However, that proportion appears to be steadily decreasing, while the proportion of NMNH descendants is staying roughly constant. It will be necessary to consider larger sets of non-Hamiltonian graphs to gain a clearer picture.

2.3 Bicubic Graphs

A bipartite graph is one whose vertices can be divided into two disjoint subsets, such that every edge in the graph is adjacent to exactly one vertex from each subset. Alternatively, we can think of bipartite graphs as being those which permit each vertex to be assigned a binary weight (either $+1$ or -1), such that the weight of every edge (equal to the sum of weights on its two incident vertices) in the graph is zero. Bipartite graphs are a commonly studied class of graphs in literature. In particular, bipartite cubic graphs (also known as bicubic graphs) were the subject of Tutte's Conjecture that all 3-connected bicubic graphs are Hamiltonian, for which a counterexample was first discovered by Horton [3]. The Horton graph is discussed in some detail at the end of this section. A refinement of Tutte's conjecture, with the additional requirement of planarity, was suggested by Barnette [2] and remains an open problem. Planarity is discussed in detail in Sect. 2.4 of the present manuscript.

In this section, we consider the construction of bicubic descendants. Certainly there exist bicubic genes – the 6-vertex gene Γ_6^* is the smallest such example. Bicubic mutants also exist (and constitute counterexamples to Tutte's Conjecture), with George's graph the smallest currently known, on 50 vertices [9]. Whether any or not any smaller bicubic mutants exist is an open problem. The 6-vertex gene and George's graph are displayed in Fig. 2.9.

To prove that a graph is bipartite, we need to demonstrate that we can assign weights of $+1$ or -1 to each vertex in such a way that all edge weights are zero. For cubic graphs (and in fact any k -regular graphs for odd k), this determination can sometimes be aided by the following result.

Lemma 2.9. *Consider a cubic graph $\Gamma = \langle V, E \rangle$. It is impossible to assign weights of $+1$ or -1 to each vertex in such a way that all but one edge has zero weight.*

Proof. The weight on each edge $e \in E$ is equal to the sum of the weights on its two adjacent vertices. That is, if w_i is the weight assigned to vertex $i \in V$, then the weight of an edge $(i, j) \in E$ is $w_i + w_j$. Note that, as vertices can only be assigned a weight of $+1$ or -1 , an edge may only have a weight of $+2$, 0 , or -2 . Since Γ is cubic, each vertex is adjacent to three edges. Therefore, the total sum W of edge weights in Γ is

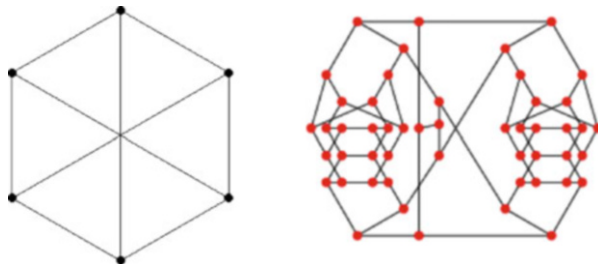


Fig. 2.9 The two smallest known Hamiltonian and non-Hamiltonian bicubic genes, Γ_6^* and George's graph respectively

$$W = 3 \sum_{i=1}^{|V|} w_i.$$

Since all of the vertex weights are integer, it is clear that W must be some factor of 3. However, if we require all but one edge in E to have zero weight, then W must be either 2 or -2 . Neither value is a factor of 3, and therefore such a requirement is not possible. $\square$

2.3.1 Type 1 Breeding

Lemma 2.9 enables us to propose the following theorem.

Theorem 2.2. *There are no bicubic bridge graphs.*

Proof. Recall that a bridge graph can only be constructed through the application of type 1 breeding, or by performing type 1 or type 3 parthenogenesis. In the latter two cases it is clear that the resultant graph will not be bipartite, as both the parthenogenic diamond and parthenogenic triangle contain odd cycles. So we can focus solely on bridge graphs containing irreducible 1-crackers. Then, assume that there exists a bicubic bridge graph $\Gamma_D = \langle V_D, E_D \rangle$, with an irreducible 1-cracker comprising an edge (v_1, v_2) . Then v_1 must be adjacent to two other vertices, say a and b , and likewise v_2 must be adjacent to two other vertices, say c and d . Such a graph has two parents $\Gamma_1 = \langle V_1, E_1 \rangle$ and $\Gamma_2 = \langle V_2, E_2 \rangle$ that we can obtain by use of the type 1 inverse breeding operation $\mathcal{B}^{-1}(\Gamma_D, (v_1, v_2))$. In the first parent it is clear that there exists an edge (a, b) , and likewise in the second parent there exists an edge (c, d) (Fig. 2.10).

Since Γ_D is bipartite, we can assign a weight of $+1$ or -1 to each vertex $v \in V_D$ in such a way that the weight of every edge is zero. Because they are both adjacent to vertex v_1 , it is clear that the weight assigned to vertices a and b must be the same. Likewise, the weight assigned to vertices c and d must also be the same (but opposite to that assigned to a and b). Every edge $e \in E_D$ other than (a, v_1) , (b, v_1) ,

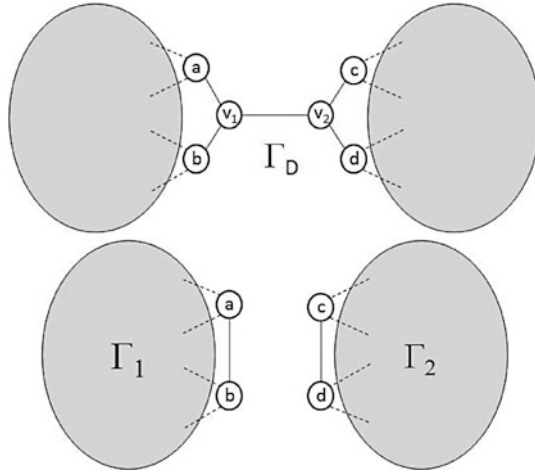


Fig. 2.10 A bicubic bridge graph, and its two parents respectively

(v_1, v_2) , (c, v_2) , (d, v_2) exists in either E_1 or E_2 . The only edges that appear in E_1 and E_2 which are not in E_D are the edges (a, b) and (c, d) . Therefore, it is possible for us to assign weights to the vertices in V_1 and V_2 in such a way that every edge in E_1 and E_2 has zero weight, other than edges $(a, b) \in E_1$ and $(c, d) \in E_2$. However, by Lemma 2.9, this is impossible, and therefore the initial assumption that there exists a bicubic bridge graph must be false. $\square$

2.3.2 Type 2 Breeding

In the proof of Theorem 2.2, it became clear that the majority of structure in a bipartite descendant must be inherited from its parents. The same theme is followed in the following three propositions, which will be used in the proof of the main theorem of this section.

Proposition 2.1. Consider two cubic graphs $\Gamma_1 = \langle V_1, E_1 \rangle$ and $\Gamma_2 = \langle V_2, E_2 \rangle$, and a descendant $\Gamma_D = \langle V_D, E_D \rangle$ obtained by performing the type 2 breeding operation $\mathcal{B}_2(\Gamma_1, \Gamma_2, a, b, c, d)$, where $(a, b) \in E_1$ and $(c, d) \in E_2$. Then, (Figs. 2.11)

- (1) if Γ_1 and Γ_2 are bipartite, Γ_D is also bipartite.
- (2) if at least one of Γ_1 and Γ_2 is non-bipartite, Γ_D is non-bipartite.

Proof. Consider first the case where Γ_1 and Γ_2 are bipartite. Then, it is possible to assign weights to the vertices in V_1 and V_2 in such a way that all edge weights in Γ_1 and Γ_2 are zero. It is clear that a must be assigned the opposite weight to

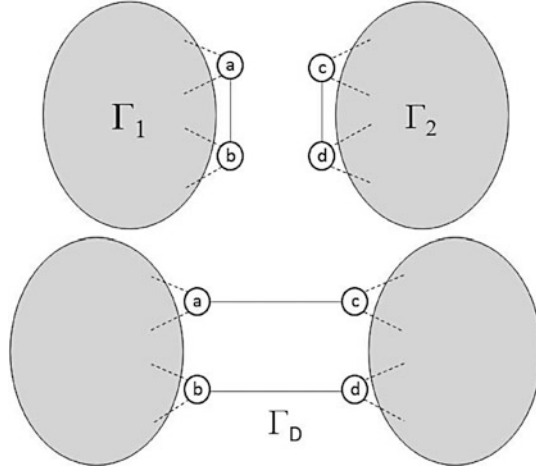


Fig. 2.11 Γ_1 , Γ_2 and Γ_D as described in Proposition 2.1

b , and likewise c must be assigned the opposite weight to d . It is then possible to additionally demand that the weight assigned to a be the opposite to the weight assigned to c .

Note that $V_D = V_1 \cup V_2$. By assigning the same weights to vertices in V_D as their corresponding vertices in V_1 and V_2 , all edge weights in Γ_D are zero, and therefore Γ_D is bipartite. This completes the proof of (1).

Next, consider the case where at least one of Γ_1 and Γ_2 are non-bipartite. Without loss of generality, assume that Γ_1 is non-bipartite. Then, assume that Γ_D is bipartite. Every edge $e \in E_1$ also exists in E_D except (a,b) . Now, it is possible to assign weights to every vertex in V_D such that every edge in E_D has weight zero. Then, every vertex $v \in V_1$ can be assigned the same weight as the corresponding vertex in V_D , and subsequently all edges in E_1 have weight zero except for edge (a,b) . From Lemma 2.9, this is impossible, and therefore the assumption that Γ_D is bipartite must be false. This completes the proof of (2). $\square$

2.3.3 Type 2 Parthenogenesis

Proposition 2.2. Consider a cubic graph $\Gamma = \langle V, E \rangle$ containing a 2-cracker $C_2 = \{(a,b), (c,d)\}$, and a descendant $\Gamma_D = \langle V_D, E_D \rangle$ obtained by performing the type 2 parthenogenic operation $\mathcal{P}_2(\Gamma, (a,b), (c,d))$. Then, (Fig. 2.12)

- (1) if Γ is bipartite, Γ_D is also bipartite.
- (2) if Γ is non-bipartite, Γ_D is also non-bipartite.

Proof. Consider first the case where Γ is bipartite. Then it is possible to assign weights to all vertices $v \in V$ such that every edge in E has weight zero. It is clear

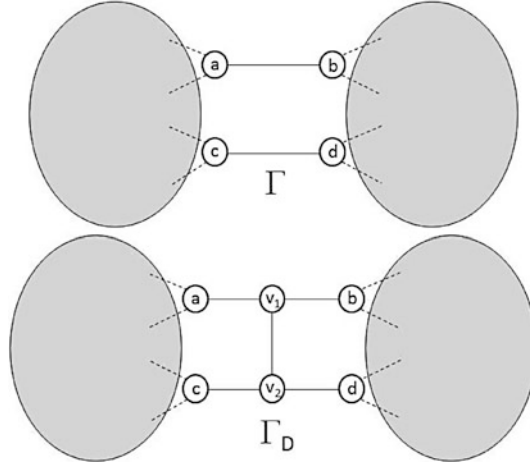


Fig. 2.12 Γ and Γ_D as described in Proposition 2.2

that a must be assigned the opposite weight to b , and likewise c must be assigned the opposite weight to d . Since C_2 is a 2-cracker, graph Γ must have been obtained from type 2 breeding. Then, from Proposition 2.1, it is clear that its two parents are also bipartite. Therefore, it is possible to request that the weight assigned to a is the opposite weight as that assigned to c , and that the weight assigned to b is the opposite weight as that assigned to d .

Since C_2 is a 2-cracker, the vertices in Γ can be divided into two subsets, V_1 and V_2 , such that the only edges incident on vertices from both subsets are (a,b) and (c,d) . Likewise, we can divide the vertices in Γ_D into three sets, $V_{D1} = V_1$, $V_{D2} = V_2$ and $V_{D3} = \{v_1, v_2\}$. Then, we assign weights to the vertices $v_d \in V_{D1}$ in Γ_D , in such a way that they are assigned the same weights as vertices $v \in V_1$ in Γ . Similarly, we can assign weights to the vertices $v_d \in V_{D2}$ in Γ_D , in such a way that they are assigned the *opposite* weights as vertices $v \in V_2$ in Γ . Therefore, vertices $a, b \in V_D$ will both be assigned the same weight, and vertices $c, d \in V_D$ will also be assigned the same weight, but the opposite of that assigned to a and b . Then, v_1 is assigned the same weight as c and d , and v_2 is assigned the same weight as a and b . It is clear that this assignment of weights induces all edge weights of zero in Γ_D , and therefore Γ_D is bipartite. This completes the proof of (1).

Next, consider the case where Γ is non-bipartite, and assume that Γ_D is bipartite. Then, we can assign weights to the vertices in V_D in order to obtain a zero weight for every edge in E_D . Recall the vertex divisions $V_1, V_2 \subset V$ and $V_{D1}, V_{D2}, V_{D3} \subset V_D$ used in the proof of (1). We can then assign weights to the vertices $v \in V_1$ in Γ , in such a way that they are assigned the same weight as vertices $v_d \in V_{D1}$ were. Similarly, we can assign weight to vertices $v \in V_2$ in Γ , in such a way that they are assigned the *opposite* weight as vertices $v_d \in V_{D2}$ were. It is clear that this assignment of weights induces a zero weight for every edge in E . This is a contradiction, as Γ is non-bipartite, and therefore the assumption that Γ_D is bipartite must be false. This completes the proof of (2). $\square$

2.3.4 Type 3 Breeding

Proposition 2.3. Consider two cubic graphs $\Gamma_1 = \langle V_1, E_1 \rangle$ and $\Gamma_2 = \langle V_2, E_2 \rangle$, and a descendant $\Gamma_D = \langle V_D, E_D \rangle$ obtained by performing the type 3 breeding operation $\mathcal{B}_3(\Gamma_1, \Gamma_2, v_1, v_2, a, b, c, d, e, f)$, where $v_1, a, b, c \in V_1$, $v_2, d, e, f \in V_2$, $(a, v_1), (b, v_1), (c, v_1) \in E_1$, and $(d, v_2), (e, v_2), (f, v_2) \in E_2$. Then, (Fig. 2.13)

- (1) if Γ_1 and Γ_2 are bipartite, Γ_D is also bipartite.
- (2) if at least one of Γ_1 and Γ_2 is non-bipartite, Γ_D is not bipartite.

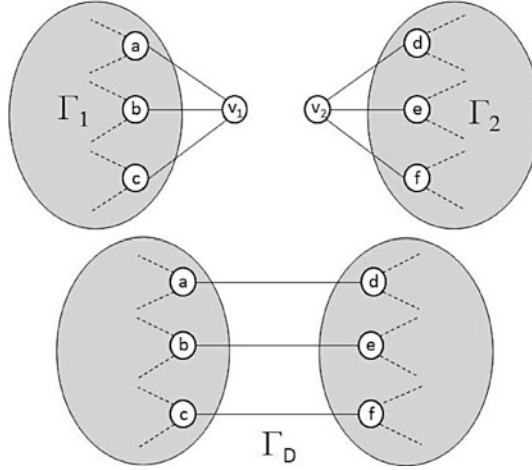


Fig. 2.13 Γ_1 , Γ_2 and Γ_D as described in Proposition 2.3

Proof. Consider first the case where both Γ_1 and Γ_2 are bipartite. Then, it is possible to assign weights to the vertices in V_1 and V_2 in such a way that the all edge weights in Γ_1 and Γ_2 are zero. It is clear that the weights assigned to vertices $a, b, c \in V_1$ must all be the same, and likewise that the weights assigned to vertices $d, e, f \in V_2$ must all be the same. Without loss of generality, assume that the weights assigned to vertices a, b and c are the opposite as those assigned to vertices d, e and f . Then, in Γ_D we can assign weights to every vertex such that the weights are the same as that applied in the corresponding vertices in V_1 and V_2 . The only edges in E_D that were not in $E_1 \cup E_2$ are (a, d) , (b, e) and (c, f) , and therefore all other edges $e \in E_D$ have zero weight. Since vertices a, b and c are assigned opposite weight than vertices d, e and f , it is clear that these new edges also have zero weight. Therefore, Γ_D is bipartite. This completes the proof of (1).

Next, consider the case where at least one of Γ_1 and Γ_2 is non-bipartite. Without loss of generality, assume that Γ_1 is non-bipartite. Then, assume that Γ_D is bipartite. Therefore, we can assign weights to the vertices in Γ_D such that all edge weights

are zero. The only vertices in $V_1 \cup V_2$ that are not in V_D are v_1 and v_2 . For all other vertices in $V_1 \cup V_2$, we can assign weights such that the weights are the same as the corresponding vertices in V_D . Then all edges in E_1 and E_2 must have zero weight, except those adjacent to vertices v_1 and v_2 , respectively.

Now, consider the possible weight assignments for vertices a , b and c in Γ_1 , as inherited from Γ_D . There are only two possibilities – either all three are assigned the same weight, or one of the three is assigned the opposite weight to the other two. If all three are assigned the same weight, then we assign v_1 the opposite weight, and consequently Γ_1 has zero weight for all edges. This is a contradiction, and so vertices a , b and c must not all be the same weight. Without loss of generality to Γ_1 being non-bipartite, assume that vertex a has opposite weight to vertices b and c . Then, if we assign v_1 the same weight as vertex a , all edges in E_1 have zero weight except edge (a, v_1) . However, by Lemma 2.9, this is impossible, and therefore the assumption that Γ_D is bipartite must be false. This completes the proof of (2). $\square$

2.3.5 Main Theorem and Discussion

From Theorem 2.2 we see that there are no bicubic bridge graphs. Then, Propositions 2.1–2.3 enable us to propose the main theorem of this section, which considers all non-bridge bicubic graphs.

Theorem 2.3. *A non-bridge descendant Γ is bicubic if and only if all of its ancestor genes are bicubic.*

Proof. Since Γ is non-bridge, no type 1 breeding operations may be used in its construction. Also, since there are no 1-crackers, no type 1 or type 3 parthenogenic operations could have been performed. Therefore, Γ can only be constructed from its ancestor genes through the combined use of type 2 breeding, type 3 breeding, and type 2 parthenogenesis. Consider the ancestor genes of Γ . If any of them are non-bipartite, then from Propositions 2.1–2.3 their descendants are also non-bipartite, and inductively we see that Γ is also non-bipartite. If all ancestor genes are bipartite, then from Propositions 2.1–2.3 their descendants are all bipartite, and inductively we see that Γ is also bipartite. $\square$

We can now see that every bicubic graph is either a gene, or is descended from bipartite genes. This enables us to reduce the study of bicubic graphs to merely the study of bipartite genes. The following discussion considers such a problem, by searching for smaller counterexamples to Tutte’s conjecture than the current known best.

Discussion 2.1 *Tutte’s conjecture was that every 3-connected bicubic graph is Hamiltonian. Horton provided the first counterexample [3], and George’s graph (see Fig. 2.9) is the smallest currently known counterexample [9]. In fact, George’s graph is a mutant. The second-smallest known counterexample, the second Ellingham-Horton graph on 54 vertices, is also a mutant.*

From Lemma 2.8 and Theorem 2.3 we know that breeding any bipartite mutant with any other bipartite genes will output a non-Hamiltonian bipartite graph. If we restrict ourselves to using only type 3 breeding operations, then we obtain 3-connected non-Hamiltonian bipartite graphs, which constitute a counterexample to Tutte's conjecture.² It was shown in Remark 1.3 that there are infinitely many genes, and the Möbius ladder graphs were given as an example of such an infinite family of genes. It is easy to see by their construction that Möbius ladder graphs of size $4m + 2$ for $m = 1, 2, \dots$ are bipartite, and therefore there are infinitely many counterexamples to Tutte's conjecture. In fact, breeding George's graph $\Gamma_G = \langle V_G, E_G \rangle$ with the 6-vertex bicubic gene $\Gamma_6^* = \langle V_6^*, E_6^* \rangle$ with the operation $\mathcal{B}_3(\Gamma_G, \Gamma_6^*, v_1, v_2, a, b, c, d, e, f)$, for any selection of $v_1, a, b, c \in V_G$, $v_2, d, e, f \in V_6^*$, $(a, v_1), (b, v_1), (c, v_1) \in E_G$ and $(d, v_2), (e, v_2), (f, v_2) \in E_6^*$ provides a 54-vertex counterexample. However, all mutant descendant counterexamples contain more vertices than their mutant ancestor gene. Therefore, it is clear that the smallest possible counterexample to Tutte's conjecture must either be a mutant, or must have all Hamiltonian genes. The following proposition eliminates the latter possibility, requiring a manual search of all descendants up to 26 vertices, rather than 48 vertices.

Proposition 2.4. *The smallest possible counterexample to Tutte's conjecture is a mutant.*

Proof. As demonstrated in Sect. 2.2.1.3, the simplest way to generate non-Hamiltonian descendants from Hamiltonian ancestor genes using type 3 breeding is to choose two genes, one containing an NH-edge, and one containing a forced edge. However, an exhaustive search of all bipartite genes up to 26 vertices found that there are no examples of either such gene for this restrictive size. Since it is impossible for type 3 breeding to introduce a new NH-edge or forced edge if the two parents do not themselves contain any, it is therefore impossible to find two such bipartite graphs whose descendant will be smaller than 50 vertices after a type 3 breeding operation is performed. $\square$

Such an exhaustive search to first identify all the bipartite genes up to 26 vertices, and then to check if they contain an NH-edge or a forced edge, is tractable on a modern computer in relatively short time. The same search on graphs up to 48 vertices is not. The number of connected cubic graphs containing up to 26 vertices is 2,220,297,317 [14]. The number of connected cubic graphs containing up to 48 vertices is not currently listed in literature, but Robinson and Wormald [14] gives the number containing up to 40 vertices as 9,155,199,875,057,748,089, or roughly 4×10^9 times as many graphs as those containing up to 26 vertices. Judging by the exponential growth in the number of graphs for larger vertices, by the time we consider 48-vertex cubic graphs it is likely that this number would increase to approximately 5×10^{15} times as many graphs. That we can avoid considering these graphs is an example of the advantage gained from the present theory.

² In fact, any breeding operations other than type 3 breeding prevents 3-connectedness in all subsequence descendants, since cubic crackers cannot be destroyed by further breeding. Therefore, if a 3-connected descendant is desired, only type 3 breeding operations may be used.

Discussion 2.2 *The famous Horton graph [3] is not a mutant, but rather a non-Hamiltonian non-bridge descendant containing four ancestor genes, three of which are a 32-vertex bipartite gene containing a forced edge, and fourth being the 6-vertex gene Γ_6^* . The first Ellingham-Horton graph is also not a mutant, but rather a descendant containing three ancestor genes, two of which are the same 32-vertex bipartite gene containing a forced edge, and an 18-vertex bipartite gene containing an NH-edge-pair (i.e. an NH-edge-set of cardinality 2). In fact, this 18-vertex gene is the smallest such bipartite gene to contain an NH-edge-pair.*

Due to computational restriction, we do not currently know whether any bipartite genes, of sizes 28 or 30, contain an NH-edge or a forced edge. If not, the 32-vertex gene used in the construction of the Horton graph and the first Ellingham-Horton graph are the smallest example of such a bipartite gene.

The Horton graph, the first Ellingham-Horton graph, and their ancestors are displayed in Figs. 2.14–2.15.

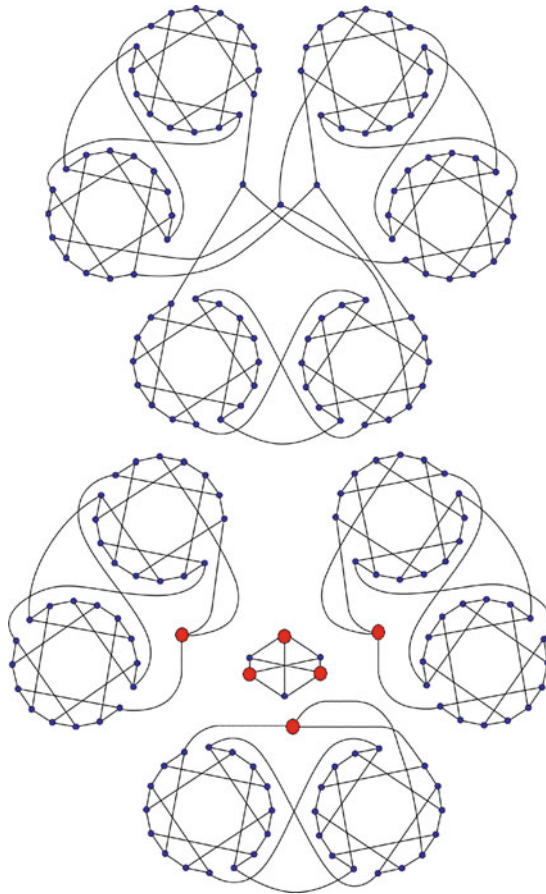


Fig. 2.14 The Horton graph, and its ancestor genes

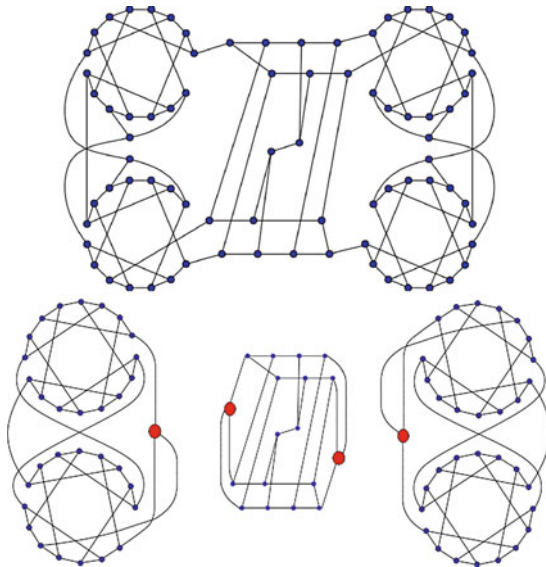


Fig. 2.15 The first Ellingham-Horton graph, and its ancestor genes

2.4 Planarity

A *planar* graph is one that can be visually represented in such a way that all edges intersect only at vertex points. That is, none of the edges cross any other edges. The inheritance of planarity in graphs from their minors has been studied [12] and many other criteria have been given (e.g. see Robertson and Seymour [13]). Planar graphs were the subject of Tait's conjecture that all 3-connected cubic planar graphs are Hamiltonian, which was first disproved by Tutte [16]. Holton and McKay proved that the smallest possible counterexamples contain 38 vertices [11]. There are six such graphs up to isomorphism, known as the Barnette-Bořak-Lederberg graphs. Unlike Tutte's conjecture, where the smallest counterexamples are mutants, the Barnette-Bořak-Lederberg graphs are all descendants. They will be discussed in some detail at the end of this section.

In this section, we demonstrate that planarity is also inherited by descendants through the six breeding operations, and retained by the six inverse operations. Certainly planar genes exist, of which Γ_4^* is the smallest. Planar mutants also exist (and constitute counterexamples to Tait's conjecture), of which the smallest given in literature is the Faulkner-Younger graph on 42 vertices [5].³ These two graphs are displayed in Fig. 2.16.

The upcoming six propositions will be aided by three simple properties of planar graphs that can be easily verified.

³ Note that the Grinberg graph on 42 vertices [10] is an equally small example of a planar mutant.

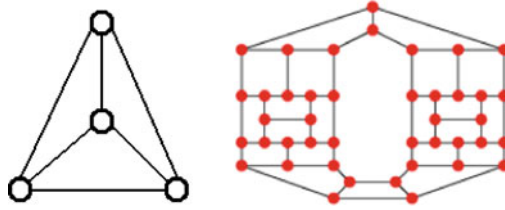


Fig. 2.16 The two smallest planar Hamiltonian and non-Hamiltonian genes, Γ_4^* and the Faulkner-Younger graph 42 respectively

- Pr.1** In a planar graph, if any edge is removed, the resulting graph is planar. Likewise, if any vertex is removed (and the adjacent edges are therefore removed as well), the resulting graph is planar.
- Pr.2** In a planar graph, a vertex may be placed inside any edge (such that if an edge (a,b) has vertex c placed inside it, the edge (a,b) is replaced with two edges (a,c) and (c,b)), and the resulting graph is planar. Equivalently, any degree 2 vertex can be removed in the sense that a single edge is placed between its adjacent vertices, and the resulting graph is planar.
- Pr.3** In a planar graph, any vertex may be replaced with a planar subgraph, in such a way that each edge that was adjacent to the replaced vertex becomes adjacent to an external vertex in the planar subgraph, and in such a way that these edges do not overlap. Then, the resulting graph is planar. Equivalently, any connected subgraph in a planar graph can be replaced with a vertex, such that all external edges to the subgraph become adjacent to the new vertex, and the resulting graph is planar.

2.4.1 Type 1 Breeding

Proposition 2.5. Consider two graphs $\Gamma_1 = \langle V_1, E_1 \rangle$ and $\Gamma_2 = \langle V_2, E_2 \rangle$. Suppose there is an edge $(a,b) \in E_1$ and an edge $(c,d) \in E_2$. Then, consider the descendant graph $\Gamma_D = \langle V_D, E_D \rangle$ (where $V_D = V_1 \cup V_2 \cup \{v_1, v_2\}$) obtained from the type 1 breeding operation $\mathcal{B}_1(\Gamma_1, \Gamma_2, (a,b), (c,d))$. These graphs are displayed in Fig. 2.17.

- (1) If Γ_1 and Γ_2 are both planar, then Γ_D is planar.
- (2) If Γ_D is planar, then Γ_1 and Γ_2 are both planar.

Proof. First, consider the case where Γ_1 and Γ_2 are both planar. Then, by Pr.2 we can add a vertex v_1 inside edge (a,b) , and another vertex v_2 inside edge (c,d) , and the two resulting graphs are planar. Call these two new graphs $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. Then, consider the introduction of a new vertex v_3 and edge (v_1, v_3) introduced to $\bar{\Gamma}_1$ to obtain $\hat{\Gamma}_1$. This is equivalent to replacing vertex v_1 with the planar subgraph $\Gamma_S = \langle \{v_1, v_3\}, \{(v_1, v_3)\} \rangle$ (where edges (a, v_1) and (b, v_1) remain unchanged), and

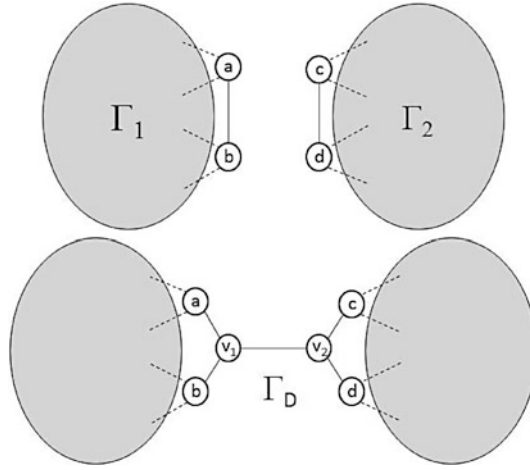


Fig. 2.17 Γ_1 , Γ_2 and Γ_D as described in Proposition 2.5

so by Pr.3, $\tilde{\Gamma}_1$ is planar. Graphs $\tilde{\Gamma}_1$ and $\bar{\Gamma}_2$ are displayed in Fig. 2.18. Then, we can replace v_3 , in $\tilde{\Gamma}_1$, with the graph $\bar{\Gamma}_2$, where edge (v_1, v_3) is replaced by the new edge (v_1, v_2) . The resultant graph is equivalent to Γ_D , and so by Pr.3, Γ_D is planar. This completes the proof of (1).

Next, consider the case where Γ_D is planar. Then by Pr.1 we can delete the 1-cracker (v_1, v_2) to separate the graph into two planar subgraphs, $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. Then, we can delete both vertices v_1 and v_2 in the sense that edges (a, v_1) and (b, v_1) are replaced with edge (a, b) in Γ_1 , and edges (c, v_2) and (d, v_2) are replaced with edge (c, d) in Γ_2 . The two resulting subgraphs are equivalent to Γ_1 and Γ_2 , and so by Pr.2, they are both planar. This completes the proof of (2). $\square$

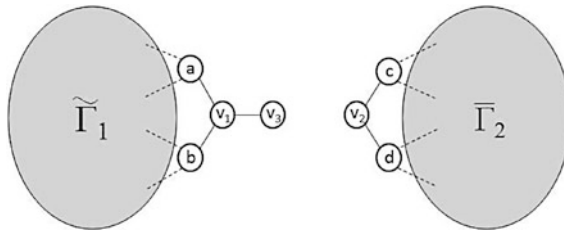


Fig. 2.18 $\tilde{\Gamma}_1$ and $\bar{\Gamma}_2$ as described in the proof of Proposition 2.5

2.4.2 Type 2 Breeding

Proposition 2.6. Consider two graphs $\Gamma_1 = \langle V_1, E_1 \rangle$ and $\Gamma_2 = \langle V_2, E_2 \rangle$. Suppose there is an edge $(a, b) \in E_1$ and an edge $(c, d) \in E_2$. Then, consider the descendant graph $\Gamma_D = \langle V_D, E_D \rangle$ (where $V_D = V_1 \cup V_2$) obtained from the type 2 breeding operation $\mathcal{B}_2(\Gamma_1, \Gamma_2, a, b, c, d)$. These graphs are displayed in Fig. 2.19.

- (1) If Γ_1 and Γ_2 are both planar, then Γ_D is planar.
- (2) If Γ_D is planar, then Γ_1 and Γ_2 are both planar.

Proof. First, consider the case where Γ_1 and Γ_2 are both planar. Then, in Γ_1 , by Pr.2, we can add a vertex, say v_1 , inside edge (a, b) in such a way that the edge is replaced by edges (a, v_1) and (b, v_1) , to obtain a planar graph $\bar{\Gamma}_1$. In Γ_2 , by Pr.1, we can delete edge (c, d) to obtain a planar graph $\tilde{\Gamma}_2$. Graphs $\bar{\Gamma}_1$ and $\tilde{\Gamma}_2$ are displayed in Fig. 2.20. Finally, we can replace vertex v_1 with the graph $\tilde{\Gamma}_2$, where edges (a, v_1) and (b, v_1) are replaced with new edges (a, c) and (b, d) respectively. It is clear that edges (a, c) and (b, d) need not intersect (if they do, $\tilde{\Gamma}_2$ can just be turned upside-down). The resultant graph is equivalent to Γ_D , and so by Pr.3, Γ_D is planar. This completes the proof of (1).

Next, consider the case where Γ_D is planar. Recall that $V_D = V_1 \cup V_2$. Consider the subgraph $\tilde{\Gamma}_2$ containing all vertices in V_2 , and all edges incident only on vertices in V_2 . It is clear that $\tilde{\Gamma}_2$ is connected. Then, by Pr.3, within Γ_D we can replace subgraph $\tilde{\Gamma}_2$ by a vertex v_1 , introducing edges (a, v_1) and (b, v_1) , to obtain a planar graph $\bar{\Gamma}_1$. Then, by removing vertex v_1 , replacing edges (a, v_1) and (b, v_1) with edge (a, b) , we obtain Γ_1 . By Pr.2, Γ_1 is planar. Using an equivalent argument it can easily be seen that Γ_2 is also planar. This completes the proof of (2). $\square$

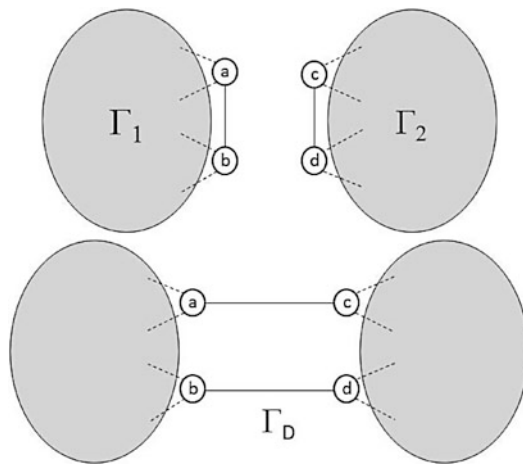


Fig. 2.19 Γ_1 , Γ_2 and Γ_D as described in Proposition 2.6

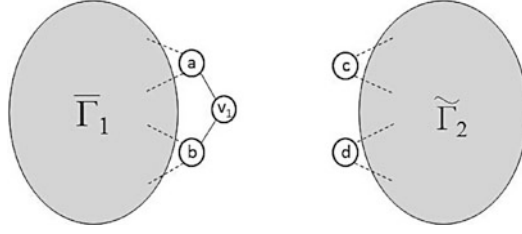


Fig. 2.20 $\bar{\Gamma}_1$ and $\tilde{\Gamma}_2$ as described in the proof of Proposition 2.6

2.4.3 Type 3 Breeding

The following lemma is used in the proof of Proposition 2.7.

Lemma 2.10. *Any 3-cracker can be drawn with nonintersecting edges.*

Proof. Consider the case where a 3-cracker (joining two subgraphs Γ_1 and Γ_2) is drawn with straight edges. If none are intersecting, we are done. Now, consider the alternative case, where at least two of the edges intersect. There are three possibilities:

- (1) All three edges intersect the other two edges.
- (2) Two edges are nonintersecting with respect to each other, but both intersect the third edge.
- (3) Two edges intersect each other, but neither edge intersects the third.

For (1), simply turning one of the subgraphs upside-down gives nonintersecting edges. For (2), the edge that intersects both other edges can be redirected around one subgraph to give nonintersecting edges. For (3), turning one of the subgraphs “upside-down” reduces the situation to (2). $\square$

The above three situations and solutions are displayed in Fig. 2.21.

Proposition 2.7. *Consider two graphs $\Gamma_1 = \langle V_1, E_1 \rangle$ and $\Gamma_2 = \langle V_2, E_2 \rangle$. Suppose there is a vertex $v_1 \in V_1$ adjacent to vertices $a, b, c \in V_1$, and a vertex $v_2 \in V_2$ adjacent to vertices $d, e, f \in V_2$. Then, consider the descendant graph $\Gamma_D = \langle V_D, E_D \rangle$ (where $V_D = \{V_1 \setminus v_1\} \cup \{V_2 \setminus v_2\}$) obtained from the type 3 breeding operation $\mathcal{B}_3(\Gamma_1, \Gamma_2, v_1, v_2, a, b, c, d, e, f)$. These graphs are displayed in Fig. 2.22.*

- (1) *If Γ_1 and Γ_2 are both planar, then Γ_D is planar.*
- (2) *If Γ_D is planar, then Γ_1 and Γ_2 are both planar.*

Proof. First, consider the case where Γ_1 and Γ_2 are both planar. Then, by Pr.1, we can remove vertex v_2 and edges (d, v_2) , (e, v_2) and (f, v_2) from Γ_2 to obtain a planar graph $\tilde{\Gamma}_2$, displayed in Fig. 2.23. Then, we can replace vertex v_1 in Γ_1 with subgraph $\tilde{\Gamma}_2$, in such a way that edges (a, v_1) , (b, v_1) and (c, v_1) are replaced with new

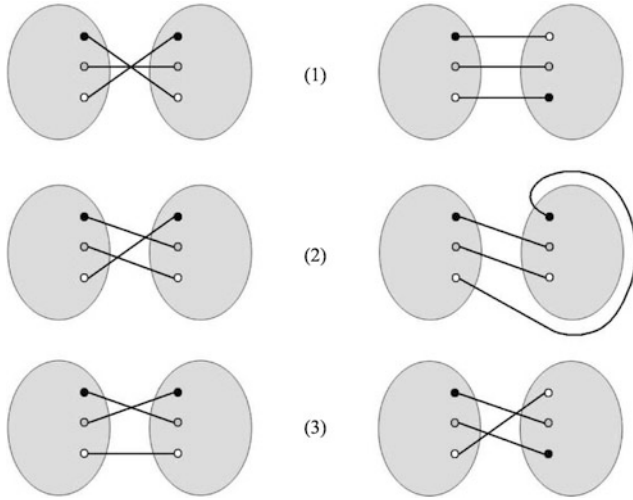


Fig. 2.21 The three possible types of intersecting 3-crackers detailed in Lemma 2.10, and how the graphs can be redrawn to avoid the intersecting edges

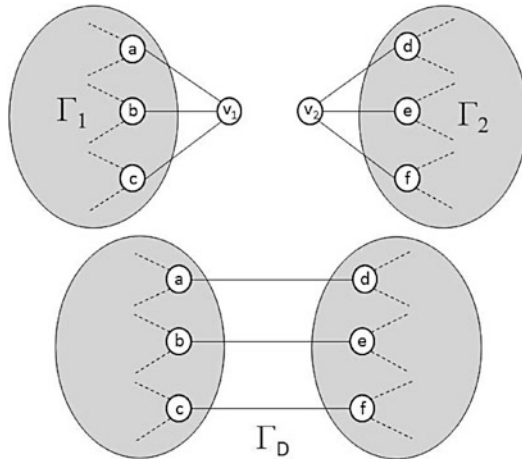


Fig. 2.22 Γ_1 , Γ_2 and Γ_D as described in Proposition 2.7

edges (a, d) , (b, e) and (c, f) . The resultant graph is equivalent to Γ_D , and from Lemma 2.10 the edges need not overlap. Therefore, by Pr.3, Γ_D is planar. This completes the proof of (3).

Next, assume that Γ_D is planar. Recall that $V_D = \{V_1 \setminus v_1\} \cup \{V_2 \setminus v_2\}$. Consider the subgraph $\tilde{\Gamma}_2$ containing all vertices in $\{V_2 \setminus v_2\}$, and all edges adjacent only to vertices in $\{V_2 \setminus v_2\}$. It is clear that $\tilde{\Gamma}_2$ is connected. Then, within Γ_D , we can replace subgraph $\tilde{\Gamma}_2$ by a vertex v_1 , replacing edges (a, d) , (b, e) and (c, f) by edges (a, v_1) ,

(b, v_1) and (c, v_1) respectively. The resultant graph is equivalent to Γ_1 , and so by Pr.3, Γ_1 is planar. Using an equivalent argument it is easy to see that Γ_2 is also planar. This completes the proof of (2). $\square$

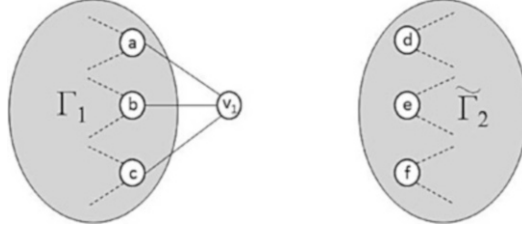


Fig. 2.23 Γ_1 and $\tilde{\Gamma}_2$ as described in the proof of Proposition 2.7

2.4.4 Type 1 Parthenogenesis

Proposition 2.8. Consider a descendant graph $\Gamma_1 = \langle V_1, E_1 \rangle$ containing a 1-cracker comprising edge (a, b) . Then, consider the descendant graph $\Gamma_D = \langle V_D, E_D \rangle$ (where $V_D = V_1 \cup \{v_1, v_2, v_3, v_4\}$) obtained from the type 1 parthenogenic operation $\mathcal{P}_1(\Gamma_1, (a, b))$. These graphs are displayed in Fig. 2.24.

- (1) If Γ_1 is planar, then Γ_D is planar.
- (2) If Γ_D is planar, then Γ_1 is planar.

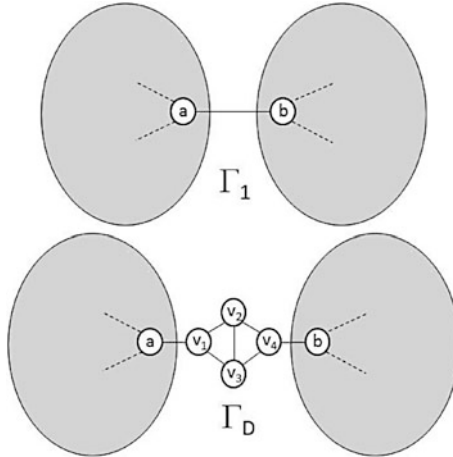


Fig. 2.24 Γ and Γ_D as described in Proposition 2.8

Proof. First, consider the case where Γ_1 is planar. Then, by Pr.2, we can introduce a vertex v inside edge (a, b) , in such a way that edge (a, b) is replaced with new edges (a, v) and (b, v) , to obtain $\bar{\Gamma}_1$, displayed in Fig. 2.25. Then, we can replace vertex v in $\bar{\Gamma}_1$ with the parthenogenic diamond, which is a planar subgraph, in such a way as to replace edges (a, v) and (b, v) with new edges (a, v_1) and (b, v_4) respectively. The resultant graph is equivalent to Γ_D , and therefore by Pr.3, Γ_D is planar. This completes the proof of (1).

Next, assume that Γ_D is planar. We can replace the parthenogenic diamond with a vertex v_1 , introducing edges (a, v_1) and (b, v_1) , to obtain $\bar{\Gamma}_1$. By Pr.3, $\bar{\Gamma}_1$ is planar. Then, by removing vertex v_1 , replacing edges (a, v_1) and (b, v_1) with edge (a, b) , we obtain Γ_1 . By Pr.2, Γ_1 is planar. This completes the proof of (2). $\square$

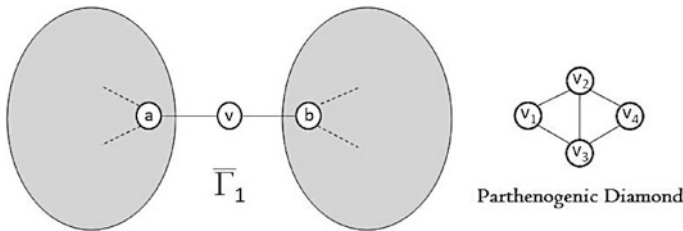


Fig. 2.25 $\bar{\Gamma}_1$ and the parthenogenic diamond as described in the proof of Proposition 2.8

2.4.5 Type 2 Parthenogenesis

Proposition 2.9. Consider a descendant graph $\Gamma_1 = \langle V_1, E_1 \rangle$ containing a 2-cracker comprising edges (a, b) and (c, d) . Then, consider the descendant graph $\Gamma_D = \langle V_D, E_D \rangle$ (where $V_D = V_1 \cup \{v_1, v_2\}$) obtained from the type 2 parthenogenic operation $\mathcal{P}_2(\Gamma_1, (a, b), (c, d))$. These graphs are displayed in Fig. 2.26.

- (1) If Γ_1 is planar, then Γ_D is planar.
- (2) If Γ_D is planar, then Γ_1 is planar.

Proof. First, consider the case where Γ_1 is planar. Then, we can introduce a vertex v_1 inside edge (a, b) , in such a way that edge (a, b) is replaced with new edges (a, v_1) and (b, v_1) , and another vertex v_2 inside edge (c, d) , in such a way that edge (c, d) is replaced with new edges (c, v_2) and (d, v_2) , to obtain $\bar{\Gamma}_1$, displayed in Fig. 2.27. By Pr.2, $\bar{\Gamma}_1$ is planar. Then, since (a, b) and (c, d) form a 2-cracker in Γ_1 , it is clear that an edge (v_1, v_2) may be introduced without intersecting any other edges in $\bar{\Gamma}_1$. The resultant graph is equivalent to Γ_D , and therefore Γ_D is planar. This completes the proof of (1).

Next, assume that Γ_D is planar. By Pr.1, we can remove edge (v_1, v_2) to obtain a planar graph $\bar{\Gamma}_1$. Then, we can remove vertices v_1 and v_2 , such that edges (a, v_1)

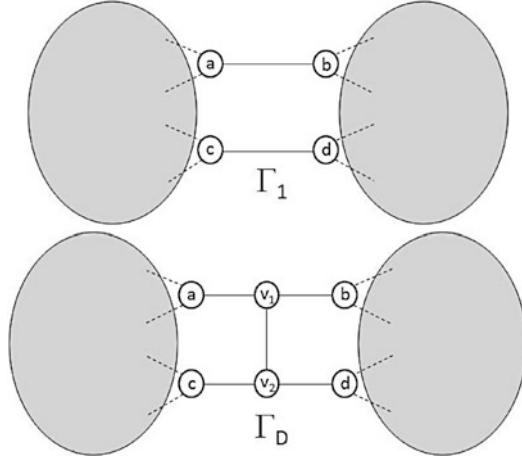


Fig. 2.26 Γ and Γ_D as described in Proposition 2.9

and (b, v_1) are replaced with edge (a, b) , and edges (c, v_2) and (d, v_2) are replaced with edge (c, d) . The resultant graph is equivalent to Γ_1 , and so by Pr.2, Γ_1 is planar. This completes the proof of (2). $\square$

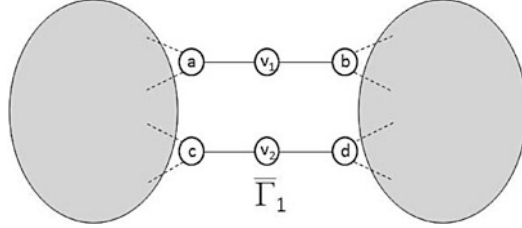


Fig. 2.27 $\bar{\Gamma}_1$ as described in the proof of Proposition 2.9

2.4.6 Type 3 Parthenogenesis

Proposition 2.10. *Consider a descendant graph $\Gamma_1 = \langle V_1, E_1 \rangle$ containing a 1-cracker comprising edge (a, b) . Suppose that vertex a is also adjacent to vertices $c, d \in V_1$. Then, consider the descendant graph $\Gamma_D = \langle V_D, E_D \rangle$ (where $V_D = V_1 \cup \{v_1, v_2\}$) obtained from the type 3 parthenogenic operation $\mathcal{P}_3(\Gamma_1, a)$. These graphs are displayed in Fig. 2.28.*

- (1) If Γ_1 is planar, then Γ_D is planar.
- (2) If Γ_D is planar, then Γ_1 is planar.

Proof. First, consider the case where Γ_1 is planar. Then, we can replace vertex a with a parthenogenic triangle subgraph, which is planar, in such a way that edges (a, c) and (a, d) are replaced by new edges (v_1, c) and (v_2, d) , and edge (a, b) remains unchanged. The resultant graph is equivalent to Γ_D , and so by Pr.3, Γ_D is planar. This completes the proof of (1).

Next, assume that Γ_D is planar. By Pr.3, we can replace the parthenogenic triangle subgraph with a vertex a , replacing edges (a, b) , (v_1, c) and (v_2, d) with edges (a, b) , (a, c) and (a, d) . The resultant graph is equivalent to Γ_1 , and so by Pr.3, Γ_1 is planar. This completes the proof of (2). $\square$

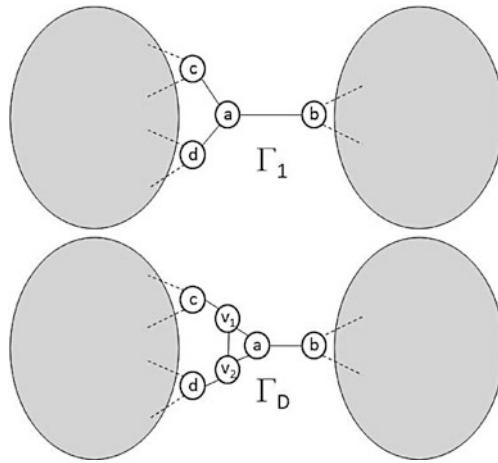


Fig. 2.28 Γ and Γ_D as described in Proposition 2.10

2.4.7 Main Theorem and Discussion

Propositions 2.5–2.10 enable us to give the main theorem of this section.

Theorem 2.4. *A descendant graph Γ_D is planar if and only if all of its ancestor genes are planar.*

Proof. From Theorem 1.2, Γ_D has a unique complete family of ancestor genes. Each of their child graphs must be constructed by one of the six breeding operations. By Propositions 2.5–2.10, these child graphs will all be planar if and only if all of the ancestor genes are planar. By induction, we can continue this argument until we arrive at Γ_D . $\square$

As discussed at the beginning of this section, the smallest counterexamples to Tait's conjecture are the Barnette-Bořak-Lederberg graphs on 38 vertices. There are

six of these graphs, and all six are descendants of the same three Hamiltonian ancestor genes. The only distinction between the six graphs is the ordering used in the type 3 breeding operation.

Two of the ancestor genes, which we call Γ_1 and Γ_2 , are in fact the same graph. As can be seen, the Barnette-Bošak-Lederberg graph displayed in Fig. 2.29 is produced by two type 3 breeding operations. First, the operation $\mathcal{B}_3(\Gamma_1, \Gamma_3, h, a, i, j, k, d, b, c)$ is performed to produce a descendant, say Γ_D , and then the operation $\mathcal{B}_3(\Gamma_2, \Gamma_D, l, e, m, n, o, g, b, f)$ is performed.

This descendant is non-Hamiltonian because two forced edges, one in each of Γ_1 and Γ_2 , are connected with a NH-edge-pair in Γ_3 . The forced edge in Γ_1 is edge (h, k) , and the forced edge in Γ_2 is (l, o) . There are many NH-edge-pairs in Γ_3 , but due to isomorphism, only one needs to be considered. In Fig. 2.29 we select the NH-edge-pair comprising edges (a, c) and (e, f) . Then, once the two type 3 breeding operations are performed, the resultant descendant contains edges (c, k) and (f, o) .

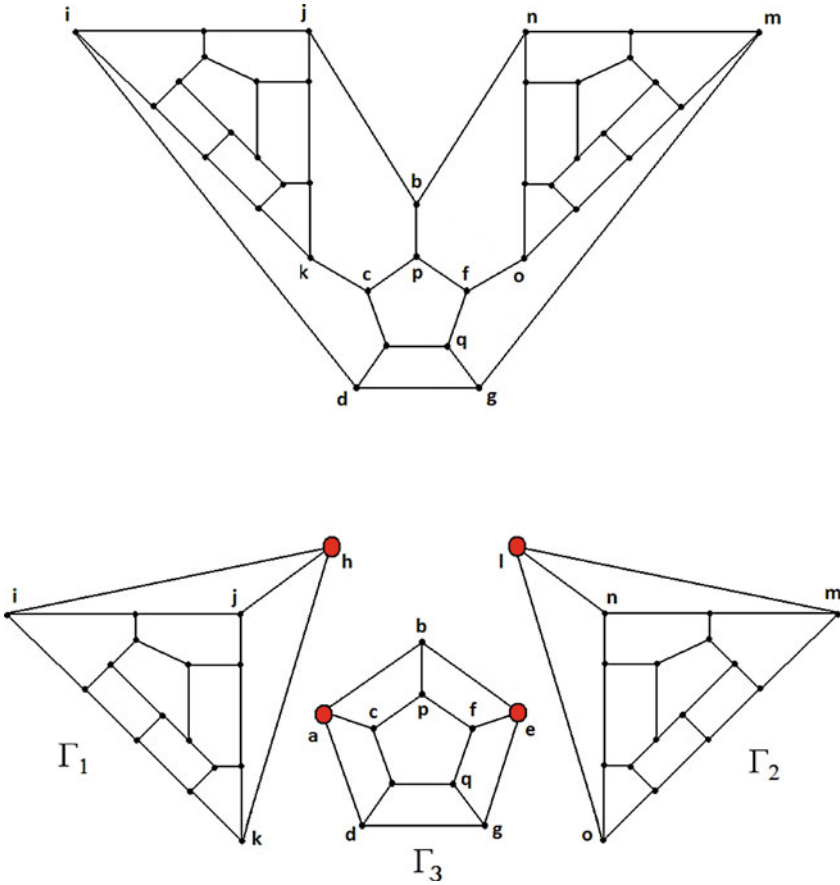


Fig. 2.29 One of the six Barnette-Bošak-Lederberg graph, and its ancestor genes

Therefore, any Hamiltonian cycle in the descendant corresponds, in part, to a Hamiltonian cycle in Γ_3 containing edges (a, c) and (e, f) , which does not exist.

The other five Barnette-Bořak-Lederberg graphs can be produced by altering the ordering as follows:

- First $\mathcal{B}_3(\Gamma_1, \Gamma_3, h, a, i, j, k, b, d, c)$, then $\mathcal{B}_3(\Gamma_2, \Gamma_D, l, e, m, n, o, b, g, f)$.
- First $\mathcal{B}_3(\Gamma_1, \Gamma_3, h, a, i, j, k, d, b, c)$, then $\mathcal{B}_3(\Gamma_2, \Gamma_D, l, e, m, n, o, b, g, f)$.
- First $\mathcal{B}_3(\Gamma_1, \Gamma_3, h, a, i, j, k, b, d, c)$, then $\mathcal{B}_3(\Gamma_2, \Gamma_D, l, f, m, n, o, p, q, e)$.
- First $\mathcal{B}_3(\Gamma_1, \Gamma_3, h, a, i, j, k, d, b, c)$, then $\mathcal{B}_3(\Gamma_2, \Gamma_D, l, f, m, n, o, q, p, e)$.
- First $\mathcal{B}_3(\Gamma_1, \Gamma_3, h, a, i, j, k, d, b, c)$, then $\mathcal{B}_3(\Gamma_2, \Gamma_D, l, f, m, n, o, p, q, e)$.

Note in the latter three graphs, vertex f is removed in Γ_D rather than vertex e . In these instances, the descendant contains edges (c, k) and (e, o) , which also demand the NH-edge-pair be traversed in any corresponding Hamiltonian cycle in Γ_3 .

Obviously there are more than six combinations available, however, due to isomorphism, any other combination produces a graph which is isomorphic to one of the six Barnette-Bořak-Lederberg graphs.

The ancestor genes Γ_1 and Γ_2 are both the same 16-vertex gene, and contain a forced edge. This gene is the only 16-vertex planar gene to contain a forced edge, and no smaller planar genes contain a forced edge. The parent gene Γ_3 is a 10-vertex planar gene that contains an NH-edge-pair. This gene is the only 10-vertex planar gene to contain a NH-edge-pair, and no smaller genes contain an NH-edge-pair.

There are no planar genes with 22 vertices or less that contain an NH-edge, so it is not possible to obtain a descendant with fewer than 38 vertices by breeding Γ_1 with a planar gene containing an NH-edge. It is therefore clear that no 3-connected planar descendants with fewer than 38 vertices are non-Hamiltonian. Of course, the result given in [11] is stronger, as it also eliminates the possibility of planar mutants with fewer than 38 vertices.

2.5 Summary and Conclusions

We can summarise the results of this chapter with the following corollary.

Corollary 2.4. *For a descendant to be:*

- **Hamiltonian:** *It is necessary for all ancestor genes to be Hamiltonian.*
- **Bipartite:** *It is necessary and sufficient for all ancestor genes to be bipartite, unless the descendant is a bridge graph, in which case it is never bipartite.*
- **Planar:** *It is necessary and sufficient for all ancestor genes to be planar.*

Proof. Follows immediately from Lemma 2.8 and Theorems 2.3 and 2.4. $\square$

To determine the bipartiteness or planarity of a descendant, one could, in theory, decompose the descendant into its ancestor genes, and determine the bipartiteness or planarity of these genes instead. Although this would likely represent a saving

in time if the ancestor genes were known in advance, the process of identifying the ancestor genes is an $O(N^4)$ process in general, and therefore such a decomposition is useful only for problems where the best known algorithms are slow, or instances in which the ancestor genes are easy to identify. Of course, all of the results in this chapter are valid for any complete family of ancestors, not only a complete family of ancestor genes. In some cases, identifying a complete family of ancestors may be much more efficient than decomposing entirely into genes, and this may still represent a large saving in time for identifying properties of the original descendant graph.

The results of Sect. 2.2 give rise to a natural decomposition-based heuristic for the Hamiltonian cycle problem. Bridge graphs can be easily detected, so only non-bridge graphs need to be considered. For a given descendant, we can identify its ancestor genes, and then solve the Hamiltonian cycle problem for each of them, using whatever state of the art algorithms are available for solving cubic graphs (e.g. see Eppstein [4] or Baniasadi et al. [1]). Although the total solving time would still be exponential, the exponent would be proportional to the size of the largest ancestor gene, which in some cases might be much smaller than the descendant. For large mutant descendants, this represents a very significant saving in time. However, non-bridge NMNH cubic graphs would remain undetected by this heuristic. How best to address these graphs, and how many of them may be detected by more direct methods, is a topic for future research. It is worth noting that non-bridge non-Hamiltonian cubic graphs (whether mutant descendants or otherwise) are conjectured to be very rare compared to the set of non-Hamiltonian cubic graphs [6], which in turn are known to be very rare compared to the set of all cubic graphs [15].

If the problem of finding *every* Hamiltonian cycle in a cubic graph is considered, a natural decomposition algorithm emerges. First, identify all the ancestor genes, then find every Hamiltonian cycle in each of the ancestor genes. If any 2-crackers exist in the descendant, cycles that do not traverse the corresponding edges in the ancestor genes can be discarded. Finally, consider all combinations of Hamiltonian cycles in the ancestor genes to see which combinations “match up”. That is, which combinations induce the same edge crossings in the descendant’s cubic crackers. Discard the combinations that do not match up, and those that remain constitute all of the Hamiltonian cycles in the descendant.

To date, Hamiltonicity, bipartiteness and planarity are the only graph theoretic properties that have been investigated in the context of descendants and their ancestor genes. The investigation of other such properties is a subject for future research.

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