

Chapter 2

Modeling of Flow and Heat Transfer in Porous Media

Abstract Modeling of flow and heat transfer in rotating porous media is introduced in this chapter. The governing equations are presented and transformed into dimensionless form in order, among others, to identifying conditions for neglecting terms in these equations. The classification of different types of rotating flows in porous media concludes the chapter.

Keywords Porous media modeling · Rotating porous media · Dimensionless forms

2.1 Modeling of Flow in Porous Media

A common equation to all models of flow in porous media is the continuity equation representing the mass balance of the fluid. Regardless of what kind of dynamic model is being used the continuity equation is to be used in conjunction with the latter. The continuity equation for a non-deformable porous medium (the porosity is constant, $\phi = \text{const.}$) is presented in the form

$$\phi \frac{\partial \rho_*}{\partial t_*} + \mathbf{V}_* \cdot \nabla_* \rho_* + \rho_* \nabla_* \cdot \mathbf{V}_* = 0 \quad (2.1)$$

where ϕ is the porosity of the porous medium defined as the ratio between the volume of effective pores to the total volume of the porous matrix (voids + solid). The definition of effective pores is applicable to pores that are not completely isolated from fluid flow, i.e. interconnected pores. In Eq. (2.1) ρ_* is the density of the fluid, $\mathbf{V}_* = u_* \hat{e}_x + v_* \hat{e}_y + w_* \hat{e}_z$ is the filtration velocity vector and u_*, v_*, w_* are its components in the x_*, y_* and z_* direction, respectively, in a Cartesian coordinate system.

When the flow is definitely incompressible the density is identically constant, i.e. $\rho_* = \rho_o = \text{const.}$ and Eq. (2.1) above becomes

$$\nabla_* \cdot \mathbf{V}_* = 0 \quad (2.2)$$

Even when the flow is only approximately incompressible Eq. (2.2) can be used as a very accurate approximation. An example of such an application that is widely used in this book refers to the Boussinesq approximation (Boussinesq 1903) that is acceptable in reference to buoyancy driven flows, and indicates that the density can be regarded as constant in all terms of the governing equations except the buoyancy terms in the momentum equation. Therefore, subject to Boussinesq approximation (Boussinesq 1903) Eq. (2.2) will be used in this book to represent a good approximation of the continuity Eq. (2.1). When relaxation of the latter approximation is applied in particular circumstances we shall revert to the accurate form of the continuity Eq. (2.1).

Dimensionless form of the continuity equation

The two forms of the continuity equation presented above can be presented in a dimensionless form by using a reference value of density say ρ_o , a characteristic velocity u_c , a characteristic length l_c and a characteristic time that can be defined in terms of the latter two in the form $t_c = l_c/u_c M_f$ (The coefficient M_f represents the ratio between the effective heat capacity of the fluid and effective heat capacity of the porous medium, a coefficient that will be defined later. The choice of including this coefficient in the characteristic time is done for later convenience in notation). By definition, any characteristic or reference values are constant. We distinguish between the two only for physical reasons. A reference value typically applies to material properties, which act as system parameters in the equations and may or may not change, while a characteristic value applies to the dependent and independent variables that naturally always change. The specific choice of ρ_o , u_c and l_c will be made in each instance when we deal with a specific problem. For now all we need to remember is that they are constant values. Then we may introduce the dimensionless position vector, the dimensionless filtration velocity, the dimensionless density and the dimensionless ∇ operator (in Cartesian coordinates) as

$$\mathbf{X} = \frac{\mathbf{X}_*}{l_c} = \frac{x_*}{l_c} \hat{\mathbf{e}}_x + \frac{y_*}{l_c} \hat{\mathbf{e}}_y + \frac{z_*}{l_c} \hat{\mathbf{e}}_z = x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y + z \hat{\mathbf{e}}_z, \quad (2.3a)$$

$$\mathbf{V} = \frac{\mathbf{V}_*}{u_c}, \quad \rho = \frac{\rho_*}{\rho_o}, \quad (2.3b)$$

$$\nabla = l_c \nabla_* = l_c \left(\frac{\partial}{\partial x_*} \hat{\mathbf{e}}_x + \frac{\partial}{\partial y_*} \hat{\mathbf{e}}_y + \frac{\partial}{\partial z_*} \hat{\mathbf{e}}_z \right) = \frac{\partial}{\partial x} \hat{\mathbf{e}}_x + \frac{\partial}{\partial y} \hat{\mathbf{e}}_y + \frac{\partial}{\partial z} \hat{\mathbf{e}}_z \quad (2.3c)$$

leading to the dimensionless form of Eq. (2.1)

$$\phi \frac{\partial \rho}{\partial t} + \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} = 0 \quad (2.4)$$

and the dimensionless form of Eq. (2.2)

$$\nabla \cdot \mathbf{V} = 0 \quad (2.5)$$

2.1.1 Darcy Model

The most fundamental model of flow in porous media is the Darcy model. It represents the momentum balance for the fluid and can be presented in the form

$$\mathbf{V}_* = -\frac{K_*}{\mu_*} [\nabla_* p_* - \rho_* g_* \hat{\mathbf{e}}_g] \quad (2.6)$$

where K_* is the permeability of the porous matrix (carrying units of m^2 or darcy, 1 darcy unit = $9.869233 \times 10^{-13} \text{m}^2$), μ_* is the dynamic viscosity of the fluid (carrying units of $\text{Pa} \cdot \text{s} = \text{N} \cdot \text{s}/\text{m}^2$), g_* is the acceleration due to gravity (carrying units of m/s^2), ρ_* is the density of the fluid carrying units of kg/m^3), and $\hat{\mathbf{e}}_g$ is a unit vector in the gravity acceleration direction. The permeability of the porous medium is a property of the matrix and is independent of the fluid flowing through it. It can be regarded as an effective cross sectional area perpendicular to the direction of the flow and hence represents a measure of how permeable the matrix is. The Darcy equation accounts for the lumped effect of the friction forces between the fluid and the solid matrix on the filtration velocity, in addition to the gravitational forces. The forces between the fluid particles within a Representative Elementary Volume are being assumed negligible and therefore neglected. The latter are of the order of magnitude of Darcy number ($Da = K_*/l_c^2$), i.e. $O(Da)$, which for typical porous media has very small values.

Dimensionless form of the Darcy equation

The dimensionless form of the Darcy equation, assuming constant values of permeability and dynamic viscosity, can be obtained by introducing a characteristic pressure difference Δp_c and by using the same characteristic and reference values as used for the continuity equation in the previous section. Again, except for the fact that the characteristic pressure difference that we introduce, Δp_c , is constant we do not need to specify anything else about Δp_c . This will be done later when dealing with specific problems. Therefore, the dimensionless form of the Darcy equation is

$$\mathbf{V} = -N_p \nabla p + Fr \rho \hat{\mathbf{e}}_g \quad (2.7)$$

where two dimensionless groups, a pressure number N_p and a porous media Froude number Fr , emerged in the form

$$N_p = \frac{K_* \Delta p_c}{\mu_* u_c l_c}, \quad Fr = \frac{g_* K_*}{v_* u_c} \quad (2.8)$$

where $v_* = \mu_*/\rho_o$ is the kinematic viscosity of the fluid.

2.1.2 Darcy Equation in a Rotating Frame of Reference and Other Extended Forms

Darcy equation can be extended to include additional body forces, such as centrifugal and Coriolis effects in the case of a rotating porous matrix, or an additional term that accounts for time dynamics of the evolving values of the filtration velocity, i.e. a time derivative term. For the former, rotation effects introduce the centrifugal and Coriolis terms as follows

$$\mathbf{V}_* = -\frac{K_*}{\mu_*} \left[\nabla_* p_* - \rho_* g_* \hat{\mathbf{e}}_g + \rho_* \boldsymbol{\omega}_* \times (\boldsymbol{\omega}_* \times \mathbf{X}_*) + \frac{2\rho_*}{\phi} \boldsymbol{\omega}_* \times \mathbf{V}_* \right] \quad (2.9)$$

where $\boldsymbol{\omega}_*$ is the angular velocity of rotation (carrying units of s^{-1}), ϕ is porosity, and $\mathbf{X}_*$ is the position vector measured from the axis of rotation. The third term in the brackets represents the centrifugal force while the fourth term represents the Coriolis acceleration. Another extension to the Darcy equation is applicable when fast transients or high frequency effects are of interest. Then, the time resolution obtained by assuming a very fast reaction of Darcy flow to changes and therefore the quasi-steady approximation which is inherent in the formulation of the Darcy law is not sufficient and a time derivative of the filtration velocity needs to be incorporated leading to

$$\frac{\rho_*}{\phi} \frac{\partial \mathbf{V}_*}{\partial t_*} + \frac{\mu_*}{K_*} \mathbf{V}_* = -\nabla_* p_* + \rho_* g_* \hat{\mathbf{e}}_g \quad (2.10)$$

Combining (2.9) with (2.10) yields the Darcy equation in a rotating frame of reference extended to include fast transients and high frequency effects.

$$\frac{\rho_*}{\phi} \frac{\partial \mathbf{V}_*}{\partial t_*} + \frac{\mu_*}{K_*} \mathbf{V}_* = -\nabla_* p_* + \rho_* g_* \hat{\mathbf{e}}_g - \rho_* \boldsymbol{\omega}_* \times (\boldsymbol{\omega}_* \times \mathbf{X}_*) - \frac{2\rho_*}{\phi} \boldsymbol{\omega}_* \times \mathbf{V}_* \quad (2.11)$$

Dimensionless forms of the extensions to Darcy equation

Assuming a constant angular velocity of rotation, $\omega_* = \text{const.}$, we can transform Eq. (2.9) into the following dimensionless form

$$\mathbf{V} = -N_p \nabla p + Fr \rho \hat{\mathbf{e}}_g - Cn \rho \hat{\mathbf{e}}_\omega \times (\hat{\mathbf{e}}_\omega \times \mathbf{X}) - \frac{\rho}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \quad (2.12)$$

where two additional dimensionless groups emerged, a centrifugal number Cn and a porous media Ekman number Ek_Δ defined in the form

$$Cn = \frac{K_* \omega_*^2 l_c}{v_* u_c}, \quad Ek_\Delta = \frac{\phi v_*}{2 \omega_* K_*} \quad (2.13)$$

and where $v_* = \mu_*/\rho_o$ is the kinematic viscosity of the fluid, and Eq. (2.11) is transformed into the dimensionless form

$$\frac{Da Re M_f}{\phi} \rho \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} = -N_p \nabla p + Fr \rho \hat{\mathbf{e}}_g - Cn \rho \hat{\mathbf{e}}_\omega \times (\hat{\mathbf{e}}_\omega \times \mathbf{X}) - \frac{\rho}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \quad (2.14)$$

where the additional dimensionless groups that emerged are the Darcy number Da , and the Reynolds number Re

$$Da = \frac{K_*}{l_c^2}, \quad Re = \frac{u_c l_c}{v_*} \quad (2.15)$$

and where the definition of M_f will follow later. Note that despite the fact that it is the porous media filtration velocity that emerged in the definition of the Reynolds number, the corresponding length scale is a macro-level length scale, not the pore-size. However the Reynolds number appears in Eq. (2.14) in a product combination with the Darcy number, bringing therefore the pore-scale effects into account too.

2.2 Modeling of Heat Transfer in Porous Media

Similarly as in fluid flow, the equation governing the heat transfer in porous media is the energy equation averaged over the phases of the porous medium. Fourier law is assumed for the conduction heat fluxes in each phase.

2.2.1 Finite Heat Transfer Between the Phases

Assuming different temperatures associated with the different phases, i.e. T_{s*} for the solid phase temperature and T_{f*} for the fluid phase temperature one obtains the following equations of energy balance

$$(1 - \phi)\rho_{s*}c_{s*}\frac{\partial T_{s*}}{\partial t_*} = (1 - \phi)\tilde{k}_{s*}\nabla_*^2 T_{s*} + h_*(T_{f*} - T_{s*}) \quad (2.16)$$

$$\phi\rho_{f*}c_{pf*}\frac{\partial T_{f*}}{\partial t_*} + \phi\rho_{pf*}c_{pf*}\mathbf{V}_* \cdot \nabla_* T_{f*} = \phi\tilde{k}_{f*}\nabla_*^2 T_{f*} - h_*(T_{f*} - T_{s*}) \quad (2.17)$$

where ρ_{s*} , c_{s*} , and $\tilde{k}_{s*}$ are the density, specific heat, and thermal conductivity of the solid phase material, and ρ_{f*} , c_{pf*} , and $\tilde{k}_{f*}$ are the density, the constant pressure specific heat, and the thermal conductivity of the fluid phase material. The following notation is useful

$$\gamma_{s*} = (1 - \phi)\rho_{s*}c_{s*}, \quad \gamma_{f*} = \phi\rho_{f*}c_{pf*} \quad (2.18)$$

representing the solid phase and fluid phase effective heat capacities, respectively, and

$$k_{s*} = (1 - \phi)\tilde{k}_{s*}, \quad k_{f*} = \phi\tilde{k}_{f*} \quad (2.19)$$

representing the effective thermal conductivities of the solid and fluid phases, respectively. The term $h_*(T_{f*} - T_{s*})$ represents the rate of heat generation per unit volume in the solid phase within the REV due to the heat transferred from the fluid over the fluid-solid interface. The coefficient $h_* > 0$, carrying units of $\text{W m}^{-3} \text{K}^{-1}$, is a macro-level integral heat transfer coefficient for the heat conduction at the fluid-solid interface (averaged over the REV) that is assumed independent of the phases' temperatures and independent of time. Note that this coefficient is conceptually distinct from the convection heat transfer coefficient and is anticipated to depend on the thermal conductivities of both phases as well as on the surface area to volume ratio (specific area) of the medium. Subject to the notation (2.18), (2.19), (2.16) and (2.17) become

$$\gamma_{s*}\frac{\partial T_{s*}}{\partial t_*} = k_{s*}\nabla_*^2 T_{s*} + h_*(T_{f*} - T_{s*}) \quad (2.20)$$

$$\gamma_{f*}\frac{\partial T_{f*}}{\partial t_*} + \gamma_{f*}\mathbf{V}_* \cdot \nabla_* T_{f*} = k_{f*}\nabla_*^2 T_{f*} - h_*(T_{f*} - T_{s*}) \quad (2.21)$$

2.2.2 Thermal Equilibrium Between the Phases

When the temperature difference between the solid and fluid phases is very small, conditions applicable to most circumstances when for example the thermal conductivity of the solid phase is not extremely small, then a sensible approximation would be to set the temperatures of the phases equal to each other, i.e. $T_* = T_{s*} \approx T_{f*}$. These conditions are identified as Local Thermal Equilibrium (LTE or Lotheq), while the former case of distinct phase temperatures is identified as Local Thermal Non Equilibrium (LTNE) or Lack of Local Thermal Equilibrium (LaLotheq). Then the two Eqs. (2.20) and (2.21) can be combined by adding them to yield

$$\gamma_{e*} \frac{\partial T_*}{\partial t_*} + \gamma_{f*} \mathbf{V}_* \cdot \nabla_* T_* = k_{e*} \nabla_*^2 T_* \quad (2.22)$$

where $\gamma_{e*} = \gamma_{s*} + \gamma_{f*}$, $k_{e*} = k_{s*} + k_{f*}$ are the effective heat capacity and effective thermal conductivity of the porous medium, and the interface heat transfer term disappeared as the amount of heat gained by the solid as it transferred from the fluid is equal to the amount of heat lost by the fluid as it transferred to the solid via the fluid-solid interface at each point in space. Dividing Eq. (2.22) by γ_{e*} yields

$$\frac{\partial T_*}{\partial t_*} + M_f \mathbf{V}_* \cdot \nabla_* T_* = \tilde{\alpha}_{e*} \nabla_*^2 T_* \quad (2.23)$$

where

$$\tilde{\alpha}_{e*} = \frac{k_{e*}}{\gamma_{e*}}, \quad M_f = \frac{\gamma_{f*}}{\gamma_{e*}} \quad (2.24)$$

are an effective thermal diffusivity of the porous medium, and the heat capacity ratio, i.e. the ratio between the effective heat capacity of the fluid phase and the effective heat capacity of the porous medium, respectively.

Dimensionless forms of the energy equation for local thermal equilibrium

The derivation of the dimensionless form of the energy Eq. (2.23) follows the definition of the dimensionless temperature

$$T = \frac{(T_* - T_o)}{\Delta T_c} \quad (2.25)$$

where T_o is a reference value of temperature and ΔT_c is a characteristic temperature difference to be explicitly defined for every specific problem. Introducing the definition of the adjusted effective thermal diffusivity $\alpha_{e*} = \tilde{\alpha}_{e*}/M_f = k_{e*}/\gamma_{f*}$ the dimensionless energy equation becomes

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = \frac{1}{Pe} \nabla^2 T \quad (2.26)$$

where Peclet number emerged as an additional dimensionless group, defined in the form

$$Pe = \frac{u_c l_c}{\alpha_{e*}} = Pr Re \quad (2.27)$$

where the Prandtl number Pr is defined by

$$Pr = \frac{v_*}{\alpha_{e*}} \quad (2.28)$$

2.2.3 Equation of State

A relationship between the density, temperature, and pressure (and solute concentration when the fluid is a solution of soluble substances, e.g. salt in water, alcohol in water, etc.) is needed in order to complete the model formulation. If the fluid is compressible such as a gas, in certain circumstances one may use as an approximation the ideal gas equation

$$\rho_* = \frac{1}{R_{g*}} \frac{p_*}{T_*} \quad (2.29)$$

where R_{g*} is the specific gas constant (i.e. the universal gas constant divided by the molecular mass of the gas). For liquids and also for gases (even when Eq. (2.29) applies) one may use a linear approximation relating the density to temperature and pressure, when temperature and pressure differences are not excessively large. Then

$$\rho_* = \rho_o [1 - \beta_{T*}(T_* - T_o) + \beta_{p*}(p_* - p_o)] \quad (2.30)$$

where ρ_o is a reference value of density obtained when the temperature is T_o and the pressure is p_o . The coefficients β_{T*} and β_{p*} are the thermal expansion coefficient (carrying units of K^{-1}) and the pressure compression coefficient (carrying units of Pa^{-1}), respectively, defined in the form

$$\beta_{T*} = -\frac{1}{\rho_*} \frac{\partial \rho_*}{\partial T_*} \approx -\frac{1}{\rho_o} \frac{\partial \rho_*}{\partial T_*} \quad (2.31)$$

$$\beta_{p*} = \frac{1}{\rho_*} \frac{\partial \rho_*}{\partial p_*} \approx \frac{1}{\rho_o} \frac{\partial \rho_*}{\partial p_*} \quad (2.32)$$

It can be observed that by using the ideal gas Eq. (2.29) with (2.31) and (2.32) it leads to the following thermal expansion and pressure compression coefficients: $\beta_{T*} = 1/T_*$ and $\beta_{p*} = 1/p_*$, respectively.

Dimensionless forms of equation of state

The dimensionless form of the linear approximation of the equation of state can be obtained by using the definition of the dimensionless temperature from Eq. (2.25) and the dimensionless pressure in the form $p = (p_* - p_o)/\Delta p_c$. Then Eq. (2.30) becomes

$$\rho = [1 - \beta_T T + \beta_p p] \quad (2.33)$$

where $\rho = \rho_*/\rho_o$ is the dimensionless density, and $\beta_T = \beta_{T*} \Delta T_c$, $\beta_p = \beta_{p*} \Delta p_c$ are the dimensionless thermal expansion and pressure compression coefficients, respectively. Typically, for most fluid flows $\beta_p \ll \beta_T$, especially for incompressible flows, i.e. flows of liquids. Therefore a common approximation of the dimensionless equation of state would be

$$\rho = [1 - \beta_T T] \quad (2.34)$$

When the fluid phase is a solution of a soluble substance in a liquid, such as salt in water, then the density would depend also on the concentration S . Then an extension of the equation of state (2.34) would be required in the form

$$\rho = [1 - \beta_T T + \beta_S S] \quad (2.35)$$

where $S = (S_* - S_o)/\Delta S_c$, and $\beta_S = \beta_{S*} \Delta S_c$ is the dimensionless compression coefficient due to solute concentration, and S_* is the solute concentration. In such cases an additional transport equation for mass (solute) transfer accounting for the molecular diffusion and the convection of the solute within the fluid phase would be required too.

2.3 Natural Convection and Buoyancy in Porous Media

2.3.1 Homogeneous Porous Media

Natural convection is the effect of flow and convection heat transfer due to the existence of density gradients in a body force field (such as gravity or centrifugal force field). As density depends on temperature as demonstrated in the derivation of the equation of state, temperature gradients may create natural convective flows when a body force field is present. What characterizes natural convection is the lack of a known value of characteristic filtration velocity that can be applied upfront in a problem. No characteristic velocity can be specified because the latter is dictated by

the temperature gradients and their resulting buoyancy rather than being known upfront. Therefore a sensible choice of u_c would be $u_c = \alpha_{e*}/l_c$. With this choice of u_c the Froude number Fr , the pressure number N_p , and the centrifugal dimensionless group Cn in Eqs. (2.12) and (2.14) become

$$Fr = \frac{g_* K_* l_c}{v_* \alpha_{e*}}, \quad N_p = \frac{K_* \Delta p_c}{\mu_* \alpha_{e*}}, \quad Cn = \frac{\omega_*^2 l_c^2 K_*}{v_* \alpha_{e*}} \quad (2.36)$$

Without loss of generality for the same reason as for the filtration velocity one can chose the characteristic pressure difference to be such that the pressure number N_p becomes unity, i.e. $\Delta p_c = \mu_* \alpha_{e*}/K_*$ leading to $N_p = 1$. Also the Reynolds number in Eq. (2.14) renders into the reciprocal Prandtl number

$$Re = \frac{\alpha_{e*}}{v_*} = \frac{1}{Pr} \quad (2.37)$$

and the Peclet number in Eqs. (2.27) and (2.26) equals one by definition

$$Pe = \frac{u_c l_c}{\alpha_{e*}} = \frac{\alpha_{e*} l_c}{l_c \alpha_{e*}} = 1 \quad (2.38)$$

One may define the effective Prandtl number in terms of the effective thermal diffusivity $\tilde{\alpha}_{e*}$ (see Eq. (2.24))

$$Pr_e = \frac{v_*}{\tilde{\alpha}_{e*}} = \frac{Pr}{M_f} \quad (2.39)$$

Then the coefficient to the time derivative term in Eq. (2.14) becomes

$$\frac{Da Re M_f}{\phi} = \frac{Da M_f}{\phi Pr} = \frac{Da}{\phi Pr_e} = \frac{1}{Va} \quad (2.40)$$

a new dimensionless group that Straughan (2001) named the Vadasz number (Va), or the Vadasz coefficient named by Straughan (2008) (see also Sheu 2006 and Govender 2010).

An extremely useful approximation that is usually applied to natural convection problems is the Boussinesq approximation (Boussinesq 1903). Boussinesq approximation states that the density can be considered constant, i.e. $\rho_* = \rho_o = \text{const.}$ (and consequently its dimensionless value is $\rho = 1$) everywhere except in the terms associated with body forces, where the variations of density are to be accounted for. According to this approximation the equation of state (2.34) will be substituted instead of ρ in the gravity and centrifugal terms in Eqs. (2.12) and (2.14), but $\rho = 1$ will be substituted everywhere else. Introducing these results from Eqs. (2.36), (2.37), (2.38), (2.39), (2.40) with $N_p = 1$ and applying the Boussinesq

approximation (Boussinesq 1903) one obtains the following equations for natural convection to replace Eqs. (2.5), (2.14), and (2.26)

$$\nabla \cdot \mathbf{V} = 0 \quad (2.41)$$

$$\mathbf{V} = -\nabla p + Fr \rho \hat{\mathbf{e}}_g - Cn \rho \hat{\mathbf{e}}_\omega \times (\hat{\mathbf{e}}_\omega \times \mathbf{X}) - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \quad (2.42)$$

$$\frac{1}{Va} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} = -\nabla p + Fr \rho \hat{\mathbf{e}}_g - Cn \rho \hat{\mathbf{e}}_\omega \times (\hat{\mathbf{e}}_\omega \times \mathbf{X}) - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \quad (2.43)$$

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = \nabla^2 T \quad (2.44)$$

It becomes appealing in the application of the porous media models presented in the previous sections for problems of natural convection where buoyancy effects are to be investigated to introduce some identities that are extremely useful in what follows. These identities are

$$\hat{\mathbf{e}}_g = \nabla (\hat{\mathbf{e}}_g \cdot \mathbf{X}) \quad (2.45)$$

$$\hat{\mathbf{e}}_\omega \times (\hat{\mathbf{e}}_\omega \times \mathbf{X}) = -\nabla \left[\frac{1}{2} (\hat{\mathbf{e}}_\omega \times \mathbf{X}) \cdot (\hat{\mathbf{e}}_\omega \times \mathbf{X}) \right] \quad (2.46)$$

$$\hat{\mathbf{e}}_\omega \times (\hat{\mathbf{e}}_\omega \times \mathbf{X}) = (\hat{\mathbf{e}}_\omega \cdot \mathbf{X}) \hat{\mathbf{e}}_\omega - \mathbf{X} \quad (2.47)$$

and their proof is provided in the Appendix. Substituting these identities into Eqs. (2.42) and (2.43) and presenting the body force terms by using the artificial representation $\rho = \rho - 1 + 1$ that will prove useful shortly, produces

$$\begin{aligned} \mathbf{V} = & -\nabla p + Fr (\rho - 1) \nabla (\hat{\mathbf{e}}_g \cdot \mathbf{X}) + Fr \nabla (\hat{\mathbf{e}}_g \cdot \mathbf{X}) \\ & - Cn (\rho - 1) [(\hat{\mathbf{e}}_\omega \cdot \mathbf{X}) \hat{\mathbf{e}}_\omega - \mathbf{X}] + Cn \nabla \left[\frac{1}{2} (\hat{\mathbf{e}}_\omega \times \mathbf{X}) \cdot (\hat{\mathbf{e}}_\omega \times \mathbf{X}) \right] - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \end{aligned} \quad (2.48)$$

$$\begin{aligned} \frac{1}{Va} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} = & -\nabla p + Fr (\rho - 1) \nabla (\hat{\mathbf{e}}_g \cdot \mathbf{X}) + Fr \nabla (\hat{\mathbf{e}}_g \cdot \mathbf{X}) \\ & - Cn (\rho - 1) [(\hat{\mathbf{e}}_\omega \cdot \mathbf{X}) \hat{\mathbf{e}}_\omega - \mathbf{X}] + Cn \nabla \left[\frac{1}{2} (\hat{\mathbf{e}}_\omega \times \mathbf{X}) \cdot (\hat{\mathbf{e}}_\omega \times \mathbf{X}) \right] - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \end{aligned} \quad (2.49)$$

Introducing $(\rho - 1) = -\beta_T T$ from (2.34) and grouping all gradient terms under a common gradient operator yields

$$\begin{aligned} \mathbf{V} = & -\nabla \left[p - Fr(\hat{\mathbf{e}}_g \cdot \mathbf{X}) - \frac{Cn}{2}(\hat{\mathbf{e}}_\omega \times \mathbf{X}) \cdot (\hat{\mathbf{e}}_\omega \times \mathbf{X}) \right] + Fr\beta_T T \hat{\mathbf{e}}_g - Cn\beta_T T [(\hat{\mathbf{e}}_\omega \cdot \mathbf{X}) \hat{\mathbf{e}}_\omega - \mathbf{X}] \\ & - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \end{aligned} \quad (2.50)$$

$$\begin{aligned} \frac{1}{Va} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} = & -\nabla \left[p - Fr(\hat{\mathbf{e}}_g \cdot \mathbf{X}) - \frac{Cn}{2}(\hat{\mathbf{e}}_\omega \times \mathbf{X}) \cdot (\hat{\mathbf{e}}_\omega \times \mathbf{X}) \right] + Fr\beta_T T \hat{\mathbf{e}}_g \\ & - Cn\beta_T T [(\hat{\mathbf{e}}_\omega \cdot \mathbf{X}) \hat{\mathbf{e}}_\omega - \mathbf{X}] - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \end{aligned} \quad (2.51)$$

The term under the common gradient operator is a reduced pressure p_r , defined as

$$p_r = p - Fr(\hat{\mathbf{e}}_g \cdot \mathbf{X}) - \frac{Cn}{2}(\hat{\mathbf{e}}_\omega \times \mathbf{X}) \cdot (\hat{\mathbf{e}}_\omega \times \mathbf{X}) \quad (2.52)$$

The product of β_T by Fr and Cn yields two new dimensionless groups in the form of the gravity related Rayleigh number and the centrifugal Rayleigh number, respectively in the form

$$Ra_g = Fr\beta_T = \frac{\beta_{T*} \Delta T_c g_* K_* l_c}{v_* \alpha_{e*}} \quad (2.53)$$

$$Ra_\omega = Cn\beta_T = \frac{\beta_{T*} \Delta T_c \omega_*^2 l_c^2 K_*}{v_* \alpha_{e*}} \quad (2.54)$$

Equations (2.52), (2.53) and (2.54) transform Eqs. (2.50) and (2.51) into the following form

$$\mathbf{V} = -\nabla p_r + Ra_g T \hat{\mathbf{e}}_g - Ra_\omega T [(\hat{\mathbf{e}}_\omega \cdot \mathbf{X}) \hat{\mathbf{e}}_\omega - \mathbf{X}] - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \quad (2.55)$$

$$\frac{1}{Va} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} = -\nabla p_r + Ra_g T \hat{\mathbf{e}}_g - Ra_\omega T [(\hat{\mathbf{e}}_\omega \cdot \mathbf{X}) \hat{\mathbf{e}}_\omega - \mathbf{X}] - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \quad (2.56)$$

The particular case when $\hat{\mathbf{e}}_g = -\hat{\mathbf{e}}_z$ and $\hat{\mathbf{e}}_\omega = \hat{\mathbf{e}}_z$ will be selected to analyze later. Subject to this orientation of the gravity and angular velocity of rotation Eqs. (2.55) and (2.56) take the form

$$\mathbf{V} = -\nabla p_r + Ra_g T \hat{\mathbf{e}}_z - Ra_\omega T (x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y) - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_z \times \mathbf{V} \quad (2.57)$$

$$\frac{1}{Va} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} = -\nabla p_r + Ra_g T \hat{\mathbf{e}}_z - Ra_\omega T (x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y) - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_z \times \mathbf{V} \quad (2.58)$$

The vector $\mathbf{r} = (x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y)$ in Eqs. (2.57) and (2.58) represents the perpendicular radius vector from the axis of rotation to any point in the flow domain.

Three dimensionless groups emerged in Eq. (2.57) when fast transients or high frequencies are not of interest. These dimensionless groups control the significance of the different phenomena. Therefore, the value of Ekman number (Ek_Δ) controls the significance of the Coriolis effect and the ratio between the gravity related Rayleigh number (Ra_g) and the centrifugal Rayleigh number (Ra_ω) controls the significance of gravity with respect to centrifugal forces as far as natural convection is concerned. This ratio is $Ra_g/Ra_\omega = g^*/\omega_*^2 l_c^2$. When fast transients or high frequencies are of interest Eq. (2.58) is to be considered. In such a case one additional dimensionless group emerged, the Va number representing the ratio between two characteristic frequencies, i.e. the fluid flow frequency $\omega_{v*} = \phi v_*/K_*$ and the thermal diffusion frequency $\omega_{\alpha*} = \tilde{\alpha}_{e*}/l_c^2$, i.e. $Va = \omega_{v*}/\omega_{\alpha*} = \phi v_* l_c^2 / K_* \tilde{\alpha}_{e*} = \phi Pr_e / Da$, or alternatively the ratio between two time scales, i.e. the thermal diffusion time scale $\tau_{\alpha*} = l_c^2 / \tilde{\alpha}_{e*}$, and the fluid flow time scale $\tau_{v*} = K_*/\phi v_*$, i.e. $Va = \tau_{\alpha*}/\tau_{v*} = \phi v_* l_c^2 / K_* \tilde{\alpha}_{e*} = \phi Pr_e / Da$. In addition to such cases Eq. (2.58) should be used also when the Prandtl number is of the order of magnitude of Darcy number, i.e. $Pr_e = O(Da)$ i.e. a very small number (as $Da < 1$ in most porous media). Such small values of the Prandtl number are typical for liquid metals. In such cases too the time derivative term in Eq. (2.58) should be retained.

Equations (2.41), (2.55) or (2.56), and (2.44) include three distinct mechanisms of coupling. Two of these couplings are linear. The first is the coupling between the pressure terms and the filtration velocity components, i.e., a coupling between Eq. (2.41) and the three components of Eq. (2.55) or (2.56). The second linear coupling is due to the Coriolis acceleration acting on the filtration velocity components in the plane perpendicular to the direction of the angular velocity. The third coupling is non-linear as it involves the gravity or centrifugal buoyancy terms in Eq. (2.55) or (2.56) and the energy Eq. (2.44). As $\mathbf{V}$ in Eq. (2.55) or (2.56) is dependent on temperature (T) due to the buoyancy terms, it follows that the convective term $\mathbf{V} \cdot \nabla T$ in the energy Eq. (2.44) causes the non-linearity. Therefore the coupling between the temperature, T , and the filtration velocity, $\mathbf{V}$, is the only source of non-linearity in the Darcy's formulation of flow and heat transfer in rotating porous media Eq. (2.55), or its extension Eq. (2.56).

2.3.2 Heterogeneous Porous Media

In heterogeneous porous media the permeability and thermal conductivity can be dependent on the space variables, i.e. $K_*(x_*, y_*, z_*)$ and possibly $k_{e*}(x_*, y_*, z_*)$.

In this book only cases when the permeability is allowed to vary in space are being considered. Then, a reference value of permeability K_o is being used in all previous definitions and the dimensionless permeability function is then defined in the form $K(x, y, z) = K_*(x_*, y_*, z_*)/K_o$. Subsequently, following the same derivations one obtains the following set of equations for the heterogeneous porous media

$$\nabla \cdot \mathbf{V} = 0 \quad (2.59)$$

$$\mathbf{V} = K \left[-\nabla p_r + Ra_g T \hat{\mathbf{e}}_g - Ra_\omega T [(\hat{\mathbf{e}}_\omega \cdot \mathbf{X}) \hat{\mathbf{e}}_\omega - \mathbf{X}] - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \right] \quad (2.60)$$

$$\frac{K}{Va} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} = K \left[-\nabla p_r + Ra_g T \hat{\mathbf{e}}_g - Ra_\omega T [(\hat{\mathbf{e}}_\omega \cdot \mathbf{X}) \hat{\mathbf{e}}_\omega - \mathbf{X}] - \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \right] \quad (2.61)$$

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = \nabla^2 T \quad (2.62)$$

Decoupling the equations is difficult without losing generality although the linear couplings can be resolved. Resolving the Coriolis related coupling is particularly useful. In doing so a Cartesian coordinate system is used and without loss of generality one can choose $\hat{\mathbf{e}}_\omega = \hat{\mathbf{e}}_z$. A further choice is made, which reduces the generality of the problem, namely that the gravity acceleration is collinear with the z axis and directed downwards, i.e., $\hat{\mathbf{e}}_g = -\hat{\mathbf{e}}_z$. Some generality is lost, as problems where the direction of rotation and gravity are not collinear will not be represented by the resulting equations. Nevertheless, it is of interest to demonstrate this particular choice of system as it represents a significant number of practical cases. Subject to these choices ($\hat{\mathbf{e}}_\omega = \hat{\mathbf{e}}_z$ and $\hat{\mathbf{e}}_g = -\hat{\mathbf{e}}_z$) Eq. (2.60) can be expressed by extending Eq. (2.57) to heterogeneous media in the following form (with the r dropped from p_r and the Δ dropped from Ek_Δ for convenience)

$$[1 + K Ek^{-1} \hat{\mathbf{e}}_z \times] \mathbf{V} = -K [\nabla p + Ra_\omega T (x \hat{\mathbf{e}}_x + y \hat{\mathbf{e}}_y) - Ra_g T \hat{\mathbf{e}}_z] \quad (2.63)$$

An important observation can be made by presenting Eq. (2.63) explicitly in terms of the three scalar components

$$u - K Ek^{-1} v = -K \left[\frac{\partial p}{\partial x} + Ra_\omega T x \right] \quad (2.64)$$

$$K Ek^{-1} u + v = -K \left[\frac{\partial p}{\partial y} + Ra_\omega T y \right] \quad (2.65)$$

$$w = -K \left[\frac{\partial p}{\partial z} + Ra_g T \right] \quad (2.66)$$

where u, v , and w are the corresponding x, y , and z components of the filtration velocity vector $\mathbf{V}$. It is now observed that the first two Eqs. (2.64) and (2.65) for the horizontal components of $\mathbf{V}$ are “Coriolis coupled”, while the third one is not. Since this type of coupling is linear it is possible to decouple them to obtain

$$\mathbf{V}_H = -\underset{\sim}{\mathbf{E}}_H \cdot [\nabla_H p + Ra_\omega T \mathbf{X}_H] \quad (2.67)$$

where $\mathbf{V}_H = u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_y$ is the horizontal filtration velocity, $\mathbf{X}_H = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$ is the horizontal position vector, $\nabla_H \equiv \partial/\partial x\hat{\mathbf{e}}_x + \partial/\partial y\hat{\mathbf{e}}_y$ is the horizontal gradient operator and $\underset{\sim}{\mathbf{E}}_H$ is the following tensor operator

$$\underset{\sim}{\mathbf{E}}_H = \frac{K}{[1 + Ek^{-2}K^2]} \begin{bmatrix} 1 & Ek^{-1}K \\ -Ek^{-1}K & 1 \end{bmatrix} \quad (2.68)$$

The definition of the tensor operator $\underset{\sim}{\mathbf{E}}_H$ is extended to include the z component of $\mathbf{V}$, thus leading to

$$\mathbf{V} = -\underset{\sim}{\mathbf{E}} \cdot [\nabla p + \mathbf{B} T] \quad (2.69)$$

where $\mathbf{B} = Ra_\omega x\hat{\mathbf{e}}_x + Ra_\omega y\hat{\mathbf{e}}_y - Ra_g\hat{\mathbf{e}}_z$ is the buoyancy coefficient vector and the tensor operator $\underset{\sim}{\mathbf{E}}$ is defined in the form

$$\underset{\sim}{\mathbf{E}} = \frac{K}{[1 + Ek^{-2}K^2]} \begin{bmatrix} 1 & Ek^{-1}K & 0 \\ -Ek^{-1}K & 1 & 0 \\ 0 & 0 & (1 + Ek^{-2}K^2) \end{bmatrix} \quad (2.70)$$

It is interesting to observe from Eqs. (2.69) and (2.70) that the Coriolis effect due to rotation is equivalent to a particular form of an anisotropic porous medium. The anisotropy is represented by the tensor operator $\underset{\sim}{\mathbf{E}}$. It should be pointed out that the analogy between the Coriolis effect and anisotropy was identified by Palm and Tyvand (1984) for the problem of gravity driven thermal convection in a rotating porous layer at marginal stability. It has been shown here and by Vadasz (1997, 2000) that this analogy is more general and applies also for centrifugally driven convection and for isothermal flows in rotating heterogeneous porous media ($Ra_\omega = Ra_g = 0 \rightarrow \mathbf{B} = 0$ in Eq. (2.69)). Subsequently Auriault et al. (2000, 2002) have obtained the same conclusion by using a method of multiple scale expansions.

2.4 Non-Darcy Models

Although for a significantly high number of practical instances Darcy's model in a rotating frame of reference is sufficient for representing the effects of rotation, non-Darcy models have been used as well. Their relevance and limitations are subject to professional discourse (e.g. Nield 1991, 1995, and Vafai and Kim 1990). Nevertheless there are circumstances where there is sufficient consensus that Darcy model falls short in representing accurately the fluid flow in porous media. In order to present the different relevant models, a dimensionless form of a general equation will be used based on a formulation by Hsu and Cheng (1990) as presented also by Nield (1991), but extended to include the centrifugal and Coriolis terms as dictated by the rotating frame of reference. Since the objective is to establish the validity conditions for different models, the basic dimensionless form of the mass balance (continuity equation), and energy equations are presented too. Therefore the mass, momentum and energy balance equations for a heterogeneous porous medium are presented in the form

$$\phi M_f \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \quad (2.71)$$

$$\begin{aligned} \rho K \left[\frac{M_f Re_\Delta}{\phi} \frac{\partial \mathbf{V}}{\partial t} + \frac{Re_\Delta}{\phi^2} (\mathbf{V} \cdot \nabla) \mathbf{V} \right] + \mathbf{V} = K \left[-\nabla p_r + \frac{Da}{\phi} \nabla^2 \mathbf{V} - \frac{F Re_\Delta}{Da^{1/2}} |\mathbf{V}| \mathbf{V} + \right. \\ \left. \frac{Re_\Delta}{Fr_g} (\rho - 1) \nabla (\hat{\mathbf{e}}_g \cdot \mathbf{X}) - \frac{Re_\Delta}{Fr_\omega} (\rho - 1) \hat{\mathbf{e}}_\omega \times (\hat{\mathbf{e}}_\omega \times \mathbf{X}) - \rho \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \right] \end{aligned} \quad (2.72)$$

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = \frac{1}{Pe} \nabla^2 T + \dot{Q} \quad (2.73)$$

where p_r is the dimensionless reduced pressure generalized to include the constant components of the centrifugal and gravity terms as presented in Eq. (2.52), F is the dimensionless Forchheimer coefficient, and $\dot{Q}$ is a heat source term included for the eventuality of dealing with heat transfer problems where heat is generated within the porous medium, e.g. an electric current passing through an electrically conducting solid phase and producing Ohm's heating. In Eqs. (2.71), (2.72) and (2.73) the following dimensionless groups or their combination emerged as the Reynolds number Re , the Darcy number Da , the Darcy-Reynolds number Re_Δ , the Peclet number Pe , the Prandtl number Pr , the gravity related Froude number Fr_g , the Rossby number Ro , the Ekman number Ek , the centrifugal Froude number Fr_ω , and the porous media Ekman number Ek_Δ defined in the form

$$Re = \frac{u_c l_c}{\nu_*}; \quad Da = \frac{K_o}{l_c^2}; \quad Re_\Delta = Re Da = \frac{u_c K_o}{\nu_* l_c}; \quad Pe = \frac{u_c l_c}{\alpha_{e*}} = Pr Re; \quad (2.74)$$

$$Pr = \frac{\nu_*}{\alpha_{e*}}$$

$$Fr_g = \frac{u_c^2}{g_* l_c}; \quad Ro = \frac{u_c}{2\omega_* l_c}; \quad Ek = \frac{\nu_*}{2\omega_* l_c^2} = \frac{Ro}{Re}; \quad Fr_\omega = \left(\frac{u_c}{\omega_* l_c} \right)^2 = 4Ro^2 \quad (2.75)$$

$$Ek_\Delta = \frac{\phi Ek}{Da} = \frac{\phi Ro}{Re Da} = \frac{\phi Ro}{Re_\Delta} = \frac{\phi \nu_*}{2\omega_* K_o} \quad (2.76)$$

$$\frac{Re_\Delta}{Fr_g} = \frac{Re Da}{Fr_g} = \frac{K_o g_*}{\nu_* u_c}; \quad \frac{Re_\Delta}{Fr_\omega} = \frac{Re Da}{Fr_\omega} = \frac{Re Da}{4Ro^2} = \frac{K_o \omega_*^2 l_c}{\nu_* u_c} \quad (2.77)$$

where ω_* is the angular velocity of rotation, ν_* is the kinematic viscosity, and K_o is a reference value of permeability used to scale the dimensional permeability function $K_*(x_*, y_*, z_*)$ of the heterogeneous porous domain. Characteristic values of filtration velocity u_c , length l_c , time $t_c = l_c/u_c M_f$, pressure difference $\mu_* \alpha_{e*}/K_*$, were used to transform the filtration velocity, space variables, time, and pressure into their dimensionless form. The characteristic value of temperature difference is ΔT_c to be explicitly defined in each particular problem, and the density is rendered dimensionless in dividing it by the reference density value ρ_o . As a result, the dimensionless groups presented in Eqs. (2.74)–(2.77) and/or their combination appear as coefficients of the different terms in the dimensionless equations.

The physical interpretation of these dimensionless groups is now being discussed in order to perform an order of magnitude analysis and derive the validity conditions for assessment of the relative significance of each one of the terms in these equations. The Reynolds number Re defined in Eq. (2.74) represents the ratio between the inertial effects and viscous forces (here the viscous forces considered are those representing the shear stress between the fluid and solid phases). It does not appear in this form in the equations as Re is a Reynolds number related to the macroscopic length scale l_c . In the equations the effect of Reynolds number is felt via a combination of micro and macro scales via the microscopic length scale $\sqrt{K_o}$ in addition to the macroscopic one introduced by the Darcy number. The Darcy number Da defined in Eq. (2.74) is the square of the ratio between the microscopic length scale $\sqrt{K_o}$ (because K_o is proportional to the square of the pore size Δ_*) and the macroscopic length scale l_c . The gravity related Froude number Fr_g expresses the ratio between inertial effects and the gravity body force. The Rossby number Ro defined in Eq. (2.75) is the ratio between convective inertial acceleration to the Coriolis force, while Ekman number Ek is a measure of how viscous forces compare to the Coriolis force. The relative significance of the centrifugal body force is defined in Eq. (2.75) by the centrifugal Froude number Fr_ω , representing the ratio between convective inertial forces and the centrifugal body force.

The dimensionless coefficients appearing in Eqs. (2.71), (2.72) and (2.73) are merely combinations of these dimensionless groups. As such Re_A defined in Eq. (2.74) is the Reynolds number related to a microscopic pore size length scale, times the ratio between the microscopic and macroscopic length scales, i.e. $Re_A = (u_c \sqrt{K_o} / \nu_*) \times (\sqrt{K_o} / l_c)$. Equation (2.76) defines the porous media Ekman number related to the microscopic pore size length scale $\sqrt{K_o}$. As the microscopic length scale $\sqrt{K_o}$ is by definition much smaller than the macroscopic length scale l_c it is evident that $Da \ll 1$ for applications of transport phenomena in porous media. Therefore, even for reasonable high values of the macroscopic Reynolds number Re , the value of the microscopic Reynolds number is still much smaller than 1, i.e. $Re_A \ll 1$ (see Eq. (2.74)). This is certainly true for Re numbers of unit order of magnitude. This situation breaks down only for values of Re as large as $Re = O(Da^{-1})$, or for values of the microscopic Reynolds number in the order of magnitude of $u_c \sqrt{K_o} / \nu_* = O(Da^{-1/2})$. Therefore, as long as these conditions are satisfied $Re_A \ll 1$ and the terms including Re_A as their **only coefficient** can be neglected when compared to the coefficient of V , which is 1. Yet, the terms including Re_A in their coefficient as a **combination of dimensionless groups** can not be neglected as for example Re_A / Fr_g is usually of a unit order of magnitude despite the fact that $Re_A \ll 1$. This is possible when $Fr_g \ll 1$ causing the combined group $Re_A / Fr_g = O(1)$. From Eq. (2.72) one can observe that the Brinkman term $(Da/\phi) \nabla^2 V$ is small and insignificant when $Da \ll 1$. As this is always the case, this term is significant only in thin layers adjacent to a rigid boundary (or to a porous medium adjacent to a pure fluid interface), i.e. for length scales in the order of magnitude $O(Da^{1/2})$ - dimensionless, or $O(K_o^{1/2})$ - dimensional. This obvious conclusion is consistent with Nield (1983, 1991). Some studies discuss the case of sparsely packed porous media when the porosity can get closer to $\phi \sim 1$. Then, substantially higher permeabilities are anticipated and one may argue that then $Da = O(1)$. This scenario is not realistic as it violates the very basic assumption made when adopting the continuum approach for a porous medium, i.e. that the size of the REV is substantially smaller than the macroscopic length scale for the statistical averages to be meaningful. When the porosity gets closer to $\phi \sim 1$ in order to account for sparsely packed porous media the latter condition is violated and all derived porous media models are then void. New and different models are then needed to be developed, tested and validated experimentally in order to replace the ones based on the porous media approach.

Of particular interest is the question related to when the inertial terms in Eq. (2.72) are significant and which one of the different inertial terms is more significant than the others. Although the centrifugal and Coriolis terms can be regarded as inertial terms they are not being considered as such for the purpose of the present analysis. There are therefore three inertial terms, i.e. the convective inertial term $(V \cdot \nabla)V$, the Forchheimer term $|V|V$, and the local time derivative term $\partial V / \partial t$. It is clear that a necessary condition for all these terms to be neglected is that $Re_A \ll 1$. When $Re_A = O(1)$ these terms are no insignificant anymore.

Nevertheless, even for $Re_A = O(1)$, when comparing the convective inertial term to the Forchheimer term one obtains the ratio between these terms as the ratio between their coefficients. Assuming $F = O(1)$ and that $\phi = O(1)$ this ratio becomes $[(\mathbf{V} \cdot \nabla)\mathbf{V} \text{ term}]/[|\mathbf{V}|\mathbf{V} \text{ term}] = O(Da^{1/2}) < 1$. Therefore, since $Da < 1$ for the continuum approach to be valid, the convective inertial term is much smaller than the Forchheimer term, irrespective of the value of Re_A . If $Re_A < 1$ both terms are insignificantly small compared to the Darcy term $\mathbf{V}$ (third term on the left hand side of Eq. (2.72)). If $Re_A = O(1)$ then the convective inertial term can be neglected when compared to the Forchheimer term because $Da^{1/2} < 1$. This conclusion is in agreement with Nield (1991) although the arguments used are different. When steady state solutions are sought, the time derivative terms can be dropped out from Eqs. (2.71), (2.72) and (2.73). However, for time dependent solutions a comparison between the time derivative terms in Eqs. (2.71), (2.72) and (2.73) is needed. To perform such a comparison the equation of state relating the density to pressure and/or temperature is required. For incompressible fluids or compressible fluids flowing at low Mach numbers and/or subjected to moderate temperature variations, a linear relationship between density, pressure and temperature as presented in Eq. (2.33) is an adequate approximation. Since in most circumstances $\beta_p < \beta_T$ the pressure related term in Eq. (2.33) can be neglected, leading to the final form of the equation of state as Eq. (2.34). Furthermore, by using the Boussinesq approximation (Boussinesq 1903) the density may be assumed constant $\rho = 1$ ($\rho_* = \rho_o$) everywhere except when it appears in a body force term in the momentum Eq. (2.72). This can be easily verified to be true for moderate temperature variations, i.e. for $\beta_T < 1$, by introducing Eq. (2.34) into (2.71), and (2.72) and performing an order of magnitude analysis. Therefore, assuming the Boussinesq approximation, $\beta_T < 1$ and $Da < 1$ Eqs. (2.71), and (2.72) take the form

$$\nabla \cdot \mathbf{V} = 0 \quad (2.78)$$

$$K \frac{M_f Re_A}{\phi} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} = K \left[-\nabla p_r - \frac{F Re_A}{Da^{1/2}} |\mathbf{V}| \mathbf{V} - \frac{Re_A \beta_T}{Fr_g} T \nabla (\hat{\mathbf{e}}_g \cdot \mathbf{X}) + \frac{Re_A \beta_T}{Fr_\omega} T \hat{\mathbf{e}}_\omega \times (\hat{\mathbf{e}}_\omega \times \mathbf{X}) - \frac{1}{Ek_A} \hat{\mathbf{e}}_\omega \times \mathbf{V} \right] \quad (2.79)$$

Therefore the time derivatives appear in the momentum Eq. (2.79) and the energy Eq. (2.73) involving two time scales, their ratio being the ratio of their coefficients in Eqs. (2.79) and (2.73) evaluated in the following form (assuming $K = O(1)$, $\phi = O(1)$, and $M_f = O(1)$) (momentum time scale)/(energy time scale) = Re_A . This result shows that the flow transients decay much faster, $O(Re_A^{-1})$, than the energy transients and can therefore be neglected as long as $Re_A < 1$. However, when $Re_A = O(1)$ both time derivatives should be retained. Of particular interest is the set of equations that apply to gravity or centrifugally driven natural convection in homogeneous porous media. For natural convection the characteristic filtration

velocity is $u_c = \alpha_{e*}/l_c$ leading to Eqs. (2.37), (2.38), and (2.40), i.e. $Re = 1/Pr$, $Pe = 1$, and $M_f Re_\Delta / \phi = M_f Da Re / \phi = Da / \phi Pr_e = 1/Va$. In addition, using this choice of characteristic filtration velocity the following dimensionless groups in Eq. (2.79) change form $Re_\Delta = Da/Pr$, the gravity related Rayleigh number $Re_\Delta \beta_T / Fr_g = Ra_g = \beta_* \Delta T_c g_* K_o l_c / \nu_* \alpha_{e*}$, the centrifugal Rayleigh number $Re_\Delta \beta_T / Fr_\omega = Ra_\omega = \beta_* \Delta T_c \omega_*^2 l_c^2 K_o / \nu_* \alpha_{e*}$, and $Re_\Delta / Da^{1/2} = Da^{1/2} / Pr$. For homogenous media $K = 1$ and Eqs. (2.79) and (2.73) take the form

$$\frac{1}{Va} \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} = - \left[\nabla p_r + \frac{F Da^{1/2}}{Pr} |\mathbf{V}| \mathbf{V} + Ra_g T \nabla (\hat{\mathbf{e}}_g \cdot \mathbf{X}) - Ra_\omega T \hat{\mathbf{e}}_\omega \times (\hat{\mathbf{e}}_\omega \times \mathbf{X}) + \frac{1}{Ek_\Delta} \hat{\mathbf{e}}_\omega \times \mathbf{V} \right] \quad (2.80)$$

$$\frac{\partial T}{\partial t} + \mathbf{V} \cdot \nabla T = \nabla^2 T + \dot{Q} \quad (2.81)$$

From these equations it is evident that for very small values of the Prandtl number, i.e. $Pr_e = O(Da)$ such that $Va = O(1)$, which is typical only for liquid metals, the time derivative term in Eq. (2.80) should be retained. This will occur in applications linked to solidification of binary alloys where the mushy layer at the interface between the liquid metal and the solidified material is considered a porous medium. However, even when $Da < < Pr_e$ such that $Va > > 1$ there might be instances when the time derivative in Eq. (2.80) cannot be neglected. In cases when instability sets in as oscillatory and the frequency of oscillations is very high, or when chaotic solutions are anticipated high frequency modes will make the time derivative term in Eq. (2.80) significant.

This exhaustive discussion of non-Darcy models became necessary as some studies, particularly on gravity-driven convection, use non-Darcy models, which are not justified and in some cases reach conclusions, which can be misleading. Nield (1983) pointed out some examples of such occurrences.

2.5 Classification of Rotating Flows in Porous Media

Rotating flows in porous media can be dealt with by classifying them first into three major categories

- Isothermal flows in heterogeneous porous media subject to rotation.*
- Convective flows in non-isothermal homogeneous porous media subject to rotation.*
- Convective flows in non-isothermal heterogeneous porous media subject to rotation.*

This book is concerned with the first two categories. The third category is not covered, simply because there are no reported research results on natural convection

in rotating heterogeneous porous media. Regarding category (a) above, it is evident from Eq. (2.60) subject to isothermal conditions, i.e. $Ra_g = Ra_\omega = 0$, that heterogeneity of the medium is an important condition for the effects of rotation to be significant. This can be observed, for example, from the basic Darcy's law, which under homogeneous conditions (i.e. $K = \text{const.} = 1$) yields irrotational types of flows, i.e. the vorticity $\nabla \times \mathbf{V} = 0$. The heterogeneity of the medium, $K \equiv K(\mathbf{X})$, introduces a non-vanishing vorticity, i.e. $\nabla \times \mathbf{V} \neq 0$. Non-isothermal flows, as a result of natural convection, allow also a non-vanishing vorticity. Convective flows in rotating porous media are further classified into three categories and each one of them can be separated into three cases. In order to justify this classification we proceed first by defining the phenomenon of natural convection. Natural convection is the phenomenon of **fluid flow** driven by **density variations** in a fluid **subject to body forces**. Therefore, there are two necessary conditions to be met in order to obtain natural convective flow; (i) *the existence of density variations within the fluid* and (ii) *the fluid must be subjected to body forces*. As density is in general a function of pressure, temperature and solute concentration (in the case of a binary mixture), i.e., $\rho \equiv \rho(p, T, S)$ and its variation with respect to pressure is much smaller than that with respect to temperature or concentration (i.e., $\beta_p < \beta_T$, leading to Eq. (2.34) $\rho = 1 - \beta_T T$ for example). Hence the convection can be driven either by temperature variations or by variations in solute concentration or by both. In the two latter cases Eq. (2.34) should be extended to include S in the form of Eq. (2.35) $\rho = 1 - \beta_T T + \beta_S S$, the solute transport equation should be included in the model and Eq. (2.60) should be extended as well to account for the solute effect on density. Gravity, centrifugal forces, electromagnetic forces (in the case of liquid metals subject to an electric field) are only examples of body forces which represent the second necessary condition for natural convection to occur. Some of these body forces are constant, such as gravity for example, while others can vary linearly with the perpendicular distance from the axis of rotation like the centrifugal force. The lack of body forces, for example under micro-gravity conditions in the outer space, prevents occurrence of convection. The two conditions mentioned above are indeed necessary for natural convection to occur; however they are not sufficient. The relative orientation of the density gradient with respect to the body force is an important factor for providing the sufficient conditions for convection to occur. This is shown graphically in Fig. 2.1 for the particular case of thermal convection, where $\mathbf{B}$ represents the body force and $\nabla \rho = -\beta_T \nabla T$ is the direction of the density gradient.

Given this basic introduction on the causes for the setup of convection the following classification of convective flows is introduced:

- (i) *Convection due to thermal buoyancy.*
- (ii) *Convection due to thermo-solutal buoyancy.*

A third category related to convection due to solutal buoyancy alone could have been introduced but it would not bring any significant contribution to (i) above as

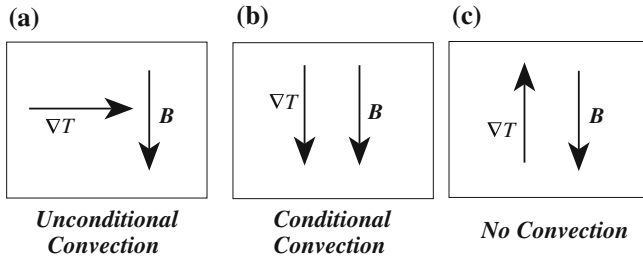


Fig. 2.1 The effect of the relative orientation of the temperature gradient with respect to the body force on the setup of convection

the results obtained for thermal buoyancy alone can be easily converted to the third case by analogy.

Three separate cases for each one of the above categories can be considered, depending on the driving body force, i.e., *convection driven by gravity*, *convection driven by the centrifugal force* and *convection driven by both gravity and centrifugal force*. In each of these cases the effect of Coriolis acceleration on the natural convective flow is another rotation effect of interest in the present book.

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