

Chapter 2

Stabilization of Saturated Systems

2.1 Introduction

Although many important problems related to two-dimensional (2-D) systems such as realization, observation, and controllability have been extensively investigated (see [1]), the stabilization problem is not fully solved for practical problems. As has been shown in Chap. 1, the stability of 2-D systems reduces to checking a 2-D characteristic polynomial [2–4], and many conditions for asymptotic stability and stabilizability have already been proposed, see [5–10, 16]; however, the control synthesis problem remains elusive, especially in the presence of control saturations. Some results of this chapter appeared for the first time in [11]. To the best of the authors' knowledge, the stabilization of saturated 2-D systems has not been considered elsewhere: only digital 2-D filters have been studied (see [12, 13]).

Thus, this chapter investigates saturated systems in the continuous and discrete cases: First, the situation when the control may reach the saturation value is studied: stabilization is then guaranteed even if the saturation is reached. For this the saturated 2-D system is described as a convex combination of 2-D linear systems using the results presented in Sect. 1.4. Then, specific quadratic Lyapunov functions are used to characterize the stability of the convex combination of 2-D feedback systems. This makes it possible to derive sufficient conditions of stabilizability under LMI form. Then the unsaturating controller case for 2-D saturated systems is considered in this chapter: stabilizability conditions are derived such that the saturation is not reached, so linear behavior is always guaranteed. Again these conditions are given in LMI form and the obtained results are extended to saturated repetitive systems.

2.2 Continuous 2-D Systems

2.2.1 Formulation of the Stabilization Problem

Consider the continuous 2-D Roesser model studied in Sect. 1.1.3.1:

$$\begin{bmatrix} \frac{\partial x^h(t_1, t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1, t_2)}{\partial t_2} \end{bmatrix} = Ax(t_1, t_2) + Bu(t_1, t_2) \quad (2.1)$$

$$y(t_1, t_2) = Cx(t_1, t_2) + Du(t_1, t_2) \quad (2.2)$$

with $x^h(t_1, t_2) \in \mathbb{R}^{n_1}$ the horizontal state, $x^v(t_1, t_2) \in \mathbb{R}^{n_2}$ the vertical state, and $u(t_1, t_2) \in \mathbb{R}^m$ the control vector. The states $x^h(t_1, t_2)$, $x^v(t_1, t_2)$ and the boundary conditions are defined in Sect. 1.1.3 of Chap. 1.

The saturation function used here is the following one (already studied in Chap. 1), defined as follows for $i = 1, \dots, m$:

$$\text{sat}(w) = (\text{sat}(w_i)) = \begin{cases} 1 & \text{if } w_i > 1 \\ w_i & \text{if } -1 \leq w_i \leq 1 \\ -1 & \text{if } w_i < -1 \end{cases} \quad (2.3)$$

Further, state-feedback control is used such that

$$u(t_1, t_2) = [K_1 \ K_2] \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}, \quad (2.4)$$

where matrix $K = [K_1 \ K_2]$ is the state-feedback gain to be designed.

The problem we address hereafter is to find stabilizing state-feedback controllers for continuous 2-D systems (1.57) with saturation on the control (2.3) using state-feedback control (2.4). We address the problem from two points of view: first, saturating controls are allowed, so nonlinear behavior may occur (thus, a saturating controller is used). Second, the behavior is limited to be linear, so saturating controls are not allowed (an unsaturating controller is designed).

2.2.1.1 Stabilization with Saturating Control

Using the results of Sect. 1.4 the state-feedback control (2.3), and the fact that $v = Hx$ with $x \in \mathcal{A}(H)$ the closed-loop 2-D saturated continuous system can be rewritten as

$$\begin{bmatrix} \frac{\partial x^h(t_1, t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1, t_2)}{\partial t_2} \end{bmatrix} = A \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} + B \sum_{s=1}^N \delta_s(t_1, t_2) (D_s K + D_s^- H) \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}.$$

That is,

$$\begin{bmatrix} \frac{\partial x^h(t_1, t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1, t_2)}{\partial t_2} \end{bmatrix} = \sum_{s=1}^N \delta_s(t_1, t_2) \tilde{A}_s \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} = \tilde{A}(\delta) \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}, \quad (2.5)$$

where matrices $\tilde{A}(\delta)$ and $\tilde{A}_s$ are given as

$$\begin{aligned} \tilde{A}(\delta) &= \sum_{s=1}^N \delta_s(t_1, t_2) \tilde{A}_s \\ \tilde{A}_s &= \begin{bmatrix} \tilde{A}_{11}^s & \tilde{A}_{12}^s \\ \tilde{A}_{21}^s & \tilde{A}_{22}^s \end{bmatrix}, \end{aligned}$$

with

$$\begin{aligned} \tilde{A}_{11}^s &= A_{11} + B_1(D_s K_1 + D_s^- H_1), \\ \tilde{A}_{12}^s &= A_{12} + B_1(D_s K_2 + D_s^- H_2), \\ \tilde{A}_{21}^s &= A_{21} + B_2(D_s K_1 + D_s^- H_1), \\ \tilde{A}_{22}^s &= A_{22} + B_2(D_s K_2 + D_s^- H_2). \end{aligned} \quad (2.6)$$

Sufficient conditions are now derived for the stabilization of continuous 2-D saturated systems. In order to allow the synthesis of stabilizing controllers some transformations to LMI form are worked out.

Theorem 2.1 *For a given scalar $\rho > 0$, if there exist matrices $H_1 \in \mathbb{R}^{m \times n_1}$, $H_2 \in \mathbb{R}^{m \times n_2}$, $K_1 \in \mathbb{R}^{m \times n_1}$, $K_2 \in \mathbb{R}^{m \times n_2}$, and symmetric positive definite matrices $P_1 \in \mathbb{R}^{n_1 \times n_1}$, $P_2 \in \mathbb{R}^{n_2 \times n_2}$, such that the following LMI conditions hold true, for $s = 1, \dots, N$:*

$$\Phi(s) = \begin{bmatrix} P_1 \tilde{A}_{11}^s + \tilde{A}_{11}^{sT} P_1 & P_1 \tilde{A}_{12}^s + \tilde{A}_{21}^{sT} P_2 \\ * & P_2 \tilde{A}_{22}^s + \tilde{A}_{22}^{sT} P_2 \end{bmatrix} < 0, \quad (2.7)$$

where matrices $\tilde{A}_{ij}^s$ are given by (2.6) and, following definitions (1.63) and (1.64),

$$\varepsilon(P, \rho) \subset \mathcal{L}(H), \quad (2.8)$$

with $P = \text{diag}(P_1, P_2)$ and $H = [H_1 \ H_2]$, then the 2-D continuous system (2.5) is asymptotically stable $\forall x_0 \in \varepsilon(P, \rho)$.

Proof Assume that condition (2.8) holds true; then the saturated system (1.57) can be written as (2.5). Consider the candidate Lyapunov function $V(x(t_1, t_2))$ given by

$$\begin{aligned}
V(x(t_1, t_2)) &= x^T(t_1, t_2) P x(t_1, t_2) \\
&= \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix}.
\end{aligned} \tag{2.9}$$

Computing its unidirectional derivative gives:

$$\begin{aligned}
\dot{V}_u(t_1, t_2) &= 2 \frac{\partial x^h T}{\partial t_1} P_1 x^h(t_1, t_2) + 2 \frac{\partial x^v T}{\partial t_2} P_2 x^v(t_1, t_2) \\
&= 2 \begin{bmatrix} \frac{\partial x^h(t_1, t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1, t_2)}{\partial t_2} \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} \\
&= x(t_1, t_2)^T \left(\tilde{A}(\delta)^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} + \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \tilde{A}(\delta) \right) x(t_1, t_2).
\end{aligned} \tag{2.10}$$

This derivative is negative if

$$\tilde{A}(\delta)^T P + P \tilde{A}(\delta) < 0. \tag{2.11}$$

It is clear that if the LMI (2.7) is satisfied, inequality (2.11) holds, then the unidirectional derivative (2.10) is negative. Hence, using Lemma 1.5, the 2-D saturated closed-loop continuous system is asymptotically stable. $\square$

The previous result states an asymptotic stabilizability condition for the closed-loop system. This condition is now transformed into an LMI test, which makes it possible to synthesize the state-feedback saturating controller.

Corollary 2.1 *For a given scalar $\rho > 0$, if there exist matrices $Z_1, Z_2, Y_1, Y_2, X_1 = X_1^T > 0$, and $X_2 = X_2^T > 0$ such that the following LMIs hold true:*

$$\Pi(s) = \begin{bmatrix} \Pi_{11}^s + \Pi_{11}^{sT} & \Pi_{12}^s + \Pi_{21}^{sT} \\ * & \Pi_{22}^s + \Pi_{22}^{sT} \end{bmatrix} < 0, \quad s = 1, \dots, N \tag{2.12}$$

$$\begin{bmatrix} \mu & (Z_1)_i & (Z_2)_i \\ * & X_1 & 0 \\ * & * & X_2 \end{bmatrix} > 0 \quad i = 1, \dots, m, \tag{2.13}$$

where $(Z_1)_i$ and $(Z_2)_i$ denote the i th row of matrices Z_1 and Z_2 respectively; $\mu = 1/\rho$, and matrices Π_{ij}^s are given by

$$\Pi_{ij}^s = A_{ij} X_j + B_i (D_s Y_j + D_s^- Z_j), \quad i, j = 1, 2, \tag{2.14}$$

then system (2.5) is asymptotically stable in closed-loop for all boundary conditions $x_0 \in \varepsilon(P, \rho)$ with $P = \text{diag}(P_1, P_2)$, when the controller gain is given as

$$K = \begin{bmatrix} Y_1 X_1^{-1} & Y_2 X_2^{-1} \end{bmatrix}. \quad (2.15)$$

Moreover, the set $\mathcal{L}(H)$ is given as (1.64) with

$$H = \begin{bmatrix} Z_1 X_1^{-1} & Z_2 X_2^{-1} \end{bmatrix}. \quad (2.16)$$

Proof The sufficient condition of stability of the 2-D saturated continuous system is given by (2.11). Pre- and post-multiplying by $X = P^{-1}$, leads to $X \tilde{A}(\delta)^T + \tilde{A}(\delta) X < 0$ and using notation (2.14) and the facts that $Y = KX$ and $Z = HX$, it is then easily obtained that (2.12) implies (2.7), which is a sufficient condition of asymptotic stability for the 2-D closed-loop continuous system for all boundary conditions inside the set $\varepsilon(P, \rho)$. Furthermore, (2.8) is equivalent to $\rho(H)_i P^{-1}(H)_i^T \leq 1$, $i = 1, \dots, m$, [14]. Developing equivalently as follows: $\rho(HX)_i X^{-1}(HX)_i^T \leq 1$, that is, $\rho(Z)_i X^{-1}(Z)_i^T \leq 1$. Using Schur complement, one obtains

$$\begin{bmatrix} \rho^{-1} & (Z)_i \\ * & X \end{bmatrix} > 0, \quad i = 1, \dots, m. \quad (2.17)$$

Finally, using $\mu = 1/\rho$, $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$ and $Z = [Z_1 \ Z_2]$, the LMIs (2.13) follow. $\square$

It is worth noting that the scalar $\mu = 1/\rho$ can also be taken as a variable of LMIs (2.13).

Example 2.1 Consider the differential Darboux equation:

$$\frac{\partial^2 q(x, t)}{\partial x \partial t} = a_1 \frac{\partial q(x, t)}{\partial t} + a_2 \frac{\partial q(x, t)}{\partial x} + a_0 q(x, t) + bu(x, t),$$

with the boundary conditions $q(x, 0) = q_1(x)$ and $q(t, 0) = q_2(t)$, where $q(x, t)$ is the variable function, a_0, a_1, a_2, b are real coefficients, and $u(x, t)$ is the input function which is assumed here to be constrained as $|u(x, t)| \leq 1$. Let us define

$$\begin{aligned} x^h(x, t) &= \frac{\partial q(x, t)}{\partial t} - a_2 q(x, t), \\ x^v(x, t) &= q(x, t). \end{aligned}$$

The following continuous 2-D system is obtained:

$$\begin{bmatrix} \frac{\partial x^h(x, t)}{\partial t} \\ \frac{\partial x^v(x, t)}{\partial t} \end{bmatrix} = A \begin{bmatrix} x^h(x, t) \\ x^v(x, t) \end{bmatrix} + Bu(x, t), \quad (2.18)$$

with

$$A = \begin{bmatrix} a_1 & a_1 a_2 + a_0 \\ 1 & a_2 \end{bmatrix}; B = \begin{bmatrix} b \\ 0 \end{bmatrix},$$

and boundary conditions given by $x^h(0, t) = \dot{q}_2(t) - a_2 q_2(t)$ and $x^v(x, 0) = q_1(x)$. Observe that in this example $t_1 = x$ is a space variable and $t_2 = t$ is a time variable.

To illustrate the results in this section, set, for example, $a_0 = -1.1$, $a_1 = -1$, $a_2 = -0.6$ and $b = 1$; the obtained system is given as $A_{11} = -1$, $A_{12} = -0.5$, $A_{21} = 1$, $A_{22} = -0.6$, $B_1 = 1$, $B_2 = 0$. For this data and $\rho = 100$, the LMIs (2.12) and (2.13) are feasible, with a solution given as

$$\begin{aligned} P_1 &= 0.2207, \\ P_2 &= 0.1642, \\ K &= [0.2773 \quad -0.2441], \\ H &= [0.0028 \quad 0.0044]. \end{aligned}$$

Figure 2.1 presents the evolution of the control $u(t_1, t_2)$ and the states $x^h(t_1, t_2)$ and $x^v(t_1, t_2)$ when these feedback gains K and H are used (For simulation the system was discretized using $T_{samp} = 0.01$ for both variables). It is clear that the continuous 2-D saturated system is asymptotically stable and converges toward zero, allowing saturation of the control.

2.2.1.2 Stabilization with Unsaturating Controller

Consider again the continuous 2-D system (1.57) with constrained control (2.3). In the previous section, the design of a saturating controller was studied (i.e., saturation of the control signal was allowed), whereas in this section saturation is not allowed, so the synthesis proposed will guarantee that the state evolves inside a region of linear behavior given (by $\mathcal{L}(F)$ with F being the controller gain). Thus, this case can be seen as a particular case of the saturating one, or an extension to the 2-D case of the approach presented in [15].

Theorem 2.2 *For a given scalar $\rho > 0$, if there exist matrices $F_1 \in \mathbb{R}^{m \times n_1}$, $F_2 \in \mathbb{R}^{m \times n_2}$, and symmetric positive definite matrices $P_1 \in \mathbb{R}^{n_1 \times n_1}$, $P_2 \in \mathbb{R}^{n_2 \times n_2}$, such that the following conditions hold true:*

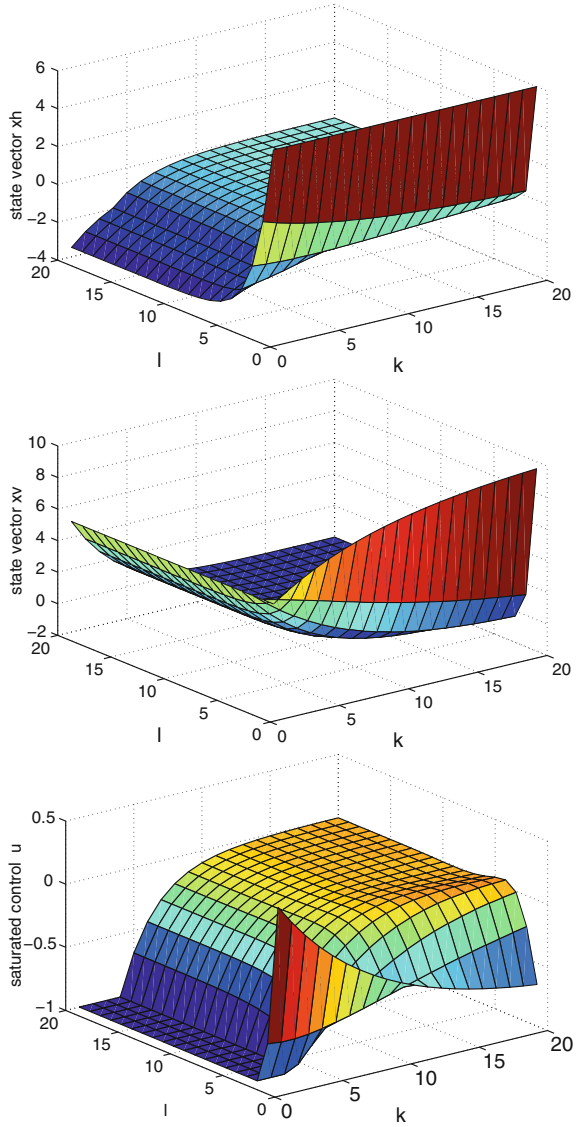
$$\begin{bmatrix} \Gamma_{11} + \Gamma_{11}^T & \Gamma_{12} + \Gamma_{21}^T \\ * & \Gamma_{22} + \Gamma_{22}^T \end{bmatrix} < 0, \quad (2.19)$$

$$\varepsilon(P, \rho) \subset \mathcal{L}(F) \quad (2.20)$$

where

$$\Gamma_{ij} = P_i A_{ij} + P_i B_i F_j, \quad i, j = 1, 2, \quad (2.21)$$

Fig. 2.1 The evolution of the states $x^h(t_1, t_2)$, $x^v(t_1, t_2)$, and the control $u(t_1, t_2)$ using the saturating controller



and $P = \text{diag}(P_1, P_2)$, then the continuous 2-D system (2.5) is asymptotically stable $\forall x_0 \in \mathcal{E}(P, \rho)$.

Proof The proof follows readily if one replaces K by F in the proof of Theorem 2.1 and removes the saturated convex writing of the control. This can be done, as in this case, the state is restricted to evolve inside the linear region of behavior given by condition (2.20). $\square$

In the next result, the LMI formulation of these conditions that enables the unsaturating state-feedback control to be derived is given as

Corollary 2.2 *For a given scalar $\rho > 0$, if there exist matrices $Y_1, Y_2, X_1 = X_1^T > 0, X_2 = X_2^T > 0$ such that the following LMIs hold true:*

$$\begin{bmatrix} \Psi_{11} + \Psi_{11}^T & \Psi_{12} + \Psi_{21}^T \\ * & \Psi_{22} + \Psi_{22}^T \end{bmatrix} < 0, \quad (2.22)$$

$$\begin{bmatrix} \mu & (Y_1)_i & (Y_2)_i \\ * & X_1 & 0 \\ * & * & X_2 \end{bmatrix} > 0, \quad i = 1, \dots, m, \quad (2.23)$$

where the Ψ_{ij} are defined by $\Psi_{ij} = A_{ij}X_j + B_iY_j$, and $\mu = 1/\rho$; then the continuous 2-D system (2.5) is asymptotically stable $\forall x_0 \in \varepsilon(P, \mu)$, with $P = \text{diag}(X_1^{-1}, X_2^{-1})$. The corresponding stabilizing controller gain is then given as

$$F = \begin{bmatrix} Y_1 X_1^{-1} & Y_2 X_2^{-1} \end{bmatrix}. \quad (2.24)$$

Proof The proof follows readily from Corollary 2.1. □

Example 2.2 Consider the system studied in Example 2.1. If no saturation is allowed in the control signal, Corollary 2.2 can be used to synthesize the controller. In this case, the LMIs (2.22) and (2.23) are feasible, with a solution given as

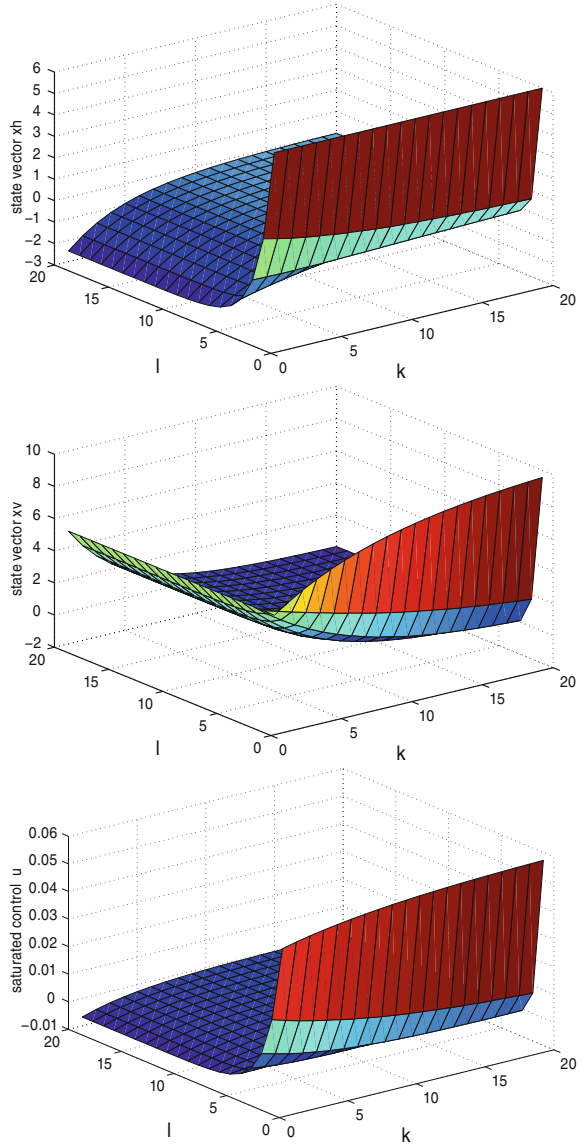
$$\begin{aligned} P_1 &= 0.1749, \\ P_2 &= 0.1243, \\ F &= [0.0065 \quad 0.0019]. \end{aligned}$$

In Fig. 2.2, the evolutions of the states $x^h(t_1, t_2)$, $x^v(t_1, t_2)$, and the control $u(t_1, t_2)$ are shown when this F is used for feedback. It is clear that the 2-D system is asymptotically stable and converges toward zero, while the control evolves without saturating.

2.2.1.3 Saturating Versus Unsaturating Controllers

It is well known in the literature of constrained control systems that both the techniques of saturating control and unsaturating control may be applied. However, the criterion for choosing between the two approaches is to make a compromise between the size of the boundary conditions set ensuring asymptotic stability ($\mathcal{L}(F)$ versus $\mathcal{L}(H)$) and the burden of computing cost ($(m+1)$ LMIs instead of $(2^m + m)$ LMIs).

Fig. 2.2 The states evolution of $x^h(t_1, t_2)$, $x^v(t_1, t_2)$, and control $u(t_1, t_2)$ obtained with the unsaturating controller



For example, for the previous example Fig. 2.3 presents the ellipsoid set, with all the states evolving inside $\mathcal{L}(K)$, whereas Fig. 2.4 presents the sets $\mathcal{L}(H)$ together with $\mathcal{L}(K)$ for the saturating controller and $\mathcal{L}(F)$ for the unsaturating controller: The technique that allows saturation provides a possible set of states $\mathcal{L}(H)$ which is significantly larger than the linear behavior set $\mathcal{L}(F)$, at the cost of solving $(2^m - 1)$ more LMIs.

Fig. 2.3 Domains $\mathcal{L}(K)$ and $\varepsilon(P, \rho)$ for the saturating controller

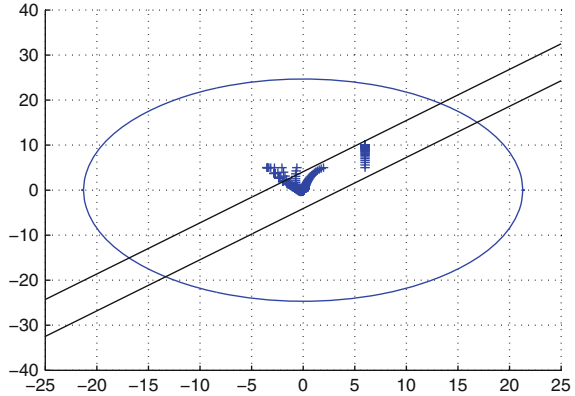
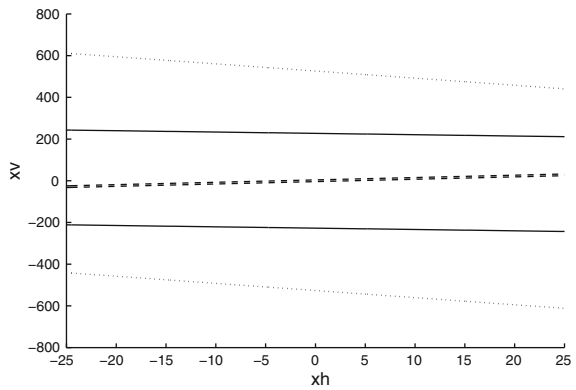


Fig. 2.4 Domains $\mathcal{L}(H)$ in dotted line, $\mathcal{L}(K)$ in dashed line, for the saturating controller and $\mathcal{L}(F)$ in solid line for the unsaturating controller



2.3 Discrete 2-D Systems

2.3.1 Stabilization with Saturating Controller

We consider now the discrete 2-D system described by the Roesser model given by (1.68) and (1.69), where the control signal is limited by the saturation function (2.3) and the controller is again the state feedback given by (2.4). The problem we address again is to find state-feedback gains K that stabilize the system even when the control signal saturates.

Using the results of Sect. 1.4, the saturating system can be written as follows:

$$\begin{bmatrix} x^h(k+1, l) \\ x^v(k, l+1) \end{bmatrix} = \sum_{s=1}^N \delta_s(t_1, t_2) \tilde{A}_s \begin{bmatrix} x^h(k, l) \\ x^v(k, l) \end{bmatrix} \quad (2.25)$$

$$= \tilde{A}(\delta) \begin{bmatrix} x^h(k, l) \\ x^v(k, l) \end{bmatrix}. \quad (2.26)$$

The states $x^h(k, l)$, $x^v(k, l)$ and the boundary conditions are defined in Sect. 1.1.2 of Chap. 1.

Matrices $\tilde{A}_s$ and $\tilde{A}(\delta)$ are given by

$$\tilde{A}_s = \begin{bmatrix} \tilde{A}_{11}^s & \tilde{A}_{12}^s \\ \tilde{A}_{21}^s & \tilde{A}_{22}^s \end{bmatrix},$$

$$\tilde{A}(\delta) = \sum_{s=1}^N \delta_s(t_1, t_2) \tilde{A}_s$$

with

$$\begin{aligned} \tilde{A}_{11}^s &= A_{11} + B_1(D_s K_1 + D_s^- H_1), \\ \tilde{A}_{12}^s &= A_{12} + B_1(D_s K_2 + D_s^- H_2), \\ \tilde{A}_{21}^s &= A_{21} + B_2(D_s K_1 + D_s^- H_1), \\ \tilde{A}_{22}^s &= A_{22} + B_2(D_s K_2 + D_s^- H_2). \end{aligned} \quad (2.27)$$

Following a similar approach to the continuous case, sufficient stabilizability condition is now derived for discrete 2-D saturated systems, which is later transformed into an LMI condition that facilitates the synthesis of stabilizing controllers.

Theorem 2.3 *If there exist matrices $H_1 \in \mathbb{R}^{m \times n_1}$, $H_2 \in \mathbb{R}^{m \times n_2}$, $K_1 \in \mathbb{R}^{m \times n_1}$, $K_2 \in \mathbb{R}^{m \times n_2}$, and symmetric positive definite matrices $P_1 \in \mathbb{R}^{n_1 \times n_1}$, $P_2 \in \mathbb{R}^{n_2 \times n_2}$, such that the following conditions hold:*

$$\begin{bmatrix} P_1 & 0 & A_{s11}^{cT} P_1 & A_{s21}^{cT} P_1 \\ * & P_2 & A_{s12}^{cT} P_2 & A_{s22}^{cT} P_2 \\ * & * & P_1 & 0 \\ * & * & * & P_2 \end{bmatrix} > 0, \quad s = 1, \dots, N$$

$$\varepsilon(P, \rho) \subset \mathcal{L}(H) \quad (2.28)$$

with $P = \text{diag}(P_1, P_2)$, then the discrete 2-D system (2.25) is asymptotically stable $\forall x_o \in \varepsilon(P, \rho)$.

Proof Assume that condition (2.28) holds true, then the saturated system (1.68) can be written as (2.25). Consider the candidate Lyapunov function $V(x(k, l))$ given by:

$$\begin{aligned} V(x(k, l)) &= \begin{bmatrix} x^h(k, l) \\ x^v(k, l) \end{bmatrix}^T \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} x^h(k, l) \\ x^v(k, l) \end{bmatrix} \\ &= x(k, l)^T P x(k, l). \end{aligned} \quad (2.29)$$

Computing its rate of variation gives

$$\begin{aligned}\Delta V(x(k, l)) &= x^{+T}(k, l) P x^+(k, l) - x^T(k, l) P x(k, l) \\ &= x^T(k, l) \left[A^c(\delta)^T P A^c(\delta) - P \right] x(k, l).\end{aligned}$$

This variation is negative if

$$A^c(\delta)^T P A^c(\delta) - P < 0, \quad (2.30)$$

where $A^c(\delta) = \sum_{s=1}^N \delta(k, l)_s A_s^c$. Using Schur complement, one obtains

$$\begin{bmatrix} P & A^c(\delta)^T P \\ * & P \end{bmatrix} > 0. \quad (2.31)$$

Substituting (2.46) into (2.31), bearing in mind that $\sum_{s=1}^N \delta_s = 1$, and that $\delta_s > 0$, $s = 1, \dots, N$, leads to

$$\sum_{s=1}^N \delta_s(k, l) \begin{bmatrix} P & [A + B(D_s K + D_s^- H)]^T P \\ * & P \end{bmatrix} > 0. \quad (2.32)$$

It is obvious that the following set of inequalities:

$$\begin{bmatrix} P & [A + B(D_s K + D_s^- H)]^T P \\ * & P \end{bmatrix} > 0, \quad (2.33)$$

for $s = 1, \dots, N$ gives a sufficient condition to have $\Delta V(x(k, l)) < 0$. Using the definition of the system (2.25) and $P = \text{diag}(P_1, P_2)$, it follows that inequalities (2.28) are sufficient conditions of asymptotic stability of the 2-D saturated discrete system (2.25). $\square$

As previously mentioned the next result provides an LMI formulation of the condition in the previous results, which makes it possible to synthesize stabilizing saturating controllers.

Corollary 2.3 *If there exist matrices $Z_1, Z_2, Y_1, Y_2, X_1 = X_1^T > 0, X_2 = X_2^T > 0$ such that the following LMIs hold:*

$$\begin{bmatrix} X_1 & 0 & \Pi_{s11}^T & \Pi_{s21}^T \\ * & X_2 & \Pi_{s12}^T & \Pi_{s22}^T \\ * & * & X_1 & 0 \\ * & * & * & X_2 \end{bmatrix} > 0, \quad s = 1, \dots, N \quad (2.34)$$

$$\begin{bmatrix} \mu & Z_{1i} & Z_{2i} \\ * & X_1 & 0 \\ * & * & X_2 \end{bmatrix} > 0, \quad i = 1, \dots, m, \quad (2.35)$$

where $\mu = 1/\rho$ and

$$\begin{aligned} \Pi_{s11} &= A_{11}X_1 + B_1(D_s Y_1 + D_s^- Z_1), \\ \Pi_{s12} &= A_{12}X_2 + B_1(D_s Y_2 + D_s^- Z_2), \\ \Pi_{s21} &= A_{21}X_1 + B_2(D_s Y_1 + D_s^- Z_1), \\ \Pi_{s22} &= A_{22}X_2 + B_2(D_s Y_2 + D_s^- Z_2), \end{aligned} \quad (2.36)$$

then the discrete 2-D system (2.25) is asymptotically stable $\forall x_o \in \varepsilon(P, \rho)$, with $P = \text{diag}(P_1, P_2)$ and

$$P_1 = X_1^{-1}, P_2 = X_2^{-1}; \quad (2.37)$$

the corresponding stabilizing controller gain is given as

$$K = [Y_1(X_1)^{-1} \quad Y_2(X_2)^{-1}], \quad (2.38)$$

and H can be obtained from

$$H = [Z_1 X_1^{-1} \quad Z_2 X_2^{-1}]. \quad (2.39)$$

Proof Post- and pre-multiplying (2.30) by $P^{-1} = X$, leads to

$$X - X [A^c(\delta)]^T X^{-1} [A^c(\delta)] X > 0.$$

Applying Schur complement gives

$$\begin{bmatrix} X & [A^c(\delta)X]^T \\ * & X \end{bmatrix} > 0. \quad (2.40)$$

Substituting (2.46) in (2.40), leads to

$$\sum_{s=1}^N \delta_s(k, l) \begin{bmatrix} X & [A_s^c X]^T \\ * & X \end{bmatrix} > 0. \quad (2.41)$$

Let $Y = KX$, $Z = HX$. A sufficient condition to have (2.41) is then

$$\begin{bmatrix} X & [AX + B(D_s Y + D_s^- Z)]^T \\ * & X \end{bmatrix} > 0, \quad s = 1, \dots, N. \quad (2.42)$$

Substituting matrices A , B according to (1.69) and letting $X = \text{diag}\{X_1, X_2\}$, $Y = [Y_1 \ Y_2]$ and $Z = [Z_1 \ Z_2]$, the LMIs (2.34) are then directly obtained. The stabilizing controller gain is given as $K = YX^{-1}$, so expression (2.38) follows.

Furthermore, inequality (2.28) is equivalent to $\rho H_i P^{-1} H_i^T \leq 1$, $i = 1, \dots, m$ [14]. Developing equivalently as follows: $\rho(HX)_i X^{-1} (HX)_i^T \leq 1$, $i = 1, \dots, m$, that is $\rho Z_i X^{-1} Z_i^T \leq 1$, $i = 1, \dots, m$.

Using Schur complement, one obtains

$$\begin{bmatrix} \rho^{-1} & Z_i \\ * & X \end{bmatrix} > 0, \quad i = 1, \dots, m. \quad (2.43)$$

Finally, using $\mu = 1/\rho$, $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$ and $Z = [Z_1 \ Z_2]$, the LMI (2.35) follows. $\square$

Example 2.3 Consider the discrete 2-D saturated system given as (1.68) where

$$A = \begin{bmatrix} -0.5 & 1.6 \\ 0.8 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 0.1 \end{bmatrix}. \quad (2.44)$$

It can be seen that for this numerical example the LMIs (2.34) and (2.35) are feasible, with the following solution:

$$\begin{aligned} P_1 &= 1.7380, \\ P_2 &= 1.9527, \\ K &= [0.6127 \quad -3.1487], \\ H &= [0.6612 \quad -1.2004]. \end{aligned}$$

Figure 2.5 plots the evolution of the states $x^h(k, l)$, $x^v(k, l)$ and the saturating control $u(k, l)$ from given boundary conditions inside the set $\varepsilon(P, \rho)$ of feasible initial conditions. It is clear that these states converge toward the origin and that the closed-loop system is asymptotically stable even if the control saturates, as required.

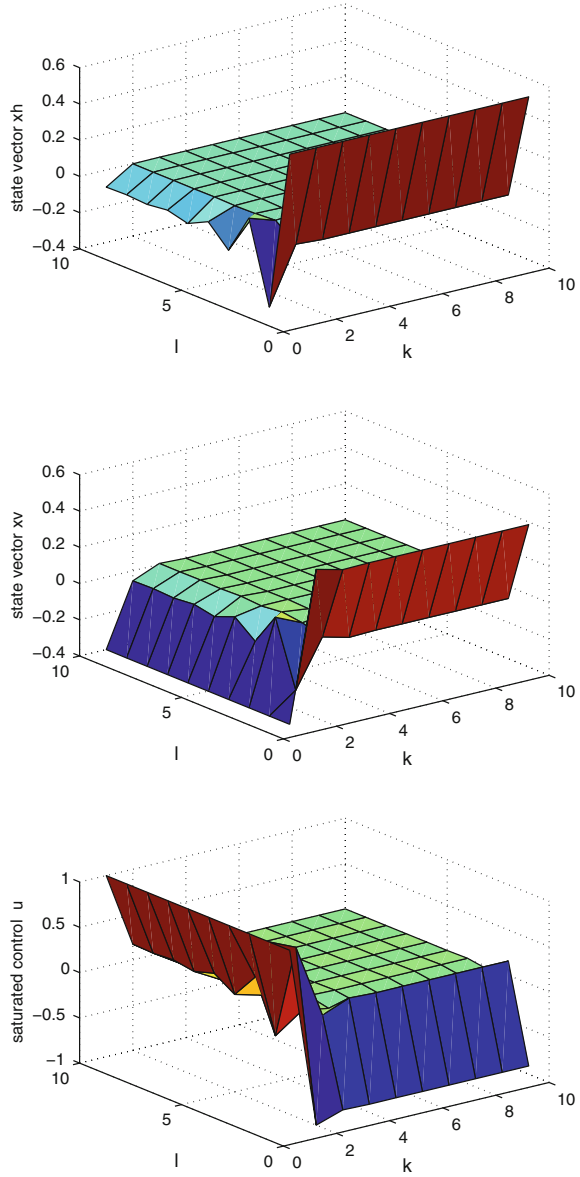
2.3.1.1 Unsaturating Controller

In the previous section, stabilization under saturating controller was studied. In this section, a controller is designed that avoids saturation; thus, the synthesis proposed guarantees that the state evolves always inside a region of linear behavior. This case is also developed as a particular case of the saturating one, with the controller gain now called F instead of K , to clearly distinguish both cases.

In the linear region of behavior $\mathcal{L}(F)$ the closed-loop system is just

$$\begin{bmatrix} x^h(k+1, l) \\ x^v(k, l+1) \end{bmatrix} = (A + BF) \begin{bmatrix} x^h(k, l) \\ x^v(k, l) \end{bmatrix}, \quad (2.45)$$

Fig. 2.5 Evolution of the states $x^h(k, l)$, $x^v(k, l)$ and the control $u(k, l)$



that is,

$$\begin{bmatrix} x^h(k+1, l) \\ x^v(k, l+1) \end{bmatrix} = A^c \begin{bmatrix} x^h(k, l) \\ x^v(k, l) \end{bmatrix}, \quad (2.46)$$

where $A^c = A + BF$,
and

$$A^c = \begin{bmatrix} A_{11}^c & A_{12}^c \\ A_{21}^c & A_{22}^c \end{bmatrix},$$

with

$$\begin{aligned} A_{11}^c &= A_{11} + B_1 F_1, \\ A_{12}^c &= A_{12} + B_1 F_2, \\ A_{21}^c &= A_{21} + B_2 F_1, \\ A_{22}^c &= A_{22} + B_2 F_2. \end{aligned} \quad (2.47)$$

The following result is then a parallel of Theorem 2.3 for the unsaturating case.

Theorem 2.4 *If there exist matrices $F_1 \in \mathbb{R}^{m \times n_1}$, $F_2 \in \mathbb{R}^{m \times n_2}$, $P_1 \in \mathbb{R}^{n_1 \times n_1}$, $P_2 \in \mathbb{R}^{n_2 \times n_2}$ such that the following conditions hold:*

$$\begin{bmatrix} P_1 & 0 & (A_{11} + B_1 F_1)^T P_1 & (A_{21} + B_2 F_1)^T P_2 \\ * & P_2 & (A_{12} + B_1 F_2)^T P_1 & (A_{22} + B_2 F_2)^T P_2 \\ * & * & P_1 & 0 \\ * & * & * & P_2 \end{bmatrix} > 0, \quad (2.48)$$

$$\varepsilon(P, \rho) \subset \mathcal{L}(F) \quad (2.49)$$

with $P = \text{diag}\{P_1, P_2\}$, then the discrete 2-D system (2.45) is asymptotically stable $\forall x_o \in \varepsilon(P, \rho)$.

Proof The results follow readily if one replaces matrix K by F in the proof of Theorem 2.3 and removes the saturated convex writing of the control, as the state is now restricted to evolve inside the region of linear behavior. $\square$

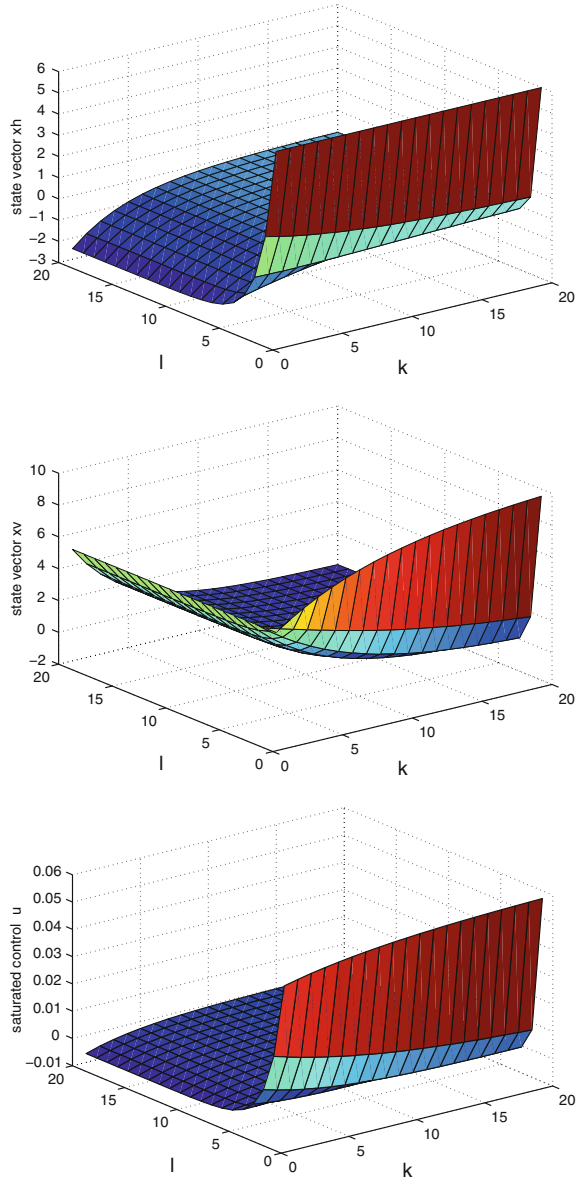
The following result provides the practical formulation as LMIs of this condition, to facilitate the derivation of the unsaturating state-feedback control gain

Corollary 2.4 *If there exist matrices $Y_1, Y_2, X_1 = X_1^T > 0, X_2 = X_2^T > 0$ such that the following LMIs hold:*

$$\begin{bmatrix} X_1 & 0 & \Psi_{11}^T & \Psi_{21}^T \\ * & X_2 & \Psi_{12}^T & \Psi_{22}^T \\ * & * & X_1 & 0 \\ * & * & * & X_2 \end{bmatrix} > 0, \quad (2.50)$$

$$\begin{bmatrix} \mu & Y_{1i} & Y_{2i} \\ * & X_1 & 0 \\ * & * & X_2 \end{bmatrix} > 0, \quad i = 1, \dots, m, \quad (2.51)$$

Fig. 2.6 Evolution of $x^h(k, l)$, $x^v(k, l)$ and the control $u(k, l)$



where

$$\begin{aligned}
 \Psi_{11} &= A_{11}X_1 + B_1Y_1, \\
 \Psi_{12} &= A_{12}X_2 + B_1Y_2, \\
 \Psi_{21} &= A_{21}X_1 + B_2Y_1, \\
 \Psi_{22} &= A_{22}X_2 + B_2Y_2,
 \end{aligned} \tag{2.52}$$

and $\mu = 1/\rho$; then, the discrete 2-D system (2.45) is asymptotically stable $\forall x_o \in \varepsilon(P, \rho)$, with the controller gain given as

$$F = [Y_1(X_1)^{-1} \quad Y_2(X_2)^{-1}] \quad (2.53)$$

and the set of valid boundary conditions $\varepsilon(P, 1/\mu)$ given as

$$P_1 = X_1^{-1}, P_2 = X_2^{-1}. \quad (2.54)$$

Proof The results follow directly from Corollary 2.3. $\square$

Example 2.4 Let us consider the numerical example studied in the saturated (Example 2.3). It can be seen that the LMIs (2.50) and (2.51) are feasible, with the obtained solution given as $P = \text{diag}(P_1, P_2)$, with

$$\begin{aligned} P_1 &= 1.8071, \\ P_2 &= 2.1043, \\ F &= [0.7321 \quad -1.1967]. \end{aligned}$$

Figure 2.6 plots the evolution of the states $x^h(k, l)$ and $x^v(k, l)$, together with the control $u(k, l)$. It is clear that the system is asymptotically stable, that the states converge towards the origin, and the control does not saturate as requested.

For comparison, Fig. 2.7 plots the polyhedral sets $\mathcal{L}(H)$, $\mathcal{L}(K)$, and the ellipsoid set $\varepsilon(P, \rho)$ for the saturating controller, together with $\mathcal{L}(F)$ for the unsaturating controller. The evolutions of the states $x^h(k, l)$ and $x^v(k, l)$ are also plotted. One can notice that all these states, once initiated inside the set of asymptotic stability $\varepsilon(P, \rho)$, remain inside this set.

It must be pointed out that in order to show the effectiveness of the approach, the boundary conditions for the simulation have been chosen inside the set $\mathcal{L}(H)$ and outside the set $\mathcal{L}(K)$. Hence, saturating control appears, as shown in Fig. 2.7.

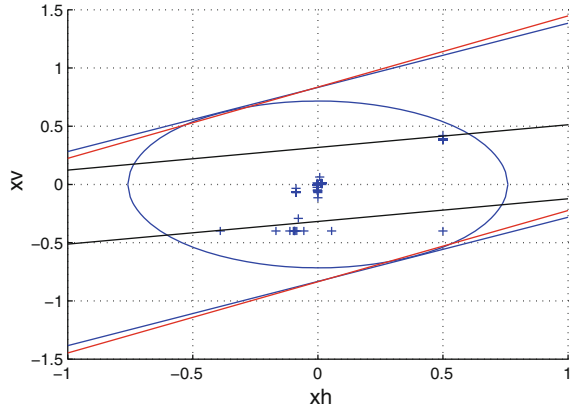
2.4 Stabilization of Saturated Repetitive Systems

2.4.1 Saturated Repetitive Systems

Consider now the differential linear repetitive process (presented in Sect. 1.5) described by the following state-space model over $0 \leq t \leq \beta$, $k \geq 0$:

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + B_0 y_k(t) + B \text{sat}(u_{k+1}(t)) \\ y_{k+1}(t) &= Cx_{k+1}(t) + D_0 y_k(t) + D \text{sat}(u_{k+1}(t)), \end{aligned} \quad (2.55)$$

Fig. 2.7 The sets $\mathcal{L}(H)$ in red, $\mathcal{L}(K)$ in black, $\varepsilon(P, \rho)$ in blue ellipsoid for the saturated control and $\mathcal{L}(F)$ in blue for the unsaturated control



where on pass k , $x_k(t) \in \mathbb{R}^n$ is the state vector and $y_k(t) \in \mathbb{R}^p$ is the pass profile vector, and $u_{k+1}(t) \in \mathbb{R}^m$ is the control vector respectively; A, B_0, B, C, D_0, D are time-invariant real matrices with appropriate dimensions.

The form of the state initial vector on each pass and the initial pass profile (on pass 0) considered here are

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0. \\ y_0(t) &= f(t). \end{aligned} \quad (2.56)$$

The following state-feedback control is used:

$$u_{k+1}(t) = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix}, \quad (2.57)$$

where $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ is the state-feedback gain matrix to be designed.

Using the state-feedback control (2.57) and the fact that $v_{k+1}(t) = H \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix}$ with $H = \begin{bmatrix} H_1 & H_2 \end{bmatrix}$ and $\xi_k(t) = \begin{bmatrix} x_{k+1}(t) \\ y_k(t) \end{bmatrix} \in \mathcal{L}(H)$ the repetitive system (2.55) can be rewritten as follows:

$$\begin{bmatrix} \dot{x}_{k+1}(t) \\ y_{k+1}(t) \end{bmatrix} = \mathcal{A} \xi_k(t) + \mathcal{B} \sum_{s=1}^N \delta_s(k, t) (D_s K + D_s^- H) \xi_k(t), \quad (2.58)$$

where matrices $\mathcal{A}$ and $\mathcal{B}$ are given as $\mathcal{A} = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}$; $\mathcal{B} = \begin{bmatrix} B \\ D \end{bmatrix}$. That is,

$$\begin{bmatrix} \dot{x}_{k+1}(t) \\ y_{k+1}(t) \end{bmatrix} = \sum_{s=1}^N \delta_s(k, t) \tilde{A}_s \xi_k(t) = \tilde{A}(\delta) \xi_k(t), \quad (2.59)$$

where matrices $\tilde{A}(\delta)$ and $\tilde{A}_s$ are given as

$$\begin{aligned}\tilde{A}(\delta) &= \sum_{s=1}^N \delta_s(k, t) \tilde{A}_s; \\ \tilde{A}_s &= \begin{bmatrix} \tilde{A}^s & \tilde{B}_0^s \\ \tilde{C}^s & \tilde{D}_0^s \end{bmatrix} = \begin{bmatrix} A + B(D_s K_1 + D_s^- H_1) & B_0 + B(D_s K_2 + D_s^- H_2) \\ C + D(D_s K_1 + D_s^- H_1) & D_0 + D(D_s K_2 + D_s^- H_2) \end{bmatrix}. \quad (2.60)\end{aligned}$$

2.4.2 Conditions for Stabilization

In parallel with the results of Sects. 2.2 and 2.3, sufficient conditions are now given for the stabilization of repetitive systems under saturation. The two cases are considered separately: saturating controller and unsaturating controller. In order to allow the synthesis of stabilizing controllers, some transformations into LMI form are then also worked out in each case.

2.4.2.1 Saturating Controller

Theorem 2.5 *For a given scalar $\rho > 0$, if there exist matrices $H_1 \in \mathbb{R}^{m \times n}$, $H_2 \in \mathbb{R}^{m \times p}$, $K_1 \in \mathbb{R}^{m \times n}$, $K_2 \in \mathbb{R}^{m \times p}$, and symmetric positive definite matrices $0 < P_1 \in \mathbb{R}^{n \times n}$ and $0 < P_2 \in \mathbb{R}^{p \times p}$ such that the following LMI conditions hold true, for $s = 1, \dots, N$:*

$$\Upsilon(s) = \begin{bmatrix} P_1 \tilde{A}^s + \tilde{A}^{sT} P_1 & P_1 \tilde{B}_0^s & \tilde{C}^{sT} P_2 \\ * & -P_2 & \tilde{D}_0^{sT} P_2 \\ * & * & -P_2 \end{bmatrix} < 0, \quad (2.61)$$

where matrices $\tilde{A}^s$, $\tilde{B}_0^s$, $\tilde{C}^s$ and $\tilde{D}_0^s$ are given by (2.60), and, following definitions (1.63) and (1.64),

$$\varepsilon(P, \rho) \subset \mathcal{L}(H) \quad (2.62)$$

with $P = \text{diag}(P_1, P_2)$, then the differential linear repetitive processes system (2.58) is stable along the pass, $\forall \xi^0 = \begin{bmatrix} x_{k+1}(0) \\ y_0(t) \end{bmatrix} \in \varepsilon(P, \rho)$.

Proof Assume that condition (2.62) holds; then using the condition of stability (1.86) for the closed-loop system given by (2.59), one obtains $\Upsilon(s) < 0$, for $s = 1, \dots, N$. $\square$

The result in Theorem 2.5 provides the stabilizability along the pass for the closed-loop system. The LMI formulation of these conditions is now derived, which makes it possible to synthesize the state-feedback saturating controller.

Corollary 2.5 For a given scalar $\rho > 0$, if there exist matrices $Z_1, Z_2, U_1, U_2, W_1 = W_1^T > 0$, and $W_2 = W_2^T > 0$ such that the following LMIs hold true:

$$\Psi(s) = \begin{bmatrix} \Psi_{11}^s + \Psi_{11}^{sT} & \Psi_{12}^s & \Psi_{13}^s \\ * & -W_2 & \Psi_{23}^s \\ * & * & -W_2 \end{bmatrix} < 0, s = 1, \dots, N \quad (2.63)$$

$$\begin{bmatrix} \mu (U_1)_i & (U_2)_i \\ * & W_1 & 0 \\ * & * & W_2 \end{bmatrix} > 0, s = 1, \dots, m, \quad (2.64)$$

where $(U_1)_i$ and $(U_2)_i$ denote for the i th row of matrices U_1 and U_2 , respectively, $\mu = 1/\rho$, and matrices $\Psi_{11}^s, \Psi_{12}^s, \Psi_{13}^s, \Psi_{23}^s$ are given by

$$\begin{aligned} \Psi_{11}^s &= A W_1 + B(D_s Z_1 + D_s^- U_1), \\ \Psi_{12}^s &= B_0 W_2 + B(D_s Z_2 + D_s^- U_2), \\ \Psi_{13}^s &= W_1 C^T + (Z_1^T D_s^T + U_1^T D_s^{-T}) D^T, \\ \Psi_{23}^s &= W_2 D_0^T + (Z_2^T D_s^T + U_2^T D_s^{-T}) D^T, \end{aligned}$$

then the repetitive system (2.58) is stable along the pass, $\forall \xi^0 = \begin{bmatrix} x_{k+1}(0) \\ y_0(t) \end{bmatrix} \in \varepsilon(P, \rho)$ with $P = \text{diag}(P_1, P_2)$, when the controller gain is given as

$$K = [Z_1 W_1^{-1} \quad Z_2 W_2^{-1}]. \quad (2.65)$$

Moreover, the set $\mathcal{L}(H)$ is given as (1.64) with

$$K = [U_1 W_1^{-1} \quad U_2 W_2^{-1}]. \quad (2.66)$$

Proof Post- and pre-multiplying $\Upsilon(s)$ by the following matrix:

$$\Theta = \text{diag}(P_1^{-1}, P_2^{-1}, P_2^{-1}),$$

and then replacing matrices $\tilde{A}^s, \tilde{B}_0^s, \tilde{C}^s$ and $\tilde{D}_0^s$ with their expressions in (2.60) $\forall s \in [1; N]$, one obtains (2.63) with $W = \text{diag}(W_1, W_2)$, $W_i = P_i^{-1}$, $Z_i = K_i W_i$ and $U_i = H_i W_i$, for $i = 1, 2$.

On the other hand, the inclusion (2.62) is equivalent to $\rho(H)_i P^{-1}(H)_i^T \leq 1$, $i = 1, \dots, l$. Developing equivalently gives $\rho(HW)_i W^{-1}(HW)_i^T \leq 1$, that is, $\rho(U)_i W^{-1}(U)_i^T \leq 1$. Then, using Schur complement, one obtains:

$$\begin{bmatrix} \mu (U)_i \\ * & W \end{bmatrix} > 0, i = 1, \dots, m. \quad (2.67)$$

Fig. 2.8 Evolution of the first component of x_{k+1} (gauge) using the saturating controller from Example 2.5

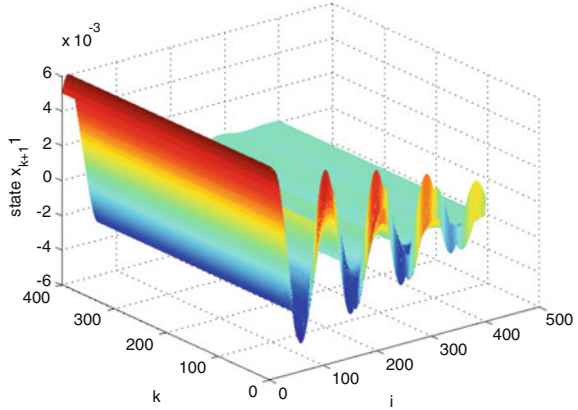
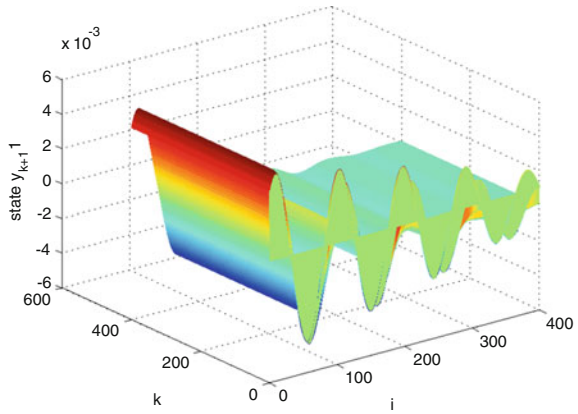


Fig. 2.9 Evolution of the second component of x_{k+1} (gauge derivative) using the saturating controller from Example 2.5



Finally, using $\mu = 1/\rho$, $W = \text{diag}(W_1, W_2)$ and $U = [U_1, U_2]$, the LMIs (2.64) follows. $\square$

Example 2.5 Consider the metal rolling plant studied in Sect. 1.5.1. LMIs (2.63) and (2.64) are feasible. The simulation results are plotted in Figs. 2.8, 2.9, 2.10, 2.11, 2.12 and 2.13.

2.4.2.2 Unsaturating Controller

Consider now the differential linear repetitive processes saturated system (2.55). In the previous section, the design of a saturating controller was studied (saturation of control was allowed), whereas in this section, saturation is not allowed and the synthesis will guarantee that the state evolves inside a region of linear behavior given by $\mathcal{L}(F)$ (F being the controller gain). Thus, this case can be seen as a particular case of the saturating one.

Fig. 2.10 Evolution of the second component of z_k (gauge derivative) using the unsaturating controller from Example 2.6

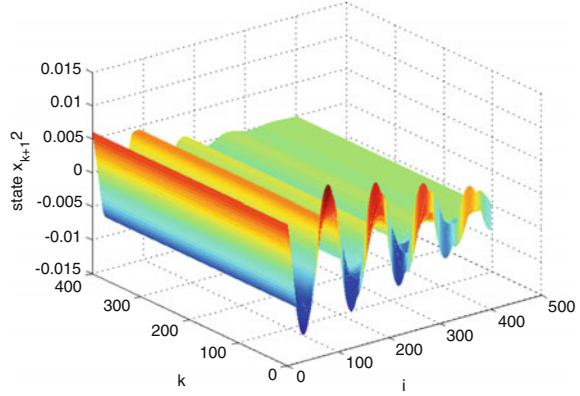
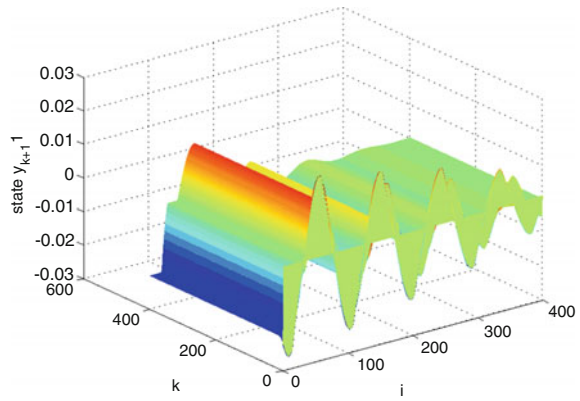


Fig. 2.11 Evolution of the second component of z_k (gauge second derivative) when using the saturating controller from Example 2.5



Theorem 2.6 For a given scalar $\rho > 0$, if there exist matrices $F_1 \in \mathbb{R}^{m \times n}$, $F_2 \in \mathbb{R}^{m \times p}$, and symmetric positive definite matrices $0 < P_1 \in \mathbb{R}^{n \times n}$ and $0 < P_2 \in \mathbb{R}^{p \times p}$ such that the following LMI conditions hold true:

$$\Upsilon(s) = \begin{bmatrix} P_1 \tilde{A} + \tilde{A}^T P_1 & P_1 \tilde{B}_0 & \tilde{C}^T P_2 \\ * & -P_2 & \tilde{D}_0^T P_2 \\ * & * & -P_2 \end{bmatrix} < 0, \quad (2.68)$$

$$\varepsilon(P, \rho) \subset \mathcal{L}(F) \quad (2.69)$$

where

$$\begin{bmatrix} \tilde{A} & \tilde{B}_0 \\ \tilde{C} & \tilde{D}_0 \end{bmatrix} = \begin{bmatrix} A + BF_1 & B_0 + BF_2 \\ C + DF_1 & D_0 + DF_2 \end{bmatrix}$$

and $P = \text{diag}(P_1, P_2)$, then the differential linear repetitive processes system (2.58)

is stable along the pass, $\forall \xi^0 = \begin{bmatrix} x_{k+1}(0) \\ y_0(t) \end{bmatrix} \in \varepsilon(P, \rho)$.

Fig. 2.12 Evolution of the first component of the control signal u_k when using the saturating controller from Example 2.5

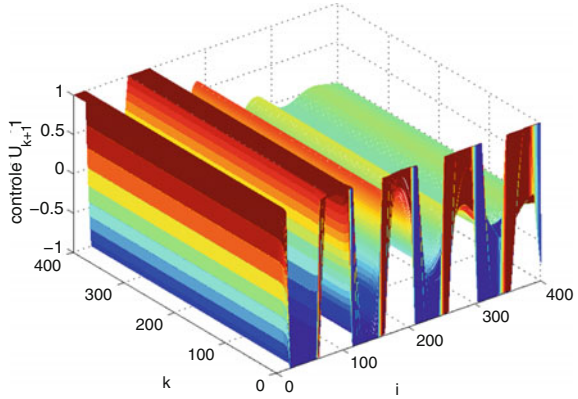
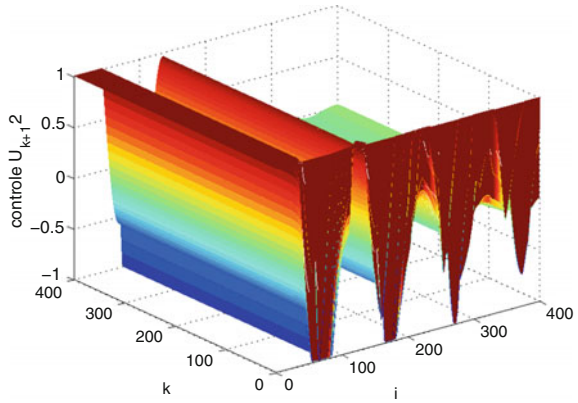


Fig. 2.13 Evolution of the second component of the control signal u_k when using the saturating controller from Example 2.5



Proof The proof follows readily if one replaces K by F in the proof of Theorem 2.5 and removes the saturated convex writing of the control. This can be done, as in this case, the state is restricted to evolve inside the linear region of behavior given by condition (2.69). $\square$

In the next result, the LMI formulation of these conditions that enables the unsaturating state-feedback control to be derived is given as

Corollary 2.6 For a given scalar $\rho > 0$, if there exist matrices $Z_1, Z_2, W_1 = W_1^T > 0$, and $W_2 = W_2^T > 0$ such that the following LMIs hold true:

$$\begin{bmatrix} \Psi_{11} + \Psi_{11}^T & \Psi_{12} & \Psi_{13} \\ * & -W_2 & \Psi_{23} \\ * & * & -W_2 \end{bmatrix} < 0, \quad (2.70)$$

Fig. 2.14 Evolution of the first component of x_{k+1} (gauge) using the unsaturating controller from Example 2.6

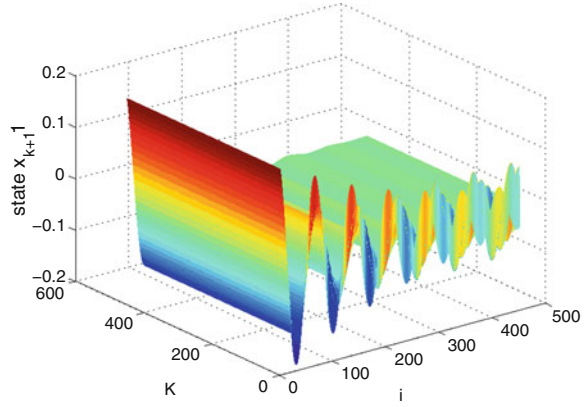


Fig. 2.15 Evolution of the second component of x_{k+1} (gauge derivative) using the unsaturating controller from Example 2.6

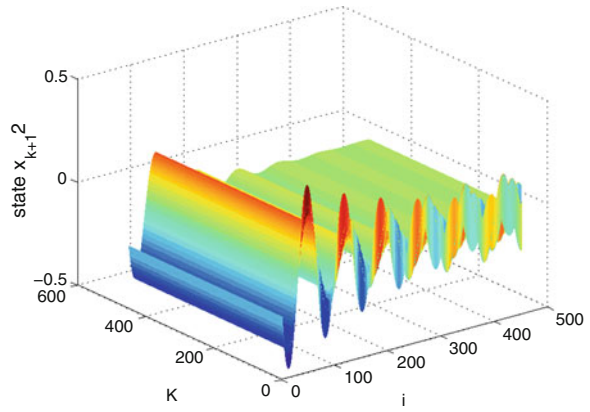
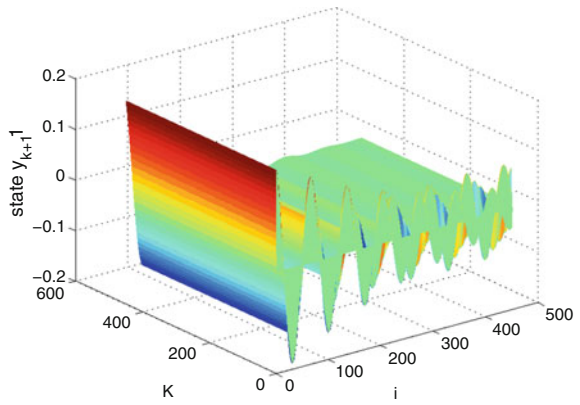


Fig. 2.16 Evolution of the second component of z_k (gauge derivative) using the unsaturating controller from Example 2.6



$$\begin{bmatrix} \mu & (Z_1)_i & (Z_2)_i \\ * & W_1 & 0 \\ * & * & W_2 \end{bmatrix} > 0, s = 1, \dots, m \quad (2.71)$$

Fig. 2.17 Evolution of the second component of z_k (gauge second derivative) when using the unsaturating controller from Example 2.6

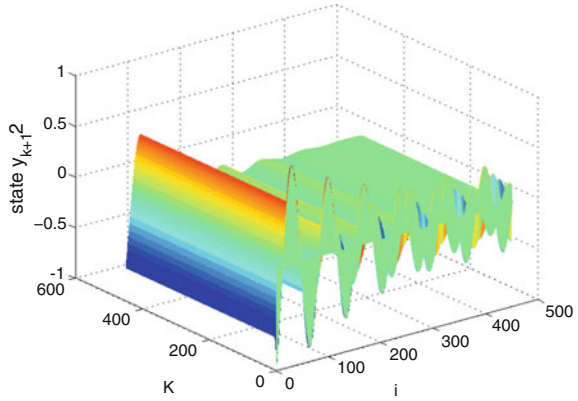


Fig. 2.18 Evolution of the first component of the control signal u_k when using the unsaturating controller from Example 2.6

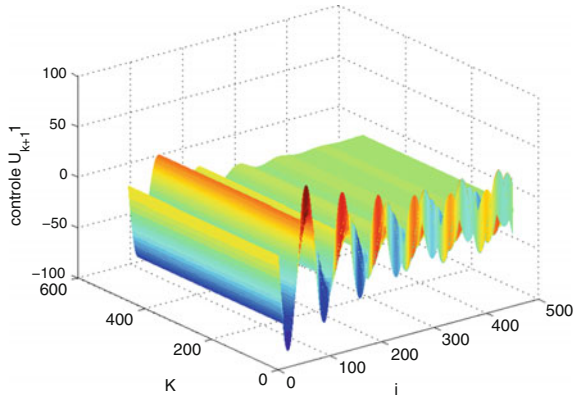
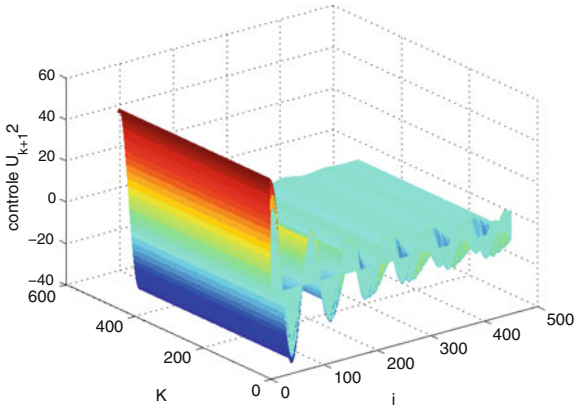


Fig. 2.19 Evolution of the second component of the control signal u_k when using the unsaturating controller from Example 2.6



where $(Z_1)_i$ and $(Z_2)_i$ hold for the i th row of matrices Z_1 and Z_2 respectively; $\mu = 1/\rho$, while matrices Ψ_{11} , Ψ_{12} , Ψ_{13} , Ψ_{23} are given as

$$\Psi_{11} = AW_1 + BZ_1$$

$$\begin{aligned}\Psi_{12} &= B_0 W_2 + B Z_2 \\ \Psi_{13} &= W_1 C^T + Z_1^T D^T \\ \Psi_{23} &= W_2 D_0^T + Z_2^T D^T,\end{aligned}$$

then the differential linear repetitive processes system (2.58) is stable along the pass, $\forall \xi^0 = \begin{bmatrix} x_{k+1}(0) \\ y_0(t) \end{bmatrix} \in \varepsilon(P, \rho)$ with $P = \text{diag}(W_1^{-1}, W_2^{-1})$, when the controller gain is given as

$$F = \begin{bmatrix} Z_1 W_1^{-1} & Z_2 W_2^{-1} \end{bmatrix}. \quad (2.72)$$

Proof The proof follows readily from Corollary 2.5.

Example 2.6 Consider the same system studied in the saturating controller case of Example 2.5. If no saturation is allowed in the control signal, Corollary 2.6 can be used to synthesize the controller. In this case, the LMIs (2.70) and (2.71) are feasible, with a solution given as

$$F_1 = [231.9679 \quad 166.5527], \quad F_2 = [627.1912 \quad 104.5319].$$

Some simulations are presented in Figs. 2.14, 2.15, 2.16, 2.17, 2.18, and 2.19 for the same boundary conditions as those presented for Example 2.5. It can be seen that the proposed controller effectively ensures the stability of the closed-loop 2-D system.

2.5 Conclusion

In this chapter, the problem of stabilization under state-feedback control of several classes of 2-D saturated systems is studied. Sufficient conditions of asymptotic stability are derived for each case using a common approach, with the synthesis of the required controllers given in LMI form.

First, the problem of stabilizability of continuous 2-D saturated systems has been studied for Roesser models. Two different cases were considered: saturating and unsaturating controllers. The first allows saturation to take effect, while the second limits the system's evolution to the region of linear behavior. Then the problem of stabilizability of discrete 2-D saturated system was studied for Roesser models, using a parallel approach, for the two cases: saturating and unsaturating controllers.

The final part of the chapter shows how the proposed approach can be extended to other classes of 2-D systems, as a relevant example, repetitive systems with saturation have been studied. Stability along the pass is guaranteed for both saturating and unsaturating controllers.

It must be pointed out that the required conditions can be easily used in practice, as the synthesis of the required controllers is given in LMI form; some numerical examples are provided to illustrate the results.

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