Chapter 2
Rate of Convergence of Basic Multivariate Neural Network Operators to the Unit

This chapter deals with the determination of the rate of convergence to the unit of some multivariate neural network operators, namely the normalized "bell" and "squashing" type operators. This is given through the multidimensional modulus of continuity of the involved multivariate function or its partial derivatives of specific order that appear in the right-hand side of the associated multivariate Jackson type inequality. It follows [3].

2.1 Introduction

The multivariate Cardaliaguet-Euvrard operators were first introduced and studied thoroughly in [4], where the authors among many other interesting things proved that these multivariate operators converge uniformly on compacta, to the unit over continuous and bounded multivariate functions. Our multivariate normalized "bell" and "squashing" type operators (2.1) and (2.16) were motivated and inspired by the "bell" and "squashing" functions of [4].

The work in [4] is qualitative where the used multivariate bell-shaped function is general. However, though our work is greatly motivated by [4], it is quantitative and the used multivariate "bell-shaped" and "squashing" functions are of compact support.

This paper is the continuation and simplification of [1, 2], in the multidimensional case. We produce a set of multivariate inequalities giving close upper bounds to the errors in approximating the unit operator by the above multidimensional neural network induced operators. All appearing constants there are well determined. These are mainly pointwise estimates involving the first multivariate modulus of continuity of the engaged multivariate continuous function or its partial derivatives of some fixed order.

G.A. Anastassiou, *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Studies in Computational Intelligence 608,
DOI 10.1007/978-3-319-20505-2_2

2.2 Convergence with Rates of Multivariate Neural Network Operators

We need the following (see [4]) definitions.

Definition 2.1 A function $b : \mathbb{R} \to \mathbb{R}$ is said to be bell-shaped if b belongs to L^1 and its integral is nonzero, if it is nondecreasing on $(-\infty, a)$ and nonincreasing on $[a, +\infty)$, where a belongs to $\mathbb{R}$. In particular $b(x)$ is a nonnegative number and at a, b takes a global maximum; it is the center of the bell-shaped function. A bell-shaped function is said to be centered if its center is zero.

Definition 2.2 (*see* [4]) A function $b : \mathbb{R}^d \to \mathbb{R}$ $(d \geq 1)$ is said to be a d-dimensional bell-shaped function if it is integrable and its integral is not zero, and for all $i = 1, \ldots, d$,

$$t \to b(x_1, \ldots, t, \ldots, x_d)$$

is a centered bell-shaped function, where $\overrightarrow{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$ arbitrary.

Example 2.3 (from [4]) Let b be a centered bell-shaped function over $\mathbb{R}$, then $(x_1, \ldots, x_d) \to b(x_1) \ldots b(x_d)$ is a d-dimensional bell-shaped function.

Assumption 2.4 Here $b\left(\overrightarrow{x}\right)$ is of compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and it may have jump discontinuities there. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a continuous and bounded function or a uniformly continuous function.

In this chapter, we study the pointwise convergence with rates over $\mathbb{R}^d$, to the unit operator, of the "normalized bell" multivariate neural network operators

$$M_n(f)\left(\overrightarrow{x}\right) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}, \tag{2.1}$$

where $0 < \alpha < 1$ and $\overrightarrow{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$, $n \in \mathbb{N}$. Clearly M_n is a positive linear operator.

The terms in the ratio of multiple sums (2.1) can be nonzero iff simultaneously

$$\left|n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right)\right| \leq T_i, \text{ all } i = 1, \ldots, d,$$

i.e., $\left|x_i - \frac{k_i}{n}\right| \leq \frac{T_i}{n^{1-\alpha}}$, all $i = 1, \ldots, d$, iff

$$nx_i - T_i n^{\alpha} \leq k_i \leq nx_i + T_i n^{\alpha}, \text{ all } i = 1, \ldots, d. \tag{2.2}$$

To have the order

$$-n^2 \leq nx_i - T_i n^\alpha \leq k_i \leq nx_i + T_i n^\alpha \leq n^2, \tag{2.3}$$

we need $n \geq T_i + |x_i|$, all $i = 1, \ldots, d$. So (2.3) is true when we take

$$n \geq \max_{i \in \{1,\ldots,d\}} (T_i + |x_i|). \tag{2.4}$$

When $\overrightarrow{x} \in \mathcal{B}$ in order to have (2.3) it is enough to assume that $n \geq 2T^*$, where $T^* := \max\{T_1, \ldots, T_d\} > 0$. Consider

$$\widetilde{I}_i := \left[nx_i - T_i n^\alpha, nx_i + T_i n^\alpha\right], i = 1, \ldots, d, \ n \in \mathbb{N}.$$

The length of $\widetilde{I}_i$ is $2T_i n^\alpha$. By Proposition 1 of [1], we get that the cardinality of $k_i \in \mathbb{Z}$ that belong to $\widetilde{I}_i := card\,(k_i) \geq \max\,(2T_i n^\alpha - 1, 0)$, any $i \in \{1, \ldots, d\}$. In order to have $card\,(k_i) \geq 1$, we need $2T_i n^\alpha - 1 \geq 1$ iff $n \geq T_i^{-\frac{1}{\alpha}}$, any $i \in \{1, \ldots, d\}$.

Therefore, a sufficient condition in order to obtain the order (2.3) along with the interval $\widetilde{I}_i$ to contain at least one integer for all $i = 1, \ldots, d$ is that

$$n \geq \max_{i \in \{1,\ldots,d\}} \left\{T_i + |x_i|, T_i^{-\frac{1}{\alpha}}\right\}. \tag{2.5}$$

Clearly as $n \to +\infty$ we get that $card\,(k_i) \to +\infty$, all $i = 1, \ldots, d$. Also notice that $card\,(k_i)$ equals to the cardinality of integers in $[\lceil nx_i - T_i n^\alpha \rceil, [nx_i + T_i n^\alpha]]$ for all $i = 1, \ldots, d$. Here, $[\cdot]$ denotes the integral part of the number while $\lceil \cdot \rceil$ denotes its ceiling.

From now on, in this chapter we will assume (2.5). Furthermore it holds

$$(M_n(f))\left(\overrightarrow{x}\right) = \frac{\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha]} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right)}{V\left(\overrightarrow{x}\right)} \cdot \tag{2.6}$$

$$b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)$$

all $\overrightarrow{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d$, where

$$V\left(\overrightarrow{x}\right) :=$$

$$\sum_{k_1=\lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha]} \cdots \sum_{k_d=\lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha]} b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right).$$

Denote by $\|\cdot\|_\infty$ the maximum norm on $\mathbb{R}^d$, $d \geq 1$. So if $\left|n^{1-\alpha}\left(x_i - \frac{k_i}{n}\right)\right| \leq T_i$, all $i = 1, \ldots, d$, we get that

$$\left\| \overrightarrow{x} - \frac{\overrightarrow{k}}{n} \right\|_\infty \leq \frac{T^*}{n^{1-\alpha}},$$

where $\overrightarrow{k} := (k_1, \ldots, k_d)$.

Definition 2.5 Let $f : \mathbb{R}^d \to \mathbb{R}$. We call

$$\omega_1(f, h) := \sup_{\substack{all\ \overrightarrow{x}, \overrightarrow{y}: \\ \|\overrightarrow{x} - \overrightarrow{y}\|_\infty \leq h}} \left| f\left(\overrightarrow{x}\right) - f\left(\overrightarrow{y}\right)\right|, \tag{2.7}$$

where $h > 0$, the first modulus of continuity of f.

Here is our first main result.

Theorem 2.6 *Let $\overrightarrow{x} \in \mathbb{R}^d$; then*

$$\left|(M_n(f))\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right)\right| \leq \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{2.8}$$

Inequality (2.8) is attained by constant functions.

Inequality (2.8) gives $M_n(f)\left(\overrightarrow{x}\right) \to f\left(\overrightarrow{x}\right)$, pointwise with rates, as $n \to +\infty$, where $\overrightarrow{x} \in \mathbb{R}^d$, $d \geq 1$.

Proof Next, we estimate

$$\left|(M_n(f))\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right)\right| \overset{(2.6)}{=}$$

$$\left| \sum_{k_1 = \lceil nx_1 - T_1 n^\alpha \rceil}^{[nx_1 + T_1 n^\alpha]} \cdots \sum_{k_d = \lceil nx_d - T_d n^\alpha \rceil}^{[nx_d + T_d n^\alpha]} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) \cdot \right.$$

$$\left. \frac{b\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} - f\left(\overrightarrow{x}\right) \right| =$$

$$\left| \frac{\sum_{\overrightarrow{k} = \left\lceil n\overrightarrow{x} - \overrightarrow{T} n^\alpha \right\rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T} n^\alpha\right]} \left(f\left(\frac{\overrightarrow{k}}{n}\right) - f\left(\overrightarrow{x}\right)\right) b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} \right| \leq$$

$$\sum_{\vec{k}=\lceil n\vec{x}-\vec{T}n^{\alpha}\rceil}^{[n\vec{x}+\vec{T}n^{\alpha}]} \frac{\left|f\left(\frac{\vec{k}}{n}\right)-f\left(\vec{x}\right)\right|}{V\left(\vec{x}\right)} b\left(n^{1-\alpha}\left(\vec{x}-\frac{\vec{k}}{n}\right)\right) \leq$$

$$\sum_{\vec{k}=\lceil n\vec{x}-\vec{T}n^{\alpha}\rceil}^{[n\vec{x}+\vec{T}n^{\alpha}]} \frac{\omega_1\left(f, \left\|\vec{x}-\frac{\vec{k}}{n}\right\|_{\infty}\right)}{V\left(\vec{x}\right)} b\left(n^{1-\alpha}\left(\vec{x}-\frac{\vec{k}}{n}\right)\right).$$

That is

$$\left|\left(M_n\left(f\right)\right)\left(\vec{x}\right)-f\left(\vec{x}\right)\right| \leq \frac{\omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right)}{V\left(\vec{x}\right)}\cdot$$

$$\sum_{k_1=\lceil nx_1-T_1n^{\alpha}\rceil}^{[nx_1+T_1n^{\alpha}]} \cdots \sum_{k_d=\lceil nx_d-T_dn^{\alpha}\rceil}^{[nx_d+T_dn^{\alpha}]} b\left(n^{1-\alpha}\left(x_1-\frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d-\frac{k_d}{n}\right)\right)$$

$$= \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right), \tag{2.9}$$

proving the claim. ■

Our second main result follows.

Theorem 2.7 *Let $\vec{x} \in \mathbb{R}^d$, $f \in C^N\left(\mathbb{R}^d\right)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are uniformly continuous or continuous are bounded. Then,*

$$\left|\left(M_n\left(f\right)\right)\left(\vec{x}\right)-f\left(\vec{x}\right)\right| \leq \tag{2.10}$$

$$\left\{\sum_{j=1}^{N} \frac{\left(T^*\right)^j}{j!n^{j(1-\alpha)}}\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f\left(\vec{x}\right)\right)\right\}+$$

$$\frac{\left(T^*\right)^N d^N}{N!n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1\left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}}\right).$$

Inequality (2.10) is attained by constant functions. Also, (2.10) gives us with rates the pointwise convergence of $M_n\left(f\right) \to f$ over $\mathbb{R}^d$, as $n \to +\infty$.

Proof Set

$$g_{\frac{\vec{k}}{n}}\left(t\right) := f\left(\vec{x}+t\left(\frac{\vec{k}}{n}-\vec{x}\right)\right), \ 0 \leq t \leq 1.$$

Then

$$g_{\frac{\overrightarrow{k}}{n}}^{(j)}(t) = \left[\left(\sum_{i=1}^{d} \left(\frac{k_i}{n} - x_i \right) \frac{\partial}{\partial x_i} \right)^j f \right] \left(x_1 + t \left(\frac{k_1}{n} - x_1 \right), \ldots, x_d + t \left(\frac{k_d}{n} - x_d \right) \right)$$

and $g_{\frac{\overrightarrow{k}}{n}}(0) = f\left(\overrightarrow{x}\right)$. By Taylor's formula, we get

$$f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) = g_{\frac{\overrightarrow{k}}{n}}(1) = \sum_{j=0}^{N} \frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)}{j!} + R_N\left(\frac{\overrightarrow{k}}{n}, 0\right),$$

where

$$R_N\left(\frac{\overrightarrow{k}}{n}, 0\right) = \int_0^1 \left(\int_0^{t_1} \ldots \left(\int_0^{t_{N-1}} \left(g_{\frac{\overrightarrow{k}}{n}}^{(N)}(t_N) - g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0) \right) dt_N \right) \ldots \right) dt_1.$$

Here we denote by

$$f_{\widetilde{\alpha}} := \frac{\partial^{\widetilde{\alpha}} f}{\partial x^{\widetilde{\alpha}}}, \ \widetilde{\alpha} := (\alpha_1, \ldots, \alpha_d), \alpha_i \in \mathbb{Z}^+,$$

$i = 1, \ldots, d$, such that $|\widetilde{\alpha}| := \sum_{i=1}^{d} \alpha_i = N$. Thus,

$$\frac{f\left(\frac{\overrightarrow{k}}{n}\right) b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} =$$

$$\sum_{j=0}^{N} \frac{g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0)}{j!} \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} + \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} \cdot R_N\left(\frac{\overrightarrow{k}}{n}, 0\right).$$

Therefore

$$\left(M_n(f)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right) =$$

$$\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha} \rceil}^{\lfloor n\overrightarrow{x} + \overrightarrow{T} n^{\alpha} \rfloor} \frac{f\left(\frac{\overrightarrow{k}}{n}\right)}{V\left(\overrightarrow{x}\right)} b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right) - f\left(\overrightarrow{x}\right) =$$

$$\sum_{j=1}^{N} \frac{1}{j!} \left(\sum_{\overrightarrow{k} = \lceil n\overrightarrow{x} - \overrightarrow{T} n^{\alpha} \rceil}^{\lfloor n\overrightarrow{x} + \overrightarrow{T} n^{\alpha} \rfloor} g_{\frac{\overrightarrow{k}}{n}}^{(j)}(0) \frac{b\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)} \right) + R^*,$$

where

$$R^{*}:=\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\cdot R_{N}\left(\frac{\overrightarrow{k}}{n},0\right).$$

Consequently, we obtain

$$\left|\left(M_{n}\left(f\right)\right)\left(\overrightarrow{x}\right)-f\left(\overrightarrow{x}\right)\right|\leq$$

$$\sum_{j=1}^{N}\frac{1}{j!}\left(\sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]}\frac{\left|g_{\frac{\overrightarrow{k}}{n}}^{(j)}\left(0\right)\right|b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\right)+\left|R^{*}\right|=:\Theta.$$

Notice that

$$\left|g_{\frac{\overrightarrow{k}}{n}}^{(j)}\left(0\right)\right|\leq\left(\frac{T^{*}}{n^{1-\alpha}}\right)^{j}\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_{i}}\right|\right)^{j}f\left(\overrightarrow{x}\right)\right)$$

and

$$\Theta\leq\left\{\sum_{j=1}^{N}\frac{1}{j!}\left(\frac{T^{*}}{n^{1-\alpha}}\right)^{j}\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_{i}}\right|\right)^{j}f\left(\overrightarrow{x}\right)\right)\right\}+\left|R^{*}\right|. \tag{2.11}$$

That is, by (2.11), we get

$$\left|\left(M_{n}\left(f\right)\right)\left(\overrightarrow{x}\right)-f\left(\overrightarrow{x}\right)\right|\leq$$

$$\left\{\sum_{j=1}^{N}\frac{\left(T^{*}\right)^{j}}{j!n^{j(1-\alpha)}}\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_{i}}\right|\right)^{j}f\left(\overrightarrow{x}\right)\right)\right\}+\left|R^{*}\right|. \tag{2.12}$$

Next, we need to estimate $|R^{*}|$. For that, we observe $(0\leq t_{N}\leq 1)$

$$\left|g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(t_{N}\right)-g_{\frac{\overrightarrow{k}}{n}}^{(N)}\left(0\right)\right|=$$

$$\left|\left(\sum_{i=1}^{d}\left(\frac{k_{i}}{n}-x_{i}\right)\frac{\partial}{\partial x_{i}}\right)^{N}f\left(\overrightarrow{x}+t_{N}\left(\frac{\overrightarrow{k}}{n}-\overrightarrow{x}\right)\right)-\right.$$

$$\left(\sum_{i=1}^{d}\left(\frac{k_i}{n}-x_i\right)\frac{\partial}{\partial x_i}\right)^N f\left(\overrightarrow{x}\right)\Bigg|$$
$$\leq \frac{(T^*)^N d^N}{n^{N(1-\alpha)}}\cdot\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}\right).$$

Thus,

$$\left|R_N\left(\frac{\overrightarrow{k}}{n},0\right)\right| \leq \int_0^1\left(\int_0^{t_1}\ldots\left(\int_0^{t_{N-1}}\left|g_{\frac{\overrightarrow{k}}{n}}^{(N)}(t_N)-g_{\frac{\overrightarrow{k}}{n}}^{(N)}(0)\right|dt_N\right)\ldots\right)dt_1$$

$$\leq \frac{(T^*)^N d^N}{N!n^{N(1-\alpha)}}\cdot\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}\right).$$

Therefore,

$$\left|R^*\right| \leq \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x}-\overrightarrow{T}n^{\alpha}\rceil}^{[n\overrightarrow{x}+\overrightarrow{T}n^{\alpha}]}\frac{b\left(n^{1-\alpha}\left(\overrightarrow{x}-\frac{\overrightarrow{k}}{n}\right)\right)}{V\left(\overrightarrow{x}\right)}\left|R_N\left(\frac{\overrightarrow{k}}{n},0\right)\right|$$

$$\leq \frac{(T^*)^N d^N}{N!n^{N(1-\alpha)}}\cdot\max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N}\omega_1\left(f_{\widetilde{\alpha}},\frac{T^*}{n^{1-\alpha}}\right). \tag{2.13}$$

By (2.12) and (2.13) we get (2.10). ■

Corollary 2.8 *Here, additionally assume that b is continuous on $\mathbb{R}^d$. Let*

$$\Gamma := \prod_{i=1}^{d}[-\gamma_i,\gamma_i]\subset\mathbb{R}^d,\ \gamma_i>0,$$

and take

$$n \geq \max_{i\in\{1,\ldots,d\}}\left(T_i+\gamma_i, T_i^{-\frac{1}{\alpha}}\right).$$

Consider $p\geq 1$. Then,

$$\|M_n f - f\|_{p,\Gamma} \leq \omega_1\left(f,\frac{T^*}{n^{1-\alpha}}\right)2^{\frac{d}{p}}\prod_{i=1}^{d}\gamma_i^{\frac{1}{p}}, \tag{2.14}$$

attained by constant functions. From (2.14), we get the L_p convergence of $M_n f$ to f with rates.

Proof By (2.8). ■

Corollary 2.9 *Same assumptions as in Corollary 2.8. Then*

$$\|M_n f - f\|_{p,\Gamma} \leq \left\{ \sum_{j=1}^{N} \frac{(T^*)^j}{j! n^{j(1-\alpha)}} \left\| \left(\sum_{i=1}^{d} \left| \frac{\partial}{\partial x_i} \right| \right)^j f \right\|_{p,\Gamma} \right\} +$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right) 2^{\frac{d}{p}} \prod_{i=1}^{d} \gamma_i^{\frac{1}{p}}, \tag{2.15}$$

attained by constants. Here, from (2.15), we get again the L_p convergence of $M_n(f)$ to f with rates.

Proof By the use of (2.10). ■

2.3 The Multivariate "Normalized Squashing Type Operators" and Their Convergence to the Unit with Rates

We give the following definition

Definition 2.10 Let the nonnegative function $S : \mathbb{R}^d \to \mathbb{R}$, $d \geq 1$, S has compact support $\mathcal{B} := \prod_{i=1}^{d} [-T_i, T_i]$, $T_i > 0$ and is nondecreasing there for each coordinate. S can be continuous only on either $\prod_{i=1}^{d} (-\infty, T_i]$ or $\mathcal{B}$ and can have jump discontinuities. We call S the multivariate "squashing function" (see also [4]).

Example 2.11 Let $\widehat{S}$ as above when $d = 1$. Then,

$$S\left(\overrightarrow{x}\right) := \widehat{S}(x_1) \ldots \widehat{S}(x_d), \ \overrightarrow{x} := (x_1, \ldots, x_d) \in \mathbb{R}^d,$$

is a multivariate "squashing function".

Let $f : \mathbb{R}^d \to \mathbb{R}$ be either uniformly continuous or continuous and bounded function.

For $\overrightarrow{x} \in \mathbb{R}^d$, we define the multivariate "normalized squashing type operator",

$$L_n(f)\left(\overrightarrow{x}\right) :=$$

$$\frac{\sum_{k_1=-n^2}^{n^2} \cdots \sum_{k_d=-n^2}^{n^2} f\left(\frac{k_1}{n}, \ldots, \frac{k_d}{n}\right) S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \ldots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right)}{W\left(\overrightarrow{x}\right)}, \tag{2.16}$$

where $0 < \alpha < 1$ and $n \in \mathbb{N}$:

$$n \geq \max_{i \in \{1,\dots,d\}} \left\{ T_i + |x_i| \,, T_i^{-\frac{1}{\alpha}} \right\}, \tag{2.17}$$

and

$$W\left(\overrightarrow{x}\right) := \sum_{k_1=-n^2}^{n^2} \dots \sum_{k_d=-n^2}^{n^2} S\left(n^{1-\alpha}\left(x_1 - \frac{k_1}{n}\right), \dots, n^{1-\alpha}\left(x_d - \frac{k_d}{n}\right)\right). \tag{2.18}$$

Obviously L_n is a positive linear operator. It is clear that

$$\left(L_n(f)\right)\left(\overrightarrow{x}\right) = \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T}n^{\alpha}\rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T}n^{\alpha}\right]} \frac{f\left(\frac{\overrightarrow{k}}{n}\right)}{\Phi\left(\overrightarrow{x}\right)} S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right), \tag{2.19}$$

where

$$\Phi\left(\overrightarrow{x}\right) := \sum_{\overrightarrow{k}=\lceil n\overrightarrow{x} - \overrightarrow{T}n^{\alpha}\rceil}^{\left[n\overrightarrow{x} + \overrightarrow{T}n^{\alpha}\right]} S\left(n^{1-\alpha}\left(\overrightarrow{x} - \frac{\overrightarrow{k}}{n}\right)\right). \tag{2.20}$$

Here, we study the pointwise convergence with rates of $\left(L_n(f)\right)\left(\overrightarrow{x}\right) \to f\left(\overrightarrow{x}\right)$, as $n \to +\infty$, $\overrightarrow{x} \in \mathbb{R}^d$.

This is given by the next result.

Theorem 2.12 *Under the above terms and assumptions, we find that*

$$\left|\left(L_n(f)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right)\right| \leq \omega_1\left(f, \frac{T^*}{n^{1-\alpha}}\right). \tag{2.21}$$

Inequality (2.21) is attained by constant functions.

Proof Similar to (2.8). ■

We also give

Theorem 2.13 *Let $\overrightarrow{x} \in \mathbb{R}^d$, $f \in C^N\left(\mathbb{R}^d\right)$, $N \in \mathbb{N}$, such that all of its partial derivatives $f_{\widetilde{\alpha}}$ of order N, $\widetilde{\alpha} : |\widetilde{\alpha}| = N$, are uniformly continuous or continuous are bounded. Then,*

$$\left|\left(L_n(f)\right)\left(\overrightarrow{x}\right) - f\left(\overrightarrow{x}\right)\right| \leq \tag{2.22}$$

$$\left\{\sum_{j=1}^{N} \frac{(T^*)^j}{j!n^{j(1-\alpha)}}\left(\left(\sum_{i=1}^{d}\left|\frac{\partial}{\partial x_i}\right|\right)^j f\left(\overrightarrow{x}\right)\right)\right\} +$$

$$\frac{(T^*)^N d^N}{N! n^{N(1-\alpha)}} \cdot \max_{\widetilde{\alpha}:|\widetilde{\alpha}|=N} \omega_1 \left(f_{\widetilde{\alpha}}, \frac{T^*}{n^{1-\alpha}} \right).$$

Inequality (2.22) is attained by constant functions. Also, (2.22) gives us with rates the pointwise convergence of $L_n(f) \to f$ *over* $\mathbb{R}^d$, *as* $n \to +\infty$.

Proof Similar to (2.10). ■

Note 2.14 We see that

$$M_n(1) = L_n(1) = 1.$$

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http://www.springer.com/978-3-319-20504-5

Intelligent Systems II: Complete Approximation by
Neural Network Operators
Anastassiou, G.A.
2016, XV, 712 p. 1 illus., Hardcover
ISBN: 978-3-319-20504-5