Chapter 2
Univariate Fractional Polya Integral Inequalities

Here we establish a series of various fractional Polya type integral inequalities with the help of generalised right and left fractional derivatives. We give an application to complex valued functions defined on the unit circle. It follows [5].

2.1 Introduction

We mention the following famous Polya's integral inequality, see [13], [14, p. 62], [15] and [16, p. 83].

Theorem 2.1 *Let $f(x)$ be differentiable and not identically a constant on $[a,b]$ with $f(a)=f(b)=0$. Then the exists at least one point $\xi \in [a,b]$ such that*

$$\left|f'(\xi)\right| > \frac{4}{(b-a)^2}\int_a^b f(x)\,dx. \tag{2.1}$$

In [17], Feng Qi presents the following very interesting Polya type integral inequality (2.2), which generalizes (2.1).

Theorem 2.2 *Let $f(x)$ be differentiable and not identically constant on $[a,b]$ with $f(a)=f(b)=0$ and $M=\sup\limits_{x\in[a,b]}\left|f'(x)\right|$. Then*

$$\left|\int_a^b f(x)\,dx\right| \le \frac{(b-a)^2}{4}M, \tag{2.2}$$

where $\frac{(b-a)^2}{4}$ in (2.2) is the best constant.

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The above motivate the current chapter.

In this chapter we present univariate fractional Polya type integral inequalities in various cases, similar to (2.2).

For this purpose we need the following fractional calculus background.

Let $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $0 < \beta < 1$, $f \in C([a, b])$, $[a, b] \subset \mathbb{R}$, $x \in [a, b]$. The gamma function Γ is given by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$. We define the left Riemann-Liouville integral

$$\left(J_\alpha^{a+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)\, dt, \tag{2.3}$$

$a \le x \le b$. We define the subspace $C_{a+}^\alpha([a.b])$ of $C^m([a, b])$:

$$C_{a+}^\alpha([a, b]) = \left\{ f \in C^m([a, b]) : J_{1-\beta}^{a+} f^{(m)} \in C^1([a, b]) \right\}. \tag{2.4}$$

For $f \in C_{a+}^\alpha([a, b])$, we define the left generalized α-fractional derivative of f over $[a, b]$ as

$$D_{a+}^\alpha f := \left(J_{1-\beta}^{a+} f^{(m)}\right)', \tag{2.5}$$

see [1], p. 24. Canavati first in [6], introduced the above over $[0, 1]$.

Notice that $D_{a+}^\alpha f \in C([a, b])$.

We need the following left fractional Taylor's formula, see [1], pp. 8–10, and in [6] the same formula over $[0, 1]$ that appeared first.

Theorem 2.3 *Let $f \in C_{a+}^\alpha([a, b])$.*

(i) If $\alpha \ge 1$, then

$$f(x) = f(a) + f'(a)(x - a) + f''(a)\frac{(x - a)^2}{2} + \cdots + f^{(m-1)}(a)\frac{(x - a)^{m-1}}{(m - 1)!} + \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \text{all } x \in [a, b]. \tag{2.6}$$

(ii) If $0 < \alpha < 1$, we have

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} \left(D_{a+}^\alpha f\right)(t)\, dt, \quad \text{all } x \in [a, b]. \tag{2.7}$$

Let again $\alpha > 0$, $m = [\alpha]$, $\beta = \alpha - m$, $f \in C([a, b])$, call the right Riemann-Liouville fractional integral operator by

$$\left(J_{b-}^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t)\, dt, \tag{2.8}$$

$x \in [a, b]$, see also [2, 9–11, 18]. Define the subspace of functions

$$C_{b-}^{\alpha}([a, b]) := \left\{ f \in C^m([a, b]) : J_{b-}^{1-\beta} f^{(m)} \in C^1([a, b]) \right\}. \tag{2.9}$$

Define the right generalized α-fractional derivative of f over $[a, b]$ as

$$D_{b-}^{\alpha} f = (-1)^{m-1} \left(J_{b-}^{1-\beta} f^{(m)} \right)', \tag{2.10}$$

see [2]. We set $D_{b-}^{0} f = f$. Notice that $D_{b-}^{\alpha} f \in C([a, b])$.

We need the following right Taylor fractional formula from [2].

Theorem 2.4 *Let $f \in C_{b-}^{\alpha}([a, b])$, $\alpha > 0$, $m := [\alpha]$. Then*

(i) If $\alpha \geq 1$, we get

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!} (x - b)^k + \left(J_{b-}^{\alpha} D_{b-}^{\alpha} f \right)(x), \quad all\, x \in [a, b]. \tag{2.11}$$

(ii) If $0 < \alpha < 1$, we get

$$f(x) = J_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} \left(D_{b-}^{\alpha} f \right)(t)\, dt, \quad all\, x \in [a, b]. \tag{2.12}$$

Definition 2.5 ([3]) Let $f \in C([a, b])$, $x \in [a, b]$, $\alpha > 0$, $m := [a]$. Assume that $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$ and $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$. We define the balanced Canavati type fractional derivative by

$$D^{\alpha} f(x) := \begin{cases} D_{b-}^{\alpha} f(x), & \text{for } \frac{a+b}{2} \leq x \leq b, \\ D_{a+}^{\alpha} f(x), & \text{for } a \leq x < \frac{a+b}{2}. \end{cases} \tag{2.13}$$

In [4] we proved the following fractional Polya type integral inequality without any boundary conditions.

Theorem 2.6 *Let $0 < \alpha < 1$, $f \in C([a, b])$. Assume $f \in C_{a+}^{\alpha}\left(\left[a, \frac{a+b}{2}\right]\right)$ and $f \in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2}, b\right]\right)$. Set*

$$M_1(f) = \max \left\{ \left\| D_{a+}^{\alpha} f \right\|_{\infty, \left[a, \frac{a+b}{2}\right]}, \left\| D_{b-}^{\alpha} f \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right\}. \tag{2.14}$$

Then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx \leq \tag{2.15}$$

$$\frac{\left(\left\|D_{a+}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}+\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma(\alpha+2)}\left(\frac{b-a}{2}\right)^{\alpha+1}\leq M_{1}(f)\,\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)\,2^{\alpha}}. \tag{2.16}$$

Inequalities (2.15), (2.16) are sharp, namely they are attained by

$$f_{*}(x)=\begin{Bmatrix}(x-a)^{\alpha}, & x\in\left[a,\frac{a+b}{2}\right]\\ (b-x)^{\alpha}, & x\in\left[\frac{a+b}{2},b\right]\end{Bmatrix},\ 0<\alpha<1. \tag{2.17}$$

Clearly here non zero constant functions f are excluded.

The last result also motivates this chapter.

Remark 2.7 (see [4]) When $\alpha\geq 1$, thus $m=[\alpha]\geq 1$, and by assuming that $f^{(k)}(a)=f^{(k)}(b)=0$, $k=0,1,\ldots,m-1$, we can prove the same statements (2.15–2.17) as in Theorem 2.6. If we set there $\alpha=1$ we derive exactly Theorem 2.2 . So we have generalized Theorem 2.2. Again here $f^{(m)}$ cannot be a constant different than zero, equivalently, f cannot be a non-trivial polynomial of degree m.

We continue here with other interesting univariate fractional Polya type inequalities.

2.2 Main Results

We present our first main result.

Theorem 2.8 *Let $\alpha\geq 1$, $m=[\alpha]$, $f\in C([a,b])$. Assume $f\in C_{a+}^{\alpha}\left(\left[a,\frac{a+b}{2}\right]\right)$ and $f\in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2},b\right]\right)$, such that $f^{(k)}(a)=f^{(k)}(b)=0$, $k=0,1,\ldots,m-1$. Set*

$$M_{2}(f)=\max\left\{\left\|D_{a+}^{\alpha}f\right\|_{L_{1}\left(\left[a,\frac{a+b}{2}\right]\right)},\left\|D_{b-}^{\alpha}f\right\|_{L_{1}\left(\left[\frac{a+b}{2},b\right]\right)}\right\}. \tag{2.18}$$

Then

$$\left|\int_{a}^{b}f(x)\,dx\right|\leq\frac{1}{\Gamma(\alpha+1)}\left(\frac{b-a}{2}\right)^{\alpha}\left\|D^{\alpha}f\right\|_{L_{1}([a,b])}\leq\frac{M_{2}(f)}{\Gamma(\alpha+1)\,2^{\alpha-1}}(b-a)^{\alpha}. \tag{2.19}$$

Here f cannot be a non-trivial polynomial of degree m.

Proof By assumption and Theorem 2.3 we have

$$f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}\left(D_{a+}^{\alpha}f\right)(t)\,dt,\quad \text{all } x\in\left[a,\frac{a+b}{2}\right], \tag{2.20}$$

also it holds, by assumption and Theorem 2.4, that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(t)\,dt, \quad \text{all} \quad x \in \left[\frac{a+b}{2}, b\right]. \tag{2.21}$$

By (2.20) we get

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} \left|\left(D_{a+}^{\alpha} f\right)(t)\right| dt$$

$$\leq \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \int_a^{\frac{a+b}{2}} \left|\left(D_{a+}^{\alpha} f\right)(t)\right| dt, \quad \text{all}\, x \in \left[a, \frac{a+b}{2}\right]. \tag{2.22}$$

By (2.21) we derive

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left|\left(D_{b-}^{\alpha} f\right)(t)\right| dt$$

$$\leq \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} \int_{\frac{a+b}{2}}^b \left|\left(D_{b-}^{\alpha} f\right)(t)\right| dt, \quad \text{all}\, x \in \left[\frac{a+b}{2}, b\right]. \tag{2.23}$$

Consequently we have

$$\int_a^{\frac{a+b}{2}} |f(x)|\,dx \leq \frac{1}{\Gamma(\alpha)} \left(\int_a^{\frac{a+b}{2}} (x-a)^{\alpha-1}\,dx\right) \left\|D_{a+}^{\alpha} f\right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)} \tag{2.24}$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha} \left\|D_{a+}^{\alpha} f\right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)},$$

and

$$\int_{\frac{a+b}{2}}^b |f(x)|\,dx \leq \frac{1}{\Gamma(\alpha)} \left(\int_{\frac{a+b}{2}}^b (b-x)^{\alpha-1}\,dx\right) \left\|D_{b-}^{\alpha} f\right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)} \tag{2.25}$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha} \left\|D_{b-}^{\alpha} f\right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)}.$$

Therefore we obtain by adding (2.24) and (2.25) that

$$\int_a^b |f(x)|\,dx \leq \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha} \left[\left\|D_{a+}^{\alpha} f\right\|_{L_1\left(\left[a, \frac{a+b}{2}\right]\right)} + \left\|D_{b-}^{\alpha} f\right\|_{L_1\left(\left[\frac{a+b}{2}, b\right]\right)}\right]$$

$$= \frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2}\right)^{\alpha} \left\|D^{\alpha} f\right\|_{L_1([a,b])} \leq$$

$$\max\left\{\left\|D_{a+}^{\alpha}f\right\|_{L_1\left(\left[a,\frac{a+b}{2}\right]\right)},\left\|D_{b-}^{\alpha}f\right\|_{L_1\left(\left[\frac{a+b}{2},b\right]\right)}\right\}\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)\,2^{\alpha-1}}, \tag{2.26}$$

proving the claim. ■

We continue with

Theorem 2.9 *Let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha>\frac{1}{q}$, $m=[\alpha]$, $f\in C([a,b])$. Assume $f\in C_{a+}^{\alpha}\left(\left[a,\frac{a+b}{2}\right]\right)$ and $f\in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2},b\right]\right)$, such that $f^{(k)}(a)=f^{(k)}(b)=0$, $k=0,1,\ldots,m-1$. When $\frac{1}{q}<\alpha<1$, the last boundary conditions are void.*

Set

$$M_3(f)=\max\left\{\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)},\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right\}. \tag{2.27}$$

Then

$$\int_a^b|f(x)|\,dx\leq\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\cdot \tag{2.28}$$

$$\left[\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]\leq$$

$$\frac{M_3(f)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}}. \tag{2.29}$$

Again here f cannot be a non-trivial polynomial of degree m.

Proof By Theorem 2.3 we have

$$|f(x)|\leq\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}\left|\left(D_{a+}^{\alpha}f\right)(t)\right|dt$$

$$\leq\frac{1}{\Gamma(\alpha)}\left(\int_a^x(x-t)^{p(\alpha-1)}dt\right)^{\frac{1}{p}}\left(\int_a^x\left|\left(D_{a+}^{\alpha}f\right)(t)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq\frac{1}{\Gamma(\alpha)}\frac{(x-a)^{\frac{(p(\alpha-1)+1)}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}}\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)},\quad\text{for all } x\in\left[a,\frac{a+b}{2}\right]. \tag{2.30}$$

That is

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \frac{(x-a)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{a+}^{\alpha} f \right\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)}, \quad \text{for all } x \in \left[a, \frac{a+b}{2}\right] \tag{2.31}$$

Similarly from Theorem 2.4 we get

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} \left| \left(D_{b-}^{\alpha} f\right)(t) \right| dt$$

$$\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (t-x)^{p(\alpha-1)} dt \right)^{\frac{1}{p}} \left(\int_x^b \left| \left(D_{b-}^{\alpha} f\right)(t) \right|^q dt \right)^{\frac{1}{q}} \tag{2.32}$$

$$\leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{b-}^{\alpha} f \right\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)}, \text{ for all } x \in \left[\frac{a+b}{2}, b\right]. \tag{2.33}$$

That is

$$|f(x)| \leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{\alpha-1+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{b-}^{\alpha} f \right\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)}, \quad \text{for all } x \in \left[\frac{a+b}{2}, b\right]. \tag{2.34}$$

Consequently we obtain by (2.31) that

$$\int_a^{\frac{a+b}{2}} |f(x)|\, dx \leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_a^{\frac{a+b}{2}} (x-a)^{\alpha-1+\frac{1}{p}} dx \right) \left\| D_{a+}^{\alpha} f \right\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)} =$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right)} \left(\frac{b-a}{2} \right)^{\alpha+\frac{1}{p}} \left\| D_{a+}^{\alpha} f \right\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)}. \tag{2.35}$$

Similarly it holds by (2.34) that

$$\int_{\frac{a+b}{2}}^b |f(x)|\, dx \leq$$

$$\frac{1}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_{\frac{a+b}{2}}^b (b-x)^{\alpha-1+\frac{1}{p}} dx \right) \left\| D_{b-}^{\alpha} f \right\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)} =$$

$$\frac{1}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}. \tag{2.36}$$

Adding (2.35) and (2.36) we have

$$\int_a^b |f(x)|\,dx \leq \frac{1}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}} \tag{2.37}$$

$$\cdot\left[\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]\leq$$

$$\frac{\max\left\{\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)},\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right\}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}}, \tag{2.38}$$

proving the claim. ■

Combining Theorem 2.6, Remark 2.7, Theorems 2.8 and 2.9, we obtain

Theorem 2.10 *Let any* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha\geq 1$, $m=[\alpha]$, $f\in C([a,b])$. *Assume* $f\in C_{a+}^{\alpha}\left(\left[a,\frac{a+b}{2}\right]\right)$ *and* $f\in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2},b\right]\right)$, *such that* $f^{(k)}(a)=f^{(k)}(b)=0$, $k=0,1,\ldots,m-1$. *Then*

$$\int_a^b |f(x)|\,dx \leq$$

$$\min\left\{\frac{\left(\left\|D_{a+}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}+\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma(\alpha+2)}\left(\frac{b-a}{2}\right)^{\alpha+1},\right.$$

$$\frac{1}{\Gamma(\alpha+1)}\left(\frac{b-a}{2}\right)^{\alpha}\left\|D^{\alpha}f\right\|_{L_1([a,b])}, \tag{2.39}$$

$$\left.\frac{\left[\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\right\}\leq$$

$$\min\left\{M_1(f)\frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2)2^{\alpha}},\frac{M_2(f)}{\Gamma(\alpha+1)2^{\alpha-1}}(b-a)^{\alpha},\right.$$

$$\left.\frac{M_3(f)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}}\right\}, \tag{2.40}$$

where $M_1(f)$ as in (2.14), $M_2(f)$ as in (2.18) and $M_3(f)$ as in (2.27).
Here f cannot be a non-trivial polynomial of degree m.

Corollary 2.11 *Here all as in Theorem 2.10. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{b-a}\int_a^b |f(x)|\,dx \leq$$

$$\min\left\{\frac{\left(\left\|D_{a+}^{\alpha}f\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}+\left\|D_{b-}^{\alpha}f\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma(\alpha+2)\,2^{\alpha+1}}(b-a)^{\alpha},\right. \tag{2.41}$$

$$\frac{1}{2^{\alpha}\Gamma(\alpha+1)}(b-a)^{\alpha-1}\left\|D^{\alpha}f\right\|_{L_1([a,b])},$$

$$\left.\frac{\left[\left\|D_{a+}^{\alpha}f\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}f\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha+\frac{1}{p}}}(b-a)^{\alpha+\frac{1}{p}-1}\right\}\leq$$

$$\min\left\{M_1(f)\frac{(b-a)^{\alpha}}{\Gamma(\alpha+2)\,2^{\alpha}},\ \frac{M_2(f)}{\Gamma(\alpha+1)\,2^{\alpha-1}}(b-a)^{\alpha-1},\right.$$

$$\left.\frac{M_3(f)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)2^{\alpha-\frac{1}{q}}}(b-a)^{\alpha+\frac{1}{p}-1}\right\}. \tag{2.42}$$

In 1938, Ostrowski [12] proved the following important inequality.

Theorem 2.12 *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\left\|f'\right\|_{\infty}:=\sup_{t\in(a,b)}\left|f'(t)\right|<+\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\cdot(b-a)\left\|f'\right\|_{\infty}, \tag{2.43}$$

for any $x\in[a,b]$. The constant $\frac{1}{4}$ is the best possible.

In (2.43) for $x = \frac{a+b}{2}$ we get

$$\left| \frac{1}{b-a} \int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \right| \leq \left(\frac{b-a}{4}\right) \|f'\|_\infty . \tag{2.44}$$

We have proved the following

Theorem 2.13 *Let $f \in C^1([a,b])$, with $f\left(\frac{a+b}{2}\right) = 0$. Then*

$$\left| \int_a^b f(t)\,dt \right| \leq \frac{(b-a)^2}{4} \|f'\|_\infty , \tag{2.45}$$

where the constant $\frac{1}{4}$ is the best possible.

So we proved once again (2.2) with only one initial condition.

2.3 Application

Inequalities for complex valued functions defined on the unit circle were studied extensively by Dragomir, see [7, 8].

We give here our version for these functions involved in a Polya type inequality, by applying a result of this chapter.

Let $t \in [a,b] \subseteq [0, 2\pi)$, the unit circle arc $A = \left\{z \in \mathbb{C} : z = e^{it},\ t \in [a,b]\right\}$, and $f : A \to \mathbb{C}$ be a continuous function. Clearly here there exist functions $u, v : A \to \mathbb{R}$ continuous, the real and the complex part of f, respectively, such that

$$f\left(e^{it}\right) = u\left(e^{it}\right) + iv\left(e^{it}\right). \tag{2.46}$$

So that f is continuous, iff u, v are continuous.

Call $\boldsymbol{g}(t) = f\left(e^{it}\right)$, $l_1(t) = u\left(e^{it}\right)$, $l_2(t) = v\left(e^{it}\right)$, $t \in [a,b]$; so that $\boldsymbol{g} : [a,b] \to \mathbb{C}$ and $l_1, l_2 : [a,b] \to \mathbb{R}$ are continuous functions in t.

If $\boldsymbol{g}$ has a derivative with respect to t, then l_1, l_2 have also derivatives with respect to t. In that case

$$f_t\left(e^{it}\right) = u_t\left(e^{it}\right) + iv_t\left(e^{it}\right), \tag{2.47}$$

(i.e. $\boldsymbol{g}'(t) = l_1'(t) + il_2'(t)$), which means

$$f_t(\cos t + i\sin t) = u_t(\cos t + i\sin t) + iv_t(\cos t + i\sin t). \tag{2.48}$$

Let us call $x = \cos t$, $y = \sin t$. Then

$$u_t\left(e^{it}\right) = u_t(\cos t + i\sin t) = u_t(x+iy) = u_t(x,y) =$$

$$\frac{\partial u}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t}=\frac{\partial u\left(e^{it}\right)}{\partial x}\left(-\sin t\right)+\frac{\partial u\left(e^{it}\right)}{\partial y}\cos t. \tag{2.49}$$

Similarly we find that

$$v_t\left(e^{it}\right)=\frac{\partial v\left(e^{it}\right)}{\partial x}\left(-\sin t\right)+\frac{\partial v\left(e^{it}\right)}{\partial y}\cos t. \tag{2.50}$$

Since $\boldsymbol{g}$ is continuous on $[a, b]$, then $\int_a^b f\left(e^{it}\right)dt$ exists. Furthermore it holds

$$\int_a^b f\left(e^{it}\right)dt=\int_a^b u\left(e^{it}\right)dt+i\int_a^b v\left(e^{it}\right)dt. \tag{2.51}$$

We have here that

$$\left|\int_a^b f\left(e^{it}\right)dt\right|\leq\int_a^b\left|f\left(e^{it}\right)\right|dt\leq \tag{2.52}$$

$$\int_a^b\left|u\left(e^{it}\right)\right|dt+\int_a^b\left|v\left(e^{it}\right)\right|dt=\int_a^b\left|l_1\left(t\right)\right|dt+\int_a^b\left|l_2\left(t\right)\right|dt. \tag{2.53}$$

We give the following application of Theorem 2.10.

Theorem 2.14 *Let $f\in C\left(A,\mathbb{C}\right)$, $[a,b]\subseteq[0,2\pi)$; any $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, $\alpha\geq 1$, $m=[\alpha]$. Assume $l_1,l_2\in C_{a+}^{\alpha}\left(\left[a,\frac{a+b}{2}\right]\right)$ and $l_1,l_2\in C_{b-}^{\alpha}\left(\left[\frac{a+b}{2},b\right]\right)$, such that $l_1^{(k)}\left(a\right)=l_2^{(k)}\left(a\right)=l_1^{(k)}\left(b\right)=l_2^{(k)}\left(b\right)=0$, $k=0,1,\ldots,m-1$. Then*

$$\left|\int_a^b f\left(e^{it}\right)dt\right|\leq\int_a^b\left|f\left(e^{it}\right)\right|dt\leq$$

$$\min\left\{\frac{\left(\left\|D_{a+}^{\alpha}l_1\right\|_{\infty,\left[a,\frac{a+b}{2}\right]}+\left\|D_{b-}^{\alpha}l_1\right\|_{\infty,\left[\frac{a+b}{2},b\right]}\right)}{\Gamma\left(\alpha+2\right)}\left(\frac{b-a}{2}\right)^{\alpha+1},\right. \tag{2.54}$$

$$\frac{1}{\Gamma\left(\alpha+1\right)}\left(\frac{b-a}{2}\right)^{\alpha}\left\|D^{\alpha}l_1\right\|_{L_1([a,b])},$$

$$\left.\frac{\left[\left\|D_{a+}^{\alpha}l_1\right\|_{L_q\left(\left[a,\frac{a+b}{2}\right]\right)}+\left\|D_{b-}^{\alpha}l_1\right\|_{L_q\left(\left[\frac{a+b}{2},b\right]\right)}\right]}{\Gamma\left(\alpha\right)\left(p\left(\alpha-1\right)+1\right)^{\frac{1}{p}}\left(\alpha+\frac{1}{p}\right)}\left(\frac{b-a}{2}\right)^{\alpha+\frac{1}{p}}\right\}+$$

$$\min \left\{ \frac{\left(\left\| D_{a+}^{\alpha} l_2 \right\|_{\infty, \left[a, \frac{a+b}{2}\right]} + \left\| D_{b-}^{\alpha} l_2 \right\|_{\infty, \left[\frac{a+b}{2}, b\right]} \right)}{\Gamma(\alpha+2)} \left(\frac{b-a}{2} \right)^{\alpha+1}, \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left(\frac{b-a}{2} \right)^{\alpha} \left\| D^{\alpha} l_2 \right\|_{L_1([a,b])},$$

$$\left. \frac{\left[\left\| D_{a+}^{\alpha} l_2 \right\|_{L_q\left(\left[a, \frac{a+b}{2}\right]\right)} + \left\| D_{b-}^{\alpha} l_2 \right\|_{L_q\left(\left[\frac{a+b}{2}, b\right]\right)} \right]}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} \left(\frac{b-a}{2} \right)^{\alpha+\frac{1}{p}} \right\} \leq$$

$$\min \left\{ M_1(l_1) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2) 2^{\alpha}}, \frac{M_2(l_1)}{\Gamma(\alpha+1) 2^{\alpha-1}} (b-a)^{\alpha}, \right.$$
$$\left. \frac{M_3(l_1)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right) 2^{\alpha-\frac{1}{q}}} (b-a)^{\alpha+\frac{1}{p}} \right\} + \tag{2.55}$$

$$\min \left\{ M_1(l_2) \frac{(b-a)^{\alpha+1}}{\Gamma(\alpha+2) 2^{\alpha}}, \frac{M_2(l_2)}{\Gamma(\alpha+1) 2^{\alpha-1}} (b-a)^{\alpha}, \right.$$
$$\left. \frac{M_3(l_2)}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}} \left(\alpha+\frac{1}{p}\right) 2^{\alpha-\frac{1}{q}}} (b-a)^{\alpha+\frac{1}{p}} \right\},$$

where $M_1(l_i)$ *as in (2.14),* $M_2(l_i)$ *as in (2.18) and* $M_3(l_i)$ *as in (2.27),* $i = 1, 2$.
Here l_1, l_2 *cannot be non-trivial polynomials of degree* m.

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