Freak-Waves: Compact Equation Versus Fully Nonlinear One

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Abstract We compare applicability of the recently derived compact equation for surface wave with the fully nonlinear equations. Strongly nonlinear phenomena, namely modulational instability and breathers with the steepness $\mu \sim 0.4$ are compared in numerical simulations using both models.

1 Introduction

A two-dimensional potential flow of an ideal incompressible fluid of infinite depth with a one-dimensional free surface (boundary) in a gravity field is described by the following well-known set of equations:

$$
\begin{aligned}
\phi_{xx} + \phi_{zz} &= 0 \qquad (\phi_z \to 0, z \to -\infty),\\
\eta_t + \eta_x \phi_x &= \phi_z \Big|_{z=\eta}\\
\phi_t + \frac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta &= 0 \Big|_{z=\eta};
\end{aligned}
\tag{1}
$$

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here $\eta \leftharpoonup x^c \psi t \leftharpoondown$ is the shape of a surface, $\phi \leftharpoonup x^c \psi z^c t \leftharpoondown$ is a potential function of the flow and g is a gravitational acceleration. As was shown in Zakharov (1968), the variables $\eta(x,t)$ and $\psi(x,t) = \phi(x,z,t)\Big|_{z=\eta}$ are canonically conjugated, and satisfy the equations

$$\frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta} \qquad \frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}.$$

Hamiltonian can be written as infinite series of powers of ψ and η, (see Zakharov 1968). Taking into account only three- and four-wave interactions, one can cut this series after fourth order term:

$$H = \frac{1}{2}\int g\eta^2 + \psi \hat{k}\psi dx - \frac{1}{2}\int \{(\hat{k}\psi)^2 - (\psi_x)^2\}\eta dx + \frac{1}{2}\int \{\psi_{xx}\eta^2\hat{k}\psi + \psi\hat{k}(\eta\hat{k}(\eta\hat{k}\psi))\}dx \tag{2}$$

We will study two strongly nonlinear problems numerically:

- modulational instability of the homogeneous wave train of the Stokes waves up to the freak-wave formation
- propagation of narrow breather with the steepness $\mu \sim 0.4 - -0.5$

The goal is to justify the applicability of approximate equation based on truncated Hamiltonian (2) for strongly nonlinear flows of fluid. The reference solutions (fully nonlinear) is performed for the Eq. (1) written in conformal variables, according to Dyachenko (2001).

2 Fully Nonlinear Conformal Equations

To study two-dimensional potential flow of fluid one can perform the conformal transformation to map the domain, filled with fluid

$$-\infty < x < \infty, \qquad -\infty < y < \eta(x,t), \qquad Z = x + iy$$

in Z-plane to the lower half-plane

$$-\infty < u < -\infty, \qquad -\infty < v < 0, \qquad W = u + iv$$

in W-plane like in Figs. 1 and 2. The shape of surface $\eta(x,t)$ is given now by parametric equations

$$y = y(u,t), \;\; x = x(u,t),$$

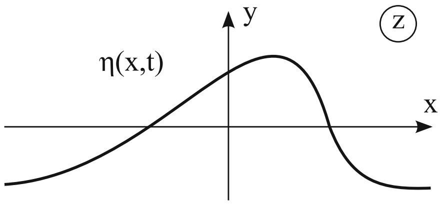

Fig. 1 Physical plane Z

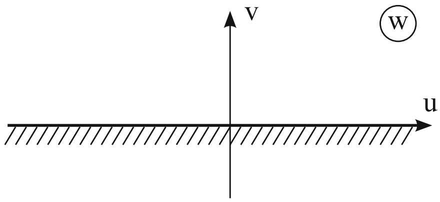

Fig. 2 Conformal plane W

As it was shown in Dyachenko (2001), Dyachenko et al. (1996a, b) the exact Eq. (1) can be written as following:

$$\begin{aligned} Z_t &= iUZ_u, \\ \Phi_t &= iU\Phi_u - B + ig(Z-u). \end{aligned} \tag{3}$$

Here, $\Phi = \psi + i\hat{H}\psi$ is complex velocity potential, and

$$U = \hat{P}\left(\frac{-\hat{H}\psi_u}{|Z_u|^2}\right), \qquad B = \hat{P}\left(\frac{|\Phi_u|^2}{|z_u|^2}\right) = \hat{P}\left(|\Phi_z|^2\right). \tag{4}$$

In (4) $\hat{P}$ is the projector operator generating a function analytical in the lower half-plane

$$\hat{P}(f) = \frac{1}{2}\left(1 + i\hat{H}\right) f.$$

$$\hat{H}(f(u)) = P \cdot V.\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(u')\mathrm{d}u'}{u'-u} \quad \text{is the Hilbert transformation.}$$

Functions Φ and Z can be easily analytically continued to the lower half-plane, just by changing u by w.

Introducing new variables (Dyachenko 2001)

$$R = \frac{1}{Z_w}, \qquad \text{and } V = i\Phi_z = i\frac{\Phi_w}{Z_w} \tag{5}$$

one can transform system (3) into the following one

$$\begin{aligned} R_t &= i(UR_w - RU_w), \\ V_t &= i(UV_w - RB_w) + g(R-1). \end{aligned} \tag{6}$$

Now U and B are the following:

$$\begin{aligned} U &= \hat{P}(V\bar{R} + \bar{V}R) \\ B &= \hat{P}(V\bar{V}). \end{aligned}$$

So, these exact (fully nonlinear) Eq. (6) give us reference solutions to compare with.

3 Compact Equation

In this section, we very briefly derive compact equation based on the truncated Hamiltonian (2). All the details of the derivation can be found in Dyachenko and Zakharov (2011, 2012). It based on the following property of one-dimensional gravity surface waves: In Dyachenko and Zakharov (1994) it was shown that four-wave interaction coefficient vanishes on the following resonant manifold

$$\begin{aligned} k + k_1 &= k_2 + k_3, \\ \omega_k + \omega_{k_1} &= \omega_{k_2} + \omega_{k_3} \end{aligned}$$

with nontrivial solution:

$$\begin{aligned} k &= a(1+\zeta)^2, \qquad & k_1 &= a(1+\zeta)^2\zeta^2, \\ k_2 &= -a\zeta^2, \qquad & k_3 &= a(1+\zeta+\zeta^2)^2. \end{aligned}$$

Than only trivial resonant interaction remains in force:

$$k = k_2, k_1 = k_3 \qquad \text{or} \qquad k = k_3, k_1 = k_2.$$

Vanishing of four-waves interaction allows us:

- to consider solutions which consist of waves propagating in the same direction (all k in the initial condition and solution can be positive only)
- drastically simplify fourth order term in the truncated Hamiltonian (2).

To make this simplification one can apply appropriate canonical transformation to the Hamiltonian. But first it is convenient to introduce complex normal canonical variables in a standard way:

$$\eta_k = \sqrt{\frac{\omega_k}{2g}}(a_k + a^*_{-k}) \qquad \psi_k = -i\sqrt{\frac{g}{2\omega_k}}(a_k - a^*_{-k}) \tag{7}$$

where $\omega_k = \sqrt{g|k|}$ and the Fourier transform is defined as follow:

$$a(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a_k(t) e^{ikx} dx \tag{8}$$

Using normal variable $a_k(t)$ the truncated Hamiltonian can be written as follows:

$$\begin{aligned} H = & \int \omega_k |a_k|^2 dk + \\ & + \int V^{k_1}_{k_2 k_3} (a^*_{k_1} a_{k_2} a_{k_3} + a_{k_1} a^*_{k_2} a^*_{k_3}) \delta_{k_1 - k_2 - k_3} dk_1 dk_2 dk_3 + \\ & + \frac{1}{3} \int U_{k_1 k_2 k_3} (a^*_{k_1} a^*_{k_2} a^*_{k_3} + a_{k_1} a_{k_2} a_{k_3}) \delta_{k_1 + k_2 + k_3} dk_1 dk_2 dk_3 + \\ & + \frac{1}{2} \int W^{k_3 k_4}_{k_1 k_2} a^*_{k_1} a^*_{k_2} a_{k_3} a_{k_4} \delta_{k_1 + k_2 - k_3 - k_4} dk_1 dk_2 dk_3 dk_4 + \\ & + \frac{1}{3} \int G^{k_4}_{k_1 k_2 k_3} (a^*_{k_1} a^*_{k_2} a^*_{k_3} a_{k_4} + a_{k_1} a_{k_2} a_{k_3} a^*_{k_4} \delta_{k_1 + k_2 + k_3 - k_4} dk_1 dk_2 dk_3 dk_4 + \\ & + \frac{1}{12} \int R_{k_1 k_2 k_3 k_4} (a^*_{k_1} a^*_{k_2} a^*_{k_3} a^*_{k_4} + a_{k_1} a_{k_2} a_{k_3} a_{k_4}) \delta_{k_1 + k_2 + k_3 + k_4} dk_1 dk_2 dk_3 dk_4. \end{aligned} \tag{9}$$

Expressions for $V^{k_1}_{k_2 k_3}$, $U_{k_1 k_2 k_3}$, $W^{k_3 k_4}_{k_1 k_2}$, $G^{k_4}_{k_1 k_2 k_3}$, $R_{k_1 k_2 k_3 k_4}$ in Appendix (see 18, 19).

Then one applies transformation from variables a_k to b_k to exclude nonresonant cubic terms along with non resonant fourth order terms. Following Zakharov et al. (1992) canonical transformation from b_k to a_k can be written as the series:

$$\begin{aligned} a_k = b_k + & \int \Big[2\tilde{V}^{k_1}_{k k_2} b_{k_1} b^*_{k_2} \delta_{k_1 - k - k_2} - \tilde{V}^{k}_{k_1 k_2} b_{k_1} b_{k_2} \delta_{k - k_1 - k_2} - \tilde{U}_{k k_1 k_2} b^*_{k_1} b^*_{k_2} \delta_{k + k_1 + k_2} \Big] dk_1 dk_2 \\ + & \int \Big[A^{k}_{k_1 k_2 k_3} b_{k_1} b_{k_2} b_{k_3} + A^{k k_1}_{k_2 k_3} b^*_{k_1} b_{k_2} b_{k_3} + A^{k k_1 k_2}_{k_3} b^*_{k_1} b^*_{k_2} b_{k_3} + A^{k k_1 k_2 k_3} b^*_{k_1} b^*_{k_2} b^*_{k_3} \Big] dk_1 dk_2 dk_3 \end{aligned} \tag{10}$$

All coefficients in (10) are derived in Appendix (see 24, 26). After the transformation (details of it are given in Dyachenko and Zakharov (2011, 2012)) Hamiltonian takes the form in x-space:

$$\mathscr{H} = \int b^* \hat{\omega}_k b dx + \frac{1}{2} \int \left| \frac{\partial b}{\partial x} \right|^2 \left[\frac{i}{2} \left(b \frac{\partial b^*}{\partial x} - b^* \frac{\partial b}{\partial x} \right) - \hat{k} |b|^2 \right] dx. \tag{11}$$

$b(x)$ can be analytically continued to $x + iy$, $y > 0$. Motion equation for $b(x,t)$ should be understood as follow:

$$i \frac{\partial b}{\partial t} = \hat{P}^+ \frac{\delta \mathscr{H}}{\delta b^*}, \tag{12}$$

here $\hat{P}^+$ projection operator to the upper half-plane.

$$\hat{P}^+ = \frac{1}{2}(1 - i\hat{H}). \tag{13}$$

This operator is the consequence of only positive k in the system of waves. Corresponding equation of motion is the following:

$$i\frac{\partial b}{\partial t} = \hat{\omega}_k b + \frac{i}{4}\hat{P}^+\left[b^*\frac{\partial}{\partial x}(b'^2) - \frac{\partial}{\partial x}(b^{*\prime}\frac{\partial}{\partial x}b^2)\right] - \\ -\frac{1}{2}\hat{P}^+\left[b\cdot\hat{k}(|b'|^2) - \frac{\partial}{\partial x}(b'\hat{k}(|b|^2))\right]. \qquad (14)$$

Transformation from $b(x, t)$ to physical variables $\eta(x, t)$ and $\psi(x, t)$ can be recovered from canonical transformation. It has been derived in the Appendix. Here, we write this transformation up to the second order:

$$\eta(x) = \frac{1}{\sqrt{2}g^{\frac{1}{4}}}(\hat{k}^{\frac{1}{4}}b(x) + \hat{k}^{\frac{1}{4}}b(x)^*) + \frac{\hat{k}}{4\sqrt{g}}[\hat{k}^{\frac{1}{4}}b(x) - \hat{k}^{\frac{1}{4}}b^*(x)]^2,$$

$$\psi(x) = -i\frac{g^{\frac{1}{4}}}{\sqrt{2}}(\hat{k}^{-\frac{1}{4}}b(x) - \hat{k}^{-\frac{1}{4}}b(x)^*) + \frac{i}{2}[\hat{k}^{\frac{1}{4}}b^*(x)\hat{k}^{\frac{3}{4}}b^*(x) - \hat{k}^{\frac{1}{4}}b(x)\hat{k}^{\frac{3}{4}}b(x)] \\ + \frac{1}{2}\hat{H}[\hat{k}^{\frac{1}{4}}b(x)\hat{k}^{\frac{3}{4}}b^*(x) + \hat{k}^{\frac{1}{4}}b^*(x)\hat{k}^{\frac{3}{4}}b(x)]. \qquad (15)$$

Here $\hat{H}$ is Hilbert transformation with eigenvalue $i\,\mathbf{sign}(k)$.

4 Modulational Instability of Wave Train

In this section, we perform numerical simulation of the modulational instability of the homogeneous wave train in the framework of compact Eq. (14). Such a wave train has two parameters: wavelength and steepness, i.e., maximal slope of the surface, μ. Initial steepness of the wave train was equal to $\mu = 0.095$ and the number of waves in the periodic domain was equal to 100. These values were chosen for comparison with the earlier simulation in the framework of fully nonlinear simulation in the works (Dyachenko and Zakharov 2005; Zakharov et al. 2006, 2008). One can see in Figs. 3 and 4 that both waves coincide in details. Different time of their appearing is due to slightly different values of perturbations. Zoomed shape of the surface is shown in the inset to Fig. 3.

Couple of snapshots of development of modulational instability is shown in Fig. 5.

Dynamics of surface in fully nonlinear Eq. (6) can be found at the address http://www.itp.ac.ru/~kachulin/MInstability/Freak-0.095-end.avi and dynamics of the surface in compact equation is at the address
http://www.itp.ac.ru/~kachulin/MInstability/Surface-end.avi.

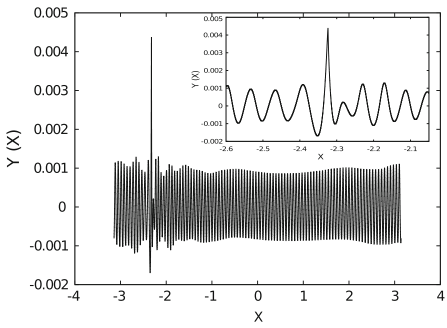

Fig. 3 Freak-wave formation after t = 802 (fully nonlinear equation)

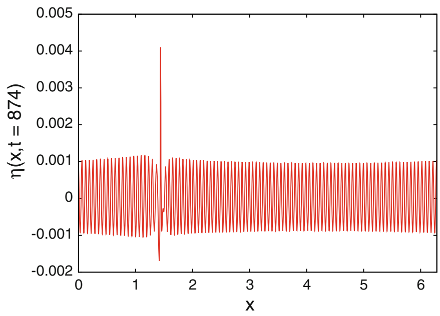

Fig. 4 Freak-wave formation after t = 874 (compact equation)

5 Breathers

Breather is the localized solution of (14) of the following type:

$$b(x,t) = B(x - Vt)e^{i(k_0 x - \omega_0 t)}, \tag{16}$$

where $B(x)$ is localized function in space, having zero asymptotic at $\pm\infty$. In K space it can be written as following:

$$b_k(t) = e^{-i(\Omega + Vk)t}\phi_k, \tag{17}$$

k_0 is the wavenumber of the carrier wave, V is the group velocity, ω_0 is the frequency close to ω_{k_0}, and Ω is close to $\frac{\omega_{k_0}}{2}$. Existence of such solution may indicate that equation is integrable one. However, in the paper (Dyachenko et al. 2013) nonintegrability of the equation was proven.

In the papers (Dyachenko et al. 2013, 2014) such solutions with different group velocities and amplitudes were found by iterative Petviashvili method. Here we tried to get numerically very narrow breather for the compact equation with carrier wavenumber $k_0 = 50$. Picture of real part of $b(x,0)$ and modulus of $b(x,0)$ is

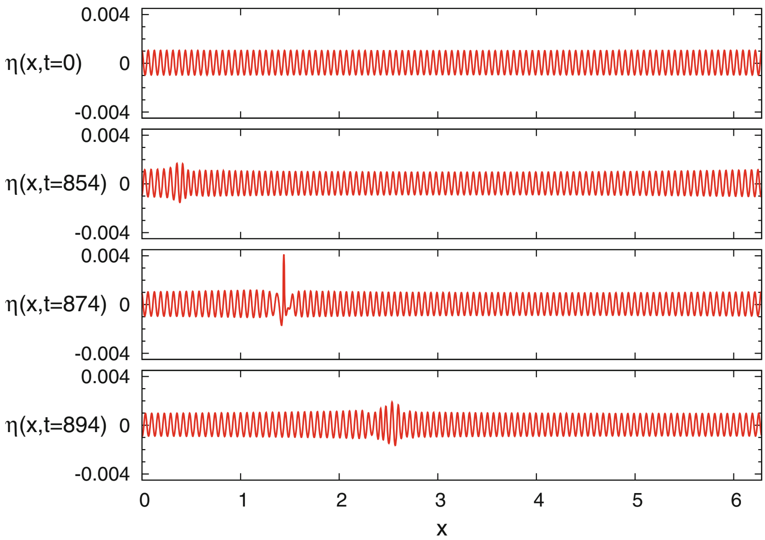

Fig. 5 Formation of the freak wave. Free surface for different times is shown (compact equation)

shown in Fig. 6. Modulus of b corresponds in some sense to the envelope in NLSE approximation. Profile of the surface calculated according to transformation (15) is shown in Fig. 7. Steepness of this solution is very high, $\mu \sim 0.45$. Profile of the

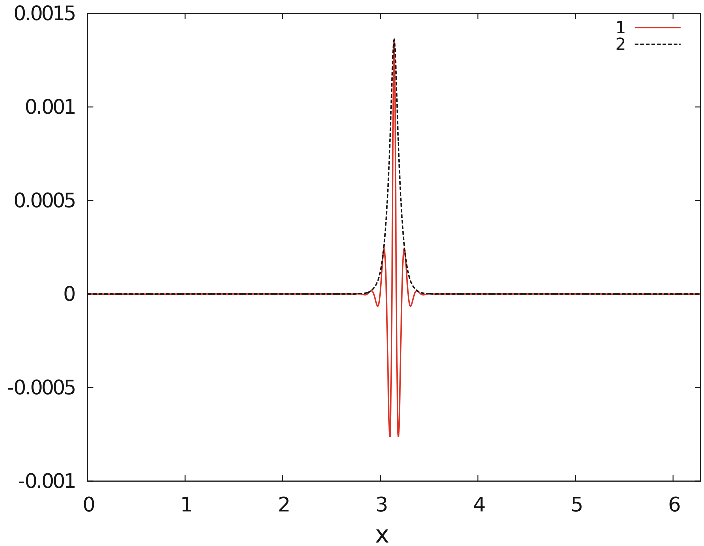

Fig. 6 Modulus of $b(x)$ and real part of $b(x)$ with $V = 1/20$ and $\Omega = 5.2$. *Solid line* (1) corresponds to the real part of $b(x)$, *dashed line* (2) corresponds to modulus of $b(x)$

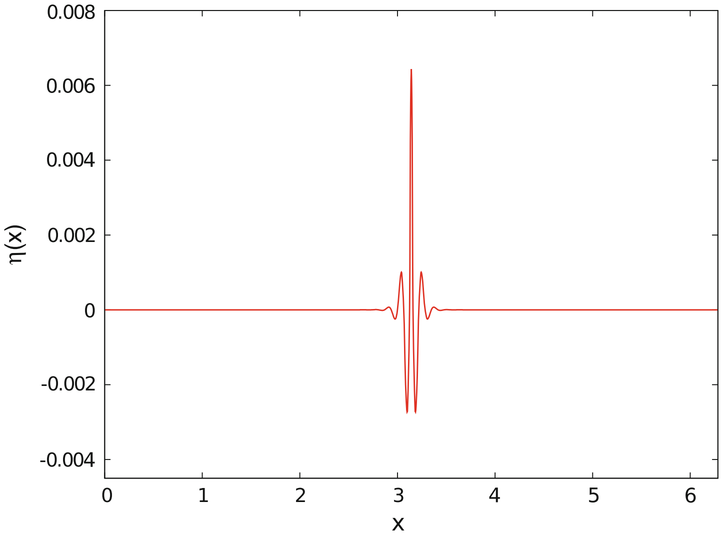

Fig. 7 Surface profile of the breather (compact equation)

steepness is shown in Fig. 8. This breather is exact solution of the compact Eq. (14) and moves on the surface without changing (see Dyachenko et al. 2013, 2014). Couple of snapshots of moving breather is given in Fig. 9. It is clearly seen that

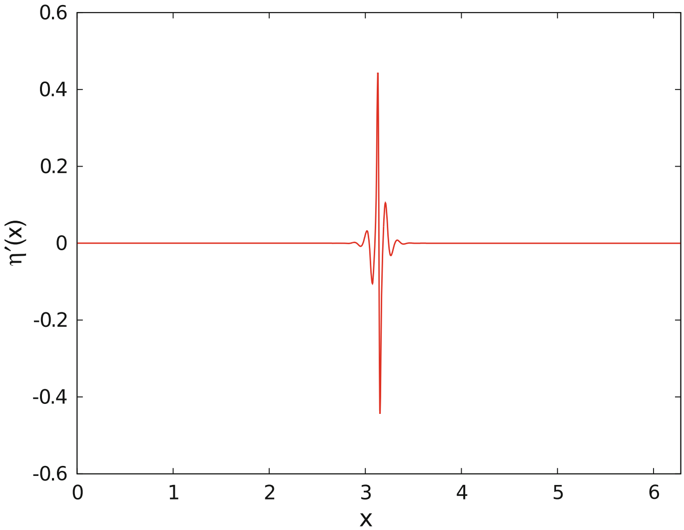

Fig. 8 Steepness of the breather (compact equation)

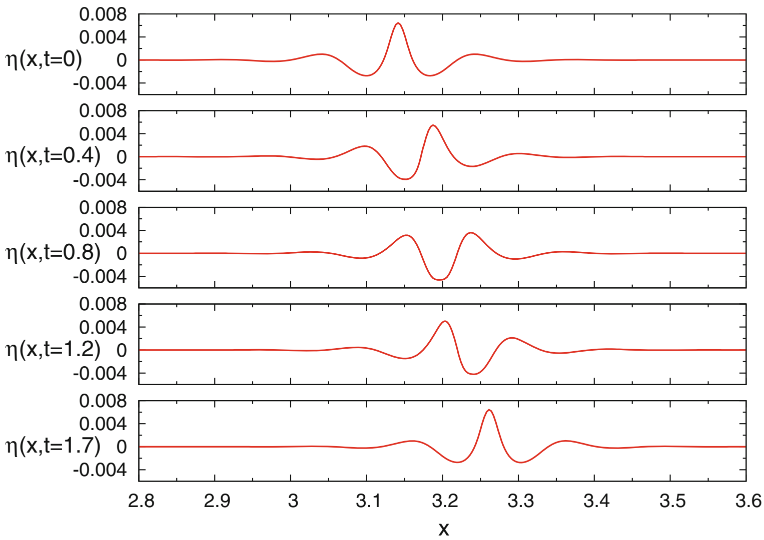

Fig. 9 Free surface corresponds to the breather solution at different times (compact equation)

breather moves with the group velocity which is $\frac{1}{2}\sqrt{\frac{g}{k}} \sim 0.0707$. Dynamics of this breather in compact equation can be found at the address http://www.itp.ac.ru/~kachulin/Breathers/k0=50/deta(x,t).avi. Breather dynamics in fully nonlinear equations can be found at
http://www.itp.ac.ru/~kachulin/Breathers/k0=50/STEEPNESS.avi.

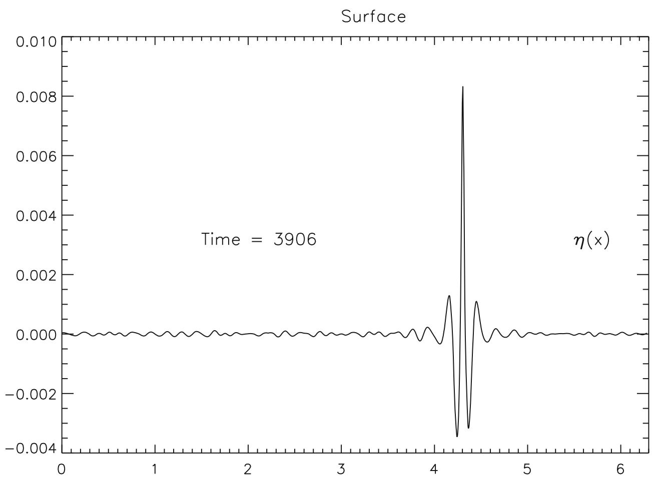

Fig. 10 Surface profile (fully nonlinear equations)

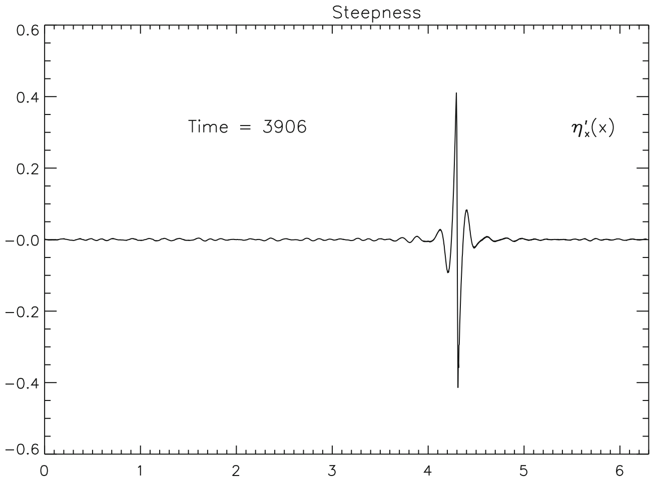

Fig. 11 Profile of steepness (fully nonlinear equations)

In the paper (Dyachenko and Zakharov 2008) we have performed similar simulations in the framework of fully nonlinear conformal equations (6). Here we present pictures from that paper, namely surface of the fluid and its steepness (Figs. 10, 11).

Figures 8 and 11 show that numerical simulation of the highly nonlinear phenomena, steep breather, in the framework of compact equation is very similar to that of fully nonlinear equation. Recent laboratory experiment (Slunyaev et al. 2013), also confirm existence of such highly narrow breathers at the surface.

6 Conclusions

We have demonstrated that compact equation although approximate, quantitatively describes strongly nonlinear phenomena at the surface of potential fluid. Especially, we have studied nonlinear stage of modulational instability up to the freak-wave formation and propagation of very steep breather. Also compact equation can be generalized for quasi one-dimensional waves propagating at the surface of 3D fluid, see Dyachenko et al. (2014). When considering waves slightly inhomogeneous in transverse direction, one can think in the spirit of Kadomtsev-Petviashvili equation for Korteveg-de-Vries equation, namely one can treat now frequency ω_k depending on both k_x and k_y as ω_{k_x,k_y}, while leaving coefficient $\tilde{T}^{kk_1}_{k_2k_3}$ not depending on y. b now depends on both x and y:

$$\mathscr{H} = \int b^* \hat{\omega}_{k_x,k_y} b dx dy + \frac{1}{2}\int |b'_x|^2 \left[\frac{i}{2}(bb'^*_x - b^* b'_x) - \hat{K}_x |b|^2\right] dx dy.$$

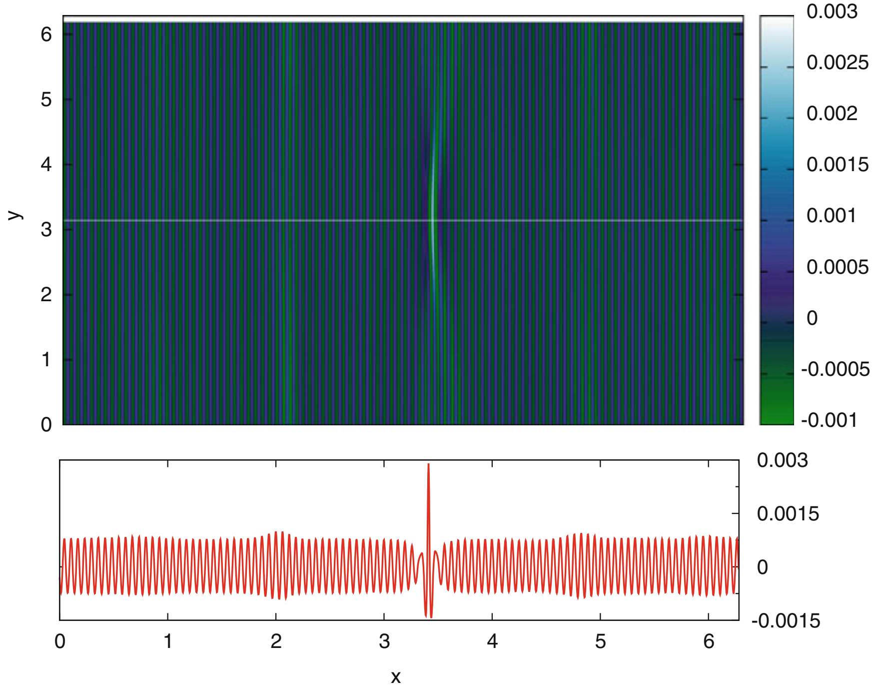

Fig. 12 2D surface with a freak-wave (2D compact equation)

Here, we can show picture of numerical simulation of quasi one-dimensional wave train. One can see in Fig. 12 top view of 100 almost 1D waves with the freak wave in some place. Profile of the surface along the white line is also shown.

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Numerical simulation was performed on the Informational Computational Center of the Novosibirsk State University.

Appendix

Coefficients in the Hamiltonian (9) can be calculated plugging expressions for complex canonical variables into the (2):

$$
\begin{aligned}
U_{k_1k_2k_3} &= \frac{1}{8}\frac{g^{\frac{1}{4}}}{\sqrt{\pi}}\left[|\frac{k_1}{k_2k_3}|^{\frac{1}{4}}L_{k_2k_3} + |\frac{k_2}{k_1k_3}|^{\frac{1}{4}}L_{k_1k_3} + |\frac{k_3}{k_1k_2}|^{\frac{1}{4}}L_{k_1k_2}\right],\\
V^{k_1}_{k_2k_3} &= \frac{1}{8}\frac{g^{\frac{1}{4}}}{\sqrt{\pi}}\left[|\frac{k_1}{k_2k_3}|^{\frac{1}{4}}L_{k_2k_3} - |\frac{k_2}{k_1k_3}|^{\frac{1}{4}}L_{-k_1k_3} - |\frac{k_3}{k_1k_2}|^{\frac{1}{4}}L_{-k_1k_2}\right]. \qquad (18)
\end{aligned}
$$

$$
\begin{aligned}
W^{k_3k_4}_{k_1k_2} &= \frac{-1}{32\pi}\left[\left|\frac{k_1k_2}{k_3k_3}\right|^{\frac{1}{4}} M^{k_1k_2}_{-k_3-k_4} + \left|\frac{k_3k_4}{k_1k_2}\right|^{\frac{1}{4}} M^{-k_3-k_4}_{k_1k_2} - \left|\frac{k_1k_3}{k_2k_4}\right|^{\frac{1}{4}} M^{k_1-k_3}_{k_2-k_4} - \left|\frac{k_2k_3}{k_1k_4}\right|^{\frac{1}{4}} M^{k_2-k_3}_{k_1-k_4} - \right.\\
&\left. - \left|\frac{k_1k_4}{k_2k_3}\right|^{\frac{1}{4}} M^{k_1-k_4}_{k_2-k_3} - \left|\frac{k_2k_4}{k_1k_3}\right|^{\frac{1}{4}} M^{k_2-k_4}_{k_1-k_3}\right]\\
G^{k_4}_{k_1k_2k_3} &= \frac{-1}{32\pi}\left[\left|\frac{k_3k_4}{k_1k_2}\right|^{\frac{1}{4}} M^{k_3-k_4}_{k_1k_2} + \left|\frac{k_2k_4}{k_1k_3}\right|^{\frac{1}{4}} M^{k_2-k_4}_{k_1k_3} + \left|\frac{k_1k_4}{k_2k_3}\right|^{\frac{1}{4}} M^{k_1-k_4}_{k_2k_3} - \left|\frac{k_1k_2}{k_3k_4}\right|^{\frac{1}{4}} M^{k_1k_2}_{k_3-k_4} - \right.\\
&\left. - \left|\frac{k_1k_3}{k_2k_4}\right|^{\frac{1}{4}} M^{k_1k_3}_{k_2-k_4} - \left|\frac{k_2k_3}{k_1k_4}\right|^{\frac{1}{4}} M^{k_2k_3}_{k_1-k_4}\right]\\
R_{k_1k_2k_3k_4} &= \frac{-1}{32\pi}\left[\left|\frac{k_3k_4}{k_1k_2}\right|^{\frac{1}{4}} M^{k_3k_4}_{k_1k_2} + \left|\frac{k_2k_4}{k_1k_3}\right|^{\frac{1}{4}} M^{k_2k_4}_{k_1k_3} + \left|\frac{k_2k_3}{k_1k_4}\right|^{\frac{1}{4}} M^{k_2k_3}_{k_1k_4} + \left|\frac{k_1k_4}{k_2k_3}\right|^{\frac{1}{4}} M^{k_1k_4}_{k_2k_3} + \right.\\
&\left. + \left|\frac{k_1k_3}{k_2k_4}\right|^{\frac{1}{4}} M^{k_1k_3}_{k_2k_4} + \left|\frac{k_1k_2}{k_3k_4}\right|^{\frac{1}{4}} M^{k_1k_2}_{k_3k_4}\right] \qquad (19)
\end{aligned}
$$

Here

$$
\begin{aligned}
L_{k_1k_2} &= |k_1k_2| + k_1k_2\\
M^{k_3k_4}_{k_1k_2} &= |k_1k_2|(|k_1+k_3| + |k_1+k_4| + |k_2+k_3| + |k_2+k_4| - 2|k_1| - 2|k_2|).
\end{aligned} \qquad (20)
$$

To construct canonical transformation of general form we follow the book (Zakharov et al. 1992) and use auxiliary Hamiltonian:

$$
\begin{aligned}
\tilde{H} = &-i\int \tilde{V}^{k_1}_{k_2k_3}(b^*_{k_1}b_{k_2}b_{k_3} - b_{k_1}b^*_{k_2}b^*_{k_3})\delta_{k_1-k_2-k_3}dk_1dk_2dk_3 -\\
&- \frac{i}{3}\int \tilde{U}_{k_1k_2k_3}(b^*_{k_1}b^*_{k_2}b^*_{k_3} - b_{k_1}b_{k_2}b_{k_3})\delta_{k_1+k_2+k_3}dk_1dk_2dk_3,\\
&+ \frac{1}{2}\int (\tilde{\tilde{W}}^{k_3k_4}_{k_1k_2} + i\tilde{W}^{k_3k_4}_{k_1k_2})b^*_{k_1}b^*_{k_2}b_{k_3}b_{k_4}\delta_{k_1+k_2-k_3-k_4}dk_1dk_2dk_3dk_4 -\\
&- \frac{i}{3}\int \tilde{G}^{k_4}_{k_1k_2k_3}(b^*_{k_1}b^*_{k_2}b^*_{k_3}b_{k_4} - b_{k_1}b_{k_2}b_{k_3}b^*_{k_4})\delta_{k_1+k_2+k_3-k_4}dk_1dk_2dk_3dk_4 -\\
&- \frac{i}{12}\int \tilde{R}_{k_1k_2k_3k_4}(b^*_{k_1}b^*_{k_2}b^*_{k_3}b^*_{k_4} - b_{k_2}b_{k_3}b_{k_4})\delta_{k_1+k_2+k_3+k_4}dk_1dk_2dk_3dk_4
\end{aligned} \qquad (21)
$$

with standard symmetry conditions for coefficients. Just mention that for $\tilde{W}^{k_3k_4}_{k_1k_2}$ this condition is the following:

$$
\tilde{W}^{k_3k_4}_{k_1k_2} = \tilde{W}^{k_3k_4}_{k_2k_1} = \tilde{W}^{k_4k_3}_{k_1k_2} = -\tilde{W}^{k_1k_2}_{k_3k_4}. \qquad (22)
$$

Again, following Zakharov et al. (1992) general canonical transformation from b_k to a_k can be written as the series:

$$a_k = b_k + \int \Big[2\tilde{V}^{k_1}_{kk_2} b_{k_1} b^*_{k_2} \delta_{k_1-k-k_2} - \tilde{V}^{k}_{k_1k_2} b_{k_1} b_{k_2} \delta_{k-k_1-k_2} - \tilde{U}_{kk_1k_2} b^*_{k_1} b^*_{k_2} \delta_{k+k_1+k_2} \Big] dk_1 dk_2$$
$$+ \int \Big[A^{k}_{k_1k_2k_3} b_{k_1} b_{k_2} b_{k_3} + A^{kk_1}_{k_2k_3} b^*_{k_1} b_{k_2} b_{k_3} + A^{kk_1k_2}_{k_3} b^*_{k_1} b^*_{k_2} b_{k_3} + A^{kk_1k_2k_3} b^*_{k_1} b^*_{k_2} b^*_{k_3} \Big] dk_1 dk_2 dk_3 \qquad (23)$$

Coefficients A with upper and lower indices are equal to:

$$\begin{aligned}
A^{k}_{k_1k_2k_3} &= \left[\frac{1}{3}\tilde{G}^{k}_{k_1k_2k_3} + \tilde{V}^{k}_{k_1k-k_1}\tilde{V}^{k_2+k_3}_{k_2k_3} - \tilde{V}^{k_1}_{kk_1-k}\tilde{U}_{-k_2-k_3k_2k_3} \right] \delta_{k-k_1-k_2-k_3}, \\
A^{kk_1}_{k_2k_3} &= \Big[-i\tilde{\tilde{W}}^{k_2k_3}_{kk_1} + \tilde{W}^{k_2k_3}_{kk_1} - 2\tilde{V}^{k}_{k_2k-k_2}\tilde{V}^{k_3}_{k_1k_3-k_1} - \tilde{V}^{k+k_1}_{kk_1}\tilde{V}^{k_2+k_3}_{k_2k_3} + 2\tilde{V}^{k_3}_{kk_3-k}\tilde{V}^{k_1}_{k_2k_1-k_2} + \\
&+ \tilde{U}_{-k-k_1kk_1}\tilde{U}_{-k_2-k_3k_2k_3} \Big] \delta_{k+k_1-k_2-k_3}, \\
A^{kk_1k_2}_{k_3} &= \Big[-\tilde{G}^{k_3}_{kk_1k_2} + \tilde{V}^{k}_{k_3k-k_3}\tilde{U}_{-k_2-k_1k_2k_1} - \tilde{V}^{k_3}_{kk_3-k}\tilde{V}^{k_1+k_2}_{k_1k_2} + 2\tilde{V}^{k+k_1}_{kk_1}\tilde{V}^{k_3}_{k_2k_3-k_2} - \\
&- 2\tilde{U}_{-k-k_1kk_1}\tilde{V}^{k_2}_{k_3k_2-k_3} \Big] \delta_{k+k_1+k_2-k_3}, \\
A^{kk_1k_2k_3} &= \left[-\frac{1}{3}\tilde{R}_{kk_1k_2k_3} - \tilde{V}^{k+k_1}_{kk_1}\tilde{U}_{-k_2-k_3k_2k_3} + \tilde{V}^{k_2+k_3}_{k_2k_3}\tilde{U}_{-k-k_1kk_1} \right] \delta_{k+k_1+k_2+k_3}.
\end{aligned} \qquad (24)$$

Let us now substitute transformation (23) into the Hamiltonian (9) and calculate second, third and fourth order terms.

Collecting all cubic terms after substitution and making symmetrization one can get:

$$H_3 = \int [V^{k_1}_{k_2k_3} - (\omega_{k_1} - \omega_{k_3} - \omega_{k_3})\tilde{V}^{k_1}_{k_2k_3}] b^*_{k_1} b_{k_2} b_{k_3} \delta_{k_1-k_2-k_3} dk_1 dk_2 dk_3 +$$
$$+ \frac{1}{3}\int [U_{k_1k_2k_3} - (\omega_{k_1} + \omega_{k_3} + \omega_{k_3})\tilde{U}_{k_1k_2k_3}] b^*_{k_1} b^*_{k_2} b^*_{k_3} \delta_{k_1+k_2+k_3} dk_1 dk_2 dk_3 + c.c. \qquad (25)$$

it is possible to cancel nonresonant both cubic and fourth order terms. If

$$\tilde{V}^{k}_{k_1k_2} = \frac{V^{k}_{k_1k_2}}{\omega_k - \omega_{k_1} - \omega_{k_2}}, \qquad \tilde{U}_{kk_1k_2} = \frac{U_{kk_1k_2}}{\omega_k + \omega_{k_1} + \omega_{k_2}}. \qquad (26)$$

than H_3 vanishes.

Counting all fourth terms, making symmetrization and calculating new H_4 one can get

$$\begin{aligned}
H_4 &= \frac{1}{2}\int [W^{k_3k_4}_{k_1k_2} + D^{k_3k_4}_{k_1k_2} + (\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4})(\tilde{W}^{k_3k_4}_{k_1k_2} - i\tilde{\tilde{W}}^{k_3k_4}_{k_1k_2})] b^*_{k_1} b^*_{k_2} b_{k_3} b_{k_4} \delta_{k_1+} \\
&\quad {}_{k_2-k_3-k_4} dk_1 dk_2 dk_3 dk_4 + \\
&+ \frac{1}{3}\int \Big[(G^{k_4}_{k_1k_2k_3} + D^{k_4}_{k_1k_2k_3} - (\omega_{k_1} + \omega_{k_2} + \omega_{k_3} - \omega_{k_4})\tilde{G}^{k_4}_{k_1k_2k_3}) b^*_{k_1} b^*_{k_2} b^*_{k_3} b_{k_4} + c.c. \Big] \delta_{k_1+} \\
&\quad {}_{k_2+k_3-k_4} dk_1 dk_2 dk_3 dk_4 +
\end{aligned}$$

$$+\frac{1}{12}\int\Big[(R_{k_1k_2k_3k_4}+D_{k_1k_2k_3k_4}-(\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{k_4})\tilde{R}_{k_1k_2k_3k_4})b^*_{k_1}b^*_{k_2}b^*_{k_3}b^*_{k_4}+c.c.\Big]\delta_{k_1+k_2+k_3+k_4}dk_1dk_2dk_3dk_4. \quad (27)$$

Here

$$\begin{aligned} D^{k_3k_4}_{k_1k_2} = {} & \tilde{V}^{k_1}_{k_3k_1-k_3}\tilde{V}^{k_4}_{k_2k_4-k_2}\left[\omega_{k_1}-\omega_{k_3}-\omega_{k_1-k_3}+\omega_{k_4}-\omega_{k_2}-\omega_{k_4-k_2}\right]+\\ &+\tilde{V}^{k_2}_{k_3k_2-k_3}\tilde{V}^{k_4}_{k_1k_4-k_1}\left[\omega_{k_2}-\omega_{k_3}-\omega_{k_2-k_3}+\omega_{k_4}-\omega_{k_1}-\omega_{k_4-k_1}\right]+\\ &+\tilde{V}^{k_1}_{k_4k_1-k_4}\tilde{V}^{k_3}_{k_2k_3-k_2}\left[\omega_{k_1}-\omega_{k_4}-\omega_{k_1-k_4}+\omega_{k_3}-\omega_{k_2}-\omega_{k_3-k_2}\right]+\\ &+\tilde{V}^{k_2}_{k_4k_2-k_4}\tilde{V}^{k_3}_{k_1k_3-k_1}\left[\omega_{k_2}-\omega_{k_4}-\omega_{k_2-k_4}+\omega_{k_3}-\omega_{k_1}-\omega_{k_3-k_1}\right]-\\ &-\tilde{V}^{k_1+k_2}_{k_1k_2}\tilde{V}^{k_3+k_4}_{k_3k_4}\left[\omega_{k_1+k_2}-\omega_{k_1}-\omega_{k_2}+\omega_{k_3+k_4}-\omega_{k_3}-\omega_{k_4}\right]-\\ &-\tilde{U}_{-k_1-k_2k_1k_2}\tilde{U}_{-k_3-k_4k_3k_4}\left[\omega_{k_1+k_2}+\omega_{k_1}+\omega_{k_2}+\omega_{k_3+k_4}+\omega_{k_3}+\omega_{k_4}\right], \end{aligned} \quad (28)$$

$$\begin{aligned} D^{k_4}_{k_1k_2k_3} = {} & \tilde{V}^{k_1+k_2}_{k_1k_2}\tilde{V}^{k_4}_{k_3k_4-k_3}(\omega_{k_1+k_2}-\omega_{k_1}-\omega_{k_2}-\omega_{k_4}+\omega_{k_3}+\omega_{k_3-k_4})+\\ &+\tilde{V}^{k_1+k_3}_{k_1k_3}\tilde{V}^{k_4}_{k_2k_4-k_2}(\omega_{k_1+k_3}-\omega_{k_1}-\omega_{k_3}-\omega_{k_4}+\omega_{k_2}+\omega_{k_2-k_4})+\\ &+\tilde{V}^{k_2+k_3}_{k_2k_3}\tilde{V}^{k_4}_{k_1k_4-k_1}(\omega_{k_2+k_3}-\omega_{k_2}-\omega_{k_3}-\omega_{k_4}+\omega_{k_1}+\omega_{k_1-k_4})+\\ &+\tilde{U}_{-k_1-k_2k_1k_2}\tilde{V}^{k_3}_{k_4k_3-k_4}(\omega_{k_1+k_2}+\omega_{k_1}+\omega_{k_2}-\omega_{k_3}+\omega_{k_4}+\omega_{k_3-k_4})+\\ &+\tilde{U}_{-k_1-k_3k_1k_3}\tilde{V}^{k_2}_{k_4k_2-k_4}(\omega_{k_1+k_3}+\omega_{k_1}+\omega_{k_3}-\omega_{k_2}+\omega_{k_4}+\omega_{k_2-k_4})+\\ &+\tilde{U}_{-k_2-k_3k_2k_3}\tilde{V}^{k_1}_{k_4k_1-k_4}(\omega_{k_2+k_3}+\omega_{k_2}+\omega_{k_3}-\omega_{k_1}+\omega_{k_4}+\omega_{k_1-k_4}), \end{aligned} \quad (29)$$

$$\begin{aligned} D_{k_1k_2k_3k_4} = {} & -\tilde{U}_{-k_1-k_2k_1k_2}\tilde{V}^{k_3+k_4}_{k_3k_4}(\omega_{k_1+k_2}+\omega_{k_1}+\omega_{k_2}+\omega_{k_3+k_4}-\omega_{k_3}-\omega_{k_4})-\\ &-\tilde{U}_{-k_1-k_3k_1k_3}\tilde{V}^{k_2+k_4}_{k_2k_4}(\omega_{k_1+k_3}+\omega_{k_1}+\omega_{k_3}+\omega_{k_2+k_4}-\omega_{k_2}-\omega_{k_4})-\\ &-\tilde{U}_{-k_1-k_4k_1k_4}\tilde{V}^{k_3+k_2}_{k_3k_2}(\omega_{k_1+k_4}+\omega_{k_1}+\omega_{k_4}+\omega_{k_3+k_2}-\omega_{k_3}-\omega_{k_2})-\\ &-\tilde{U}_{-k_2-k_3k_2k_3}\tilde{V}^{k_1+k_4}_{k_1k_4}(\omega_{k_2+k_3}+\omega_{k_2}+\omega_{k_3}+\omega_{k_1+k_4}-\omega_{k_1}-\omega_{k_4})-\\ &-\tilde{U}_{-k_2-k_4k_2k_4}\tilde{V}^{k_1+k_3}_{k_1k_3}(\omega_{k_2+k_4}+\omega_{k_2}+\omega_{k_4}+\omega_{k_1+k_3}-\omega_{k_1}-\omega_{k_3})-\\ &-\tilde{U}_{-k_3-k_4k_3k_4}\tilde{V}^{k_1+k_2}_{k_1k_2}(\omega_{k_3+k_4}+\omega_{k_3}+\omega_{k_4}+\omega_{k_1+k_2}-\omega_{k_1}-\omega_{k_2}). \end{aligned} \quad (30)$$

To cancel nonresonant fourth order terms in (27) relations given below must be valid:

$$\begin{aligned} \tilde{G}^{k_4}_{k_1k_2k_3} &= \frac{1}{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}-\omega_{k_4}}(G^{k_4}_{k_1k_2k_3}+D^{k_4}_{k_1k_2k_3}),\\ \tilde{R}_{k_1k_2k_3k_4} &= \frac{1}{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{k_4}}(R_{k_1k_2k_3k_4}+D_{k_1k_2k_3k_4}). \end{aligned} \quad (31)$$

Now the Hamiltonian has only resonant four-wave interaction term ($2 \Leftrightarrow 2$):

$$
\begin{aligned}
H = & \int \omega_k |b_k|^2 dk + \\
& + \frac{1}{2} \int [W_{k_1 k_2}^{k_3 k_4} + D_{k_1 k_2}^{k_3 k_4} + (\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4})(\tilde{W}_{k_1 k_2}^{k_3 k_4} \\
& - i \tilde{\tilde{W}}_{k_1 k_2}^{k_3 k_4})] b_{k_1}^* b_{k_2}^* b_{k_3} b_{k_4} \delta_{k_1 + k_2 - k_3 - k_4} dk_1 dk_2 dk_3 dk_4
\end{aligned}
\tag{32}
$$

If we put

$$
\tilde{W}_{k_1 k_2}^{k_3 k_4} - i \tilde{\tilde{W}}_{k_1 k_2}^{k_3 k_4} = 0, \tag{33}
$$

we obtain so-called Zakharov equation with the following Hamiltonian:

$$
\begin{aligned}
H &= \int \omega_k |b_k|^2 dk + \frac{1}{2} \int T_{k_1 k_2}^{k_3 k_4} b_{k_1}^* b_{k_2}^* b_{k_3} b_{k_4} \delta_{k_1 + k_2 - k_3 - k_4} dk_1 dk_2 dk_3 dk_4 \\
T_{k_1 k_2}^{k_3 k_4} &= W_{k_1 k_2}^{k_3 k_4} + D_{k_1 k_2}^{k_3 k_4}
\end{aligned}
\tag{34}
$$

At this moment the key point of the transformation takes place: we explicitly use property of vanishing of $T_{k_1 k_2}^{k_3 k_4}$ on the resonant manifold and consider waves propagating in the same direction. Then we chose instead of (33) the following expression:

$$
\tilde{W}_{k_1 k_2}^{k_3 k_4} - i \tilde{\tilde{W}}_{k_1 k_2}^{k_3 k_4} = \frac{1}{\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}} (\tilde{T}_{k_1 k_2}^{k_3 k_4} - W_{k_1 k_2}^{k_3 k_4} - D_{k_1 k_2}^{k_3 k_4}), \tag{35}
$$

here

$$
\begin{aligned}
\tilde{T}_{k_2 k_3}^{k k_1} = & \frac{\theta(k)\theta(k_1)\theta(k_2)\theta(k_3)}{8\pi} [(k k_1 (k + k_1) + k_2 k_3 (k_2 + k_3)) - \\
& - (k k_2 |k - k_2| + k k_3 |k - k_3| + k_1 k_2 |k_1 - k_2| + k_1 k_3 |k_1 - k_3|)],
\end{aligned}
\tag{36}
$$

This coefficient $\tilde{T}_{k_2 k_3}^{k k_1}$ gives us simple Hamiltonian (11).

Now we can calculate symmetrized coefficients A of the cubic part of the transformation:

$$
\begin{aligned}
A_{k_3 k_4}^{k_1 k_2} = & \frac{1}{\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}} \Big[\tilde{T}_{k_1 k_2}^{k_3 k_4} - W_{k_1 k_2}^{k_3 k_4} + 2(U_{-k_1 - k_2 k_1 k_2} \tilde{U}_{-k_3 - k_4 k_3 k_4} + V_{k_1 k_2}^{k_1 + k_2} \tilde{V}_{k_3 k_4}^{k_3 + k_4} \\
& - V_{k_3 k_1 - k_3}^{k_1} \tilde{V}_{k_2 k_4 - k_2}^{k_4} - \tilde{V}_{k_3 k_2 - k_3}^{k_2} V_{k_1 k_4 - k_1}^{k_4} - V_{k_4 k_1 - k_4}^{k_1} \tilde{V}_{k_2 k_3 - k_2}^{k_3} - \tilde{V}_{k_4 k_2 - k_4}^{k_2} V_{k_1 k_3 - k_1}^{k_3}) \Big]
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
A^{k_1 k_2 k_3 k_4} = & \frac{1}{3(\omega_{k_1} + \omega_{k_2} + \omega_{k_3} + \omega_{k_4})} \Big[-R_{k_1 k_2 k_3 k_4} + 2(U_{-k_1 - k_2 k_1 k_2} \tilde{V}_{k_3 k_4}^{k_3 + k_4} + U_{-k_1 - k_3 k_1 k_3} \tilde{V}_{k_2 k_4}^{k_2 + k_4} \\
& + U_{-k_1 - k_4 k_1 k_4} \tilde{V}_{k_2 k_3}^{k_2 + k_3} + \tilde{U}_{-k_2 - k_3 k_2 k_3} V_{k_1 k_4}^{k_1 + k_4} + \tilde{U}_{-k_2 - k_4 k_2 k_4} V_{k_1 k_3}^{k_1 + k_3} + \tilde{U}_{-k_3 - k_4 k_3 k_4} V_{k_1 k_2}^{k_1 + k_2}) \Big]
\end{aligned}
\tag{38}
$$

$$A_{k_4}^{k_1k_2k_3} = \frac{-1}{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}-\omega_{k_4}}\Big[G_{k_1k_2k_3}^{k_4} + 2(V_{k_1k_2}^{k_1+k_2}\tilde{V}_{k_3k_4-k_3}^{k_4} + V_{k_1k_3}^{k_1+k_3}\tilde{V}_{k_2k_4-k_2}^{k_4}$$
$$+ U_{-k_1-k_2k_1k_2}\tilde{V}_{k_4k_3-k_4}^{k_3} + U_{-k_1-k_3k_1k_3}\tilde{V}_{k_4k_2-k_4}^{k_2} - \tilde{V}_{k_2k_3}^{k_2+k_3}V_{k_1k_4-k_1}^{k_4} - \tilde{U}_{-k_2-k_3k_2k_3}V_{k_4k_1-k_4}^{k_1})\Big] \tag{39}$$

$$A_{k_2k_3k_4}^{k_1} = \frac{-1}{3(\omega_{k_1}-\omega_{k_2}-\omega_{k_3}-\omega_{k_4})}\Big[G_{k_2k_3k_4}^{k_1} - 2(\tilde{V}_{k_2k_3}^{k_2+k_3}V_{k_4k_1-k_4}^{k_1} + \tilde{V}_{k_2k_4}^{k_2+k_4}V_{k_3k_1-k_3}^{k_1}$$
$$+\tilde{V}_{k_3k_4}^{k_3+k_4}V_{k_2k_1-k_2}^{k_1} + \tilde{U}_{-k_2-k_3k_2k_3}V_{k_1k_4-k_1}^{k_4} + \tilde{U}_{-k_2-k_4k_2k_4}V_{k_1k_3-k_1}^{k_3} + \tilde{U}_{-k_3-k_4k_3k_4}V_{k_1k_2-k_1}^{k_2})\Big]. \tag{40}$$

Below we calculate $A_{k_2k_3k_4}^{k_1}$, $A^{k_1k_2k_3k_4}$, $A_{k_4}^{k_1k_2k_3}$ and $A_{k_1k_2}^{k_3k_4}$ for the case when canonical variable b_k has harmonics with positive k only.

Let us start with $A_{k_2k_3k_4}^{k_1}$, expression (40). According to δ-function in (24) k_1 is also positive. One can finally get:

$$A_{k_2k_3k_4}^{k_1} = \frac{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{k_4}}{48\pi g}k_1(k_1k_2k_3k_4)^{\frac{1}{4}}. \tag{41}$$

Coefficient $A^{k_1k_2k_3k_4}$ has to be calculated for negative k_1 (according to δ-function in (24), so we will calculate it as $A^{-k_1k_2k_3k_4}$.

$$A^{-k_1k_2k_3k_4} = \frac{\omega_{k_1}-\omega_{k_2}-\omega_{k_3}-\omega_{k_4}}{48\pi g}k_1(k_1k_2k_3k_4)^{\frac{1}{4}}. \tag{42}$$

Coefficient $A_{k_4}^{k_1k_2k_3}$ has to be calculated both for positive and negative k_1. For $k_i > 0$ the following is valid:

$$A_{k_4}^{k_1k_2k_3} = \frac{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}+\omega_{k_4}}{16\pi g}k_1(k_1k_2k_3k_4)^{\frac{1}{4}}. \tag{43}$$

For $k_1 < 0$ we will calculate it as $A_{k_4}^{-k_1k_2k_3}$. Let us start with the case $k_4 > k_2$, $k_3 > k_1$:

$$A_{k_4}^{-k_1k_2k_3} = \frac{\omega_{k_4}+\omega_{k_3}+\omega_{k_2}-\omega_{k_1}}{16\pi g}(k_1k_2k_3k_4)^{\frac{1}{4}}k_1\frac{3\sqrt{k_1k_4}-\sqrt{k_2k_3}}{\sqrt{k_1k_4}+\sqrt{k_2k_3}} \tag{44}$$

In the case $k_2 > k_1, k_4 > k_3$:

$$A_{k_4}^{-k_1k_2k_3} = \frac{\omega_{k_4}+\omega_{k_3}+\omega_{k_2}-\omega_{k_1}}{16\pi g}(k_1k_2k_3k_4)^{\frac{1}{4}}k_1\frac{\sqrt{k_1k_4}(2k_3+k_1)-\sqrt{k_2k_3}(2k_3-k_1)}{\sqrt{k_1k_4}+\sqrt{k_2k_3}} \tag{45}$$

In the case $k_3 > k_1, k_4 > k_2$:

$$A_{k_4}^{-k_1k_2k_3} = \frac{\omega_{k_4} + \omega_{k_3} + \omega_{k_2} - \omega_{k_1}}{16\pi g}(k_1k_2k_3k_4)^{\frac{1}{4}} k_1 \frac{\sqrt{k_1k_4}(2k_2 + k_1) - \sqrt{k_2k_3}(2k_2 - k_1)}{\sqrt{k_1k_4} + \sqrt{k_2k_3}} \tag{46}$$

In the case $k_1 > k_2, k_3 > k_4$:

$$A_{k_4}^{-k_1k_2k_3} = \frac{\omega_{k_4} + \omega_{k_3} + \omega_{k_2} - \omega_{k_1}}{16\pi g}(k_1k_2k_3k_4)^{\frac{1}{4}} k_1 \frac{\sqrt{k_1k_4}(2k_4 + k_1) - \sqrt{k_2k_3}(2k_4 - k_1)}{\sqrt{k_1k_4} + \sqrt{k_2k_3}} \tag{47}$$

Coefficient $A_{k_3k_4}^{k_1k_2}$ has to be calculated both for positive and negative k_1. Below we calculate $A_{k_3k_4}^{k_1k_2}$ for the case $k_1, k_2, k_3, k_4 > 0$. Let us start with the case $k_2 > k_3, k_4 > k_1$:

$$A_{k_3k_4}^{k_1k_2} = \frac{1}{\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}}\left[\tilde{T}_{k_1k_2}^{k_3k_4} - \frac{(k_1k_2k_3k_4)^{\frac{1}{4}}}{8\pi} k_1 \left(3\sqrt{k_1k_2} + \sqrt{k_3k_4}\right)\right] \tag{48}$$

In the case $k_1 > k_3, k_4 > k_2$:

$$\begin{aligned} A_{k_3k_4}^{k_1k_2} = {} & \frac{1}{\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}} \times \\ & \times \left[\tilde{T}_{k_1k_2}^{k_3k_4} - \frac{(k_1k_2k_3k_4)^{\frac{1}{4}}}{8\pi}\left(\sqrt{k_1k_2}(2k_2 + k_1) + \sqrt{k_3k_4}(2k_2 - k_1)\right)\right] \end{aligned} \tag{49}$$

In the case $k_4 > k_1, k_2 > k_3$:

$$\begin{aligned} A_{k_3k_4}^{k_1k_2} = {} & \frac{1}{\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}} \times \\ & \times \left[\tilde{T}_{k_1k_2}^{k_3k_4} - \frac{(k_1k_2k_3k_4)^{\frac{1}{4}}}{8\pi}\left(\sqrt{k_1k_2}(2k_3 + k_1) + \sqrt{k_3k_4}(2k_3 - k_1)\right)\right] \end{aligned} \tag{50}$$

In the case $k_3 > k_1, k_2 > k_4$:

$$\begin{aligned} A_{k_3k_4}^{k_1k_2} = {} & \frac{1}{\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}} \times \\ & \times \left[\tilde{T}_{k_1k_2}^{k_3k_4} - \frac{(k_1k_2k_3k_4)^{\frac{1}{4}}}{8\pi}\left(\sqrt{k_1k_2}(2k_4 + k_1) + \sqrt{k_3k_4}(2k_4 - k_1)\right)\right] \end{aligned} \tag{51}$$

For $k_1 < 0$ we will calculate it as $A_{k_3k_4}^{-k_1k_2}$ and $k_1, k_2, k_3, k_4 > 0$:

$$A_{k_3k_4}^{-k_1k_2} = \frac{1}{\omega_{k_1}+\omega_{k_2}-\omega_{k_3}-\omega_{k_4}}\left[\frac{(k_1k_2k_3k_4)^{\frac{1}{4}}}{8\pi}k_1\left(\sqrt{k_1k_4}+\sqrt{k_1k_3}-\sqrt{k_3k_4}\right)\right] = \\ = \frac{(k_1k_2k_3k_4)^{\frac{1}{4}}}{16\pi g}k_1\left(\omega_{k_2}+\omega_{k_3}+\omega_{k_4}-\omega_{k_1}\right) \tag{52}$$

It appears that if spectrum of $b(x)$ consists of harmonics with positive k only, transformation from b_k to η_k and ψ_k can be considerably simplified. To prove that, let us calculate η_k and ψ_k for positive k using transformations (23) and (7). To recover η_k and ψ_k for negative k one can use the following relations:

$$\eta_{-k} = \eta_k^*, \qquad \psi_{-k} = \psi_k^*. \tag{53}$$

But first let us write η_k and ψ_k as a power series of b_k up to the third order:

$$\eta_k = \eta_k^{(1)} + \eta_k^{(2)} + \eta_k^{(3)}, \qquad \psi_k = \psi_k^{(1)} + \psi_k^{(2)} + \psi_k^{(3)}. \tag{54}$$

Obviously

$$\eta_k^{(1)} = \sqrt{\frac{\omega_k}{2g}}[b_k + b_{-k}^*], \qquad \psi_k^{(1)} = -i\sqrt{\frac{g}{2\omega_k}}[b_k - b_{-k}^*]. \tag{55}$$

Or

$$\eta^{(1)}(x) = \frac{1}{\sqrt{2}g^{\frac{1}{4}}}(\hat{k}^{\frac{1}{4}}b(x) + \hat{k}^{\frac{1}{4}}b(x)^*), \qquad \psi^{(1)}(x) = -i\frac{g^{\frac{1}{4}}}{\sqrt{2}}(\hat{k}^{-\frac{1}{4}}b(x) - \hat{k}^{-\frac{1}{4}}b(x)^*). \tag{56}$$

Operators $\hat{k}^\alpha$ act in Fourier space as multiplication by $|k|^\alpha$.

Quadratic terms in (54) are the following:

$$\eta_k^{(2)} = \sqrt{\frac{\omega_k}{2g}}\left[2\int(\tilde{V}_{kk_1}^{k_2} + \tilde{V}_{-kk_2}^{k_1})b_{k_1}^*b_{k_2}\delta_{k+k_1-k_2}dk_1dk_2 - \right. \\ \left. -\int(\tilde{V}_{k_1k_2}^{k} + \tilde{U}_{-kk_1k_2})b_{k_1}b_{k_2}\delta_{k-k_1-k_2}dk_1dk_2\right], \\ \psi_k^{(2)} = -i\sqrt{\frac{g}{2\omega_k}}\left[2\int(\tilde{V}_{kk_1}^{k_2} - \tilde{V}_{-kk_2}^{k_1})b_{k_1}^*b_{k_2}\delta_{k+k_1-k_2}dk_1dk_2 - \right. \\ \left. -\int(\tilde{V}_{k_1k_2}^{k} - \tilde{U}_{-kk_1k_2})b_{k_1}b_{k_2}\delta_{k-k_1-k_2}dk_1dk_2\right]. \tag{57}$$

All coefficients in (57) can be easily calculated using expressions (18), (26), properties (53) and little algebra. The following formulae are valid for both positive and negative k:

$$
\begin{aligned}
\eta_k^{(2)} &= \frac{|k|}{4\sqrt{2g\pi}}\left[\int k_1^{\frac{1}{4}} b_{k_1} k_2^{\frac{1}{4}} b_{k_2} \delta_{k-k_1-k_2} dk_1 dk_2 + \int k_1^{\frac{1}{4}} b_{k_1}^* k_2^{\frac{1}{4}} b_{k_2}^* \delta_{k+k_1+k_2} dk_1 dk_2 \right.\\
&\left. - 2\int k_1^{\frac{1}{4}} b_{k_1}^* k_2^{\frac{1}{4}} b_{k_2} \delta_{k+k_1-k_2} dk_1 dk_2 \right],\\
\psi_k^{(2)} &= -\frac{i}{4\sqrt{2\pi}}\left[\int (\sqrt{k_1}+\sqrt{k_2}) k_1^{\frac{1}{4}} b_{k_1} k_2^{\frac{1}{4}} b_{k_2} \delta_{k-k_1-k_2} dk_1 dk_2 - \right.\\
&- \int (\sqrt{k_1}+\sqrt{k_2}) k_1^{\frac{1}{4}} b_{k_1}^* k_2^{\frac{1}{4}} b_{k_2}^* \delta_{k+k_1+k_2} dk_1 dk_2 -\\
&\left. -2\,\mathbf{sign}(k)\int (\sqrt{k_1}+\sqrt{k_2}) k_1^{\frac{1}{4}} b_{k_1}^* k_2^{\frac{1}{4}} b_{k_2} \delta_{k+k_1-k_2} dk_1 dk_2 \right].
\end{aligned}
\tag{58}
$$

Applying Fourier transformation to (58) one can get

$$
\begin{aligned}
\eta^{(2)}(x) &= \frac{\hat{k}}{4\sqrt{g}}[\hat{k}^{\frac{1}{4}} b(x) - \hat{k}^{\frac{1}{4}} b^*(x)]^2,\\
\psi^{(2)}(x) &= \frac{i}{2}[\hat{k}^{\frac{1}{4}} b^*(x)\hat{k}^{\frac{3}{4}} b^*(x) - \hat{k}^{\frac{1}{4}} b(x)\hat{k}^{\frac{3}{4}} b(x)]+\\
&+ \frac{1}{2}\hat{H}[\hat{k}^{\frac{1}{4}} b(x)\hat{k}^{\frac{3}{4}} b^*(x) + \hat{k}^{\frac{1}{4}} b^*(x)\hat{k}^{\frac{3}{4}} b(x)].
\end{aligned}
\tag{59}
$$

Here $\hat{H}$—is Hilbert transformation with eigenvalue $i\,\mathbf{sign}(k)$.

Cubic terms in (54) are the following (k, k_1, k_2 and k_3 are positive):

$$
\begin{aligned}
\eta_k^{(3)} &= \sqrt{\frac{\omega_k}{2g}}\left[\int (A^{k}_{k_1k_2k_3} + A_{-kk_1k_2k_3}) b_{k_1} b_{k_2} b_{k_3} \delta_{k-k_1-k_2-k_3} dk_1 dk_2 dk_3 + \right.\\
&+ \int (A^{kk_1}_{k_2k_3} + A^{-kk_2k_3}_{k_1}) b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 +\\
&\left. + \int (A^{kk_1k_2}_{k_3} + A^{-kk_3}_{k_1k_2}) b_{k_1}^* b_{k_2}^* b_{k_3} \delta_{k+k_1+k_2-k_3} dk_1 dk_2 dk_3 \right],\\
\psi_k^{(3)} &= -i\sqrt{\frac{g}{2\omega_k}}\left[\int (A^{k}_{k_1k_2k_3} - A_{-kk_1k_2k_3}) b_{k_1} b_{k_2} b_{k_3} \delta_{k-k_1-k_2-k_3} dk_1 dk_2 dk_3 + \right.\\
&+ \int (A^{kk_1}_{k_2k_3} - A^{-kk_2k_3}_{k_1}) b_{k_1}^* b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 +\\
&\left. + \int (A^{kk_1k_2}_{k_3} - A^{-kk_3}_{k_1k_2}) b_{k_1}^* b_{k_2}^* b_{k_3} \delta_{k+k_1+k_2-k_3} dk_1 dk_2 dk_3 \right]
\end{aligned}
\tag{60}
$$

Some of coefficients in (60) can be easily calculated using expressions for A and little algebra :

$$
\begin{aligned}
A^{k}_{k_1k_2k_3} + A_{-kk_1k_2k_3} &= \frac{\omega_k}{24\pi g} k(kk_1k_2k_3)^{\frac{1}{4}}\\
A^{k}_{k_1k_2k_3} - A_{-kk_1k_2k_3} &= \frac{\omega_{k_1}+\omega_{k_2}+\omega_{k_3}}{24\pi g} k(kk_1k_2k_3)^{\frac{1}{4}}
\end{aligned}
\tag{61}
$$

$$
\begin{aligned}
A^{kk_1k_2}_{k_3} + A^{-kk_3}_{k_1k_2} &= \frac{\omega_{k_1} + \omega_{k_2} + \omega_{k_3}}{8\pi g} k(kk_1k_2k_3)^{\frac{1}{4}} \\
A^{kk_1k_2}_{k_3} - A^{-kk_3}_{k_1k_2} &= \frac{\omega_k}{8\pi g} k(kk_1k_2k_3)^{\frac{1}{4}}
\end{aligned} \tag{62}
$$

For $k, k_1, k_2, k_3 > 0$

$$
\begin{aligned}
& A^{kk_1}_{k_2k_3} + A^{-kk_2k_3}_{k_1} = \frac{\tilde{T}^{k_2k_3}_{kk_1}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}} - \frac{\omega_k}{8\pi g}(kk_1k_2k_3)^{\frac{1}{4}} k - \\
& - \frac{(kk_1k_2k_3)^{\frac{1}{4}}}{8\pi g} min(k, k_1, k_2, k_3) \times \\
& \times \left[\frac{\sqrt{kk_1} + \sqrt{k_2k_3}}{\sqrt{kk_1} - \sqrt{k_2k_3}} \left(\omega_k + \omega_{k_1} + \omega_{k_2} + \omega_{k_3}\right) + \frac{\sqrt{kk_1} - \sqrt{k_2k_3}}{\sqrt{kk_1} + \sqrt{k_2k_3}} \left(\omega_k - \omega_{k_1} - \omega_{k_2} - \omega_{k_3}\right) \right] \\
& A^{kk_1}_{k_2k_3} - A^{-kk_2k_3}_{k_1} = \frac{\tilde{T}^{k_2k_3}_{kk_1}}{\omega_k + \omega_{k_1} - \omega_{k_2} - \omega_{k_3}} - \frac{\omega_{k_1} + \omega_{k_2} + \omega_{k_3}}{8\pi g}(kk_1k_2k_3)^{\frac{1}{4}} k - \\
& - \frac{(kk_1k_2k_3)^{\frac{1}{4}}}{8\pi g} min(k, k_1, k_2, k_3) \times \\
& \times \left[\frac{\sqrt{kk_1} + \sqrt{k_2k_3}}{\sqrt{kk_1} - \sqrt{k_2k_3}} \left(\omega_k + \omega_{k_1} + \omega_{k_2} + \omega_{k_3}\right) - \frac{\sqrt{kk_1} - \sqrt{k_2k_3}}{\sqrt{kk_1} + \sqrt{k_2k_3}} \left(\omega_k - \omega_{k_1} - \omega_{k_2} - \omega_{k_3}\right) \right]
\end{aligned} \tag{63}
$$

Using properties (53) expressions for $\eta^{(3)}_k$ and $\psi^{(3)}_k$ can be extended for negative k, so that the following formulae are valid for both positive and negative k:

$$
\begin{aligned}
\eta^{(3)}_k = & \frac{k^2}{24\pi g^{\frac{3}{4}}\sqrt{2}} \int k_1^{\frac{1}{4}} b_{k_1} k_2^{\frac{1}{4}} b_{k_2} k_3^{\frac{1}{4}} b_{k_3} \delta_{k-k_1-k_2-k_3} dk_1 dk_2 dk_3 + \\
& + \frac{k^2}{24\pi g^{\frac{3}{4}}\sqrt{2}} \int k_1^{\frac{1}{4}} b^*_{k_1} k_2^{\frac{1}{4}} b^*_{k_2} k_3^{\frac{1}{4}} b^*_{k_3} \delta_{k+k_1+k_2+k_3} dk_1 dk_2 dk_3 + \\
& + \int \left[\sqrt{\frac{\omega_k}{2g}} (A^{kk_1}_{k_2k_3} + A^{-kk_2k_3}_{k_1}) + \frac{k^{\frac{3}{2}} (k_1k_2k_3)^{\frac{1}{4}} \left(k_1^{\frac{1}{2}} + k_2^{\frac{1}{2}} + k_3^{\frac{1}{2}}\right)}{8\pi g^{\frac{3}{4}}\sqrt{2}} \right] \times \\
& \times b^*_{k_1} b_{k_2} b_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3 + \\
& + \int \left[\sqrt{\frac{\omega_k}{2g}} (A^{-kk_3}_{k_2k_1} + A^{kk_2k_1}_{k_3}) + \frac{k^{\frac{3}{2}} (k_1k_2k_3)^{\frac{1}{4}} \left(k_1^{\frac{1}{2}} + k_2^{\frac{1}{2}} + k_3^{\frac{1}{2}}\right)}{8\pi g^{\frac{3}{4}}\sqrt{2}} \right] \times \\
& \times b^*_{k_1} b^*_{k_2} b_{k_3} \delta_{k+k_1+k_2-k_3} dk_1 dk_2 dk_3
\end{aligned} \tag{64}
$$

$$\begin{aligned}
\psi_k^{(3)} = &-i\frac{|k|}{24\pi g^{\frac{1}{4}}\sqrt{2}}\int\left(k_1^{\frac{3}{4}}k_2^{\frac{1}{4}}k_3^{\frac{1}{4}}+k_1^{\frac{1}{4}}k_2^{\frac{3}{4}}k_3^{\frac{1}{4}}+k_1^{\frac{1}{4}}k_2^{\frac{1}{4}}k_3^{\frac{3}{4}}\right)b_{k_1}b_{k_2}b_{k_3}\delta_{k-k_1-k_2-k_3}dk_1dk_2dk_3+\\
&+i\frac{|k|}{24\pi g^{\frac{1}{4}}\sqrt{2}}\int\left(k_1^{\frac{3}{4}}k_2^{\frac{1}{4}}k_3^{\frac{1}{4}}+k_1^{\frac{1}{4}}k_2^{\frac{3}{4}}k_3^{\frac{1}{4}}+k_1^{\frac{1}{4}}k_2^{\frac{1}{4}}k_3^{\frac{3}{4}}\right)b^*_{k_1}b^*_{k_2}b^*_{k_3}\delta_{k+k_1+k_2+k_3}dk_1dk_2dk_3+\\
&+\int\left[-i\sqrt{\frac{g}{2\omega_k}}(A^{kk_1}_{k_2k_3}-A^{-kk_2k_3}_{k_1})+i\frac{k^{\frac{3}{2}}(k_1k_2k_3)^{\frac{1}{4}}}{8\pi g^{\frac{1}{4}}\sqrt{2}}\right]b^*_{k_1}b_{k_2}b_{k_3}\delta_{k+k_1-k_2-k_3}dk_1dk_2dk_3+\\
&+\int\left[i\sqrt{\frac{g}{2\omega_k}}(A^{-kk_3}_{k_2k_1}-A^{kk_2k_1}_{k_3})-i\frac{k^{\frac{3}{2}}(k_1k_2k_3)^{\frac{1}{4}}}{8\pi g^{\frac{1}{4}}\sqrt{2}}\right]b^*_{k_1}b^*_{k_2}b_{k_3}\delta_{k+k_1+k_2-k_3}dk_1dk_2dk_3
\end{aligned} \tag{65}$$

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