Chapter 2
Introduction: Digital Filters and Filter Banks

Abstract A basic operation of spectr is filterin. In this introductory chapter, processing of one- and two-dimensional signals by digital filters and filter banks is outlined. A polyphase implementation of multirate filtering is described. The application of filtering with IIR filters, whose transfer functions are rational, is described. Bases and frames in the signals' space that are generated by perfect reconstruction filter banks are discussed. The Butterworth filters, which are used in further constructions, are introduced.

We collect in this chapter some known facts about filtering discrete-time signals. For a detailed presentation of this topic we refer to [3–5].

2.1 Filtering Decaying Signals

Recall that under a discrete-time signal is a sequence $\mathbf{x} \stackrel{\text{def}}{=} \{x[k]\}$ of real numbers, where the norm $\|\mathbf{x}\|_1 \stackrel{\text{def}}{=} \sum_{k\in\mathbb{Z}} |x[k]| < \infty$.

2.1.1 Filters

A linear operator $H: \ \mathbf{y} = H\mathbf{x}$ in the space of decaying signals is time-invariant if the integer shift $\mathbf{x}_d \stackrel{\text{def}}{=} \{x[k+d]\}$ of the input signal results in the same shift $\mathbf{y}_d \stackrel{\text{def}}{=} \{y[k+d]\}$ of the output signal. Such operators are called digital filters. The symbol $\delta[k]$:

$$\delta[k] \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{otherwise,} \end{cases}$$

denotes the Kronecker delta (impulse).

If the input signal $\mathbf{i} = \{\delta[k]\}$, then the output signal $\mathbf{h} = \{h[k]\}$ is called the impulse response (IR) of the filter H. Any discrete-time signal $\mathbf{x}$ can be represented as a linear combination of impulses by $x[k] = \sum_{l\in\mathbb{Z}} x[l]\,\delta[k-l] = \mathbf{x} \star \mathbf{i}$. Then, due to the time-invariance of the filter H, the output signal becomes

A.Z. Averbuch et al., *Spline and Spline Wavelet Methods with Applications to Signal and Image Processing*, DOI 10.1007/978-3-319-22303-2_2

$$\mathbf{y} = H\mathbf{x} = \mathbf{x} \star \mathbf{h} \Longleftrightarrow y[k] = \sum_{l \in \mathbb{Z}} h[k-l]x[l]. \tag{2.1}$$

Thus, a filter is completely defined by its impulse response (IR) **h** and application of the filter to a discrete-time signal is implemented via a discrete convolution. If the IR sequence **h** is finite then the filter H is called a finite IR (FIR) filter. Otherwise it is called an infinite IR (IIR) filter. A filter $\mathbf{h} = \{h[k]\}$ is called causal if $h[k] = 0,\ k < 0$ and anticausal if $h[k] = 0,\ k > 0$. Other ways to characterize a filter are to define either its frequency response (FR)

$$\hat{h}(\omega) = \sum_{k \in \mathbb{Z}} h[k]\, e^{-i\omega k}, \tag{2.2}$$

which is the DTFT of the IR, or the transfer function(TF)

$$h(z) \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} h[k]\, z^{-k}, \tag{2.3}$$

which is the z-transform of the IR. We will refer to a filter by its IR $\mathbf{h} = \{h[k]\}$ or by its transfer function $h(z)$. Recall that filtering of a signal reduces to multiplication in the frequency and z-domains:

$$\mathbf{x} \star \mathbf{h} \Longleftrightarrow \hat{h}(\omega)\, \hat{x}(\omega) \Longleftrightarrow h(z)\, x(z). \tag{2.4}$$

The frequency response of a filter can be represented in a polar form

$$\hat{h}(\omega) = \left|\hat{h}(\omega)\right|\, e^{i\ \arg(\hat{h}(\omega))}, \tag{2.5}$$

where the positive 2π-periodic function $\left|\hat{h}(\omega)\right|$ is referred as the magnitude response (MR) of **h** and the real 2π-periodic function $\arg(\hat{h}(\omega))$, whose values lie within the interval $[-\pi, \pi]$, is called the phase response of **h**. If the phase response of a filter is a linear function of ω then the filter is referred to as a linear phase one. If the impulse response of a filter **h** is symmetric or antisymmetric, then **h** is a linear phase filter.

A filter **h** is called low-pass if it passes low frequencies of signals but attenuates or completely rejects frequencies higher than the cutoff frequency. On the contrary, a high-pass filter attenuates or rejects frequencies lower than the cutoff frequency. A band-pass filter passes frequencies within a certain range (passband) and attenuates (rejects) frequencies outside that range (stopband). Examples of low-pass, band-pass and high-pass filters are displayed in Fig. 2.1.

Allpass filters

The filters, whose magnitude responses are constant and equal to 1, are referred to as allpass filters. Application of such a filter to a signal does not change the magnitudes

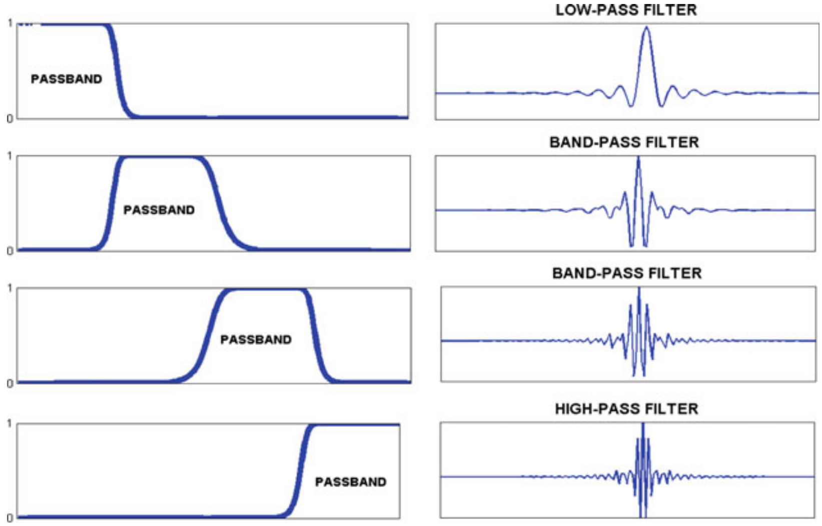

Fig. 2.1 Examples of low-pass, band-pass and high-pass filters. *Left frames* represent magnitude responses of the filters. *Right frames* represent their impulse responses

of all the signal's frequencies. We list a few specific properties of the allpass filters here.

Proposition 2.1 *Assume that* $\mathbf{h} = \{h[k]\}$, $k \in \mathbb{Z}$, *is an allpass filter, that is* $\left|\hat{h}(\omega)\right| \equiv 1$. *Then:*

1. *If a signal* $\mathbf{y} = \mathbf{h} \star \mathbf{x}$ *then* $\|\mathbf{y}\| = \|\mathbf{x}\|$ *(Energy conservation).*
2. *The signals* $\mathbf{h}_l = \{h[k-l]\}$, $k, l \in \mathbb{Z}$, *form a basis in the signal space.*
3. *The basis* $\{\mathbf{h}_l\}$, $l \in \mathbb{Z}$, *is orthonormal in* $l_2(\mathbb{Z})$.

Proof 1. Equations (1.26), (1.23) and (1.24) imply

$$\|\mathbf{h} \star \mathbf{x}\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|\hat{h}(\omega)\,\hat{x}(\omega)\right|^2 d\omega = \frac{1}{2\pi}\int_0^{2\pi} \left|\hat{x}(\omega)\right|^2 d\omega = \|\mathbf{x}\|^2.$$

2. We show that any signal $\mathbf{x}$ can be uniquely represented as a linear combination $\mathbf{x} = \sum_{l\in\mathbb{Z}} \xi[l]\mathbf{h}_l$.
The function $|\hat{h}(\omega)| \equiv 1$ on the real line, thus the same is true for the function $1/|\hat{h}(\omega)|$. Therefore, the filter $\mathbf{g} = \{g[k]\}$, $l \in \mathbb{Z}$, whose FR is $\hat{g}(\omega) = 1/\hat{h}(\omega)$, is all-pass. Apply the filter $\mathbf{g}$ to the signal $\mathbf{x}$ to get

$$\xi = \mathbf{g} \star \mathbf{x} \Longleftrightarrow \xi[k] = \sum_{l\in\mathbb{Z}} x[l]\,g[k-l] \Longleftrightarrow \hat{\xi}(\omega) = \frac{\hat{x}(\omega)}{\hat{h}(\omega)}.$$

Then, the convolution

$$\mathbf{y} = \mathbf{h} \star \xi = \sum_{l \in \mathbb{Z}} \xi[l]\, h[k-l] \Longleftrightarrow \hat{y}(\omega) = \hat{\xi}(\omega)\, \hat{h}(\omega) = \hat{x}(\omega).$$

This means that $\mathbf{x} = \mathbf{y} = \sum_{l \in \mathbb{Z}} \xi[l] \mathbf{h}_l$.

3. Using Eq. (1.38) and the Parseval identities, we write the inner product

$$\langle \mathbf{h}_l,\, \mathbf{h}_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i\omega(l-m)} \left| \hat{h}(\omega) \right|^2 d\omega = \frac{1}{2\pi} \int_0^{2\pi} e^{i\omega(l-m)}\, d\omega = \delta[l-m].$$

■

2.1.1.1 Multirate Filtering

Suppose $\mathbf{x} \stackrel{\text{def}}{=} \{x[k]\}$ is a decaying signal and $M > 1$ is a natural number. The operation $(\downarrow M)\mathbf{x} = \{x[Mk]\}$ is called downsampling the signal $\mathbf{x}$ by factor M. The operation

$$(\uparrow M)\mathbf{x} = \tilde{\mathbf{x}}, \quad \tilde{x}[k] = \begin{cases} x[l], \text{ if } l = kM; \\ 0, \quad \text{otherwise.} \end{cases}, \quad l \in \mathbb{Z}, \tag{2.6}$$

is called upsampling the signal $\mathbf{x}$ by factor M.

If filtering a signal is accompanied by either downsampling or upsampling, then it is called multirate filtering. Let $\mathbf{h} = \{h[k]\}$ and $\tilde{\mathbf{h}} = \left\{\tilde{h}[k]\right\}$ be some filters. Denote by $\bar{\mathbf{h}} = \left\{\tilde{h}[-k]\right\}$ the time-reversed filter $\tilde{\mathbf{h}}$. Application of the filter $\bar{\mathbf{h}}$ to a signal $\mathbf{x}$, which is followed by downsampling by factor M, produces the signal

$$\mathbf{y} = \{y[k]\}, \quad y[k] = \sum_{l \in \mathbb{Z}} \tilde{h}[l - Mk]\, x[l].$$

Application of the filter $\mathbf{h}$ to a signal $\mathbf{y} = \{y[k]\}$, which is upsampled by factor M, produces the signal

$$\begin{aligned} &\check{\mathbf{x}} = \left\{\check{x}[k]\right\}, \qquad \check{x}[k] = \textstyle\sum_{l \in \mathbb{Z}} h[k - Ml]\, y[l] \\ &\Longleftrightarrow \check{x}(z) = h(z)\, y(z^M),\ \hat{\check{x}}(\omega) = \hat{h}(\omega)\, \hat{y}(M\omega). \end{aligned} \tag{2.7}$$

2.1.2 Filter Banks

The set of filters $\tilde{\mathbf{H}} \stackrel{\text{def}}{=} \left\{\tilde{\mathbf{h}}^k\right\}$, $k = 1, \ldots, K$, which, being time-reversed and applied to an input signal $\mathbf{x}$, produces a set of output signals $\{\tilde{\mathbf{y}}^k\}_{k=0}^{K-1}$ downsampled by factor M:

$$y^k[l] = \sum_{n\in\mathbb{Z}} \tilde{h}^k[n - Ml]\, x[n], \quad k = 1, \ldots, K, \; l \in \mathbb{Z}, \tag{2.8}$$

is called the $K-$channel analysis filter bank. The set of filters $\mathbf{H} \stackrel{\text{def}}{=} \left\{\mathbf{h}^r\right\}$, $k = 1, \ldots, K$, which, being applied to a set of input signals $\tilde{\mathbf{y}}^k, \; k = 1, \ldots, K$, that are upsampled by factor M, produces the output signal $\check{\mathbf{x}}$

$$\check{x}[l]) = \sum_{k=1}^{K} \sum_{n\in\mathbb{Z}} h^k[l - Mn]\, y^k[n], \quad l \in \mathbb{Z}, \tag{2.9}$$

is called the $K-$channel synthesis filter bank. If the upsampled signals $\tilde{\mathbf{y}}^k, \; k = 1, \ldots, K$, which are defined in Eq. (2.8), are used as an input to the synthesis filter bank and the output signal $\check{\mathbf{x}} = \mathbf{x}$, then the pair of analysis–synthesis filter banks form a perfect reconstruction (PR) filter bank.

If the number of channels K equals the downsampling factor M, then the filter bank is said to be critically sampled. If $K > M$ then the filter bank is oversampled. Critically-sampled PR filter banks are used in wavelet analysis, while oversampled PR filter banks serve as a source for the frames' design.

2.1.3 Polyphase Representation

Polyphase representation provides tools to handle multirate filtering.

2.1.3.1 DTFT and z-Transform

A signal $\mathbf{x} \stackrel{\text{def}}{=} \{x[k]\}$ can be decomposed into a superposition of M signals, which are downsampled by the factor M, such that

$$\mathbf{x} = \bigcup_{m=0}^{M-1} \mathbf{x}_{m,M}, \quad \mathbf{x}_{m,M} \stackrel{\text{def}}{=} \{x[m + kM]\}, \; k \in \mathbb{Z}. \tag{2.10}$$

This representation is referred to as the polyphase decomposition of the signal $\mathbf{x}$. The signals $\mathbf{x}_{m,M}$ are called the polyphase components of $\mathbf{x}$. Their z-transforms are denoted by $x_{m,M}(z)$. The $z-$transform of $\mathbf{x}$ can represented as follows:

$$x(z) = \sum_{k\in\mathbb{Z}} z^{-k}x[k] = \sum_{k\in\mathbb{Z}} z^{-Mk}x[Mk] + \sum_{k\in\mathbb{Z}} z^{-Mk+1}x[Mk+1] + \cdots$$
$$+ \sum_{k\in\mathbb{Z}} z^{-Mk+M-1}x[Mk+M-1] = \sum_{m=0}^{M-1} z^{-m}\, x_{m,M}(z^M). \tag{2.11}$$

The DTFT of the signal $\mathbf{x}$ is represented as

$$\hat{x}(\omega) = \sum_{l\in\mathbb{Z}} e^{-i\omega l}x[k] = \sum_{m=0}^{M-1} e^{im\omega}\,\hat{x}_{m,M}(M\omega). \tag{2.12}$$

Proposition 2.2 *The DTFT and the z-transform of the zero-polyphase component $\mathbf{x}_{0,M}$ are*

$$\hat{x}_{0,M}(M\omega) = \frac{\sum_{k=0}^{M-1}\hat{x}(\omega + 2\pi k/M)}{M}, \quad x_{k,0}(z^M) = \frac{\sum_{k=0}^{M-1} x\left(z\, e^{2\pi ik/M}\right)}{M}. \tag{2.13}$$

Proof The function $\hat{x}_{m,M}(M\omega) = \sum_{l\in\mathbb{Z}} e^{-ilM\omega}\, x_{m,M}[l]$ is $2\pi/M$-periodic with respect to ω. Thus,

$$\sum_{k=0}^{M-1}\hat{x}(\omega + 2\pi k/M) = \sum_{k=0}^{M-1}\sum_{m=0}^{M-1} e^{im(\omega+2\pi k/M)}\,\hat{x}_{m,M}(M\omega)$$
$$= \sum_{m=0}^{M-1} e^{im\omega}\,\hat{x}_{m,M}(M\omega)\sum_{k=0}^{M-1} e^{2\pi imk/M}.$$

The sum $\sum_{k=0}^{M-1} e^{2\pi imk/M}$ is equal to zero for $m = 1, \ldots M-1$ and to M as $m = 0$. Thus, the first equation in Eq. (2.13) follows. The second one is a direct consequence of the relation $x(e^{i\omega}) = \hat{x}(\omega)$. ■

Examples:

$M = 2$. Then, $\mathbf{x}_{0,2} = \{x[2k]\}$ and $\mathbf{x}_{1,2} = \{x[2k+1]\}$ are the even and odd subsequences of the signal $\mathbf{x}$. The z-transforms are

$$x(z) = x_{0,2}(z^2) + z^{-1}x_{1,2}(z^2), \quad \hat{x}(\omega) = \hat{x}_{0,2}(2\omega) + e^{-i\omega}\,\hat{x}_{1,2}(2\omega)$$
$$x_{0,2}(z^2) = \frac{x(z) + x(-z)}{2}, \quad \hat{x}_{0,2}(2\omega) = \frac{\hat{x}(\omega) + \hat{x}(\omega+\pi)}{2}, \tag{2.14}$$
$$x_{1,2}(z^2) = \frac{x(z) - x(-z)}{2z^{-1}}, \quad \hat{x}_{1,2}(2\omega) = \frac{\hat{x}(\omega) - \hat{x}(\omega+\pi)}{2\, e^{-i\omega}}. \tag{2.15}$$

$M = 3$. Then, $\mathbf{x}_{0,3} = \{x[3k]\}$, $\mathbf{x}_{1,3} = \{x[3k+1]\}$, $\mathbf{x}_{2,3} = \{x[3k+2]\}$. The z-transforms are

$$\begin{aligned} x(z) &= x_{0,3}(z^3) + z^{-1}x_{1,3}(z^3) + z^{-2}x_{2,3}(z^3), \\ x_{0,3}(z^3) &= \frac{x(z) + x(e^{2\pi i/3}z) + x(e^{4\pi i/3}z)}{3}, \\ x_{1,3}(z^3) &= \frac{x(z) + e^{-4\pi i/3}x(e^{2\pi i/3}z) + e^{-2\pi i/3}x(e^{4\pi i/3}z)}{3z^{-1}}, \\ x_{2,3}(z^3) &= \frac{x(z) + e^{-2\pi i/3}x(e^{2\pi i/3}z) + e^{-4\pi i/3}x(e^{4\pi i/3}z)}{3z^{-2}}. \end{aligned} \tag{2.16}$$

Explicit expressions for the DTFT and the z-transforms of the polyphase components $\mathbf{x}_{m,M}$, $m = 1, \ldots, M-1$, for any $M \in \mathbb{N}$, are given in the forthcoming Remark 3.3.1.

When it does not lead to confusion, $\mathbf{x}_m$ and $x_m(z)$ stand for $\mathbf{x}_{m,M}$ and $x_{m,M}(z)$, respectively.

2.1.3.2 Filtering in a Polyphase Form

Filtering a signal can be implemented in a polyphase form. Let $\left\{\tilde{\mathbf{h}}_m\right\}$, $m = 0, \ldots, M-1$, be the polyphase components of a filter $\tilde{\mathbf{h}}$, and $\left\{\tilde{h}_m(z)\right\}$ be their z-transforms. We apply the time-reversed filter $\tilde{\mathbf{h}}$ to a signal $\mathbf{x}$. Then, the z-transform of the output signal $\mathbf{y}$ is

$$\begin{aligned} y(z) = \tilde{h}(z^{-1})\,x(z) &= \sum_{m,n=0}^{M-1} z^{m-n}\tilde{h}_m(z^{-M})x_n(z^M) \\ &= \sum_{r=0}^{M-1} z^{-r}\left(\sum_{m=0}^{M-1} \tilde{h}_m(z^{-M})x_{m+r}(z^M)\right). \end{aligned}$$

Hence, it follows that z-transforms of the polyphase components $\mathbf{y}_{r,M}$ of the output signal $\mathbf{y}$ are

$$y_{r,M}(z) = \sum_{m=0}^{M-1} \tilde{h}_m(1/z)\,x_{m+r}(z), \;\; r = 0, \ldots, M-1.$$

Downsampling the filtered signal $\mathbf{y}$ by factor M means that only the zero polyphase component $\mathbf{y}_0 = \mathbf{y}_{0,M}$ is retained, while the others are discarded:

$$y_0[l] = \sum_{n\in\mathbb{Z}} \tilde{h}[n - Ml]\, x[n] \Longleftrightarrow y_0(z) = \sum_{m=0}^{M-1} \tilde{h}_m(1/z) x_m(z) \tag{2.17}$$
$$= \left(\tilde{h}_0(1/z)\ \tilde{h}_1(1/z)\ \cdots\ \tilde{h}_{M-1}(1/z)\right) \cdot \begin{pmatrix} x_0(z) \\ x_1(z) \\ \dots \\ x_{M-1}(z) \end{pmatrix}.$$

Using the relations of type (2.17), we can express the application of a K-channel analysis filter bank $\tilde{\mathbf{H}} = \left\{\tilde{\mathbf{h}}^k\right\},\ k = 1, \dots, K$ with the downsampling factor M to the signal $\mathbf{x}$ (see Eq. (2.8)) via matrix multiplication

$$\begin{pmatrix} y^0(z) \\ y^1(z) \\ \dots \\ y^{K-1}(z) \end{pmatrix} = \tilde{\mathbf{P}}(1/z) \cdot \begin{pmatrix} x_{0,M}(z) \\ x_{1,M}(z) \\ \dots \\ x_{M-1,M}(z) \end{pmatrix}, \tag{2.18}$$

where the $K \times M$ analysis polyphase matrix is

$$\tilde{\mathbf{P}}(z) \stackrel{\text{def}}{=} \begin{pmatrix} \tilde{h}_0^0(z) & \tilde{h}_1^0(z) & \cdots\ \tilde{h}_{M-1}^0(z) \\ \tilde{h}_0^1(z) & \tilde{h}_1^1(z) & \cdots\ \tilde{h}_{M-1}^1(z) \\ \dots & \dots & \dots \\ \tilde{h}_0^{K-1}(z) & \tilde{h}_1^{K-1}(z) & \cdots\ \tilde{h}_{M-1}^{K-1}(z) \end{pmatrix}. \tag{2.19}$$

Here, $\tilde{h}_m^k(z)$ denotes the z-transform of the m-th polyphase component of the filter $\tilde{\mathbf{h}}^k,\ k = 1, \dots, K,\ m = 0, \dots, M-1$.

If filtering is applied to an upsampled signal $\mathbf{y}$ as in Eq. (2.7), then

$$\check{x}(z) = h(z)y(z^M) = \sum_{m=0}^{M-1} h_{m,M}(z)y(z^M).$$

Thus, the polyphase components of the output signal are

$$\check{x}_m(z) = h_m(z)y(z),\ m = 0, \dots, M-1. \tag{2.20}$$

Equation (2.20) allows to apply the synthesis filter bank $\mathbf{H} = \left\{\mathbf{h}^k\right\},\ k = 1, \dots, K$ to the set of upsampled signals y^k (see Eq. (2.9)) via matrix multiplication

$$\begin{pmatrix} \check{x}_0(z) \\ \check{x}_1(z) \\ \dots \\ \check{x}_{M-1}(z) \end{pmatrix} = \mathbf{P}(z) \cdot \begin{pmatrix} y^0(z) \\ y^1(z) \\ \dots \\ y^{K-1}(z) \end{pmatrix}, \tag{2.21}$$

where the $M \times K$ synthesis polyphase matrix is

$$\mathbf{P}(z) \stackrel{\text{def}}{=} \begin{pmatrix} h_0^0(z) & h_0^1(z) & \cdots & h_0^{K-1}(z) \\ h_1^0(z) & h_1^1(z) & \cdots & h_1^{K-1}(z) \\ \cdots & \cdots & \cdots & \\ h_{M-1}^0(z) & h_{M-1}^1(z) & \cdots & h_{M-1}^{K-1}(z) \end{pmatrix}. \tag{2.22}$$

Here, $h_m^k(z)$ denotes the z-transform of the m-th polyphase component of the filter $\mathbf{h}^k$, and $k = 1, \ldots, K,\ m = 0, \ldots, M-1$. If the relation

$$\mathbf{P}(z) \cdot \tilde{\mathbf{P}}(z^{-1}) = \mathbf{I}_M, \tag{2.23}$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix, holds then

$$\mathbf{P}(z) \cdot \tilde{\mathbf{P}}(1/z) \cdot \begin{pmatrix} x_{0,M}(z) \\ x_{1,M}(z) \\ \cdots \\ x_{M-1,M}(z) \end{pmatrix} = \begin{pmatrix} x_{0,M}(z) \\ x_{1,M}(z) \\ \cdots \\ x_{M-1,M}(z) \end{pmatrix}.$$

Thus, Eq. (2.23) is the condition for the analysis–synthesis pair $\left\{\tilde{\mathbf{H}}, \mathbf{H}\right\}$ of filter banks to form a PR filter bank.

Remark 2.1.1 Polyphase implementation of filter banks is computationally advantageous over direct implementation. In the analysis phase, the polyphase components $\mathbf{x}_m$ of $\mathbf{x}$, which are M times shorter than the original signal, are processed separately by the polyphase components $\tilde{\mathbf{h}}_m^k$ of $\tilde{\mathbf{h}}^k$, which are simpler than the original filter. In the synthesis phase, instead of processing the M-fold upsampled signals $\tilde{\mathbf{y}}^k$ by the filters $\mathbf{h}^k$, the non-upsampled signals $\tilde{\mathbf{y}}^k$ are processed by simpler filters $\mathbf{h}_m^k$.

2.1.4 Interpolating Filters

If the zero polyphase component of a filter $\mathbf{h}$ is constant, i.e. $h_0(z) \equiv C$, then the filter is called interpolating. In this case, Eq. (2.20) implies that the z-transform of the zero polyphase component of the output signal $\check{\mathbf{x}}$ is $\check{x}_0(z) = C y(z)$. This means that $\check{x}[kM] = C y[k],\ k \in \mathbb{Z}$.

Example: Linear interpolation, *M=2*

FIR filter $\mathbf{h}$, whose impulse response and transfer function are

$$h[k] = \begin{cases} 1, & \text{if } k = 0; \\ 1/2, & \text{if } k = -1, 1; \\ 0, & \text{otherwise,} \end{cases} \qquad h(z) = 1 + z^{-1}\frac{1+z^2}{2},$$

respectively, provides the linear interpolation of signals upsampled by factor 2 in the following way:

$$\check{x}[k] = \sum_{l\in\mathbb{Z}} h[k-2l]y[l] \Longleftrightarrow \check{x}(z) = y(z^2) + \frac{z^{-1}+z}{2}y(z^2)$$
$$\Longleftrightarrow x[k] = \begin{cases} y[l], & \text{if } k = 2l; \\ (y[l]+y[l+1])/2, & \text{if } k = 2l+1. \end{cases}$$

Example: Linear interpolation, *M=3*

FIR filter $\mathbf{h}$, whose impulse response and transfer function are

$$h[k] = \begin{cases} 1, & \text{if } k = 0; \\ 2/3, & \text{if } k = -1, 1; \\ 1/3, & \text{if } k = -2, 2; \\ 0, & \text{otherwise}, \end{cases} \qquad h(z) = 1 + z^{-1}\frac{2+z^3}{3} + z^{-2}\frac{2z^3+1}{3}, \tag{2.24}$$

respectively, provides the linear interpolation of signals upsampled by factor 3:

$$\check{x}[k] = \sum_{l\in\mathbb{Z}} h[k-3l]y[l] \Longleftrightarrow \check{x}(z) = \left(1 + z^{-1}\frac{2+z^3}{3} + z^{-2}\frac{2z^3+1}{3}\right) y(z^3)$$
$$\Longleftrightarrow x[k] = \begin{cases} y[l], & \text{if } k = 3l; \\ (2(y[l]+y[l+1])/3, & \text{if } k = 3l+1; \\ ((y[l]+2y[l+1])/3, & \text{if } k = 3l+2. \end{cases}$$

Interpolation of a signal upsampled by factor 3 is illustrated in Fig. 2.2.

Example: Critically sampled PR filter bank, *M=2*

Assume that the transfer functions of a two-channels analysis filter bank with down-sampling by factor 2 are

$$\tilde{h}^0(z) = \frac{1}{\sqrt{2}}\left(-\left(\frac{z-1/z}{2}\right)^2 + z^{-1}\frac{1+z^2}{2}\right),$$
$$\tilde{h}^1(z) = \frac{1}{\sqrt{2}}\left(-\frac{1+z^{-2}}{2} + z^{-1}\right) = z^{-1}\frac{-z+2-z^{-1}}{2\sqrt{2}},$$

and the transfer functions of the synthesis filter bank are

$$h^0(z) = \frac{1}{\sqrt{2}}\left(1 + z^{-1}\frac{1+z^2}{2}\right) = \frac{z+2+z^{-1}}{2\sqrt{2}},$$
$$h^1(z) = \frac{1}{\sqrt{2}}\left(-\frac{1+z^{-2}}{2} + z^{-1}\left(\frac{z-1/z}{2}\right)^2\right).$$

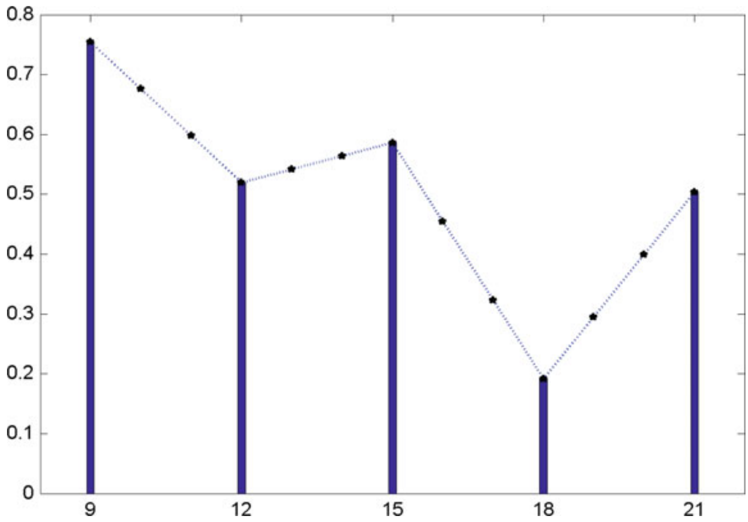

Fig. 2.2 Interpolation of a signal upsampled by factor 3

The filter $\mathbf{h}^0$ is interpolating. Its frequency response is

$$\hat{h}^0(\omega) = \frac{e^{-i\omega} + 2 + e^{i\omega}}{2\sqrt{2}} = \sqrt{2}\cos^2\frac{\omega}{2},$$

whicht means that it possesses a zero phase. All the other filters in the filter bank have a linear phase.

Example: Oversampled PR filter bank, ***M=2***

Assume that the transfer functions of the three-channels analysis and synthesis filter banks with downsampling by factor 2 are

$$\begin{aligned} \tilde{h}^0(z) = h^0(z) &= \frac{z + 2 + z^{-1}}{2\sqrt{2}}, \\ \tilde{h}^1(z) = h^1(z) &= \frac{z^{-1} - z}{2} \\ \tilde{h}^2(z) = h^2(z) &= \frac{-z + 2 - z^{-1}}{2\sqrt{2}}. \end{aligned} \tag{2.25}$$

All the filters in the filter banks have a constant phase and, except for $\tilde{\mathbf{h}}^1$ and $\mathbf{h}^1$, they are interpolating.

2.2 Bases and Frames Generated by Filter Banks

Oversampled PR filter banks generate specific types of frames in the signals' space, whereas critically-sampled PR filter banks generate biorthogonal bases.

Definition 2.1 A system $\tilde{\Phi} \stackrel{\text{def}}{=} \{\tilde{\phi}_j\}_{j\in\mathbb{Z}}$ of signals forms a frame of the signal space if there exist positive constants A and B such that for any signal $\mathbf{x} = \{x[k]\}_{k\in\mathbb{Z}}$

$$A\|\mathbf{x}\|^2 \le \sum_{j\in\mathbb{Z}} |\langle \mathbf{x}, \tilde{\phi}_j\rangle|^2 \le B\|\mathbf{x}\|^2.$$

If the frame bounds A and B are equal to each other then the frame is called tight.

If the system $\tilde{\Phi}$ is a frame then there exists another frame $\Phi \stackrel{\text{def}}{=} \{\phi_i\}_{i\in\mathbb{Z}}$ of the signals' space such that any signal $\mathbf{x}$ can be expanded into the sum $\mathbf{x} = \sum_{i\in\mathbb{Z}} \langle \mathbf{x}, \tilde{\phi}_i\rangle\, \phi_i$. The analysis $\tilde{\Phi}$ and the synthesis Φ frames can be interchanged. Together, they form a bi-frame. If the frame is tight then Φ can be chosen as $\Phi = c\tilde{\Phi}$.

If the elements $\{\tilde{\phi}_j\}$ of the analysis frame $\tilde{\Phi}$ are linearly independent then the synthesis frame Φ is unique, its elements $\{\phi_j\}$ are linearly independent and the frames $\tilde{\Phi}$ and Φ form a biorthogonal basis of the signal space. Otherwise, many synthesis frames can be associated with a given analysis frame.

Assume that an analysis filter bank $\tilde{\mathbf{H}} = \left\{\tilde{\mathbf{h}}^k\right\}_{k=0}^{K-1}$ with downsampling by factor $M \le K$ and a synthesis filter bank $\mathbf{H} = \left\{\mathbf{h}^k\right\}_{k=0}^{K-1}$ form a PR filter bank. Then, after subsequent application of the filter banks $\tilde{\mathbf{H}}$ and $\mathbf{H}$ to a signal $\mathbf{x}$, we have

$$x[l] = \sum_{k=0}^{K-1}\sum_{n\in\mathbb{Z}} h^k[l - Mn]\, y^k[n], \quad l \in \mathbb{Z}, \tag{2.26}$$

where

$$y^k[l] = \sum_{n\in\mathbb{Z}} \tilde{h}^k[n - Ml]\, x[n], \quad k = 0, \ldots, K-1. \tag{2.27}$$

For $k = 0, \ldots, K-1$, we denote

$$\tilde{\varphi}^k \stackrel{\text{def}}{=} \{\tilde{\varphi}^k[l] = \tilde{h}^k[l]\}_{l\in\mathbb{Z}}, \quad \varphi^k \stackrel{\text{def}}{=} \{\varphi^k[l] = h^k[l]\}_{l\in\mathbb{Z}},$$

where $\{\tilde{h}^k[l]\}$ and $\{h^k[l]\}$ are the impulse responses of the filters $\tilde{\mathbf{h}}^k$ and $\mathbf{h}^k$, respectively. Then, Eqs. (2.26) and (2.27) can be rewritten as

$$\mathbf{x} = \sum_{k=0}^{K-1} \mathbf{x}^k, \quad \mathbf{x}^k \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} y^k[n]\, \varphi^k[\cdot - Mn], \tag{2.28}$$

$$y^k[n] = \langle \mathbf{x}, \tilde{\varphi}^k[\cdot - Mn]\rangle, \quad k = 0, \ldots, K-1. \tag{2.29}$$

A condition for Eqs. (2.28) and (2.29) to be a frame expansion of signals $\mathbf{x}$ is established in [2]. Following [2], we call a rational function stable if it does not have poles on the unit circle.

Theorem 2.1 ([2]) *A filter bank* $\tilde{\mathbf{H}} = \left\{\tilde{\mathbf{h}}^k\right\}_{k=0}^{K-1}$*, whose polyphase matrix* $\tilde{\mathbf{P}}(z)$ *consists of stable rational functions, provides a frame expansion of signals if and only if there exists a matrix* $\mathbf{P}(z)$ *of stable rational functions, which is the left inverse of* $\tilde{\mathbf{P}}(z)$ *such that*

$$\mathbf{P}(z)\,\tilde{\mathbf{P}}(z) = c\mathbf{I}, \tag{2.30}$$

where $\mathbf{I}$ *is the identity matrix.*

Theorem 2.2 ([2]) *A filter bank* $\tilde{\mathbf{H}} = \left\{\tilde{\mathbf{h}}^k\right\}_{k=0}^{K-1}$*, whose polyphase matrix* $\tilde{\mathbf{P}}(z)$ *consists of stable rational functions, provides a tight frame expansion of signals if and only if*

$$\tilde{\mathbf{P}}^T(z)\,\tilde{\mathbf{P}}(z^{-1}) = c\mathbf{I}, \tag{2.31}$$

where $\tilde{\mathbf{P}}^T(z)$ *is the transposed matrix* $\tilde{\mathbf{P}}(z)$.

If the conditions of Theorem 2.1 are satisfied, then the system $\tilde{\Phi} \stackrel{\text{def}}{=} \{\tilde{\varphi}^k(\cdot - Mn)\}_{n\in\mathbb{Z}}$, $k = 0, \ldots, K-1$, of $M-$sample translations of the signals $\tilde{\varphi}^k$ forms an analysis frame of the signal space. The system $\Phi \stackrel{\text{def}}{=} \{\varphi^k(\cdot - Mn)\}_{n\in\mathbb{Z}}$, $k = 0, \ldots, K-1$, of $M-$sample translations of the signals φ^k forms a synthesis frame of the signal space. Together $\tilde{\Phi}$ and Φ form a bi-frame. In a special case when the filter banks are critically sampled, that is $K = M$, the pair $\tilde{\Phi}$ and Φ forms a biorthogonal basis. If the condition (2.31) is satisfied, then the synthesis filter bank can be chosen to be equal to the analysis one (up to a constant factor).

2.2.1 *Examples of Filter Banks Implementation*

In this section we discuss, in details, two typical examples.

2.2.1.1 Example 1: Oversampled PR Filterbank with FIR Filters, *M*=2

Assume that the transfer functions of the three-channels filter banks with downsampling by factor 2 is given by Eq. (2.25) (Fig. 2.3):

$$\tilde{h}^0(z) = \frac{z + 2 + z^{-1}}{2\sqrt{2}}, \quad \tilde{h}^1(z) = \frac{z^{-1} - z}{2} \quad \tilde{h}^2(z) = \tilde{h}^0(-z). \tag{2.32}$$

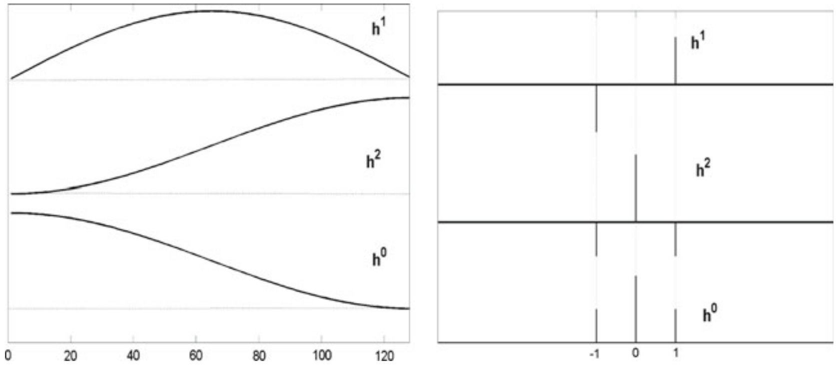

Fig. 2.3 *Left* frequency responses of the filters defined in Eq. (2.32). *Right* their impulse responses. Top panels: FR and IR of the filter $\tilde{\mathbf{h}}^1$ (band-pass), *center panels* FR and IR of the filter $\tilde{\mathbf{h}}^2$ (high-pass), *bottom panels* FR and IR of the filter $\tilde{\mathbf{h}}^0$ (low-pass)

The polyphase matrix of the filter bank

$$\tilde{\mathbf{P}}(z) = \frac{1}{2}\begin{pmatrix} \sqrt{2} & (1+z)/\sqrt{2} \\ 0 & 1-z \\ \sqrt{2} & -(1+z)/\sqrt{2} \end{pmatrix}$$

consists of polynomials that, certainly, are stable. The low-pass $\tilde{\mathbf{h}}^0$ and the high-pass $\tilde{\mathbf{h}}^2$ filters are interpolating. All the filters have a constant phase. The product

$$\begin{aligned} &\tilde{\mathbf{P}}^T(z)\,\tilde{\mathbf{P}}(z^{-1}) \\ &= \frac{1}{4}\begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} \\ (1+z)/\sqrt{2} & 1-z & -(1+z)/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} \sqrt{2} & (1+1/z)/\sqrt{2} \\ 0 & 1-1/z \\ \sqrt{2} & -(1+1/z)/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

According to Eq. (2.18), application of the analysis filter bank to a signal $\mathbf{x}$ provides the output signals:

$$\begin{pmatrix} y^0(z) \\ y^1(z) \\ y^2(z) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & (1+1/z)/\sqrt{2} \\ 0 & 1-1/z \\ \sqrt{2} & -(1+1/z)/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} x_0(z) \\ x_1(z) \end{pmatrix} \iff$$

$$\begin{aligned} y^0[k] &= (x[2k] + (x[2k-1] + x[2k+1])/2)\,\sqrt{2}, \\ y^1[k] &= (x[2k+1] - x[2k-1])/2, \\ y^2[k] &= (x[2k] + (x[2k-1] + x[2k+1])/2)\,\sqrt{2}. \end{aligned}$$

We emphasize that the even and odd polyphase components of the input signal $\mathbf{x}$ are processed separately. To derive the output signals $\mathbf{y}^0$ and $\mathbf{y}^2$, the even component is simply divided by $\sqrt{2}$, while the odd component is processed by a moving average. When deriving the signal $\mathbf{y}^1$, the even component is canceled, while the odd component is processed by a moving difference.

According to Eq. (2.21), application of the synthesis filter bank to the signals $\mathbf{y}^k,\ k = 0, 1, 2$, provides the polyphase components of the input signal $\mathbf{x}$:

$$\begin{pmatrix} \hat{x}_0(z) \\ \hat{x}_1(z) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} \\ (1+z)/\sqrt{2} & 1-z & -(1+z)/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} y^0(z) \\ y^1(z) \\ y^2(z) \end{pmatrix} \Longleftrightarrow$$

$$x[2k] = (y^0[k] + y^2[k])/\sqrt{2},$$
$$x[2k+1] = [(y^0[k] + y^0[k+1] - y^2[k] - y^2[k+1])\sqrt{2} + y^1[k] - y^1[k+1]]/2.$$

We emphasize that even and odd polyphase components of the signal $\mathbf{x}$ are derived separately. To derive the even component, we simply put together the output signals $\mathbf{y}^0$ and $\mathbf{y}^2$. The odd component is derived by apparent arithmetic operations over moving averages and differences of the signals $\mathbf{y}^k,\ k = 0, 1, 2$.

2.2.1.2 Example 2: Oversampled PR Filterbank with IIR Filters, *M*=2

The transfer functions of a three-channel filter bank with downsampling by factor 2 are

$$\tilde{h}^0(z) = \frac{(z+2+z^{-1})^2}{\sqrt{2}\left(z^{-2}+6+z^2\right)}, \quad \tilde{h}^1(z) = \frac{2z^{-1}(z-z^{-1})^2}{z^{-2}+6+z^2}, \quad \tilde{h}^2(z) = \tilde{h}^0(-z). \tag{2.33}$$

All the three filters have a linear phase. Denote $q(z) \stackrel{\text{def}}{=} z+6+1/z$. The function $q(z)$ has two negative real-valued roots $r_1 = -\alpha,\ \alpha = 3 - 2\sqrt{2} \approx 0.172,\ r_2 = 1/r_1$, which lie far away from the unit circle. Therefore, the polyphase matrix of the filter bank

$$\tilde{\mathbf{P}}(z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 4(1+z)/q(z) \\ 0 & \sqrt{2}(z-2+1/z)/q(z) \\ 1 & -4(1+z)/q(z) \end{pmatrix}$$

consists of stable rational functions. The low-pass $\tilde{\mathbf{h}}^0$ and the high-pass $\tilde{\mathbf{h}}^2$ filters are interpolating. The product $\tilde{\mathbf{P}}^T(z)\,\tilde{\mathbf{P}}(z^{-1}) = \mathbf{I}$, therefore the filter bank generates a tight frame in the signal space. Figure 2.4 displays the impulse and frequency responses of the filters $\tilde{\mathbf{h}}^k$ defined in Eq. (2.33). We observe that the impulse responses $\left\{\tilde{h}[n]\right\}$ are symmetric. Their supports are infinite but they decay rapidly ($\sim 0.172^n$ as n grows). Note that the frequency responses of the filters $\tilde{\mathbf{h}}^0$ and $\tilde{\mathbf{h}}^2$ mirror each other.

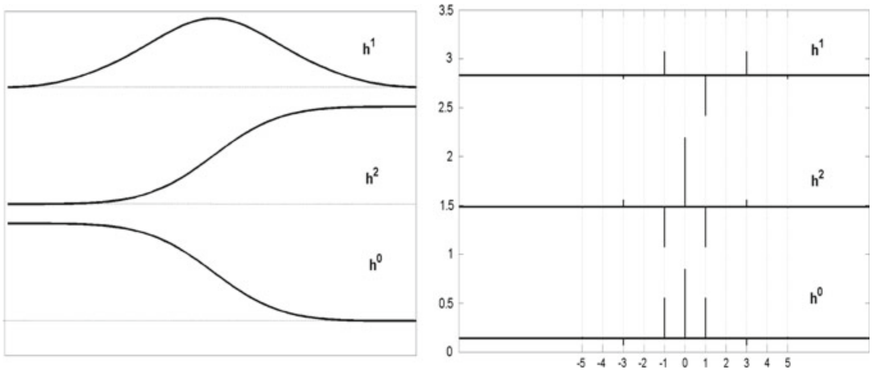

Fig. 2.4 *Left* frequency responses of the filters defined in Eq. (2.33). Right: their impulse responses. *Top panels* FR and IR of the filter $\tilde{\mathbf{h}}^1$ (band-pass), *center panels* FR and IR of the filter $\tilde{\mathbf{h}}^2$ (high-pass), *bottom panels* FR and IR of the filter $\tilde{\mathbf{h}}^0$ (low-pass)

Aplication of the analysis filter bank to the signal $\mathbf{x}$ provides the output signals:

$$\begin{pmatrix} y^0(z) \\ y^1(z) \\ y^2(z) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 4(1+1/z)/q(z) \\ 0 & \sqrt{2}(z-2+1/z)/q(z) \\ 1 & -4(1+1/z)/q(z) \end{pmatrix} \cdot \begin{pmatrix} x_0(z) \\ x_1(z) \end{pmatrix} \Longleftrightarrow$$
$$y^{0/2}(z) = \frac{1}{\sqrt{2}} \left(x_0(z) \pm 4 \frac{1+1/z}{z+6+1/z} x_1(z) \right),$$
$$y^1(z) = \sqrt{2} \frac{z-2+1/z}{z+6+1/z} x_1(z).$$

While the even polyphase component of $\mathbf{x}$ either remains intact (up to a constant factor) or becomes discarded, the odd component is subjected to filtering by filters, whose transfer functions are rational functions. The denominator $q(z) = z+6+1/z$ of these rational functions is invariant about inversion $z \to 1/z$. It has two simple real-valued roots $r_1 = \alpha,\ r_2 = 1/r_1$, where $\alpha = 3 - 2\sqrt{2} \approx 0.172 < 1$. Thus,

$$q(z) = \frac{1}{\alpha}(1+\alpha z)(1+\alpha z^{-1}), \quad 0 < \alpha < 1. \tag{2.34}$$

Many filters, which are to be designed and utilized in the subsequent sections can be factorized into products of elementary rational functions, whose denominators have a structure similar to Eq. (2.34). Therefore, we will discuss in Sect. 2.2.2 the implementation of such IIR filters $\mathbf{f}$, whose transfer function is

$$f(z) = \frac{a+z}{(1+\alpha z)(1+\alpha z^{-1})}, \quad 0 < \alpha < 1.$$

2.2.2 Implementation of Causal—Anticausal IIR Filters with a Rational Transfer Function

Assume that the transfer function of a filter $\mathbf{f}$ is

$$f(z) = \frac{a+z}{(1+\alpha z)(1+\alpha z^{-1})}, \quad 0 < \alpha < 1. \tag{2.35}$$

and

$$\mathbf{y} = \mathbf{f} \star \mathbf{x} \Longleftrightarrow y(z) = f(z)\, x(z), \tag{2.36}$$

where $\mathbf{f} = \{f[k]\}$ and $f(z)$ is defined in Eq. (2.35).

FIR approximation of IIR filters

The function $f(z)$ defined in Eq. (2.35) can be expanded into a Laurent series, which converges on the unit circle $z = e^{i\omega}$. For this, we denote $b = (a-\alpha)/(1-\alpha^2)$, $c = (1-a\alpha)/(1-\alpha^2)$ and represent $f(z)$ as the sum

$$f(z) = \frac{b}{1+\alpha z^{-1}} + \frac{cz}{1+\alpha z} = b \sum_{k=0}^{\infty} \left(\frac{-\alpha}{z}\right)^k + c \sum_{k=1}^{\infty} (-\alpha z)^k . \tag{2.37}$$

Since $0 < \alpha < 1$, the terms in the series decay exponentially on the unit circle. Therefore, the series can be truncated to

$$f(z) \approx \sum_{k=0}^{M} b \left(\frac{-\alpha}{z}\right)^k + c \sum_{k=1}^{M} (-\alpha z)^k ,$$

where M is chosen in a way that the tail of the series that satisfies

$$\left| \sum_{k=M+1}^{\infty} b \left(\frac{-\alpha}{z}\right)^k + c\,(-\alpha z)^k \right| < \frac{(|b|+|c|)\alpha^{M+1}}{1-\alpha}$$

becomes negligible. Then, the filtering is implemented via the convolution

$$y[l] \approx b \sum_{k=0}^{M} (-\alpha)^k\, x[l-k] \text{ (causal) } + c \sum_{k=1}^{M} (-\alpha)^k\, x[l+k] \text{ (anticausal)}. \tag{2.38}$$

Because of presence of anticausal component in (2.38), processing is implemented via a moving average with the delay M.

In practice, signals have a finite duration $\mathbf{x} = \{x[k]\}$, $K_0 \geq k \geq K_1$. Thus, in order to derive the values $y[l]$, $l = K_0, \ldots, K_0 + M$, the input signal $\mathbf{x}$ is extended beyond the initial value $x[K_0]$. Different ways for the extension are discussed in [1]. Most common in signal and, especially, in image processing is mirroring around the point $K_0 - 1/2$. To derive the values $y[l],\ l = K_1 - M + 1, \ldots, K_1$, we mirror the input signal $\mathbf{x}$ around the point $K_1 + 1/2$. Thus,

$$\check{x}[k] = \begin{cases} x[K_0 + l - 1] & \text{if } k = K_0 - l,\ l = 1, \ldots, M; \\ x[k], & \text{if } K_0 \geq k \geq K_1; \\ x[K_1 - l + 1] & \text{if } k = K_1 + l,\ l = 1, \ldots, M; \\ 0 & \text{otherwise.} \end{cases} \tag{2.39}$$

Parallel recursive filtering Denote

$$y_1(z) \stackrel{\text{def}}{=} \frac{b}{1 + \alpha z^{-1}} x(z), \quad y_2(z) \stackrel{\text{def}}{=} \frac{cz}{1 + \alpha z} x(z). \tag{2.40}$$

Then

$$y_1(z)(1 + \alpha z^{-1}) = b\, x(z) \Longleftrightarrow y_1[k] = b\, x[k] - \alpha\, y_1[k-1], \tag{2.41}$$
$$y_2(z)(1 + \alpha z) = c\, z x(z) \Longleftrightarrow y_2[k] = c\, x[k+1] - \alpha\, y_1[k+1]. \tag{2.42}$$

We are interested in the values $y[k] = y_1[k] + y_2[k],\ K_0 \geq k \geq K_1$. To start the computations in Eqs. (2.40) and (2.41), we need to know the values $y_1[K_1]$ and $y_2[K_2]$, respectively. These values can be evaluated using Eqs. (2.38) and (2.39) such that

$$y_1[K_0] \approx b \sum_{k=0}^{M} (-\alpha)^k\, \check{x}[K_0 - k], \quad y_2[K_1] \approx c \sum_{k=1}^{M} (-\alpha)^k\, \check{x}[K_1 + k]. \tag{2.43}$$

Then, the values $y_1[k],\ k > K_1$ are calculated by causal (left–to–right) recursion as in Eq. (2.41), while the values $y_2[k],\ k > K_1$ are calculated by anticausal (right–to–left) recursion as in Eq. (2.42).

Cascade recursive filtering For this, the filter $\mathbf{f}$ is represented as a cascade of simple filters which are being implemented subsequently as $\mathbf{y} = \mathbf{f} \star \mathbf{x} = \mathbf{f}_3 \star \mathbf{f}_2 \star \mathbf{f}_1 \star \mathbf{x}$, where

$$f_1(z) = a + z, \quad f_2(z) = \frac{1}{1 + \alpha z^{-1}}, \quad f_3(z) = \frac{1}{1 + \alpha z}.$$

The cascade algorithm is implemented by the following steps:

1. Apply FIR filtering:

$$\mathbf{x}_1 \stackrel{\text{def}}{=} \mathbf{f}_1 \star \mathbf{x} \Longleftrightarrow x_1[k] = a x[k] + x[k+1], \quad x_1[K_1] = (1 + a)\, x[K_1].$$

2. Apply the causal recursion

$$\mathbf{x}_2 \stackrel{\text{def}}{=} \mathbf{f}_2 \star \mathbf{x}_1 \Longleftrightarrow x_2[k] = x_1[k] - \alpha\, x_2[k-1], \quad x_2[K_0] \approx \sum_{k=0}^{M} (-\alpha)^k\, x_1[K_0 + k].$$

3. Apply the anticausal recursion

$$\mathbf{y} \stackrel{\text{def}}{=} \mathbf{f}_3 \star \mathbf{x}_2 \Longleftrightarrow y[k] = x_2[k] - \alpha\, y[k+1], \quad y[K_1] \approx \sum_{k=0}^{M} (-\alpha)^k\, x_2[K_1 - k].$$

In a more general case, the transfer function can be represented as a product of the elementary blocks:

$$f(z) = \frac{P(z)}{(1+\alpha_1 z)(1+\alpha_1 z^{-1}) \cdot \ldots \cdot (1+\alpha_m z)(1+\alpha_m z^{-1})}, \tag{2.44}$$

where $P(z)$ is a Laurent polynomial and $\alpha_1,\ \alpha_2, \ldots, \alpha_m$ are positive numbers smaller than 1. Then, the filter $\mathbf{f}$ can be implemented via a cascade of successive causal – anticausal recursions preceded by the application of FIR filtering with the transfer function $P(z)$. Another option is to split $f(z)$ into a sum of elementary fractions

$$f(z) = Q(z) + \frac{b_1}{1+\alpha_1 z^{-1}} + \frac{c_1 z}{1+\alpha_1 z} + \cdots + \frac{b_m}{1+\alpha_m z^{-1}} + \frac{c_m z}{1+\alpha_m z},$$

where $Q(z)$ is a Laurent polynomial, and implement the elementary IIR filters in parallel.

2.3 Discrete-Time Butterworth Filters

The Butterworth filters are widely used in signal processing [3]. Their characteristic feature is the maximal flatness of the magnitude response in their passband. The magnitude response is monotonic in the passband and in the stopband. When the orders of the filters increase, the shape of their magnitude responses tend to rectangles. In Chap. 12, we discluss a close relation between half-band Butterworth filters and discrete splines. Here, we briefly outline some properties of these filters.

The magnitude-squared frequency responses $\widehat{F}_l^r(\omega)$ and $\widehat{F}_h^r(\omega)$ of the low- and high-pass digital Butterworth filters of order r, respectively, are given by

$$|\hat{F}_l^r(\omega)|^2 = \frac{1}{1 + ((\tan \omega/2)/(\tan \omega_c/2))^{2r}},$$
$$|\hat{F}_h^r(\omega)|^2 = 1 - |\widehat{F}_l^r(\omega)|^2 = \frac{1}{1 + ((\tan \omega/2)/(\tan \omega_c/2))^{2r}},$$

where ω_c is the so-called cutoff frequency.

We are interested in the half-band Butterworth filters where $\omega_c = \pi/2$. In this case

$$|\widehat{F}_l^r(\omega)|^2 = \frac{1}{1+\tan^{2r}\omega/2} = \frac{\cos^{2r}\omega/2}{\cos^{2r}\omega/2+\sin^{2r}\omega/2}, \tag{2.45}$$
$$|\hat{F}_h^r(\omega)|^2 = 1-|\widehat{F}_l^r(\omega)|^2 = \frac{1}{1+\cot^{2r}\omega/2} = \frac{\sin^{2r}\omega/2}{\cos^{2r}\omega/2+\sin^{2r}\omega/2}.$$

If $z = e^{i\omega}$ then we obtain the magnitude-squared transfer functions of the low-pass and high-pass filters. The notation $\rho(z) \stackrel{\text{def}}{=} z+2+z^{-1}$ will be repeatedly used. The transfer functions are

$$|F_l^r(z)|^2 = \frac{\rho^r(z)}{\rho^r(z)+\rho^r(-z)} = \frac{\left(1+z^{-1}\right)^{2r}}{\left(1+z^{-1}\right)^{2r}+(-1)^r\left(1-z^{-1}\right)^{2r}}, \tag{2.46}$$
$$|F_h^r(z)|^2 = \frac{\rho^r(-z)}{\rho^r(z)+\rho^r(-z)} = \frac{(-1)^r\left(1-z^{-1}\right)^{2r}}{\left(1+z^{-1}\right)^{2r}+(-1)^r\left(1-z^{-1}\right)^{2r}}.$$

Figure 2.5, which is produced by the MATLAB code `plot_buttN`, displays the magnitude-squared frequency responses $\hat{F}_l^r(\omega)$ and $\hat{F}_h^r(\omega)$ of the half-band low- and high-pass Butterworth filters of orders $r = 1, 2, 5$. These frequency responses, especially of the order -five filters, are flat within their pass-band, while practically vanish at the stop-band.

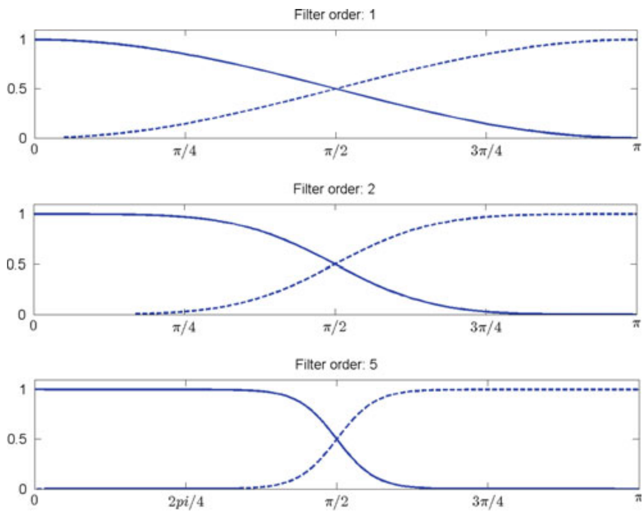

Fig. 2.5 The magnitude-squared frequency responses $\hat{F}_l^r(\omega)$ (*solid lines*) and $\hat{F}_h^r(\omega)$ (*dashed lines*) of the low- and high-pass Butterworth filters of orders $r = 1, 2, 5$

The following property of the denominators

$$D_r(z) \stackrel{\text{def}}{=} (z+1)^{2r} + (-1)^r (z-1)^{2r} \tag{2.47}$$

of the rational functions $|F_l^r(z)|^2$ and $|F_h^r(z)|^2$ is important for the constructions in Chap. 8.

Proposition 2.3 *If the filter's order is $r = 2p + 1$, then*

$$D_r(z) = 4r(\alpha_1^r \alpha_2^r \dots \alpha_p^r)^{-1} z^r \prod_{k=1}^{p} (1 + \alpha_k^r z^{-2})(1 + \alpha_k^r z^2), \tag{2.48}$$

where

$$\alpha_k^r = \cot^2 \frac{(p+k)\pi}{2r} < 1, \ \ k = 1, \dots, p.$$

If $r = 2p$ then

$$D_r(z) = 2(\alpha_1^r \alpha_2^r \dots \alpha_p^r)^{-1} z^r \prod_{k=1}^{p} (1 + \alpha_k^r z^{-2})(1 + \alpha_k^r z^2), \tag{2.49}$$

where

$$\alpha_k^r = \cot^2 \frac{(2p+2k-1)\pi}{4r} < 1, \ \ k = 1, \dots, p.$$

Proof Assume that $r = 2p + 1$. The equation $D_r(z) = 0$ is equivalent to $(z + 1)^{2r} = (z - 1)^{2r}$. The roots of this latter equation can be found from the relation $z_k + 1 = e^{2\pi i k/2r}(z_k - 1), \ k = 1, 2, \dots, 2r - 1$. Consequently,

$$z_k = \frac{e^{2\pi i k/2r} + 1}{e^{2\pi i k/2r} - 1} = -i \cot \frac{k\pi}{2r}, \ \ k = 1, 2, \dots, 2r - 1. \tag{2.50}$$

The points $x_k = \cot \frac{k\pi}{2r}$ are symmetric about zero and $x_{2p+1-k} = x_{r-k} = 1/x_k$. Therefore,

$$D_r(z) = 4rz \prod_{k=1}^{2p} (z^2 + x_k^2) = 4rz \prod_{k=1}^{p} (z^2 + \alpha_k^r)(z^2 + (\alpha_k^r)^{-1}), \tag{2.51}$$

where $\alpha_k^r = x_{p+k}^2$. Hence, Eq. (2.48) follows.

When $r = 2p$, the roots are derived from the equation $z_k + 1 = e^{2\pi i (k-1/2)/2r}(z_k - 1)$. Thus,

$$z_k = -i \cot \frac{(2k+1)r\pi}{4r}, \ \ k = 0, 1, \dots, 2r - 1. \tag{2.52}$$

Hence, Eq. (2.49) follows. ■

References

1. C.M. Brislawn, Classification of nonexpansive symmetric extension transforms for multirate filter banks. Appl. Comput. Harmon. Anal. **3**(4), 337–357 (1996)
2. Z. Cvetković, M. Vetterli, Oversampled filter banks. IEEE Trans. Signal Process. **46**(5), 1245–1255 (1998)
3. A.V. Oppenheim, R.W. Schafer, *Discrete-Time Signal Processing*, 3rd edn. (Prentice Hall, New York, 2010)
4. A. Papoulis, *Signal Analysis* (McGraw-Hill, New York, 1977)
5. P.P. Vaidyanathan, *Multirate Systems and Filter Banks* (Prentice Hall, Englewood Cliffs, 1993)

http://www.springer.com/978-3-319-22302-5

Spline and Spline Wavelet Methods with Applications to Signal and Image Processing
Volume II: Non-Periodic Splines
Averbuch, A.Z.; Neittaanmaki, P.; Zheludev, V.A.
2016, XXV, 426 p. 129 illus., 87 illus. in color.,
Hardcover
ISBN: 978-3-319-22302-5