

Chapter 2

Lagrangians, Hamiltonians and Noether's Theorem

Abstract This chapter is intended to remind the basic notions of the Lagrangian and Hamiltonian formalisms as well as Noether's theorem. We shall first start with a discrete system with N degrees of freedom, state and prove Noether's theorem. Afterwards we shall generalize all the previously introduced notions to continuous systems and prove the generic formulation of Noether's Theorem. Finally we will reproduce a few well known results in Quantum Field Theory.

2.1 Lagrangian Formalism

As it is irrelevant for this first part (the discrete case), we shall drop the super-index notation for coordinates or vectors that we have introduced in the previous chapter.

The action associated to a discrete system with N degrees of freedom ($i = 1, \dots, N$) reads:

$$S(q_i) = \int_{t_1}^{t_2} dt L(q_i, \dot{q}_i, t), \quad (2.1)$$

where $L = L(q_i, \dot{q}_i, t)$ is the Lagrangian of the system and where $\{q_i\}_{i=1}^N$ are the generalized coordinates and $\dot{q}_i \equiv dq_i/dt$ the generalized velocities. In order to obtain the Euler-Lagrange equations of motion we consider small variations of the generalized coordinates q_i keeping the extremes fixed:

$$\boxed{q'_i = q_i + \delta q_i, \quad \delta q_i(t_1) = \delta q_i(t_2) = 0}. \quad (2.2)$$

The first order Taylor expansion of L then gives

$$\begin{aligned} L(q_i + \delta q_i, \dot{q}_i + \delta \dot{q}_i, t) &= L(q_i, \dot{q}_i, t) + \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \\ &\equiv L(q_i, \dot{q}_i, t) + \delta L, \end{aligned} \quad (2.3)$$

where summation over repeated indices is also understood. It is straightforward to demonstrate that the *variation* and the *differentiation* operators commute:

$$\delta q_i(t) = q_i'(t) - q_i(t) \Rightarrow \frac{d}{dt}(\delta q_i) = \dot{q}_i'(t) - \dot{q}_i(t) = \delta \dot{q}_i(t). \quad (2.4)$$

Thus, we obtain the following expression for δL :

$$\begin{aligned} \delta L &= \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \\ &= \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt}(\delta q_i) \\ &= \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right). \end{aligned} \quad (2.5)$$

In order to obtain the equations of motion we apply the **Stationary Action Principle**: *For the physical paths, the action must be a maximum, a minimum or an inflexion point.* This translates mathematically into:

$$\boxed{\delta S = \delta \int_{t_1}^{t_2} dt L = \int_{t_1}^{t_2} dt \delta L = 0}. \quad (2.6)$$

Expanding δL we get:

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i + \int_{t_1}^{t_2} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) = 0. \quad (2.7)$$

Because $\delta q_i(t_1) = \delta q_i(t_2) = 0$ the second integral vanishes:

$$\int_{t_1}^{t_2} dt \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) = \int_{t_1}^{t_2} d \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) = \left[\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right]_{t_1}^{t_2} = 0. \quad (2.8)$$

Therefore, we are left with

$$\delta S = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i = 0, \quad (2.9)$$

for arbitrary δq_i . Thus the following equations must hold

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0}, \quad (2.10)$$

$\forall q_i$. These equations are called the Euler-Lagrange equations of motion.

From (2.8) we can also deduce an important aspect of Lagrangians, that they are not uniquely defined:

$$L(q_i, \dot{q}_i, t) \text{ and } \tilde{L}(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{dF(q_i, t)}{dt} \quad (2.11)$$

generate the same equations of motion. We have an alternative way to directly check that adding a function of the form $dF(q_i, t)/dt$ to the Lagrangian, doesn't alter the equations of motion. Applying (2.10) to $dF(q_i, t)/dt$ we obtain:

$$\frac{\partial}{\partial q_i} \left(\frac{dF(q_i, t)}{dt} \right) - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{q}_i} \left(\frac{dF(q_i, t)}{dt} \right) \right) = 0. \quad (2.12)$$

Next we will present one of the most important theorems of analytical mechanics, a powerful tool that allows us to relate the symmetries of a system with conserved quantities.

2.2 Noether's Theorem

There is a conserved quantity associated with every symmetry of the Lagrangian of a system.

Let's consider a transformation of the type

$$q_i \rightarrow q'_i = q_i + \delta q_i, \quad (2.13)$$

so that the variation of the Lagrangian can be written as the exact differential of some function F :

$$L(q'_i, \dot{q}'_i, t) = L(q_i, \dot{q}_i, t) + \frac{dF(q_i, \dot{q}_i, t)}{dt} \Rightarrow \delta L = \frac{dF(q_i, \dot{q}_i, t)}{dt}. \quad (2.14)$$

Note that here we allow F to also depend on \dot{q}_i (that was not the case for (2.11)). On the other hand, we know that we can write δL as:

$$\delta L = \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) \delta q_i + \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right) = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \delta q_i \right). \quad (2.15)$$

To get to the last equality we used the equations of motion. Let's now write δq_i as an infinitesimal variation of the form

$$q'_i = q_i + \delta q_i = q_i + \epsilon f_i, \quad (2.16)$$

with $|\epsilon| \ll 1$ a constant, and f a smooth, well behaved function. Obviously, in the limit $\epsilon \rightarrow 0$ we obtain

$$\lim_{\epsilon \rightarrow 0} q_i' = q_i \Rightarrow \lim_{\epsilon \rightarrow 0} \delta L = 0. \quad (2.17)$$

Thus, necessarily F must be of the form $F = \epsilon \tilde{F}$, and

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i}(\epsilon f_i) \right) = \epsilon \frac{d\tilde{F}(q_i, \dot{q}_i, t)}{dt}. \quad (2.18)$$

Integrating in t we obtain

$$\frac{\partial L}{\partial \dot{q}_i} f_i = \tilde{F}(q_i, \dot{q}_i, t) + C, \quad (2.19)$$

with C an integration constant. We therefore conclude, that the conserved quantity associated to our infinitesimal symmetry is:

$$C = \frac{\partial L}{\partial \dot{q}_i} f_i - \tilde{F}(q_i, \dot{q}_i, t). \quad (2.20)$$

2.3 Examples

Next, we are going to apply this simple formula to a few interesting cases and reproduce some typical results such as energy and momentum conservation, angular momentum conservation, etc.

2.3.1 Time Translations

Let's consider an infinitesimal time shift: $t \rightarrow t + \epsilon$. The first order Taylor expansion of q_i and \dot{q}_i is given by:

$$\begin{aligned} \delta q_i &= q_i(t + \epsilon) - q_i(t) = \epsilon \dot{q}_i(t) + O(\epsilon^2), \\ \delta \dot{q}_i &= \dot{q}_i(t + \epsilon) - \dot{q}_i(t) = \epsilon \ddot{q}_i(t) + O(\epsilon^2) = \frac{d}{dt}(\delta q_i). \end{aligned} \quad (2.21)$$

If the Lagrangian does not exhibit an explicit time dependence ($\partial L / \partial t = 0$) then

$$\delta L = \epsilon \frac{\partial L}{\partial q_i} \dot{q}_i + \epsilon \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i = \epsilon \frac{dL}{dt} \Rightarrow \tilde{F} = L. \quad (2.22)$$

Thus, the conserved quantity is given by the following

$$\boxed{\frac{\partial L}{\partial \dot{q}_i} \dot{q}_i - L = E}, \quad (2.23)$$

where E is the associated energy of the system.

2.3.2 Spatial Translations

Let's consider a Lagrangian of the form $L = T - V$, where T is the kinetic energy of the system and V a central potential. In this case the canonical momentum p_i defined as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad (2.24)$$

obeys $p_i = \partial T / \partial \dot{q}_i$. Due to the fact that the potential is central and $T \neq T(q_i)$ the Lagrangian obeys

$$L(\mathbf{r}_\alpha + \epsilon \mathbf{n}, \mathbf{v}_\alpha) = L(\mathbf{r}_\alpha, \mathbf{v}_\alpha), \quad (2.25)$$

with \mathbf{r}_α the coordinates of the particle α and \mathbf{n} an arbitrary spatial direction with $|\mathbf{n}| = 1$. We conclude that $\delta L = 0$. Under this spatial translation the coordinates of the particle α transform the following way:

$$\mathbf{r}_\alpha \rightarrow \mathbf{r}'_\alpha = \mathbf{r}_\alpha + \epsilon \mathbf{n}, \quad (2.26)$$

that is

$$r_{\alpha j} \rightarrow r'_{\alpha j} = r_{\alpha j} + \epsilon n_j, \quad (2.27)$$

with $j = 1, 2, 3$. Therefore $f_j = n_j$. The conserved quantity is straightforwardly obtained

$$\boxed{C = \sum_{\alpha} \frac{\partial L}{\partial \dot{q}_{\alpha j}} n_j = \sum_{\alpha} p_{\alpha j} n_j = \sum_{\alpha} \mathbf{p}_{\alpha} \mathbf{n} = \mathbf{P} \mathbf{n}}, \quad (2.28)$$

for an arbitrary \mathbf{n} . Thus, the constant associated to this transformations is the total momentum \mathbf{P} of the system.

2.3.3 Rotations

Again, let's consider a Lagrangian with the same properties as in the previous example. Under an infinitesimal rotation we have

$$\mathbf{r}'_\alpha = \mathbf{r}_\alpha - \epsilon \mathbf{n} \times \mathbf{r}_\alpha, \quad r'_{\alpha j} = r_{\alpha j} + \epsilon \epsilon_{jkm} n_m r_{\alpha k}, \quad (2.29)$$

and just as previously $\delta L = 0$. It is straightforward to observe that $f_j = \epsilon_{jkm} n_m r_{\alpha k}$ (where ϵ_{jkm} is the totally antisymmetric three-dimensional Levi-Civita tensor density). The conserved quantity is therefore (remember that summation over all repeated indices is understood):

$$C = \frac{\partial L}{\partial \dot{q}_{\alpha j}} \epsilon_{jkm} n_m r_{\alpha k} = p_{\alpha j} \epsilon_{jkm} n_m r_{\alpha k} = (\mathbf{p}_\alpha \times \mathbf{r}_\alpha) \mathbf{n} = -\mathbf{L} \mathbf{n}. \quad (2.30)$$

Again, this holds for an arbitrary \mathbf{n} , thus, the conserved quantity is the total angular momentum \mathbf{L} of the system.

2.3.4 Galileo Transformations

For this last example we shall consider the same type of Lagrangian as in the previous cases. A Galileo transformation reads

$$\mathbf{r}_\alpha \rightarrow \mathbf{r}'_\alpha = \mathbf{r}_\alpha + \mathbf{v}t, \quad (2.31)$$

with \mathbf{v} a constant velocity vector, therefore:

$$\dot{\mathbf{r}}_\alpha \rightarrow \dot{\mathbf{r}}'_\alpha = \dot{\mathbf{r}}_\alpha + \mathbf{v}. \quad (2.32)$$

Under these transformations $\delta L = \delta T$. Let's calculate T' explicitly:

$$\begin{aligned} T' &= \frac{1}{2} m_\alpha (\dot{\mathbf{r}}_\alpha + \mathbf{v})^2 = T + m_\alpha \dot{\mathbf{r}}_\alpha \mathbf{v} + \sum_\alpha \frac{1}{2} m_\alpha \mathbf{v}^2 \\ &= T + \frac{1}{2} M \mathbf{v}^2 + \frac{d}{dt} (m_\alpha \mathbf{r}_\alpha \mathbf{v}) = T + \frac{1}{2} M \mathbf{v}^2 + \frac{d}{dt} (M \mathbf{R} \mathbf{v}). \end{aligned} \quad (2.33)$$

Considering an infinitesimal transformation $\mathbf{v} = \epsilon \mathbf{n}$ with $|\epsilon| \ll 1$ and ignoring terms of $O(\epsilon^2)$ we have

$$\delta L = \delta T = \epsilon \frac{d}{dt} (M \mathbf{R} \mathbf{n}). \quad (2.34)$$

The conserved quantity is then given by:

$$C = \sum_{\alpha} p_{\alpha j} n_{j t} - M \mathbf{R} \mathbf{n} = (\mathbf{P} t - M \mathbf{R}) \mathbf{n}, \quad (2.35)$$

for an arbitrary \mathbf{n} . The conserved quantity associated to this transformation is then $\mathbf{P} t - M \mathbf{R}$.

2.4 Hamiltonian Formalism

We define the Hamiltonian functional of a physical system as

$$H(q_i, p_i, t) \equiv p_i \dot{q}_i - L, \quad (2.36)$$

where p_i is called the canonical conjugated momentum

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}, \quad (2.37)$$

as it was already introduced in (2.24). If the Euler-Lagrange equations (2.10) are satisfied then:

$$\dot{p}_i = \frac{\partial L}{\partial q_i}. \quad (2.38)$$

The Hamiltonian equations of motion are obtained just as before by applying the principle of the stationary action:

$$\begin{aligned} \delta S &= \int_{t_1}^{t_2} dt \delta L \\ &= \int_{t_1}^{t_2} dt \delta (p_i \dot{q}_i - H) \\ &= \int_{t_1}^{t_2} dt \left(\delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \\ &= \int_{t_1}^{t_2} dt \left(\delta p_i \dot{q}_i + \frac{d}{dt} (p_i \delta q_i) - \dot{p}_i \delta q_i - \frac{\partial H}{\partial q_i} \delta q_i - \frac{\partial H}{\partial p_i} \delta p_i \right) \\ &= \int_{t_1}^{t_2} dt \left(\delta p_i \left[\dot{q}_i - \frac{\partial H}{\partial p_i} \right] + \delta q_i \left[-\dot{p}_i - \frac{\partial H}{\partial q_i} \right] \right) + \int_{t_1}^{t_2} d(p_i \delta q_i) \end{aligned}$$

$$= \int_{t_1}^{t_2} dt \left(\delta p_i \left[\dot{q}_i - \frac{\partial H}{\partial p_i} \right] + \delta q_i \left[-\dot{p}_i - \frac{\partial H}{\partial q_i} \right] \right) = 0. \quad (2.39)$$

This must hold for arbitrary δp_i y δq_i , therefore, the Hamiltonian equations of motion are simply given by:

$$\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}}. \quad (2.40)$$

If the Hamiltonian exhibits an explicit time dependence, it can be easily related to the time dependence of the Lagrangian

$$\begin{aligned} \frac{dH}{dt} &= \frac{d}{dt}(p_i \dot{q}_i - L) \\ &= \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial t} \\ &= \dot{q}_i \dot{p}_i - \dot{p}_i \dot{q}_i + \frac{\partial H}{\partial t} \\ &= \dot{p}_i \dot{q}_i + p_i \ddot{q}_i - \frac{\partial L}{\partial q_i} \dot{q}_i - \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i - \frac{\partial L}{\partial t}. \end{aligned} \quad (2.41)$$

Therefore we get to the following simple relation in partial derivatives:

$$\boxed{\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}. \quad (2.42)$$

2.5 Continuous Systems

Until now we have considered discrete systems characterized by N (finite) degrees of freedom. Let's consider now that the system depends on an infinite number of degrees of freedom $N \rightarrow \infty$. It no longer makes any sense to talk about discrete coordinates q_i . Instead we have to replace them by a continuous field that is defined for every point in space and that can also vary with time

$$q_i(t) \quad \rightarrow \quad \phi(\mathbf{x}, t) \equiv \phi(x^\mu) \equiv \phi(x). \quad (2.43)$$

Because now we also have spatial dependence besides time dependence, the following replacement is also justified:

$$\dot{q}_i(t) \rightarrow (\partial_t \phi(x), \partial_k \phi(x)) \equiv \partial_\mu \phi(x). \quad (2.44)$$

Notice that we have introduced the compact relativistic notation (from Chap. 1) and we have supposed that the partial derivatives of the fields are a Lorentz (or Poincaré) covariant quantity (of the form $\partial_\mu \phi(x)$),¹ with

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\partial_t, \nabla) \equiv (\partial_t, \partial_k). \quad (2.45)$$

It will also be useful to define the following contravariant quantity

$$\partial^\mu \equiv g^{\mu\nu} \partial_\nu = (\partial_t, -\nabla) \equiv (\partial_t, -\partial_k). \quad (2.46)$$

Also, we are only interested in Lagrangians that are invariant under space-time translations besides Lorentz transformations (Poincaré group), therefore they cannot depend explicitly on x^μ . The most generic Lagrangian that exhibits all the properties we have just described can be written as:

$$L = \int_V d^3x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)). \quad (2.47)$$

where \mathcal{L} is called a Lagrangian density (which we will shortly end up calling Lagrangian). Because a system can depend in general on more than one field, we have written our Lagrangian density as a functional of M (with M finite) fields $\{\phi_i\}_{i=1}^M$. Thus, the action can simply be written as an integral of the Lagrangian density

$$\begin{aligned} S &= \int_{t_1}^{t_2} dt L \\ &= \int_{t_1}^{t_2} dt \int_V d^3x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \\ &= \int_{x_1}^{x_2} d^4x \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)). \end{aligned} \quad (2.48)$$

Just as in the discrete case, in order to obtain the Euler-Lagrange equations of motion we will consider small variations of the fields, keeping the extremes fixed

$$\phi'_i(x) = \phi_i(x) + \delta\phi_i(x); \quad \delta\phi_i(x_1) = \delta\phi_i(x_2) = 0. \quad (2.49)$$

¹This is not the most general case, of course, but as we are interested in applying field theory to Special Relativity we shall only restrict our study to this case.

Under these variations, we define

$$\delta \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \equiv \mathcal{L}(\phi_i(x) + \delta \phi_i(x), \partial_\mu \phi_i(x) + \delta[\partial_\mu \phi_i(x)]) - \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)), \quad (2.50)$$

thus, we obtain the following

$$\begin{aligned} \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta \phi_i(x) + \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i(x)]} \delta [\partial_\mu \phi_i(x)] \\ &= \frac{\partial \mathcal{L}}{\partial \phi_i(x)} \delta \phi_i(x) + \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i(x)]} \partial_\mu [\delta \phi_i(x)] \\ &= \left(\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i(x)]} \right) \delta \phi_i(x) + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i(x)]} \delta \phi_i(x) \right), \end{aligned} \quad (2.51)$$

where summation over all repeated indices is understood. Similar to (2.4), the *variation* and *derivation* operators commute. Applying the principle of the Stationary Action we obtain the Euler-Lagrange equations for continuous systems as follows:

$$\begin{aligned} \delta S &= \int_{x_1}^{x_2} \left(\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i(x)]} \right) \delta \phi_i(x) + \int_{x_1}^{x_2} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i(x)]} \delta \phi_i(x) \right) \\ &= \int_{x_1}^{x_2} \left(\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i(x)]} \right) \delta \phi_i(x) = 0, \end{aligned} \quad (2.52)$$

for arbitrary $\delta \phi_i(x)$, therefore, the equations we are looking for take the form

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi_i(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i(x)]} = 0}, \quad (2.53)$$

$\forall \phi_i, i = 1, \dots, M$. Let's now take another look at (2.52). Because $\delta \phi_i(x_1) = \delta \phi_i(x_2) = 0$, we have found that

$$\int_{x_1}^{x_2} \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i(x)]} \delta \phi_i(x) \right) = 0. \quad (2.54)$$

Thus, if we consider an arbitrary functional of the form $b^\mu(\phi_i(x))$, then

$$\delta \int_{x_1}^{x_2} \partial_\mu b^\mu(\phi_i(x)) = \int_{x_1}^{x_2} \partial_\mu \left(\frac{\partial b^\mu}{\partial \phi_i(x)} \delta [\phi_i(x)] \right) = 0. \quad (2.55)$$

We conclude that a Lagrangian density is not uniquely defined. Similar to the discrete case, one can always add a functional of the form $\partial_\mu b^\mu(\phi_i(x))$ without altering the equations of motion. Therefore

$$\boxed{\mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) \text{ and } \mathcal{L}(\phi_i(x), \partial_\mu \phi_i(x)) + \partial_\mu b^\mu(\phi_i(x))}, \quad (2.56)$$

render the same equations of motion.

2.6 Hamiltonian Formalism

We define the Hamiltonian density as

$$\mathcal{H}(\pi_i(x), \phi_i(x), \nabla \phi_i(x)) \equiv \dot{\phi}_i(x) \pi_i(x) - \mathcal{L}, \quad (2.57)$$

where $\dot{\phi}_i(x) \equiv \partial_t \phi_i(x)$ and $\pi_i(x)$ is the canonical momentum associated to the field $\phi_i(x)$:

$$\pi_i(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_i(x)}. \quad (2.58)$$

The action can be written in terms of the Hamiltonian density as

$$S = \int_{x_1}^{x_2} d^4x \mathcal{L} = \int_{x_1}^{x_2} d^4x \left(\dot{\phi}_i(x) \pi_i(x) - \mathcal{H} \right). \quad (2.59)$$

When applying the principle of the stationary action we obtain

$$\delta S = \int_{x_1}^{x_2} d^4x \left[\delta \pi_i \left\{ \dot{\phi}_i - \frac{\partial \mathcal{H}}{\partial \pi_i} \right\} - \delta \phi_i \left\{ \dot{\pi}_i + \frac{\partial \mathcal{H}}{\partial \phi_i} - \partial_k \frac{\partial \mathcal{H}}{\partial (\partial_k \phi_i)} \right\} \right] = 0, \quad (2.60)$$

for arbitrary $\delta \pi_i$ and $\delta \phi_i$. Thus the equations of motion simply read

$$\boxed{\dot{\phi}(x) = \frac{\partial \mathcal{H}}{\partial \pi(x)}, \quad \dot{\pi}(x) = -\frac{\partial \mathcal{H}}{\partial \phi(x)} + \partial_k \frac{\partial \mathcal{H}}{\partial (\partial_k \phi(x))}}. \quad (2.61)$$

where ∂_k are the spatial derivatives ($k = 1, 2, 3$).

2.7 Noether's Theorem (The General Formulation)

Until now we have only introduced a global variation of a field, which is defined as the variation of the shape of the field without changing the space-time coordinates x^μ :

$$\boxed{\delta\phi_i(x) \equiv \phi'_i(x) - \phi_i(x)}. \quad (2.62)$$

Besides this, we can define another type of variation which is closely related, a local variation. It is defined as the difference between the fields evaluated in the same space-time point but in two different coordinates systems:

$$\boxed{\bar{\delta}\phi_i(x) \equiv \phi'_i(x') - \phi_i(x)}. \quad (2.63)$$

Let's now consider a continuous space-time transformation of the type

$$\boxed{x^\mu \rightarrow x'^\mu = x^\mu + \Delta x^\mu}, \quad (2.64)$$

which can be a proper orthochronous Lorentz transformation or a space-time translation.² At first order in Δx , $\bar{\delta}\phi_i(x)$ reads:

$$\begin{aligned} \bar{\delta}\phi_i(x) &= \phi'_i(x') - \phi_i(x) \\ &= \phi'_i(x + \Delta x) - \phi_i(x) \\ &\approx \phi'_i(x) + \left(\partial_\mu \phi'_i(x)\right) \Delta x^\mu - \phi_i(x) \\ &\approx \phi'_i(x) + \left(\partial_\mu \phi_i(x)\right) \Delta x^\mu - \phi_i(x) \\ &= \delta\phi_i(x) + \left(\partial_\mu \phi_i(x)\right) \Delta x^\mu. \end{aligned} \quad (2.65)$$

We therefore, have found the following relation between $\delta\phi(x)$ and $\bar{\delta}\phi(x)$ for an infinitesimal transformation of the type (2.64):

$$\boxed{\bar{\delta}\phi_i(x) = \delta\phi_i(x) + \left(\partial_\mu \phi_i(x)\right) \Delta x^\mu}. \quad (2.66)$$

We can draw the following conclusion. If $\phi'_i(x') = \phi_i(x)$ (which is in general the case for a scalar field; it is also the case for spinor fields under space-time translations) then

$$\delta\phi_i(x) = -\left(\partial_\mu \phi_i(x)\right) \Delta x^\mu. \quad (2.67)$$

²See Chap. 1 for details.

thus, in this case, an equivalent way of making a transformation of the type (2.64), which acts on the coordinates, is by making an *opposite* transformation on the field:

$$\phi_i(x) \rightarrow \phi'_i(x) = \phi_i(x - \Delta x). \quad (2.68)$$

Let us now deduce how the Lagrangian transforms under these type of variations. In order to keep the notation short, we shall introduce the following short-hand notations:

$$\begin{aligned} \mathcal{L}(x) &\equiv \mathcal{L}\left(\phi_i(x), \partial_\mu \phi_i(x)\right), & \mathcal{L}'(x) &\equiv \mathcal{L}\left(\phi'_i(x), \partial_\mu \phi'_i(x)\right), \\ \mathcal{L}'(x') &\equiv \mathcal{L}\left(\phi'_i(x'), \partial'_\mu \phi'_i(x')\right), & b^\mu(x) &\equiv b^\mu(\phi(x)), \end{aligned} \quad (2.69)$$

where $\partial'_\mu \equiv \frac{\partial}{\partial x'^\mu}$. Keeping only terms up to $O(\Delta x)$ we can calculate $\bar{\delta}\mathcal{L}(x)$ under (2.64):

$$\begin{aligned} \bar{\delta}\mathcal{L}(x) &= \mathcal{L}'(x') - \mathcal{L}(x) \\ &= \mathcal{L}\left(\phi_i(x) + \bar{\delta}\phi_i(x), \partial_\mu \phi_i(x) + \bar{\delta}[\partial_\mu \phi_i(x)]\right) - \mathcal{L}(x) \\ &\approx \mathcal{L}(x) + \frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)} \bar{\delta}\phi_i(x) + \frac{\partial \mathcal{L}(x)}{\partial [\partial_\mu \phi_i(x)]} \bar{\delta}[\partial_\mu \phi_i(x)] - \mathcal{L}(x) \\ &\approx \delta\mathcal{L}(x) + \left(\partial_\mu \mathcal{L}(x)\right) \Delta x^\mu, \end{aligned} \quad (2.70)$$

where we have introduced the following notation

$$\partial_\mu \mathcal{L}(x) \equiv \frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)} \partial_\mu \phi_i(x) + \frac{\partial \mathcal{L}(x)}{\partial [\partial_\nu \phi_i(x)]} \partial_\mu \partial_\nu \phi_i(x). \quad (2.71)$$

Also we have used the following approximation:

$$\bar{\delta}[\partial_\mu \phi_i(x)] \approx \partial_\mu [\bar{\delta}\phi_i(x)] - \left(\partial_\nu \phi_i(x)\right) \frac{\partial \Delta x^\nu}{\partial x^\mu} = \partial_\mu [\bar{\delta}\phi_i(x)]. \quad (2.72)$$

For the last equality we have used that for a Lorentz (Poincaré) transformation $\partial_\mu \Delta x^\nu = 0$ (as it was already mentioned in Chap. 1). Thus, the new *variation* operator $\bar{\delta}$ also commutes with the *derivation* operator when restricting ourselves to Lorentz (Poincaré) continuous transformations. We have therefore obtained an expression similar to (2.66) for $\bar{\delta}\mathcal{L}$:

$$\boxed{\bar{\delta}\mathcal{L}(x) = \delta\mathcal{L}(x) + \left(\partial_\mu \mathcal{L}(x)\right) \Delta x^\mu}. \quad (2.73)$$

Let's now consider the transformation of the action. A transformation that leaves the equations of motion invariant is a symmetry of the system. Under such a symmetry the action S will mostly transform as $S \rightarrow S'$ with S' given by

$$\begin{aligned} S' &= \int_{\Omega'} d^4x' \mathcal{L}'(x') \\ &= \int_{\Omega} d^4x \mathcal{L}(x) + \int_{\Omega} d^4x \partial_{\mu} b^{\mu}(x) \\ &= S + \int_{\Omega} d^4x \partial_{\mu} b^{\mu}(x), \end{aligned} \quad (2.74)$$

so that $\delta S' = \delta S$ (thus generating the same equations of motion). Introducing the Jacobian matrix we have

$$\int_{\Omega} \left| \frac{\partial x'}{\partial x} \right| d^4x \mathcal{L}'(x') = \int_{\Omega} d^4x \mathcal{L}(x) + \int_{\Omega} d^4x \partial_{\mu} b^{\mu}(x). \quad (2.75)$$

This must hold for all space-time volumes Ω , therefore:

$$\left| \frac{\partial x'}{\partial x} \right| \mathcal{L}'(x') = \mathcal{L}(x) + \partial_{\mu} b^{\mu}(x). \quad (2.76)$$

The determinant of the Jacobian matrix is equal to 1 for a proper orthochronous Lorentz transformation or a space-time translation, thus

$$\boxed{\delta \mathcal{L}(x) - \partial_{\mu} b^{\mu}(x) = 0}. \quad (2.77)$$

Introducing (2.73) in (2.77) we obtain:

$$\boxed{\delta \mathcal{L}(x) + \partial_{\mu} \left[\mathcal{L}(x) \Delta x^{\mu} - b^{\mu}(x) \right] = 0}. \quad (2.78)$$

Inserting the explicit form of $\delta \mathcal{L}$ from (2.51) in the last expression, we obtain

$$\begin{aligned} &\left(\frac{\partial \mathcal{L}(x)}{\partial \phi_i(x)} - \partial_{\mu} \frac{\partial \mathcal{L}(x)}{\partial [\partial_{\mu} \phi_i(x)]} \right) \delta \phi_i(x) + \\ &+ \partial_{\mu} \left(\frac{\partial \mathcal{L}(x)}{\partial [\partial_{\mu} \phi_i(x)]} \delta \phi_i(x) \right) + \partial_{\mu} \left[\mathcal{L}(x) \Delta x^{\mu} - \partial_{\mu} b^{\mu}(x) \right] = 0. \end{aligned} \quad (2.79)$$

Using the Euler-Lagrange equations of motion we finally get to the conservation law we were looking for

$$\boxed{\partial_{\mu} j^{\mu}(x) = 0, \quad j^{\mu}(x) = \frac{\partial \mathcal{L}(x)}{\partial [\partial_{\mu} \phi_i(x)]} \delta \phi_i(x) + \mathcal{L}(x) \Delta x^{\mu} - b^{\mu}(x)}, \quad (2.80)$$

with $j^\mu(x)$ the conserved Noether current. Note that our result is completely general, in the sense that it holds for continuous space-time transformations of the type (2.64) and also for transformations that only imply a field variation without modifying the space-time configuration. In this last case we would simply set $\Delta x^\mu = 0$ in (2.80). As it is usual, we can also define a **conserved charge** \mathcal{Q} associated to the conserved current j^μ as:

$$\mathcal{Q} = \int d^3x j^0, \quad \frac{d\mathcal{Q}}{dt} = \int d^3x \partial_0 j^0 = - \int d^3x \nabla \mathbf{j} = 0. \quad (2.81)$$

Next, we shall take a few illustrative examples.

2.8 Examples

2.8.1 Space-time Translations

Consider the following infinitesimal space-time translation:

$$x^\mu \rightarrow x'^\mu = x^\mu - \epsilon^\mu, \quad (2.82)$$

with ϵ^μ real constants. For scalar or spinor fields we have $\phi_i(x) = \phi'_i(x')$ thus $\bar{\delta}\phi_i(x) = 0$. Under this type of transformation our Lagrangians remain unchanged so $\mathcal{L}'(x') = \mathcal{L}(x)$, therefore, by taking a look at (2.77) we conclude that $\partial_\mu b^\mu = 0$. We can thus, eliminate the b^μ term from (2.80) and the conserved current is simply given by:

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial_\nu \phi_i \epsilon^\nu - \mathcal{L} \epsilon^\mu = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\nu \phi_i - \mathcal{L} g^{\mu\nu} \right) \epsilon_\nu. \quad (2.83)$$

The conservation law $\partial_\mu j^\mu = 0$ holds for any arbitrary constants ϵ_ν , therefore we actually have four conserved currents:

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \partial^\nu \phi_i - \mathcal{L} g^{\mu\nu}, \quad (2.84)$$

with $T^{\mu\nu}$ the four-momentum tensor. The conserved Noether charges are then given by

$$\begin{aligned} \mathcal{P}^\nu &\equiv \int d^3x T^{0\nu} \\ &= \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}_i} \partial^\nu \phi_i - g^{0\nu} \mathcal{L} \right) \end{aligned}$$

$$\begin{aligned}
&= \int d^3x (\pi_i \partial^\nu \phi_i - g^{0\nu} \mathcal{L}) \\
&= \int d^3x (\pi_i \dot{\phi}_i - g^{00} \mathcal{L}, -\pi_i \nabla \phi_i) \\
&= \int d^3x (\mathcal{H}, \mathbf{P}) \\
&= (H, \mathbf{P}),
\end{aligned} \tag{2.85}$$

where we have used (2.46). As we can see, the conserved charges are the Hamiltonian and three-momentum operators.

2.8.2 Phase Redefinition

Consider a Lagrangian that depends on the fields ϕ_1 and ϕ_2 with $\phi_1 = \phi$ and $\phi_2 = \phi^\dagger$. If we perform an infinitesimal global phase redefinition of the field

$$\phi(x) \rightarrow \phi'(x) = e^{-i\theta} \phi(x), \tag{2.86}$$

with $\theta \ll 1$ (and where global means that the phase does not depend on the space-time coordinates $\theta \neq \theta(x)$), then we find:

$$\delta\phi(x) = -i\theta\phi(x), \quad \delta\phi^\dagger(x) = i\theta\phi^\dagger(x). \tag{2.87}$$

As this transformation doesn't involve the space-time coordinates we can already set $\Delta x^\mu = 0$ in (2.80). Therefore $\delta\mathcal{L} = \bar{\delta}\mathcal{L} = \partial_\mu b^\mu$. Again, if we only consider the free Dirac or Klein-Gordon Lagrangians then $\delta\mathcal{L} = 0 = \partial_\mu b^\mu$, so we can also eliminate b^μ from (2.80). The conserved current is then given by

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\delta\phi^\dagger = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}(-i\phi)\theta + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}i\phi^\dagger\theta, \tag{2.88}$$

for an arbitrary θ . Thus, redefining the current without the θ multiplying term we find

$$\partial_\mu j^\mu = 0, \quad j^\mu(x) = -i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \phi + i \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)} \phi^\dagger. \tag{2.89}$$

In particular, for the free Dirac Lagrangian $\mathcal{L}_D = \bar{\psi}(x)(i\gamma_\mu\partial^\mu - m)\psi(x)$ we obtain the well known result:

$$\partial_\mu (\bar{\psi}\gamma^\mu\psi) = 0. \tag{2.90}$$

2.8.3 Lorentz Transformations

Consider the following infinitesimal proper orthochronous³ Lorentz transformation:

$$x^\mu \rightarrow x'^\mu = x^\mu + \Delta\omega^\mu{}_\nu x^\nu, \quad (2.91)$$

with $\Delta\omega^\mu{}_\nu = -\Delta\omega^\nu{}_\mu$ real constants. Defining $\Delta\omega_{\mu\nu} \equiv g_{\mu\alpha} \Delta\omega^\alpha{}_\nu$, it is easy to show that the field transformation reads

$$\phi'_i(x') = \phi_i(x) + \frac{1}{2} \Sigma_{(i)}^{\mu\nu} \Delta\omega_{\mu\nu} \phi_i(x), \quad (2.92)$$

with $\Sigma_{(i)}^{\mu\nu} = -\frac{i}{2} \sigma^{\mu\nu} = \frac{1}{4} [\gamma^\mu, \gamma^\nu]$ for spinorial⁴ fields and, $\Sigma_{(i)}^{\mu\nu} = 0$ for scalar fields (no summation over the “ i ” index must be understood in (2.92) nor in the following expression). Using (2.66) we easily find:

$$\begin{aligned} \delta\phi_i(x) &= \frac{1}{2} \Sigma_{(i)}^{\alpha\beta} \Delta\omega_{\alpha\beta} \phi_i(x) - \partial^\alpha \phi_i(x) \Delta\omega_{\alpha\beta} x^\beta \\ &= \frac{1}{2} \Sigma_{(i)}^{\alpha\beta} \Delta\omega_{\alpha\beta} \phi_i(x) - \frac{1}{2} \left(\partial^\alpha \phi_i(x) x^\beta - \partial^\beta \phi_i(x) x^\alpha \right) \Delta\omega_{\alpha\beta} \\ &= \frac{1}{2} \left[\Sigma_{(i)}^{\alpha\beta} + (x^\alpha \partial^\beta - x^\beta \partial^\alpha) \right] \phi_i(x) \Delta\omega_{\alpha\beta}. \end{aligned} \quad (2.93)$$

On the other hand, our Lagrangians are all Lorentz invariant, thus $\bar{\delta}\mathcal{L} = 0$ and so again, we can eliminate b^μ in (2.80) just as in the previous examples. We obtain that the expression for our conserved current reads:

$$\begin{aligned} j^\mu &= \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i]} \left(\Sigma_{(i)}^{\alpha\beta} + x^\alpha \partial^\beta - x^\beta \partial^\alpha \right) \phi_i \frac{1}{2} \Delta\omega_{\alpha\beta} + \mathcal{L} \Delta\omega^\mu{}_\beta x^\beta \\ &= \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i]} \left(\Sigma_{(i)}^{\alpha\beta} + x^\alpha \partial^\beta - x^\beta \partial^\alpha \right) \phi_i \frac{1}{2} \Delta\omega_{\alpha\beta} + \\ &\quad + \frac{1}{2} \mathcal{L} (g^{\mu\alpha} x^\beta - g^{\mu\beta} x^\alpha) \Delta\omega_{\alpha\beta}, \end{aligned} \quad (2.94)$$

for arbitrary $\Delta\omega_{\alpha\beta}$. Thus, we obtain

$$\partial_\mu \mathcal{J}^{\mu, \alpha\beta} = 0, \quad \mathcal{J}^{\mu, \alpha\beta} = x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha} + \frac{\partial \mathcal{L}}{\partial [\partial_\mu \phi_i]} \Sigma_{(i)}^{\alpha\beta} \phi_i, \quad (2.95)$$

³See Chap. 1 for more details.

⁴See Chap. 5 for details on spinor algebra and for the proof of this statement.

which is the conservation law of the angular momentum *pseudo*⁵ tensor $\mathcal{J}^{\mu,\alpha\beta}$ (obviously, for the previous expression, summation over all repeated indices must be understood).

Further Reading

- A. Pich, *Class Notes on Quantum Field Theory*. <http://eeemaster.uv.es/course/view.php?id=6>
 W. Greiner, J. Reinhardt, D.A. Bromley (Foreword), *Field Quantization*
 E.L. Hill, Hamilton's principle and the conservation theorems of mathematical physics. *Rev. Mod. Phys.* **23**, 253
 J.A. de Azcárraga, J.M. Izquierdo, *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics*. Cambridge Monographs in Mathematical Physics
 J.A. Oller, *Mecánica Terica*, <http://www.um.es/oller/docencia/versionmteor.pdf>
 M. Kaksu, *Quantum Field Theory: A Modern Introduction*
 M. Srednicki, *Quantum Field Theory*
 D.E. Soper, *Classical Field Theory*
 D.V. Galtsov, Iu.V. Grats, Ch. Zhukovski, *Campos Clásicos*
 S. Noguera, *Class Notes*

⁵I am calling it pseudo tensor because it is obviously not invariant under translations!.

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