Multidimensional Discrete-Time Fractional Calculus of Variations

Agnieszka B. Malinowska and Tatiana Odzijewicz

Abstract In this paper a discrete-time multidimensional fractional calculus of variations is introduced. The fractional operators are defined in the sense of Grünvald–Letnikov. We derive necessary optimality conditions and then give examples illustrating the use of obtained results.

Keywords Backward fractional difference · Forward fractional difference · Grünvald–Letnikov fractional difference, Euler–Lagrange equations

1 Introduction

The continuous-time fractional calculus of variations (CFCV) has been widely developed since 1996 when the seminal paper [1], by Fred Riewe, about this subject was published. The literature on the CFCV is vast and covers problems with different types of fractional integrals and/or derivatives. We refer the reader to [2, 3] for a general treatment of CFCV. Significantly less works is devoted to the discrete-time fractional calculus of variations (DFCV). Bastos et al. [4, 5] published in 2011 papers introducing the DFCV. They proved first and second order necessary optimality conditions for the basic problems of the calculus of variations depending on the right and left Riemann–Liouville fractional differences [6, 7]. The fractional difference considered in this paper is based on the Grünvald–Letnikov fractional derivative

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[8–10] and it is defined in the following way: let function $f : \{0, 1, \dots, k\} \to \mathbb{R}$, then the fractional α-order (backward) difference on f is given by

$$_0\Delta_k^\alpha f(k) := \sum_{i=0}^{k} (-1)^i \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!} f(k-i).$$

An interest in the DFCV with the fractional α-order (backward) difference has been shown by Bourdin et al. [11, 12], who discussed in [11] the Gauss Grünvald–Letnikov embedding and the corresponding variational integrators on fractional Lagrangian systems. In this context they defined the forward fractional difference of order α as follows:

$$_k\Delta_N^\alpha f(k) := \sum_{i=0}^{N-k} (-1)^i \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!} f(k+i),$$

where $f : \{0, 1, \dots, N\} \to \mathbb{R}$, (note that here we assume that $h = 1$). In [12] a fractional Noether-type theorem without transformation of time was proved. Those two papers coincide with an opinion presented in [13] by Ortigueira: *"most of the articles that appear in the scientific literature, in the framework of the fractional calculus and their applications, the authors use those derivatives (Riemann–Liouville and/or Caputo) but at the end they contrast their model using a numerical approach based in a finite number of terms from the series that define the Grünvald–Letnikov derivative"*. This can be also observed in framework of the CFCV (see, e.g., [14]). Therefore, we consider pertinent to develop the theory of the DFCV with the fractional α-order difference.

The paper is organized as follows. In Sect. 2, we define partial backward and forward fractional differences, and remind results that will be useful in the sequel. Our results are presented in Sect. 3: we prove necessary and sufficient optimality conditions for basic and isoperimetric two-dimensional problems of the DFCV. Clearly, those results can be easily generalized to the high-dimensional case. Finally, in Sect. 4, we illustrate our results through examples.

2 Preliminaries

Let $K_1 = \{0, 1, 2, \dots, N\}$, $K_2 = \{0, 1, 2, \dots, M\}$ be two given subsets of $\mathbb{Z}$ and put $D = \{(k_1, k_2) : k_1 \in K_1, k_2 \in K_2\}$, which is a complete matric space with the metric d defined by

$$d((k_1, k_2), (k_1', k_2')) = \sqrt{(k_1' - k_1)^2 + (k_2' - k_2)^2}$$

for (k_1, k_2), $(k_1', k_2') \in D$. For a given δ, the δ-neighborhood of (k_1', k_2') is given by

$$U_\delta(k_1', k_2') = \{(k_1, k_2) \in D : d((k_1', k_2'), (k_1, k_2)) < \delta\}.$$

In what follows $\alpha, \beta \in \mathbb{R}$ and $0 < \alpha, \beta \leq 1$. Moreover, we set

$$a_i^{(\alpha)} = \begin{cases} 1, & \text{if } i = 0 \\ (-1)^i \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}, & \text{if } i = 1, 2, \ldots \end{cases}$$

and

$$b_j^{(\beta)} = \begin{cases} 1, & \text{if } j = 0 \\ (-1)^j \frac{\beta(\beta-1)\cdots(\beta-j+1)}{j!}, & \text{if } j = 1, 2, \ldots . \end{cases}$$

The definition of the fractional backward (or forward) difference can be extended to discrete functions of two variables.

Definition 1 The first-order partial backward fractional difference of order α with respect to k_1 of function $f : D \to \mathbb{R}$ is defined by

$$_0\Delta_{k_1}^\alpha f(k_1, k_2) := \sum_{i=0}^{k_1} a_i^{(\alpha)} f(k_1 - i, k_2),$$

while

$$_0\Delta_{k_2}^\beta f(k_1, k_2) := \sum_{j=0}^{k_2} b_j^{(\beta)} f(k_1, k_2 - j)$$

is first-order partial backward fractional difference of order β with respect to k_2 of function f.

Next we define partial forward fractional differences.

Definition 2 The first-order partial forward fractional difference of order α with respect to k_1 of function $f : D \to \mathbb{R}$ is defined by

$$_{k_1}\Delta_N^\alpha f(k_1, k_2) := \sum_{i=0}^{N-k_1} a_i^{(\alpha)} f(k_1 + i, k_2),$$

while

$$_{k_2}\Delta_M^\beta f(k_1, k_2) := \sum_{j=0}^{M-k_2} b_j^{(\beta)} f(k_1, k_2 + j)$$

is first-order partial forward fractional difference of order β with respect to k_2 of function f.

Example 1 (cf. [10]) Let

$$f(k_1,k_2)=\begin{cases}0, & \text{if } k_1,k_2<0\\ 1, & \text{if } k_1,k_2\geq 0.\end{cases}$$

Then

$${}_0\Delta^{\alpha}_{k_1}f(k_1,k_2)=\sum_{i=0}^{k_1}a_i^{(\alpha)},\quad {}_0\Delta^{\beta}_{k_2}f(k_1,k_2)=\sum_{j=0}^{k_2}b_j^{(\beta)},$$

and

$${}_{k_1}\Delta^{\alpha}_{N}f(k_1,k_2)=\sum_{i=0}^{N-k_1}a_i^{(\alpha)},\quad {}_{k_2}\Delta^{\beta}_{M}f(k_1,k_2)=\sum_{j=0}^{M-k_2}b_j^{(\beta)}.$$

Fractional backward (or forward) differences are linear operators.

Theorem 1 (cf. [10]) *Let f,g be two real functions defined on D and $a,b\in\mathbb{R}$. Then*

$${}_0\Delta^{\alpha}_{k_1}[af(k_1,k_2)+bg(k_1,k_2)]=a\,{}_0\Delta^{\alpha}_{k_1}f(k_1,k_2)+b\,{}_0\Delta^{\alpha}_{k_1}g(k_1,k_2).$$

Similar results hold for ${}_0\Delta^{\beta}_{k_2}$, ${}_{k_1}\Delta^{\alpha}_{N}$ and ${}_{k_2}\Delta^{\beta}_{M}$.

In order to obtain an analogue of the Euler–Lagrange equation for fractional problems we need the following formula of the summation by parts for one dimensional fractional operators.

Lemma 1 (cf. [11]) *Let f, g be two real functions defined on K_1. Then*

$$\sum_{k=0}^{N}g(k)\,{}_0\Delta^{\alpha}_{k}f(k)=\sum_{k=0}^{N}f(k)\,{}_k\Delta^{\alpha}_{N}g(k).$$

If $f(0)=f(N)=0$ or $g(0)=g(N)=0$, then

$$\sum_{k=1}^{N}g(k)\,{}_0\Delta^{\alpha}_{k}f(k)=\sum_{k=0}^{N-1}f(k)\,{}_k\Delta^{\alpha}_{N}g(k).\tag{1}$$

3 Main Result

Let a function $L : \mathbb{R}^3 \times D \to \mathbb{R}$, $(x, w, v, k_1, k_2) \mapsto L(x, w, v, k_1, k_2)$ be given. We assume that L has continuous first order partial derivatives with respect to x, w, v, those derivatives we denote by: L_x, L_w, L_v. Function L is called a Lagrangian. The problem under our consideration is to extremize (minimize or maximize) functional

$$\mathcal{L}[x] = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} L(x(k_1, k_2), {}_0\Delta^{\alpha}_{k_1} x(k_1, k_2), {}_0\Delta^{\beta}_{k_2} x(k_1, k_2), k_1, k_2) \tag{2}$$

on $\mathcal{D} = \{x : D \to \mathbb{R} : x|_{\mathrm{bd}D} = g\}$, where g is a fixed function defined on bd$D$.

Definition 3 A function $\tilde{x} \in \mathcal{D}$ is called a local minimizer (or maximizer) for functional $\mathcal{L}$ on $\mathcal{D}$ provided there exists $\delta > 0$ such that $\mathcal{L}[\tilde{x}] \leq \mathcal{L}[x]$ (or $\mathcal{L}[\tilde{x}] \geq \mathcal{L}[x]$) for all $x \in \mathcal{D}$ such that $\|\tilde{x} - x\| < \delta$, where $\|f\| = \max_{(k_1,k_2)\in D} |f(k_1, k_2)|$.

Definition 4 A function $\eta : D \to \mathbb{R}$ is called an admissible variation provided $\eta \neq 0$ and $\eta|_{\mathrm{bd}D} = 0$.

For a fixed function $x \in \mathcal{D}$ and a fixed admissible variation η we define a function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(\varepsilon) = \Phi(\varepsilon; x, \eta) = \mathcal{L}[x + \varepsilon\eta].$$

By assumptions imposed on L, Φ is continuously differentiable. The first variation of the functional $\mathcal{L}$ at x we define by

$$\mathcal{L}_1[x, \eta] = \Phi'(0; x, \eta).$$

It follows that

$$\begin{aligned} \mathcal{L}_1[x, \eta] &= \Phi'(0) \\ &= \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \left(L_x(\cdot)\eta(k_1, k_2) + L_w(\cdot){}_0\Delta^{\alpha}_{k_1} \eta(k_1, k_2) + L_v(\cdot){}_0\Delta^{\beta}_{k_2} \eta(k_1, k_2) \right), \end{aligned} \tag{3}$$

where $(\cdot) = (x(k_1, k_2), {}_0\Delta^{\alpha}_{k_1} x(k_1, k_2), {}_0\Delta^{\beta}_{k_2} x(k_1, k_2), k_1, k_2)$. The next standard theorem serves a necessary optimality condition for local minimizers (or maximizers) of $\mathcal{L}$.

Theorem 2 *If $\tilde{x} \in \mathcal{D}$ is a local minimizer (or maximizer) of $\mathcal{L}$, then $\mathcal{L}_1[\tilde{x}, \eta] = 0$ for all admissible variations.*

Now we can derive a necessary optimality condition of the Euler–Lagrange type.

Theorem 3 *If $\tilde{x} \in \mathcal{D}$ is a local minimizer (or maximizer) of $\mathcal{L}$, then it satisfies the Euler–Lagrange equation of the following form:*

$$L_x(\cdot) + {}_{k_1}\Delta_N^{\alpha} L_w(\cdot) + {}_{k_2}\Delta_M^{\beta} L_v(\cdot) = 0 \tag{4}$$

for all $(k_1, k_2) \in \{1, 2, \ldots, N-1\} \times \{1, 2, \ldots, M-1\}$, *where* $(\cdot) = (x(k_1, k_2), {}_0\Delta_{k_1}^{\alpha} x(k_1, k_2), {}_0\Delta_{k_2}^{\beta} x(k_1, k_2), k_1, k_2)$.

Proof By Theorem 2 we have $\mathcal{L}_1[\tilde{x}, \eta] = 0$ for all admissible variations. Therefore,

$$0 = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \left(L_x(\star)\eta(k_1, k_2) + L_w(\star)_0\Delta_{k_1}^{\alpha} \eta(k_1, k_2) + L_v(\star)_0\Delta_{k_2}^{\beta} \eta(k_1, k_2) \right), \tag{5}$$

where $(\star) = (\tilde{x}(k_1, k_2), {}_0\Delta_{k_1}^{\alpha} \tilde{x}(k_1, k_2), {}_0\Delta_{k_2}^{\beta} \tilde{x}(k_1, k_2), k_1, k_2)$. Since $\eta|_{\mathrm{bd}D} = 0$, using (1) we have

$$\sum_{k_1=1}^{N} \sum_{k_2=1}^{M} L_w(\star)_0\Delta_{k_1}^{\alpha} \eta(k_1, k_2) = \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{M} {}_{k_1}\Delta_N^{\alpha} L_w(\star)\eta(k_1, k_2)$$

and

$$\sum_{k_1=1}^{N} \sum_{k_2=1}^{M} L_v(\star)_0\Delta_{k_2}^{\beta} \eta(k_1, k_2) = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M-1} {}_{k_2}\Delta_M^{\beta} L_v(\star)\eta(k_1, k_2).$$

Consequently, remembering that $\eta|_{\mathrm{bd}D} = 0$, we get from (5):

$$0 = \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{M-1} \left(L_x(\star) + {}_{k_1}\Delta_N^{\alpha} L_w(\star) + {}_{k_2}\Delta_M^{\beta} L_v(\star) \right) \eta(k_1, k_2).$$

Since the value of η is arbitrary on $\{1, 2, \ldots, N-1\} \times \{1, 2, \ldots, M-1\}$, the Euler–Lagrange equation (4) holds along $\tilde{x}$.

Definition 5 A function $\tilde{x} \in \mathcal{D}$ that is a solution to the Euler–Lagrange equation (4) we call an extremal of $\mathcal{L}$.

The next theorem provides a sufficient condition for an extremal to be a global minimaizer (maximizer).

Theorem 4 *Let function L in* (2) *be jointly convex (concave) with respect* (x, w, v) *for all* $(k_1, k_2) \in D$. *If* $\tilde{x} \in \mathcal{D}$ *is a solution to the Euler–Lagrange equation* (4)*, then it is a global minimaizer (maximizer) of functional* (2) *on* $\mathcal{D}$.

Proof Assume that L is jointly convex with respect (x, w, v) for all $(k_1, k_2) \in D$, then for any h such that $\tilde{x} + h \in \mathcal{D}$, we have

$$\mathcal{L}[\tilde{x}+h] - \mathcal{L}[\tilde{x}] = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \Big[L(\tilde{x}(k_1,k_2) + h(k_1,k_2), {}_0\Delta_{k_1}^{\alpha}(\tilde{x}(k_1,k_2) + h(k_1,k_2)),$$

$${}_0\Delta_{k_2}^{\beta}(\tilde{x}(k_1,k_2), k_1, k_2) + h(k_1,k_2)) - L(\tilde{x}(k_1,k_2), {}_0\Delta_{k_1}^{\alpha}\tilde{x}(k_1,k_2), {}_0\Delta_{k_2}^{\beta}\tilde{x}(k_1,k_2), k_1, k_2) \Big]$$

$$\geq \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \Big(L_x(\star) h(k_1,k_2) + L_w(\star) {}_0\Delta_{k_1}^{\alpha} h(k_1,k_2) + L_v(\star) {}_0\Delta_{k_2}^{\beta} h(k_1,k_2) \Big),$$

where $(\star) = (\tilde{x}(k_1,k_2), {}_0\Delta_{k_1}^{\alpha}\tilde{x}(k_1,k_2), {}_0\Delta_{k_2}^{\beta}\tilde{x}(k_1,k_2), k_1, k_2)$. Proceeding as in the proof of Theorem 3, we obtain

$$\mathcal{L}[\tilde{x}+h] - \mathcal{L}[\tilde{x}] \geq \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{M-1} \Big(L_x(\star) + {}_{k_1}\Delta_N^{\alpha} L_w(\star) + {}_{k_2}\Delta_M^{\beta} L_v(\star) \Big) h(k_1,k_2).$$

As $\tilde{x}$ satisfies Eq. (4) we have $\mathcal{L}(\tilde{x}+h) - \mathcal{L}(\tilde{x}) \geq 0$.

Now we shall consider the isoperimetric problem, one of the oldest and interesting class of variational problems with roots in the Queen Dido problem of the calculus of variations. The discrete fractional isoperimetric problem is defined in the following way: extremize (minimize or maximize) functional

$$\mathcal{L}[x] = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} L(x(k_1,k_2), {}_0\Delta_{k_1}^{\alpha} x(k_1,k_2), {}_0\Delta_{k_2}^{\beta} x(k_1,k_2), k_1, k_2) \tag{6}$$

on $\mathcal{D} = \{x : D \to \mathbb{R} : x|_{\mathrm{bd}D} = g\}$, where g is a fixed function defined on bd$D$ and subject to the isoperimetric constraint

$$\mathcal{I}[x] = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} G(x(k_1,k_2), {}_0\Delta_{k_1}^{\alpha} x(k_1,k_2), {}_0\Delta_{k_2}^{\beta} x(k_1,k_2), k_1, k_2) = \xi, \tag{7}$$

where $\xi \in \mathbb{R}$ is given, and L, G have continuous first order partial derivatives with respect to x, w, v.

Theorem 5 *If $\tilde{x} \in \mathcal{D}$ is a local minimizer (or maximizer) of (6) subject to the isoperimetric constraint (7), then there exist two real constants, λ_0 and λ, not both zero, such that $\tilde{x}$ satisfies the following equation:*

$$H_x(\cdot) + {}_{k_1}\Delta_N^{\alpha} H_w(\cdot) + {}_{k_2}\Delta_M^{\beta} H_v(\cdot) = 0$$

for all $(k_1, k_2) \in \{1, 2, \ldots, N-1\} \times \{1, 2, \ldots, M-1\}$, where $(\cdot) = (x(k_1,k_2), {}_0\Delta_{k_1}^{\alpha} x(k_1,k_2), {}_0\Delta_{k_2}^{\beta} x(k_1,k_2), k_1, k_2)$ and $H = \lambda_0 L + \lambda G$.

Proof Can be done by using the abnormal Lagrange multiplier rule [[15], Theorem 4.1.3].

Remark 1 If $\tilde{x}$ is a normal extremizer to the isoperimetric problem, that is, $\tilde{x}$ is not a solution to equation

$$G_x(\cdot) + {}_{k_1}\Delta_N^{\alpha} G_w(\cdot) + {}_{k_2}\Delta_M^{\beta} G_v(\cdot) = 0,$$

where $(\cdot) = (x(k_1,k_2), {}_0\Delta_{k_1}^{\alpha} x(k_1,k_2), {}_0\Delta_{k_2}^{\beta} x(k_1,k_2), k_1, k_2)$, then we can choose $\lambda_0 = 1$ in Theorem 5. For abnormal extremizers, Theorem 5 holds with $\lambda_0 = 0$. The condition $(\lambda_0, \lambda) \neq 0$ guarantees that Theorem 5 is a useful necessary optimality condition.

4 Example

In this section we present three illustrative examples. In the first example the two-dimensional problem is considered. We show that the Lagrangian is invariant under a gauge transformation. Therefore, we can expect that the Euler–Lagrange equations do not uniquely determine a solution to this problem (see, e.g., [16]). The next example shows that the solutions of the fractional problems coincide with the solutions of the corresponding non-fractional variational problems when the order of the discrete derivatives is an integer value [17]. Moreover, we observe that, in this particular case, solutions obtained by us are very similar to those presented in [4], where problems with the Riemann–Liouville fractional differences were considered. In Example 4 the isoperimetric problem is considered. This problem leads us to the discrete fractional Sturm–Liouville eigenvalue problem.

Example 2 Let us consider the following problem: minimize

$$\mathcal{L}[x_1,x_2] = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \left({}_0\Delta_{k_1}^{\alpha} x_2(k_1,k_2) - {}_0\Delta_{k_2}^{\beta} x_1(k_1,k_2) \right)^2 \tag{8}$$

on $\mathcal{D} = \{(x_1,x_2) : D \to \mathbb{R}^2 : (x_1,x_2)|_{\mathrm{bd}D} = (g_1,g_2)\}$, where g_1, g_2 are fixed real functions defined on bd$D$. Necessary conditions for a minimizer are as follows:

$$\begin{aligned} {}_{k_1}\Delta_N^{\alpha} \left({}_0\Delta_{k_1}^{\alpha} x_2(k_1,k_2) - {}_0\Delta_{k_2}^{\beta} x_1(k_1,k_2) \right) &= 0, \\ {}_{k_2}\Delta_M^{\beta} \left({}_0\Delta_{k_1}^{\alpha} x_2(k_1,k_2) - {}_0\Delta_{k_2}^{\beta} x_1(k_1,k_2) \right) &= 0 \end{aligned}$$

for all $(k_1,k_2) \in \{1,2,\dots,N-1\} \times \{1,2,\dots,M-1\}$. Note that, the Lagrangian is invariant with respect to a gauge transformation:

$$\bar{x}_1(k_1,k_2) = x_1(k_1,k_2) + {}_0\Delta^{\alpha}_{k_1} f(k_1,k_2),$$
$$\bar{x}_2(k_1,k_2) = x_2(k_1,k_2) + {}_0\Delta^{\beta}_{k_2} f(k_1,k_2)$$

where $f : D \to \mathbb{R}$ is an arbitrary function. Indeed,

$$\begin{aligned}
&\left({}_0\Delta^{\alpha}_{k_1}\bar{x}_2(k_1,k_2) - {}_0\Delta^{\beta}_{k_2}\bar{x}_1(k_1,k_2)\right)^2 \\
&\quad= \left({}_0\Delta^{\alpha}_{k_1}(x_2(k_1,k_2) + {}_0\Delta^{\beta}_{k_2} f(k_1,k_2)) - {}_0\Delta^{\beta}_{k_2}(x_1(k_1,k_2) + {}_0\Delta^{\alpha}_{k_1} f(k_1,k_2))\right)^2 \\
&= \left({}_0\Delta^{\alpha}_{k_1}x_2(k_1,k_2) + {}_0\Delta^{\alpha}_{k_1}{}_0\Delta^{\beta}_{k_2} f(k_1,k_2) - {}_0\Delta^{\beta}_{k_2}x_1(k_1,k_2) - {}_0\Delta^{\beta}_{k_2}{}_0\Delta^{\alpha}_{k_1} f(k_1,k_2)\right)^2 .
\end{aligned}$$

Since

$${}_0\Delta^{\alpha}_{k_1}{}_0\Delta^{\beta}_{k_2} f(k_1,k_2) = {}_0\Delta^{\beta}_{k_2}{}_0\Delta^{\alpha}_{k_1} f(k_1,k_2),$$

by Theorem 5.2.1 [10], the desired equality holds.

Example 3 Let us consider the following problem: minimize

$$\mathcal{L}[x] = \sum_{k=1}^{N} \left({}_0\Delta^{\alpha}_{k} x(k)\right)^2 \tag{9}$$

subject to $\mathcal{D} = \{x : D \to \mathbb{R} : x(0) = A, x(N) = B\}$, where $D = \{0,\ldots,N\}$ and N, A, B are fixed. In this case the Euler–Lagrange equation takes the form

$${}_k\Delta^{\alpha}_{N}{}_0\Delta^{\alpha}_{k} x(k) = 0, \quad k = 1,\ldots,N-1.$$

For $N = 2$, the solution to the considered problem is:

$$x(0) = 0, \quad x(1) = \frac{\alpha A + \alpha B + 1/2\alpha^2(\alpha-1)A}{1+\alpha^2}, \quad x(2) = B.$$

Observe that for $\alpha = 1$: $x(1) = \frac{A+B}{2}$, as one can expect. Indeed, for $\alpha = 1$ our problem coincides with a discrete problem: minimize

$$\sum_{k=1}^{2} (x(k) - x(k-1))^2$$

on $\mathcal{D} = \{x : D \to \mathbb{R} : x(0) = A, x(2) = B\}$.
Let us choose now $A = 0$, $B = 1$ and $N = 4$. Table 1 and Fig. 1 present solutions to the considered problem for different values of α.

Table 1 Minimizer values of Example 3 with $A = 0$, $B = 1$, $N = 4$, and different α's: $1/4, 1/2, 3/4, 1$

α	$\tilde{x}(1)$	$\tilde{x}(2)$	$\tilde{x}(3)$	$\mathcal{L}[\tilde{x}]$
0.25	0.007181731484	0.1454300624	0.2668462100	0.9185395467
0.50	0.1293800539	0.3234501348	0.5229110512	0.6788674886
0.75	0.2267883116	0.4544228483	0.68702895690	0.4255906641
1	0.25	0.50	0.75	0.25

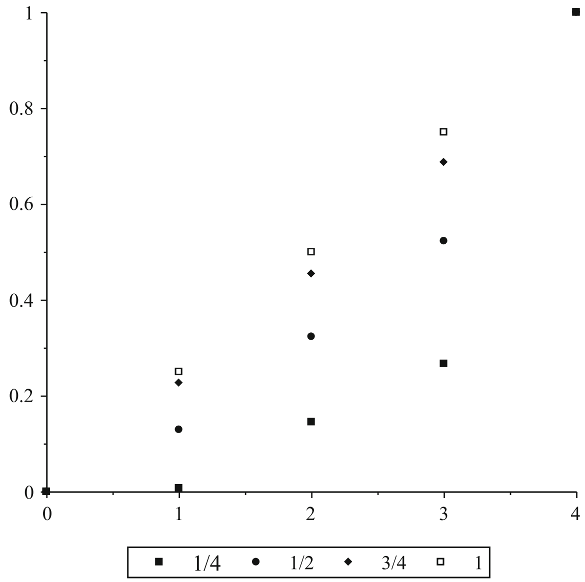

Fig. 1 Minimizer $\tilde{x}$ of Example 3 with $A = 0$, $B = 1$, $N = 4$, and different α's: $1/4, 1/2, 3/4, 1$

Note that the smallest value of $\mathcal{L}$ occurs for $\alpha = 1$ (for the classical non-fractional case). However, the values of function $\tilde{x}$ are respectively the biggest in the case of $\alpha = 1$.

Example 4 As the third example we consider the isoperimetric problem: minimize

$$\mathcal{L}[x] = \sum_{k=1}^{N} \left[p(k) \left({}_0\Delta_k^\alpha x(k) \right)^2 + q(k) x^2(k) \right] \tag{10}$$

on $\mathcal{D} = \{x : D \to \mathbb{R} : x(0) = 0, x(N) = 0\}$, and

$$\sum_{k=1}^{N} r(k) \left(x(k) \right)^2 = 1 \tag{11}$$

where $D = \{0, \ldots, N\}$, N is fixed, and $p, r : D \to \mathbb{R}_+$, $q : D \to \mathbb{R}$. By Theorem 5 and Remark 1 every nontrivial solution to this problem has to satisfy the following equation:

$$ {}_k\Delta_N^\alpha \left(p(k)_0\Delta_k^\alpha x(k)\right) + q(k)x(k) = \lambda r(k)x(k), \quad k = 1, \ldots, N-1, \tag{12}$$

for some λ. It is easily seen that Eq. (12) together with the boundary condition:

$$x(0) = 0, \quad x(N) = 0,$$

is a kind of the Sturm–Liouville eigenvalue problem (see, e.g., [18–20]). Let us choose $D = \{0, \ldots, 3\}$, $p = r = 1$, $q = 0$. Then extremizers for considered isoperimetric problem have to satisfy the following conditions:

$$ {}_k\Delta_3^\alpha {}_0\Delta_k^\alpha x(k) = \lambda x(k), \quad k = 1, 2, \tag{13}$$

$$\sum_{k=1}^{3} (x(k))^2 = 1, \tag{14}$$

$$x(0) = 0, \quad x(3) = 0. \tag{15}$$

Note that solutions to the system (13)–(15) depend on λ. Table 2 presents examples of solutions for different values of α and for chosen values of λ.

Table 2 Examples of solutions to to system of Eqs. (13)–(15) for different values of α's: $1/4, 1/2, 3/4, 1$

α	$x(1)$	$x(2)$	λ
0.25	−0.7002167868	−0.7139302847	0.8402894156
0.50	−0.7007659037	−0.71339130098	0.8202427511
0.75	−0.7048172168	−0.7093889560	0.8871928247
1	−0.7071067812	−0.7071067812	1

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