# Chapter 2
# Strain Measures

**Abstract** Strain measures for hyperelastic materials must model the effect of finite deformations. They are single-based second-order tensors, either Eulerian or Lagrangian, and are defined in terms of the Cauchy-Green deformation tensors, which are derived from the deformation gradient. The Green-Lagrange strain tensor is Lagrangian based, while the Almansi strain tensor is Eulerian based. Both of these strain measures are described in detail. The Green-Lagrange strain tensor is in terms of the right Cauchy-Green deformation tensor, while the Almansi strain tensor is in terms of the left Cauchy-Green deformation tensor. The reduced invariants of the right and left Cauchy-Green deformation tensors, known as the invariants of the right and left Cauchy-Green distortion tensors, are introduced, and the derivation of the reduced invariants is presented and defined. Since the strain measures are derived from the deformation gradient, they are related, the relationship is easily demonstrated. An additional strain measure, one which is less commonly employed, is the Biot strain tensor. The different strain measures can be formally reduced to those of linear elastic systems, this being demonstrated.

The important *right Cauchy-Green deformation tensor*, or the *Green deformation tensor*, is defined by

$$C_{IJ} = \frac{\partial x_k}{\partial X_I}\frac{\partial x_k}{\partial X_J} = F_{kI}F_{kJ} \quad \text{or} \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} \tag{2.1}$$

or, in terms of the *distortion gradient*,

$$C_{IJ} = J^{2/3}\overline{F}_{kI}\overline{F}_{kJ} \quad \text{or} \quad \mathbf{C} = J^{2/3}\overline{\mathbf{F}}^T\overline{\mathbf{F}} \tag{2.2}$$

Then, with

$$\overline{C}_{IJ} = \overline{F}_{kI}\overline{F}_{kJ} \quad \text{or} \quad \overline{\mathbf{C}} = \overline{\mathbf{F}}^T\overline{\mathbf{F}} \tag{2.3}$$



R.M. Hackett, *Hyperelasticity Primer*, DOI 10.1007/978-3-319-23273-7_2

we also have the relationship

$$\overline{C}_{IJ} = J^{-2/3} C_{IJ} \quad \text{or} \quad \overline{\mathbf{C}} = J^{-2/3} \mathbf{C} \tag{2.4}$$

The inverse of $C_{IJ}$, $C_{IJ}^{-1}$, is called the *Piola deformation tensor*.

The also important *left Cauchy-Green deformation tensor*, which is also sometimes called the *Finger tensor*, is defined by

$$b_{ij} = \frac{\partial x_i}{\partial X_K} \frac{\partial x_j}{\partial X_K} = F_{iK} F_{jK} \quad \text{or} \quad \mathbf{b} = \mathbf{F}\mathbf{F}^T \tag{2.5a}$$

where, also

$$\overline{b}_{ij} = \overline{F}_{iK} \overline{F}_{jK} \quad \text{or} \quad \overline{\mathbf{b}} = \overline{\mathbf{F}}\,\overline{\mathbf{F}}^T \tag{2.5b}$$

The inverse of $b_{ij}$, $b_{ij}^{-1}$, is called the *Cauchy deformation tensor*.

We can also note here that

$$\bar{I}_1 = \overline{b}_{ii} = \overline{C}_{II} \quad \text{or} \quad \bar{I}_1 = \text{tr}\,\overline{\mathbf{b}} = \text{tr}\,\overline{\mathbf{C}} \tag{2.6a}$$

$$\bar{I}_2 = \frac{1}{2}\left(\bar{I}_1^2 - \bar{I}_2'\right) \tag{2.6b}$$

where

$$\bar{I}_2' = \overline{b}_{ik} \overline{b}_{ki} = \overline{C}_{IK} \overline{C}_{KI} \quad \text{or} \quad \bar{I}_2' = \text{tr}\left(\overline{\mathbf{b}}\,\overline{\mathbf{b}}\right) = \text{tr}\left(\overline{\mathbf{C}}\,\overline{\mathbf{C}}\right) \tag{2.6c}$$

and

$$\bar{I}_3 = \det \overline{b}_{ij} = \det \overline{C}_{IJ} \quad \text{or} \quad \bar{I}_3 = \det \overline{\mathbf{b}} = \det \overline{\mathbf{C}} \tag{2.6d}$$

where $\bar{I}_1, \bar{I}_2, \bar{I}_3$ are the conventionally defined principal invariants of the right and left Cauchy-Green distortion, or reduced, tensors. Sometimes the alternative definition $\bar{I}_2 = \bar{I}_2'$ is used (Bonet and Wood 2008).

We can write the *Green*, or *Green-Lagrange*, *strain tensor* components as

$$E_{IJ} = \frac{1}{2}\left(F_{kI} u_{k,J} + F_{kJ} u_{k,I}\right) \tag{2.7a}$$

or

$$E_{IJ} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_J} + \frac{\partial u_j}{\partial X_I} + \frac{\partial u_k}{\partial X_I} \frac{\partial u_k}{\partial X_J}\right) \tag{2.7b}$$

or, considering Equation (2.1), as

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \quad \text{or} \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}) \tag{2.8}$$

We can note that by dropping the third term on the right-hand side of Equation (2.7b) we have, significantly, the standard small strain relationship for linear elasticity which can also be written as

$$\boldsymbol{\varepsilon} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) - \mathbf{1} \tag{2.9}$$

The *Eulerian strain tensor*, or *Almansi strain tensor* (Almansi 1911), is defined in terms of the left Cauchy-Green deformation tensor, as

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - b_{ij}^{-1}\right) \tag{2.10a}$$

or, in terms of displacement as

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right) \tag{2.10b}$$

We note that the two defined strain states are related through the expressions

$$E_{IJ} = F_{kI} e_{kl} F_{lJ} \quad \text{or} \quad \mathbf{E} = \mathbf{F}^T \mathbf{e} \mathbf{F} \tag{2.11a}$$

and

$$e_{ij} = F_{Ki} E_{KL} F_{Lj} \quad \text{or} \quad \mathbf{e} = \mathbf{F}^{-T} \mathbf{E} \mathbf{F}^{-1} \tag{2.11b}$$

The Green-Lagrange and Almansi strains are the two classical strain measures.

Now, let us consider *deviatoric* strain. We can write for the deviatoric Green-Lagrange strain

$$\overline{E}_{IJ} = \frac{1}{2}\left(\overline{F}_{kI}\overline{F}_{kJ} - \delta_{IJ}\right) \tag{2.12}$$

and, for the deviatoric Almansi strain,

$$\overline{e}_{ij} = \frac{1}{2}\left(\delta_{ij} - \overline{F}_{Ki}\overline{F}_{Kj}\right) \tag{2.13}$$

and, defining

$$\overline{B}_{iJ} = \overline{F}_{Ji}^{T} \quad \text{or} \quad \overline{\mathbf{B}} = \overline{\mathbf{F}}^{-T} \tag{2.14}$$

where $\overline{F}_{Ji}^{T}$ is the transpose of the inverse of the distortion gradient tensor, we can also write for the deviatoric Almansi strain,

$$\overline{e}_{ij} = \frac{1}{2}\left(\delta_{ij} - \overline{B}_{iK}\overline{B}_{jK}\right) \tag{2.15}$$

Another strain measure, one which is not too commonly employed, is the *Biot strain tensor* (Biot 1939). It is defined as

$$\overline{U}_{IJ} = U_{IJ} - \delta_{IJ} \quad \text{or} \quad \overline{\mathbf{U}} = \mathbf{U} - \mathbf{1} \tag{2.16}$$

$U_{IJ}$ being the *right stretch tensor* which is obtained from *polar decomposition*, as shown in the next chapter.

Springer

http://www.springer.com/978-3-319-23272-0

Hyperelasticity Primer
Hackett, R.M.
2016, XVI, 170 p. 4 illus., 3 illus. in color., Hardcover
ISBN: 978-3-319-23272-0