

Chapter 2

Heat and Fluid Flow in Single Microchannels

This chapter illustrates the analytical and hybrid numerical–analytical treatment of fluid flow and heat transfer in single microchannels, under different conditions and configurations, starting with transient slip flow and steady, transient, and periodic convection, also in the slip flow regime, for simple parallel plates and circular capillary tube geometries. Then, the situation of convection with electroosmotic flow is considered more closely. After establishing the basic solution for the direct problems, an inverse analysis via Bayesian inference is provided, for estimating the coefficients associated with the slip and temperature jump conditions within the slip flow regime. Also, the situation of longitudinally irregular walls, such as in the case of corrugated or rough channel walls, are handled both by solving the full set of Navier–Stokes equations for the flow problem and by simplifying the flow-governing equations for very low Reynolds numbers and mild geometric variations. Conjugated heat transfer in microchannels, which occurs quite frequently at this scale, is then analyzed accounting for the full energy equations in both the solid and the fluid, through the single domain formulation. Finally, microchannels with arbitrary cross-sectional shapes are considered, again via the single domain formulation, including conjugation effects with the surrounding walls. This chapter thus provides the background on the physical analysis and solution methodologies for heat and fluid flow in single microchannels, so as to advance into the analysis of microsystems in the next chapter.

2.1 Transient Slip Flow in Microchannels

In recent years attention has been directed to the analysis of transient flow in microchannels with wall slip effects (Wiwatanapatapee et al. 2009; Wu et al. 2008; Colin 2005; Sabry et al. 2004; De Socio and Marino 2002; Yang et al. 2002; Khaled and Vafai 2004; Aubert 1999; Bestman et al. 1995). Unsteady one-dimensional models have been extended from classical works, and analytical solutions have been sought for fully developed flows in simple geometries and for well-defined functional forms of the prescribed pressure gradient time variation or for the time dependence of the wall-imposed velocity, in the case of a Couette flow application.

Such solutions are relevant not only from the Microfluidics point of view, but also in the analysis of transient and periodic convection regimen in microchannels, as discussed in Castellões et al. (2007). The goal of this section is to solve a one-dimensional slip flow model for transient laminar incompressible flow in microchannels, such as circular tubes and parallel-plate channels, which accounts for a pressure gradient time variation in any functional form, by making use of the classical integral transform method, and thus yielding analytical expressions for the time and space dependence of the velocity fields in the fully developed region.

Therefore, fully developed incompressible laminar flow with slip effects at the walls is considered, inside a circular microtube or a parallel-plate microchannel, subjected to a pressure gradient dp/dz that varies in an arbitrary functional form with the time variable. The velocity field is represented by $u(r, t)$, which varies with the transversal coordinate, r , and time, t . The related time-dependent axial momentum equation (z -direction) is then written in dimensionless form as (Castellões et al. 2007)

$$R^n \frac{\partial U(R, \tau)}{\partial \tau} = \frac{\partial}{\partial R} \left[R^n \frac{\partial U(R, \tau)}{\partial R} \right] + R^n P(\tau), \quad \text{in } 0 < R < 1 \quad (2.1)$$

$$\left. \frac{\partial U(R, \tau)}{\partial R} \right|_{R=0} = 0; \quad \beta^* \left. \frac{\partial U(R, \tau)}{\partial R} \right|_{R=1} + U(1, \tau) = 0 \quad (2.2a, b)$$

$$U(R, 0) = U_0(R) \quad (2.3)$$

where $n = 0$ for a parallel-plate channel, and $n = 1$ for a circular tube, and adopting the following dimensionless groups:

$$R = \frac{r}{r_1}; \quad \tau = \frac{vt}{r_1^2}; \quad U(R, \tau) = \frac{u(r, t)}{u_m}; \quad P(\tau) = \frac{-(dp/dz)r_1^2}{\mu u_m}; \quad (2.4)$$

$$\beta^* = 2Kn\beta_v; \quad Kn = \lambda/2r_1; \quad \beta_v = (2 - \alpha_m)/\alpha_m$$

The classical integral transform method is here employed to provide an exact solution to the transient slip flow problem in Eqs. (2.1)–(2.3). One important aspect in this kind of eigenfunction expansion approach is the convergence enhancement achievable by introducing analytical solutions that filter the original problem source terms, which are responsible for an eventual slow convergence behavior. Thus, we start the integral transformation process by obtaining the filtering solution based on the quasi-steady version of the present problem:

$$U(R, \tau) = U_p(R; \tau) + U_h(R, \tau) \quad (2.5)$$

The quasi-steady solution of problem (2.1)–(2.3), essentially removing the transient term in Eq. (2.1), is thus considered as

$$\frac{d}{dR} \left(R^n \frac{dU_P}{dR} \right) + R^n P(\tau) = 0 \quad (2.6)$$

$$\left. \frac{dU_P}{dR} \right|_{R=0} = 0; \quad \beta^* \left. \frac{\partial U_P(R; t)}{\partial R} \right|_{R=1} + U_P(1; t) = 0 \quad (2.7a, b)$$

The above ODE is directly integrated to yield the analytical filter in terms of the dimensionless time-variable pressure gradient:

$$U_P(R; \tau) = P(\tau) \frac{(2\beta^* + 1 - R^2)}{2(n+1)} \quad (2.8)$$

The resulting formulation for the filtered potential U_h , is then given as

$$R^n \frac{\partial U_h}{\partial \tau} = \frac{\partial}{\partial R} \left(R^n \frac{\partial U_h}{\partial R} \right) + R^n P^*(R, \tau) \quad (2.9)$$

$$\left. \frac{\partial U_h}{\partial R} \right|_{R=0} = 0; \quad \beta^* \left. \frac{\partial U_h(R, t)}{\partial R} \right|_{R=1} + U_h(1, t) = 0 \quad (2.10a, b)$$

$$U_h(R, 0) = U_0^*(R) = U_0(R) - U_P(R; 0) \quad (2.11)$$

where the resulting source term for the filtered system becomes

$$P^*(R, \tau) = -\frac{\partial U_P}{\partial \tau} \quad (2.12)$$

The following simple eigenvalue problem is obtained from separation of variables applied to the homogeneous version of Eq. (2.1), and employed for the integral transformation pair construction:

$$\frac{d}{dR} \left(R^n \frac{d\varphi(R)}{dR} \right) + R^n \lambda^2 \varphi(R) = 0 \quad (2.13)$$

with boundary conditions:

$$\left. \frac{d\varphi(R)}{dR} \right|_{R=0} = 0; \quad \beta^* \left. \frac{d\varphi(R)}{dR} \right|_{R=1} + \varphi(1) = 0 \quad (2.14a, b)$$

The eigenfunctions $\varphi_m(R)$ are readily obtained and given as

$$\varphi_m(R) = \cos(\lambda_m R), \text{ for } n = 0, \quad \varphi_m(R) = J_0(\lambda_m R), \text{ for } n = 1 \quad (2.15a, b)$$

and the related eigenvalues are computed from satisfaction of the boundary condition Eq. (2.14b), while the normalization integral is analytically computed from the definition

$$N_m = \int_0^1 R^n \varphi_m^2(R) dR \quad (2.16)$$

The integral transform pair is written as

$$U_h(R, \tau) = \sum_{m=1}^{\infty} \frac{1}{N_m} \varphi_m(R) \bar{U}_m(\tau), \quad \text{inverse} \quad (2.17)$$

$$\bar{U}_m(\tau) = \int_0^1 R^n \varphi_m(R) U_h(R, \tau) dR, \quad \text{transform} \quad (2.18)$$

Operating the filtered momentum balance Eq. (2.9) with $\int_0^1 \varphi_m(R)(\cdot) dR$ and transforming all the original potentials with the aid of the inversion formula, we obtain the following ordinary differential equations:

$$\frac{d\bar{U}_m(\tau)}{d\tau} + \lambda_m^2 \bar{U}_m(\tau) = \bar{g}_m(\tau), \quad \tau > 0, \quad m = 1, 2, 3, \dots \quad (2.19)$$

where the transformed source term is computed from

$$\bar{g}_m(\tau) = \int_0^1 R^n \varphi_m(R) P^*(R, \tau) dR \quad (2.20)$$

Similarly, the filtered initial condition (2.11) is operated on with $\int_0^1 R^n \phi_m(R)(\cdot) dR$, to yield:

$$U_m(0) = \bar{f}_m, \quad \bar{f}_m = \int_0^1 R^n \varphi_m(R) U_0^*(R) dR \quad (2.21)$$

Equations (2.19)–(2.21) are readily solved to yield the analytical expression for the transformed velocity field:

$$\bar{U}_m(\tau) = \bar{f}_m \exp(-\lambda_m^2 \tau) + \int_0^{\tau} \exp[-\lambda_m^2(\tau - \tau')] \bar{g}_m(\tau') d\tau' \quad (2.22)$$

Once the above solution is obtained for the transformed potential, the inversion formula, Eq. (2.17), can be used to evaluate the filtered velocity, and then the original field from Eq. (2.5). For computational purposes, the infinite series is evaluated to a sufficiently large finite order so as to achieve the user's requested accuracy target.

For the transient flow analysis, we considered both geometries (parallel-plate channel and circular tube) under two different and representative transient situations: flow start-up with a step change or a periodic time variation of the pressure gradient (Mikhailov and Ozisik 1984). Here, due to space limitations, we present only a few of the parallel-plate channel case results ($n = 0$). By assigning numerical values to the parameter, $\beta^* = 0.1$, according to the chosen dimensionless formulation, the pressure gradient for the start-up case with a unit step change is written as

$$P(\tau) = \frac{3}{3\beta^* + 1} \quad (2.23)$$

For the periodic case, the definition of the dimensionless time-variable source term is given for the parallel-plates geometry as

$$P(\tau) = \frac{3}{3\beta^* + 1} \left(1 - \frac{\sin(\Omega\tau)}{2}\right) \quad (2.24)$$

with $\Omega = \pi/15$ for the reported example.

Table 2.1 illustrates the excellent convergence characteristics of the proposed eigenfunction expansion, for the case of a periodic pressure gradient in a parallel-plate channel with $\beta^* = 0.1$, and considering four different values of the dimensionless time. Truncation orders $N = 10$ and 30 are explicitly shown, demonstrating that six converged significant digits at least are achieved for N as low as 10 . Also presented are the numerical results obtained via the Method of Lines implemented in the built in routine NDSolve of the *Mathematica* system (Wolfram 2008). These results agree to within four significant digits. To the graph scale, the results from the integral transform solution and from the numerical built in routine are essentially coincident, since one can only observe numerical deviations in the last two significant digits. The analytical solution is also observed to be fully converged even with less than 10 terms in the expansion.

Table 2.1 Convergence behavior of eigenfunction expansion for the dimensionless velocity and comparison with routine NDSolve (Wolfram 2008) (parallel-plate channel, periodic flow, $\beta^* = 0.1$)

$U(R, \tau)$; GITT with $N = 10$, $N = 30$, and NDSolve (Wolfram 2008)					
Solution	R	$\tau = 5$	$\tau = 10$	$\tau = 15$	$\tau = 20$
GITT— $N = 10$	0.0	0.827503	0.755419	1.31248	1.94167
GITT— $N = 30$		0.827503	0.755419	1.31248	1.94167
NDSolve		0.827530	0.755389	1.31251	1.94160
GITT— $N = 10$	0.6	0.577521	0.529949	0.921623	1.36090
GITT— $N = 30$		0.577521	0.529949	0.921623	1.36090
NDSolve		0.577539	0.529928	0.921641	1.36084
GITT— $N = 10$	1.0	0.136928	0.126572	0.220405	0.324602
GITT— $N = 30$		0.136928	0.126572	0.220405	0.324602
NDSolve		0.136932	0.126567	0.220408	0.324585

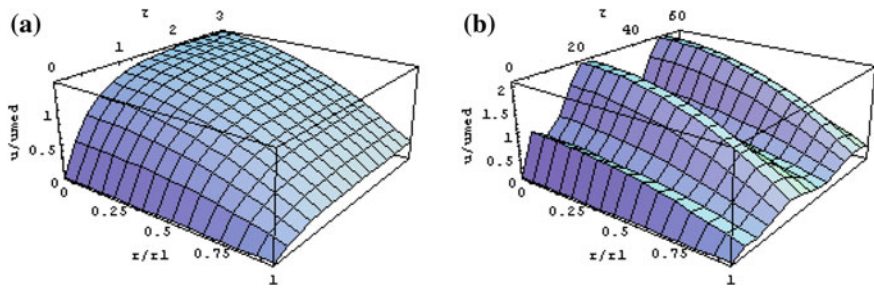


Fig. 2.1 Transient evolution of dimensionless velocity profile for parallel-plates channel ($n = 0$), for (a) step change and (b) periodic variation in pressure gradient, $\beta^* = 0.1$

For the start-up flow case, we obtain the following set of curves of the dimensionless velocity profiles evolution shown in Fig. 2.1a, where the increase with time in the fluid velocity at the wall can be clearly observed. The three-dimensional plot for the velocity distribution is given in Fig. 2.1b for the periodic case, and we can observe the quasi-steady-state (periodic state) establishment, and the time variation of the dimensionless slip velocity.

2.2 Steady, Transient, and Periodic Convection with Slip Flow

After the early work of Sparrow and Lin (1962) on laminar forced convection with slip flow, only much more recently, and especially along the last two decades, the heat transfer literature has demonstrated a vivid and growing interest in thermal analysis of slip flows, motivated by the booming research on microchannels and microsystems, both through experimental and analytical approaches, as pointed out in various reviews (Bayazitoglu and Tunc 2001; Rostami et al. 2002; Morini 2004; Yener et al. 2005; Sobhan and Peterson 2008; Kakaç et al. 2014). Since the available analytical information on heat transfer in ducts could not be directly extended to flows within microchannels with wall slip and temperature jump, a number of contributions have been directed toward the analysis of internal forced convection in the microscale. In the papers (Barron et al. 1996, 1997), the original approach in the classical work of Graetz (1883) is used to evaluate the eigenvalues for the Graetz problem extended to slip flow. The problem considered in that paper has also an exact solution in terms of the confluent hypergeometric function, explored in Mikhailov and Cotta (1997, 2005) to develop *Mathematica* rules (Wolfram 2008) for computing the desired eigenvalues with user-specified working precision. Following the work in Barron et al. (1996, 1997), the same technique was employed to solve the laminar flow heat convection problem in a cylindrical microchannel with constant uniform temperature at the boundary (Ameel et al.

1997 and Larrodé et al. 2000), taking into account both the velocity slip and temperature jump at the tube wall. Later on (Yu and Ameen 2001, 2002; Tunc and Bayazitoglu 2002a, b) the analytical contributions were directed toward more general problem formulations, including viscous dissipation in the fluid and two-dimensional flow geometries, such as rectangular channels. For this purpose, a more flexible hybrid numerical–analytical approach was employed, based on the ideas of the Generalized Integral Transform Technique, GITT (Cotta 1990, 1993, 1994, 1998; Cotta and Mikhailov 1997, 2001, 2006), thus avoiding more involved analysis in relation to the eigenvalue problem.

All such analysis are restricted to steady-state situations, and very little is apparently available on transient convective heat transfer within microchannels. Nevertheless, the analysis of unsteady phenomena in applications with microsystems is relevant and in many situations is required (Jiang et al. 2000; Castellões et al. 2007). Then, the ability of predicting unsteady temperature fields is essential in the controlled temperature variation within the system. Nevertheless, only in Tunc and Bayazitoglu (2002b), an approximate analytical solution was first presented for transient convection in microchannels with slip flow, for a step change on wall temperature, based on a previously proposed hybrid approach that combines the Laplace and Integral transforms concepts (Cotta and Ozisik 1986).

In this context, the goal of this section is thus to illustrate the results obtained from a hybrid numerical–analytical solution for transient temperature distributions in a fluid flowing through parallel-plate microchannels, taking into account the velocity and temperature jumps at the duct wall surface (Cotta et al. 2004a, 2007). The approach makes use of the GITT, but invoking the exact analytical solution of the corresponding eigenvalue problem in terms of confluent hypergeometric functions (Mikhailov and Cotta 2005), to eliminate the transversal coordinate in the original formulation. Instead of recalling the Laplace transform approach as in Cotta and Ozisik (1986), Tunc and Bayazitoglu (2002b), the GITT is employed in partial transformation mode (Cotta and Gerk 1994; Castellões and Cotta 2006; Cotta et al. 2014a, b), as discussed in Sect. 1.3.1, when the spatial variable with dominant convection effects is not eliminated through integral transformation, and the resulting transformed one-dimensional partial differential system is here numerically solved by the Method of Lines, implemented within the routine NDSolve of the *Mathematica* system (Wolfram 2008). This combination of solution methodologies provides a very effective eigenfunction expansion behavior, through the fast converging analytical representation in the transversal coordinate, together with a flexible and reliable numerical approach for the transient and longitudinal behavior of the coupled transformed potentials, allowing for accurately handling transient convective heat transfer under any ratio of convection and diffusion effects (Cotta et al. 2014a, b).

Therefore, consider transient-state heat transfer in thermally developing, hydrodynamically developed forced laminar flow inside a microchannel under the following additional assumptions:

- The flow is incompressible with constant physical properties.

- Free convection of heat is negligible.
- The temperature of the channel wall is prescribed and uniform, and equal to initial temperature.

The specific boundary condition of prescribed wall temperature is adopted to illustrate the approach, but the method can similarly handle other cases. The temperature $T(y, x, t)$ of a fluid with developed velocity profile $u(y, t)$, flowing along the channel in the region $0 < y < r_1, x > 0$, is then described by the following problem in dimensionless form:

$$\frac{\partial \theta(\xi, \eta, \tau)}{\partial \tau} + U(\eta, \tau) \frac{\partial \theta(\xi, \eta, \tau)}{\partial \xi} = \frac{\partial^2 \theta(\xi, \eta, \tau)}{\partial \eta^2} + \frac{1}{\text{Pe}^{*2}} \frac{\partial^2 \theta(\xi, \eta, \tau)}{\partial \xi^2} + \text{Br} \left(\frac{\partial U(\eta, \tau)}{\partial \eta} \right)^2 \quad (2.25)$$

in $0 < \xi < A, 0 < \eta < 1, \tau > 0$, and subjected to the following boundary and initial conditions:

$$\theta(0, \eta, \tau) = \theta_e(\eta, \tau); \quad \left. \frac{\partial \theta(\xi, \eta, \tau)}{\partial \xi} \right|_{\xi=A} = 0 \quad (2.26a, b)$$

$$\left. \frac{\partial \theta(\xi, \eta, \tau)}{\partial \eta} \right|_{\eta=0} = 0; \quad \theta(\xi, 1, \tau) = -2\beta\beta_m \text{Kn} \left. \frac{\partial \theta(\xi, \eta, \tau)}{\partial \eta} \right|_{\eta=1} \quad (2.27a, b)$$

$$\theta(\xi, \eta, 0) = 0 \quad (2.28)$$

where we have considered the following dimensionless groups:

$$\begin{aligned} \xi &= \frac{\alpha}{u_m r_1^2} x = \frac{1}{\text{Pe}^*} \frac{x}{r_1}; \quad \eta = \frac{y}{r_1}; \quad \tau = \frac{\alpha t}{r_1^2}; \quad U(\eta, \tau) = \frac{u(y, t)}{u_m}; \quad \text{Pe}^* = \frac{u_m r_1}{\alpha}; \\ \text{Br} &= \frac{\mu u_m^2}{k \Delta T_c}; \quad \theta(\xi, \eta, \tau) = \frac{T(x, y, t) - T_w}{\Delta T_c}; \quad \beta = \frac{\beta_t}{\beta_m}; \end{aligned} \quad (2.29)$$

The dimensionless velocity profile is given, employing the steady-state velocity as filter, as

$$U(\eta, \tau) = U_{\text{perm}}(\eta) + U^*(\eta, \tau) \quad (2.30)$$

with the steady and transient components of the velocity, as special case of the solution derived in Sect. 2.1.

$$U_{\text{perm}}(\eta) = \frac{6\beta_m \text{Kn} + \frac{3}{2}(1 - \eta^2)}{1 + 6\beta_m \text{Kn}} \quad (2.31)$$

$$U^*(\eta, \tau) = \sum_{n=1}^{\infty} \left(\bar{U}_{o,n} e^{-\zeta_n^2 \tau} + \int_0^{\tau} \bar{g}v_n(\tau') e^{-\zeta_n^2(\tau-\tau')} d\tau' \right) \cos(\zeta_n \eta) / N_n \quad (2.32)$$

where the norm is defined as in Eq. (2.16), and

$$\bar{U}_{o,n} = \int_0^1 -U_{\text{perm}}(\eta) \cos(\zeta_n \eta) d\eta \quad (2.33)$$

$$\bar{g}v_n(\tau) = \int_0^1 \left(\frac{d^2 U_{\text{perm}}}{d\eta^2} + P(\tau) \right) \cos(\zeta_n \eta) d\eta \quad (2.34)$$

Here ζ_n are the positive roots of the transcendental equation

$$\cos(\zeta) - 2\text{Kn}\beta_m \zeta \sin(\zeta) = 0 \quad (2.35)$$

The above exact solution is obtained by the Classical Integral Transform method, as presented in Sect. 2.1.

The integral transform solution of Eqs. (2.25)–(2.28) shall now consider a Sturm–Liouville problem that includes the steady component of the velocity profile, $U_{\text{perm}}(\eta)$, in its formulation (Mikhailov and Cotta 1997, 2005; Cotta et al. 2004b), where $\psi_i(\eta)$ are the eigenfunctions, in $0 < \eta < 1$:

$$\frac{d^2 \psi_i(\eta)}{d\eta^2} + U_{\text{perm}}(\eta) \mu_i^2 \psi_i(\eta) = 0 \quad (2.36)$$

$$\left. \frac{d\psi_i(\eta)}{d\eta} \right|_{\eta=0} = 0; \quad 2\beta_m \text{Kn} \left. \frac{d\psi_i(\eta)}{d\eta} \right|_{\eta=1} + \psi_i(1) = 0 \quad (2.37a, b)$$

and defining the norm and normalized eigenfunction as

$$N_i = \int_0^1 U_{\text{perm}}(\eta) \psi_i^2(\eta) d\eta; \quad \tilde{\psi}_i(\eta) = \frac{\psi_i(\eta)}{N_i^{1/2}} \quad (2.38a, b)$$

Equation (2.36) can thus be analytically solved, as discussed in Mikhailov and Cotta (1997, 2005), in terms of the confluent hypergeometric function, also known as Kummer function ${}_1F_1[a, b; z]$, readily available in the *Mathematica* system (Wolfram 2008), as

$$\psi_i(\eta) = {}_1F_1\left[\frac{1 - v_i(1 + 4\text{Kn}\beta_m)}{4}, \frac{1}{2}; v_i\eta^2\right] e^{v_i\frac{\eta^2}{2}} \quad (2.39)$$

where

$$\mu_i = v_i \sqrt{\frac{2}{3}(1 + 6\text{Kn}\beta_m)} \quad (2.40)$$

Equation (2.39) satisfies the first two Eqs. (2.36)–(2.37a), while Eq. (2.37b) gives an eigencondition which provides the eigenvalues for the expansion.

The next step is thus the definition of the transform-inverse pair for the dimensionless temperature, given, respectively, by

$$\bar{\theta}_i(\xi, \tau) = \int_0^1 U_{\text{perm}}(\eta) \theta(\xi, \eta, \tau) \tilde{\psi}_i(\eta) d\eta \quad (2.41)$$

$$\theta(\xi, \eta, \tau) = \sum_{i=1}^{\infty} \bar{\theta}_i(\xi, \tau) \tilde{\psi}_i(\eta) \quad (2.42)$$

Here it has been chosen to apply the GITT on Eqs. (2.25)–(2.28) in the partial transformation strategy (Cotta and Gerk 1994), as discussed in Sect. 1.3.1, resulting in the one-dimensional partial differential system below:

$$\begin{aligned} \sum_{j=1}^{\infty} A_{ij} \frac{\partial \bar{\theta}_j(\xi, \tau)}{\partial \tau} + \frac{\partial \bar{\theta}_i(\xi, \tau)}{\partial \xi} + \sum_{j=1}^{\infty} B_{ij}(\tau) \frac{\partial \bar{\theta}_j(\xi, \tau)}{\partial \xi} \\ = -\mu_i^2 \bar{\theta}_i(\xi, \tau) + \frac{1}{\text{Pe}^{*2}} \sum_{j=1}^{\infty} A_{ij} \frac{\partial^2 \bar{\theta}_j(\xi, \tau)}{\partial \xi^2} + \bar{g}_i(\tau) \end{aligned} \quad (2.43)$$

in $0 < \xi < A$, $\tau > 0$, and subjected to the following boundary and initial conditions:

$$\bar{\theta}_i(0, \tau) = \bar{\theta}_{e,i}(\tau); \quad \left. \frac{\partial \bar{\theta}_i(\xi, \tau)}{\partial \xi} \right|_{\xi=A} = 0 \quad (2.44a, b)$$

$$\bar{\theta}_i(\xi, 0) = 0 \quad (2.45)$$

where

$$A_{ij} = \int_0^1 \tilde{\psi}_i(\eta) \tilde{\psi}_j(\eta) d\eta; \quad B_{ij}(\tau) = \int_0^1 U^*(\eta, \tau) \tilde{\psi}_i(\eta) \tilde{\psi}_j(\eta) d\eta \quad (2.46a, b)$$

$$\bar{g}_i(\tau) = \text{Br} \int_0^1 \left(\frac{\partial U(\eta, \tau)}{\partial \eta} \right)^2 \tilde{\psi}_i(\eta) d\eta; \quad \bar{\theta}_{e,i}(\tau) = \int_0^1 U_{\text{perm}}(\eta) \theta_e(\eta, \tau) \tilde{\psi}_i(\eta) d\eta \quad (2.47a, b)$$

The numerical Method of Lines as implemented in the routine NDSolve of the *Mathematica* system (Wolfram 2008) deals with system (2.43)–(2.45) by employing a default high-order finite difference discretization in the spatial variable ξ , and creating a coupled system of ordinary differential equations for the transformed dimensionless temperature evaluated on the knots of the created mesh. This resulting system is internally solved with a *stiffness switching* method for stiff ODE systems. Once numerical results have been obtained and automatically interpolated by NDSolve, one can apply the inverse expression (2.42) to obtain the full dimensionless temperature field, $\theta(\xi, \eta, \tau)$. The dimensionless bulk temperature, $\theta_m(\xi, \tau)$, and the local Nusselt number, $\text{Nu}(\xi, \tau)$, can be found from

$$\theta_m(\xi, \tau) = \frac{\int_0^1 U(\eta, \tau) \theta(\xi, \eta, \tau) d\eta}{\int_0^1 U(\eta, \tau) d\eta}; \quad \text{Nu} = \frac{hD_h}{k} = - \frac{4}{\theta_m(\xi, \tau)} \frac{\partial \theta}{\partial \eta} \bigg|_{\eta=1} \quad (2.48a, b)$$

Before proceeding to the analysis of transient convection with slip flow and temperature jump, we first verify the present integral transform approach in partial transformation scheme, and inspect the convergence behavior in both the eigenfunction expansion and the numerical procedure for the transformed partial differential system. Therefore, the test case by Gondim et al. (2003, 2007) for a regular parallel-plate channel ($\text{Kn} = 0$), with steady velocity profile, is here considered for different and representative values of the Péclet number. It should be noted that gas flows in microchannels are likely to result in relatively low values of Reynolds number, in the range of incompressible flow modeling here adopted, which then produce Péclet numbers in a not very wide range. Therefore, Fig. 2.2a, b, respectively, for $\text{Pe} = 1$ and $\text{Pe} = 10$, show the excellent agreement between the present results and the total integral transformation in Gondim et al. (2003, 2007), where a double integral transformation in both transversal and longitudinal coordinates is employed. A truncation order of just $N = 15$ terms was considered sufficient for convergence in the present verification, as will be examined in what follows, since we are dealing with a single integral transformation, which is performed along the most diffusive direction (η) and exactly transforming the transversal convection term, as opposed to the double transformation in Gondim et al. (2003, 2007), which requires larger truncation orders.

Transient forced convection in microchannels is then studied for typical values of the accommodation factors ($\alpha_m = 1.0$ and $\alpha_t = 0.92$) and considering air as the working fluid ($\text{Pr} = 0.7$ and $\gamma = 1.4$). Numerical solutions were generated for

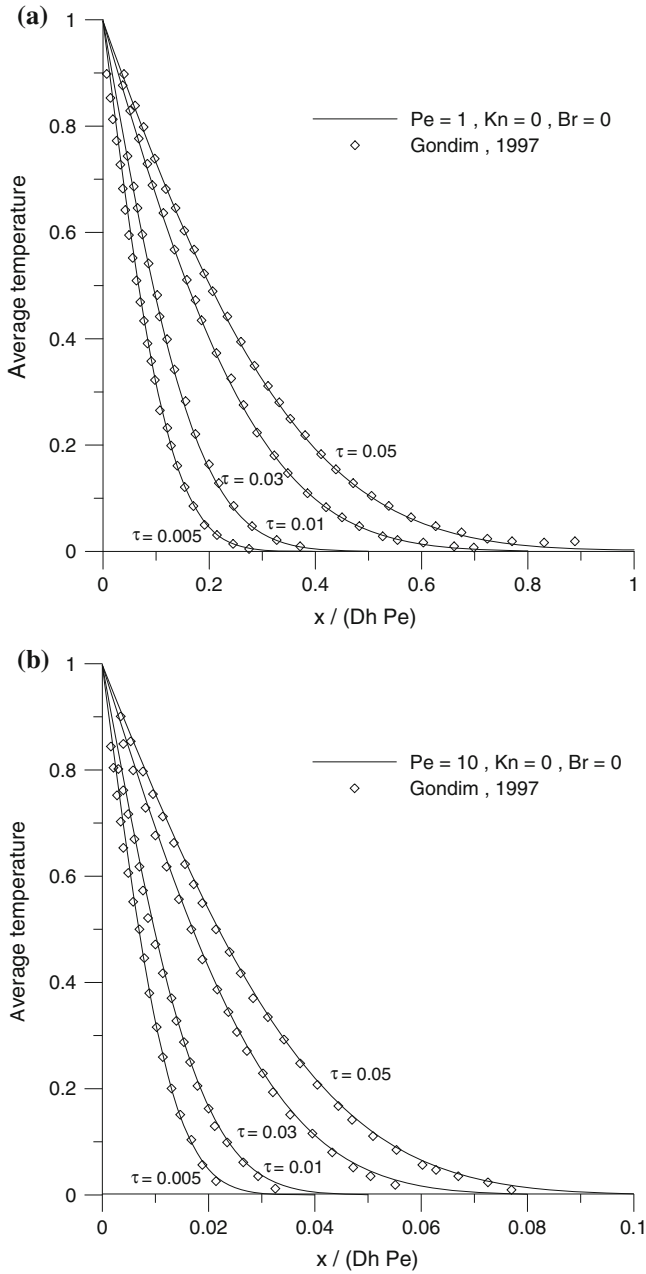


Fig. 2.2 Transient evolution of dimensionless average temperature and verification with Gondim et al. (2003, 2007) for parallel-plates channel and step change in inlet temperature, $Kn = 0$, $Br = 0$, and $S = 15$. In (a) $Pe = 1$, in (b) $Pe = 10$

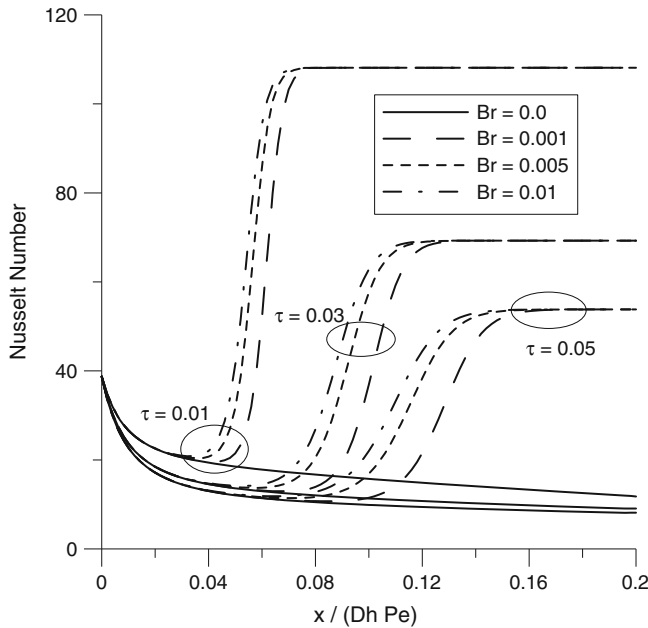


Fig. 2.3 Influence of Brinkman number on local Nusselt number evolution (parallel-plate channel, $Pe = 10$, $Kn = 0.01$, $Br = 0, 0.001, 0.005$, and 0.01 , $S = 15$)

typical parameter values, $Kn = 0.01$ and $Pe = 10$, with $\beta_t = 1.957$ and $\beta_m = 1.0$. For such physical situation, the values for the remaining governing parameters when working with air are: $Re = 14.3$ (Reynolds number), $Ma = 0.02$ (Mach number), $D_h = 50 \mu m$. Figure 2.3 shows the effect of Brinkman number on the transient behavior of the local Nusselt number, for the following governing parameter values, $Pe = 10$, $Kn = 0.01$, and $Br = 0, 0.001, 0.005$, and 0.01 , with $N = 15$. The effect of increasing the Nusselt number while increasing the internal heat generation via larger values of the Brinkman number, as also evident in previous steady-state analysis, is here reproduced, while the transient solutions approach such steady configurations.

Figure 2.4a, b presents the deviations encountered in the local Nusselt number within the entrance region with and without considering axial conduction in the formulation, for the transient and steady states, respectively, with $Pe = 10$, $Kn = 0.01$, $Br = 0$, and $N = 15$. Clearly, neglecting axial diffusion along the fluid in this case, causes a noticeable difference on the local Nusselt number, for the two values of dimensionless time ($\tau = 0.03$ and 0.05) and for the steady situation. The inclusion of axial diffusion in the model leads to higher heat transfer coefficients, with a marked difference from the formulation without axial diffusion. This comparison was particularly plotted taking the dimensionless physical dimension x/D_h , removing the Péclet number from the abscissa definition.

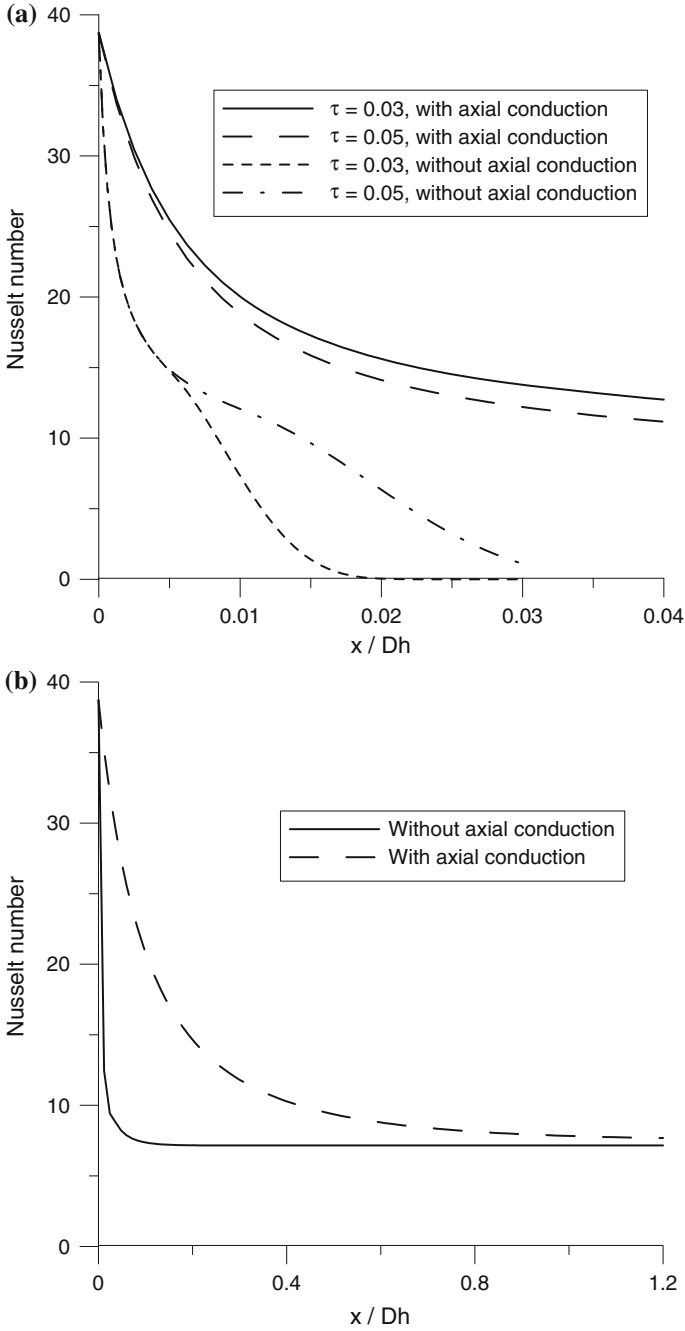


Fig. 2.4 Influence of axial conduction on the local Nusselt number for (a) transient state and (b) steady state (parallel-plate channel, $Pe = 10$, $Kn = 0.01$, $Br = 0$, $N = 15$)

Figure 2.5 presents the influence of Knudsen number (Kn) on the local Nusselt number, for $Pe = 10$, $Kn = 0, 0.001, 0.01$, and 0.1 , $Br = 0$, $N = 15$, along the entrance region of the parallel-plates channel during both the transient and steady regimen, respectively. It can be observed that variation of the local Nusselt number in such different levels of the microscale effect is remarkable, with a significant increase in Nu for decreasing Kn . This set of results also allows for the inspection of the comparative transient behavior, which indicates the less pronounced transient phenomena when the Knudsen number is increased. The microscale effects practically cease for $Kn = 0.001$, all along the transient behavior.

Figure 2.6a, b presents the variation of average temperature in time for fixed axial positions, considering a constant pressure gradient and both a sinusoidal and a sawtooth variation in time of inlet temperature, respectively, with a frequency of $\pi/15$. Also, it is presented the influence of Brinkman number in the sinusoidal situation. It should be noted that the transient effects in the microscale occur within very short time periods, and the quasi-steady thermal response is rapidly established. In the sinusoidal case, one may observe the periodic system response to the inlet perturbation, with amplitude and phase lag noticeably modified by the viscous dissipation effect.

The bulk temperature behavior encountered for the periodic variation of inlet temperature with a different functional form (sawtooth shape) is presented in Fig. 2.6b, considering again a frequency of $\pi/15$, but disregarding the viscous dissipation effects. Three different axial positions are considered for the average temperature. Such periodic response may only be obtained through the full transient analysis, by afterward considering sufficiently large elapsed times, as opposed to the simpler sinusoidal case, which allows for a direct quasi-steady analysis (Cotta et al. 2003).

Results for time-variable pressure gradients in microchannels convection are presented in Fig. 2.7. A sinusoidal time varying pressure gradient with a frequency of $\pi/15$ is considered, combined with a unit step variation of inlet temperature, neglecting viscous dissipation. The presence of a unit step variation of inlet temperature is noted in the fast initial transient period. After it, the periodic behavior imposed in the convection by the pressure gradient is dominant, leading to a quasi-steady state.

Figure 2.7 compares the behavior of the local Nusselt number in a fixed axial position along the channel when considering the periodic variation of pressure gradient and the Nusselt number when the solution is obtained with a constant pressure gradient. One can observe the behavior of Nusselt number for the periodic situation oscillating around the solution for constant pressure gradient, characterizing the periodic system response, again after a very brief initial transient region.

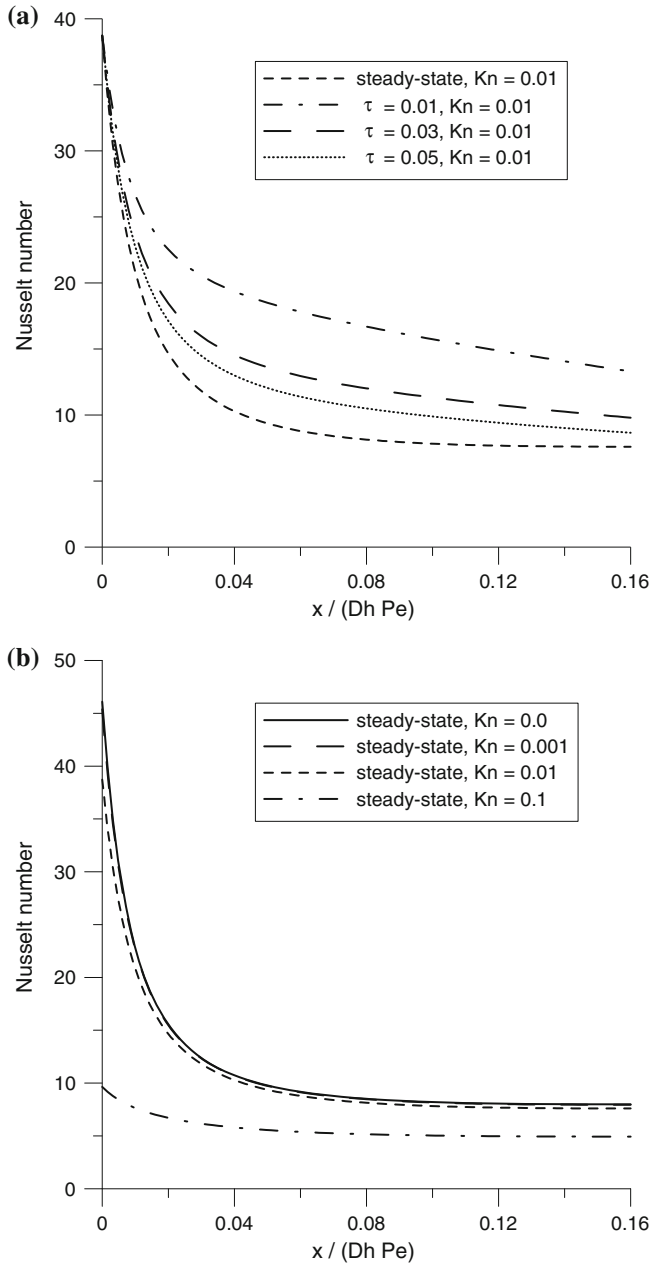


Fig. 2.5 Influence of Knudsen number on the (a) transient and (b) steady behaviors of local Nusselt number (parallel plates, $Pe = 10$, $Kn = 0, 0.001, 0.01$, and 0.1 , $Br = 0$, $N = 15$)

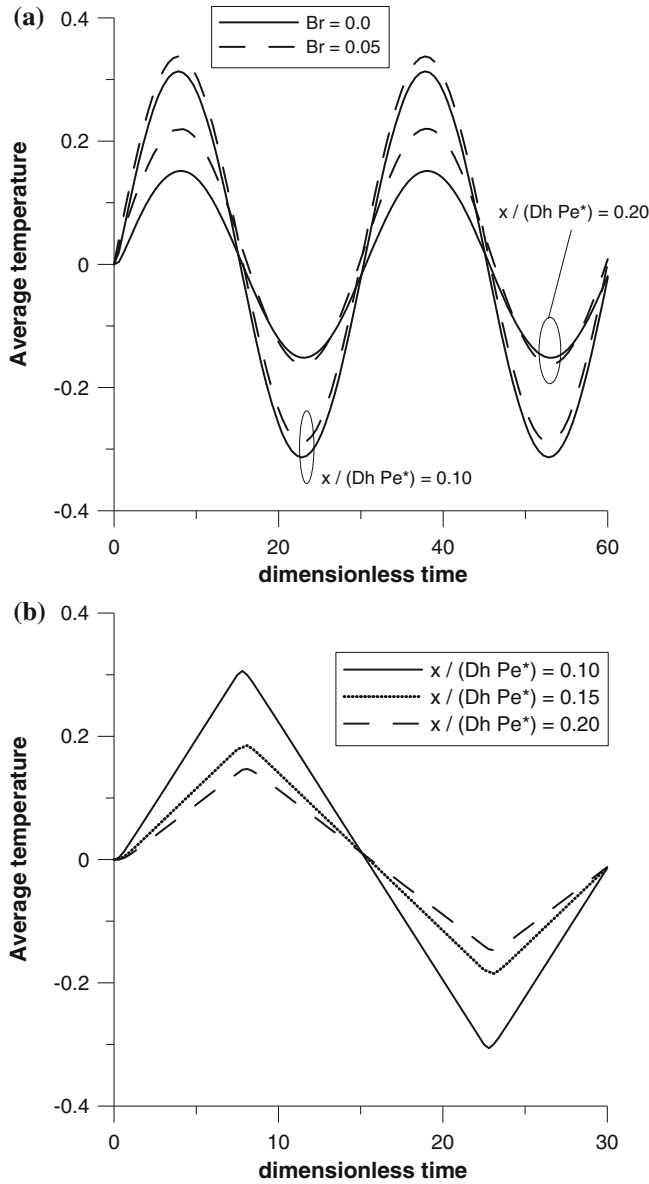


Fig. 2.6 Dimensionless bulk temperature for (a) sinusoidal time variation of inlet temperature and influence of Brinkman number; (b) different axial positions with sawtooth time variation of inlet temperature

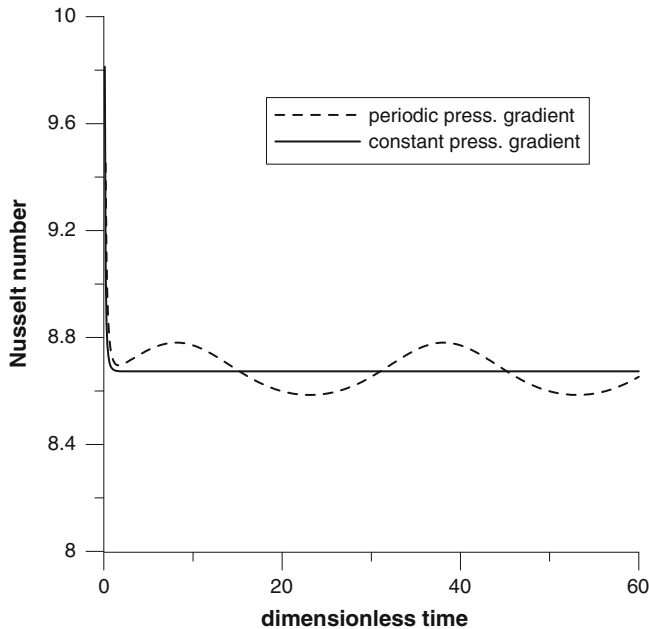


Fig. 2.7 Local Nusselt number for periodic variation of pressure gradient (step change of inlet temperature and no viscous dissipation)

2.3 Convection in Electroosmotic Flow

In the present section, forced convection under electroosmotic flow in microchannels is illustrated (Soares et al. 2005, 2008), aiming at a better understanding of the involved phenomena. The Generalized Integral Transform Technique (GITT), under the partial transformation mode, as discussed in Sect. 1.3.1, is employed to solve the convection problem and to provide converged results for bulk temperature and Nusselt numbers along the channel length. The GITT is also employed to solve the corresponding eigenvalue problem, as shown in Sect. 1.3.4. A pseudo-transient formulation of the problem is considered, which yields steady-state results with significant computational savings.

For illustration of the proposed hybrid solution, the case of an actual rectangular microchannel with $10\ \mu\text{m} \times 200\ \mu\text{m} \times 2\ \text{cm}$ (height $\times$ width $\times$ length), with one surface made of glass and the other made of silicon, is here considered (Soares et al. 2008). As the width is 20 times greater than the height, it is then modeled as a parallel-plates microchannel, as depicted in Fig. 2.8.

The following assumptions are considered for the hydrodynamic problem formulation: (a) Newtonian fluid, with constant physical properties; (b) incompressible, laminar, and fully developed flow; (c) steady-state flow; (d) purely electroosmotic flow (which means that there is no pressure gradient); (e) electric

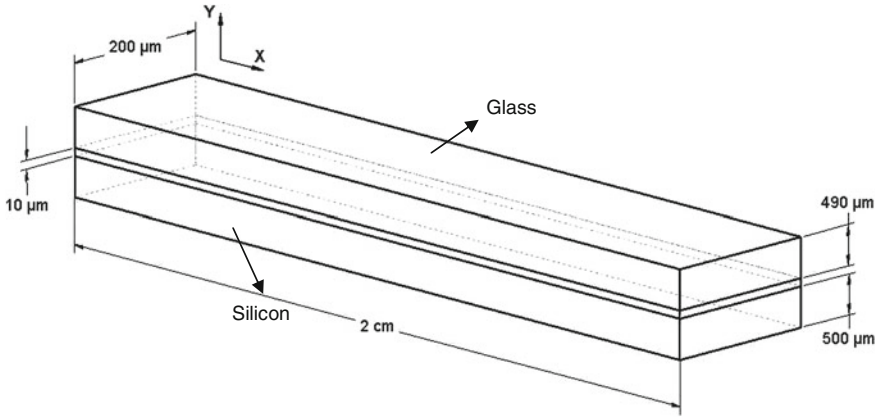


Fig. 2.8 Schematic illustration of the parallel-plates microchannel with electroosmotic flow

field externally applied on the fluid, in the x -direction. Under these assumptions, the flow problem formulation simply results in $u = u(y)$. After some manipulation, the Navier–Stokes equations take the following dimensionless form, with no-slip boundary conditions on both surfaces:

$$\frac{d^2 U(Y)}{dY^2} - G E_x^* \frac{d^2 \bar{\psi}(Y)}{dY^2} = 0, \quad 0 < Y < 1 \quad (2.49a)$$

$$U(0) = 0, \quad U(1) = 0 \quad (2.49b, c)$$

with

$$G = \frac{\varepsilon \varepsilon_0 k_b T \zeta_m}{L z q \mu u_m} \quad (2.50)$$

where E_x^* is the dimensionless externally applied electric field, $\bar{\psi}(Y)$ is the dimensionless electric potential, ε the dielectric constant of the medium, ε_0 the permittivity of vacuum, k_b the Boltzmann constant, T is the absolute temperature, ζ_m a reference zeta potential, L the length of the microchannel, z the ion charge, q the elemental charge, μ the fluid viscosity, and u_m a reference velocity.

The dimensionless electric potential is obtained by a combination of Poisson and Boltzmann distribution equations (Yang et al. 2001), and is given as

$$\frac{d^2 \bar{\psi}(Y)}{dY^2} = Z^2 \bar{\psi}(Y) \quad 0 < Y < 1 \quad (2.51a)$$

$$\bar{\psi}(0) = \bar{\varsigma}_1, \quad \bar{\psi}(1) = \bar{\varsigma}_2 \quad (2.51b, c)$$

where $Z = \kappa d = d/\lambda$ is a parameter that relates the length of the channel and the EDL length, λ , that is computed as the inverse of the Debye–Hückel parameter, written as

$$\kappa^2 = \frac{2n_0 z^2 q^2}{\varepsilon \varepsilon_0 k_b T} \quad (2.52)$$

where n_0 is bulk concentration of ions, and $\bar{\varsigma}_1$ and $\bar{\varsigma}_2$ are dimensionless zeta potentials (EDL field at the top of the Stern layer). Both systems, Eqs. (2.49)–(2.52), are easily solved by integrating and substituting boundary conditions, to yield

$$\bar{\psi}(Y) = \frac{e^{-YZ} [\bar{\varsigma}_1 (e^{2Z} - e^{2YZ}) + \bar{\varsigma}_2 (e^{Z+2YZ} - e^Z)]}{e^{2Z} - 1} \quad (2.53)$$

$$U(Y) = \frac{e^{-YZ} E_x^* G [\bar{\varsigma}_1 (e^{2Z} + e^{YZ} - e^{2YZ} - e^{2Z+YZ} - Y e^{YZ} + Y e^{2Z+YZ})]}{e^{2Z} - 1} + \frac{e^{-YZ} E_x^* G [\bar{\varsigma}_2 (e^{Z+2YZ} - e^Z + Y e^{YZ} - Y e^{2Z+YZ})]}{e^{2Z} - 1} \quad (2.54)$$

where the dimensionless parameters considered are written as

$$X = \frac{x}{d} \frac{1}{\text{Pe}}; \quad Y = \frac{y}{d}; \quad H = \frac{L}{d \text{Pe}}; \quad U(Y) = \frac{u(y)}{u_m}; \quad (2.55)$$

$$\bar{\psi}(Y) = \frac{z q}{k_b T} \psi(y); \quad E_x^* = \frac{E_x L}{\varsigma_m}; \quad \text{Pe} = \frac{u_m d}{\alpha}; \quad \varsigma_m = \frac{\varsigma_1 + \varsigma_2}{2}$$

In the present example, the microchannel is subjected to a uniform and constant prescribed heat flux at the silicon surface, and to natural convection with the ambient air at the glass surface. The additional following assumptions are considered: (a) transient or pseudo-transient state; (b) axial heat diffusion (low Peclet numbers); (c) all heat that is applied to the silicon surface is transferred to the fluid; (d) non-negligible thermal resistance of glass surface; (e) negligible Joule heating effects (Horiuchi and Dutta 2004). For this problem, the energy conservation equation can be written, in dimensionless form, as

$$\frac{\partial \theta(X, Y, \tau)}{\partial \tau} + U(Y) \frac{\partial \theta(X, Y, \tau)}{\partial X} = \frac{1}{\text{Pe}^2} \frac{\partial^2 \theta(X, Y, \tau)}{\partial X^2} + \frac{\partial^2 \theta(X, Y, \tau)}{\partial Y^2}, \quad (2.56a)$$

$$0 < X < H, \quad 0 < Y < 1, \quad \tau > 0$$

$$\theta(0, Y, \tau) = 0; \quad \left. \frac{\partial \theta(X, Y, \tau)}{\partial X} \right|_{X=H} = 0 \quad (2.56b, c)$$

$$\left. \frac{\partial \theta(X, Y, \tau)}{\partial Y} \right|_{Y=0} = -\Phi_w; \quad \left. \frac{\partial \theta(X, Y, \tau)}{\partial Y} \right|_{Y=1} = -\text{Bi} \theta(X, 1, \tau) \quad (2.56d, e)$$

$$\theta(X, Y, 0) = 0 \quad (2.56f)$$

where the following additional parameters are considered:

$$\begin{aligned} \tau &= \frac{\alpha}{d^2} t; \quad \theta(X, Y, \tau) = \frac{T(x, y, t) - T_\infty}{\Delta T}; \quad \Phi_w = \frac{\phi_w d}{k_f \Delta T}; \\ \text{Bi} &= \frac{h_{\text{ef}} d}{k_f}; \quad h_{\text{ef}} = \frac{k_v h_\infty}{h_\infty e_v + k_v} \end{aligned} \quad (2.57)$$

In Eq. (2.57), T_∞ is the temperature of the ambient air, ΔT is a reference temperature difference, ϕ_w is the applied heat flux at the silicon surface, k_f the thermal conductivity of the fluid, h_{ef} the effective heat transfer coefficient that considers the transversal thermal resistance of the glass surface, k_v the thermal conductivity of glass, h_∞ the convection heat transfer coefficient, and e_v the thickness of the glass surface. The thermal capacitance of the wall is disregarded, since the main goal is to obtain steady-state solutions from the pseudo-transient formulation.

The bulk temperature and the local Nusselt numbers for both surfaces are defined as

$$\theta_m(X, \tau) = \frac{\int_{Y=0}^1 U(Y) \theta(X, Y, \tau) dY}{\int_{Y=0}^1 U(Y) dY} \quad (2.58)$$

$$\text{Nu}_1 = \frac{2 \Phi_w}{\theta(X, 0, \tau) - \theta_m(X, \tau)}; \quad \text{Nu}_2 = 2 \text{Bi} \frac{\theta(X, 1, \tau)}{\theta_m(X, \tau) - \theta(X, 1, \tau)} \quad (2.59a, b)$$

Before proceeding to the integral transformation of problem (2.56), a filtering solution can be employed to homogenize the boundary conditions, in the form:

$$\theta(X, Y, \tau) = \theta^*(X, Y, \tau) + \theta_f(Y) \quad (2.60)$$

The filtering solution is obtained from the following ordinary differential equation:

$$\frac{d^2 \theta_f(Y)}{dY^2} = 0, \quad \text{in } 0 < Y < 1 \quad (2.61)$$

$$\left. \frac{d\theta_f(Y)}{dY} \right|_{Y=0} + \Phi_w = 0; \quad \left. \frac{d\theta_f(Y)}{dY} \right|_{Y=1} + \text{Bi} \theta_f(1) = 0 \quad (2.62a, b)$$

which is readily solved as

$$\theta_f(Y) = \frac{\Phi_w(1 + \text{Bi})}{\text{Bi}} - \Phi_w Y \quad (2.63)$$

The filtered problem, after substitution of Eq. (2.60) into Eq. (2.56), is then obtained as

$$\frac{\partial \theta^*(X, Y, \tau)}{\partial \tau} + U(Y) \frac{\partial \theta^*(X, Y, \tau)}{\partial X} = \frac{1}{\text{Pe}^2} \frac{\partial^2 \theta^*(X, Y, \tau)}{\partial X^2} + \frac{\partial^2 \theta^*(X, Y, \tau)}{\partial Y^2}, \quad (2.64)$$

$$0 < X < H, 0 < Y < 1, \tau > 0$$

$$\theta^*(0, Y, \tau) = -\theta_f(Y); \quad \left. \frac{\partial \theta^*(X, Y, \tau)}{\partial X} \right|_{X=H} = 0 \quad (2.65a, b)$$

$$\left. \frac{\partial \theta^*(X, Y, \tau)}{\partial Y} \right|_{Y=0} = 0; \quad \left. \frac{\partial \theta^*(X, Y, \tau)}{\partial Y} \right|_{Y=1} + \text{Bi} \theta^*(X, 1, \tau) = 0 \quad (2.66a, b)$$

$$\theta^*(X, Y, 0) = -\theta_f(Y) \quad (2.67)$$

The Generalized Integral Transform Technique (GITT) is employed to solve Eqs. (2.64)–(2.67) with the velocity profile defined by Eq. (2.54). The first step in the application of this method is to define an eigenvalue problem. In the present situation, we have chosen to define an eigenvalue problem that includes the velocity profile (Soares et al. 2008):

$$\frac{d^2 \Gamma_i(Y)}{dY^2} + \delta_i^2 U(Y) \Gamma_i(Y) = 0 \quad 0 < Y < 1 \quad (2.68a)$$

$$\left. \frac{d\Gamma_i(Y)}{dY} \right|_{Y=0} = 0; \quad \left. \frac{d\Gamma_i(Y)}{dY} \right|_{Y=1} + \text{Bi} \Gamma_i(1) = 0 \quad (2.68b, c)$$

which is solved by GITT itself, as discussed in Sect. 1.3.4. For this purpose, we define an auxiliary problem for the original eigenvalue problem, Eq. (2.68), in the form:

$$\frac{d^2 \Omega_m(Y)}{dY^2} + v_m^2 \Omega_m(Y) = 0 \quad 0 < Y < 1 \quad (2.69a)$$

$$\left. \frac{d\Omega_m(Y)}{dY} \right|_{Y=0} = 0; \quad \left. \frac{d\Omega_m(Y)}{dY} \right|_{Y=1} + \text{Bi} \Omega_m(1) = 0 \quad (2.69b, c)$$

and the following inverse transform pair,

$$\Gamma_i(x) = \sum_{m=1}^{\infty} \tilde{\Omega}_m(x) \bar{\Gamma}_{i,m}, \quad \text{inverse} \quad (2.70)$$

$$\bar{\Gamma}_{i,m} = \int_0^1 \tilde{\Omega}_m(x) \Gamma_i(x) dx, \quad \text{transform} \quad (2.71)$$

where $\tilde{\Omega}_m(x)$ is the normalized eigenfunction from the auxiliary problem, given as $\tilde{\Omega}_m(x) = \frac{\Omega_m(x)}{N_m^{1/2}}$, and $N_m = \int_0^1 [\Omega_m(x)]^2 dx$ is the norm of the auxiliary eigenfunctions. The analytical solution of Eq. (2.69) is given as

$$\Omega_m(Y) = \cos(v_m Y) \quad (2.72)$$

where v_m are the eigenvalues of the auxiliary problem, given by the positive roots of

$$\sin(v_m) - \frac{\text{Bi}}{v_m} \cos(v_m) = 0 \quad (2.73)$$

Following the formalism in the GITT approach in solving eigenvalue problems, described in Sect. 1.3.4, one then proceeds to obtain the eigenvectors $\bar{\Gamma}_{i,m}$, $m = 1, 2, \dots$, up to a finite-order S_{aux} , and to reconstruct $\Gamma_i(x)$ using Eq. (2.70), considering S terms in the series solution, sufficiently large to achieve the desired number of significant digits in the final result.

One may now define the inverse transform pair for the dimensionless temperature integral transformation, in terms of the original eigenfunctions:

$$\theta^*(X, Y, \tau) = \sum_{i=1}^{\infty} \tilde{\Gamma}_i(Y) \bar{\theta}_i(X, \tau), \quad \text{inverse} \quad (2.74)$$

$$\bar{\theta}_i(X, \tau) = \int_{Y=0}^1 U(Y) \tilde{\Gamma}_i(Y) \theta^*(X, Y, \tau) dY, \quad \text{transform} \quad (2.75)$$

The norm is then given as

$$N_i = \int_{Y=0}^1 U(Y) [\Gamma_i(Y)]^2 dY \quad (2.76)$$

To proceed with the integral transformation process, one operates on Eq. (2.64) with $\int_{Y=0}^1 (\cdot) \tilde{\Gamma}_i(Y) dY$ and on Eqs. (2.65) and (2.67) with $\int_{Y=0}^1 (\cdot) U(Y) \tilde{\Gamma}_i(Y) dY$ (boundary and initial conditions), and after some manipulations, the following transformed partial differential system is obtained:

$$\sum_{j=1}^{\infty} A_{ij} \frac{\partial \bar{\theta}_j(X, \tau)}{\partial \tau} + \frac{\partial \bar{\theta}_i(X, \tau)}{\partial X} = \frac{1}{\text{Pe}^2} \sum_{j=1}^{\infty} A_{ij} \frac{\partial^2 \bar{\theta}_j(X, \tau)}{\partial X^2} - \delta_i^2 \bar{\theta}_i(X, \tau) \quad (2.77a)$$

$$0 < X < H, \tau > 0$$

$$\bar{\theta}_i(0, \tau) = -\bar{\theta}_f; \quad \left. \frac{\partial \bar{\theta}_i(X, \tau)}{\partial X} \right|_{X=H} = 0 \quad (2.77b, c)$$

$$\bar{\theta}_i(X, 0) = -\bar{\theta}_f \quad (2.77d)$$

where

$$A_{ij} = \int_0^1 \tilde{\Gamma}_i(\eta) \tilde{\Gamma}_j(\eta) d\eta \quad (2.78)$$

$$\bar{\theta}_f = \int_{Y=0}^1 U(Y) \tilde{\Gamma}_i(Y) \theta_f(Y) dY \quad (2.79)$$

The transformed coupled system of partial differential equations is numerically solved by the Method of Lines, implemented in routine NDSolve of the *Mathematica* platform, truncating the infinite sums to a number S of terms, sufficient to ensure convergence to within the desired number of significant digits. Once the transformed potentials ($\bar{\theta}_i$) are numerically computed, this routine gives the solution as interpolating functions, that allow for the direct application of the inverse formula, Eq. (2.74), to obtain the filtered dimensionless temperature field.

To generate a few representative results to be here presented, an electrolytic solution of KCl in water is considered ($\varepsilon = 80$, $\mu = 0.9 \times 10^{-3}$ kg/ms, $\rho = 998$ kg/m³, $k_f = 0.6$ W/mK, $\alpha = 1.44 \times 10^{-7}$ m²/s, $z = 1$), with molar concentration of 10^{-4} M. Also, considering as an example $u_m = 2$ mm/s (Karniadakis et al. 2005), it is possible to estimate $Pe = 0.139$ (Peclet number) and $Re = 0.022$ (Reynolds number). With such low values of Peclet number, axial heat diffusion along the fluid cannot possibly be disregarded. At the molar concentration above considered, the zeta potential is equal to -100 mV at the silicon surface (Mala et al. 1997a, b) and equal to -59 mV at the glass surface (Erickson et al. 2000). Using this data, at ambient temperature ($T = 298$ K), the Debye–Hückel parameter is $\kappa = 3.25625 \times 10^6$ m⁻¹; consequently, the length of the EDL is $\lambda = 307.1$ nm and $Z = 32.562$. It has been considered $Z = 25$, which is a typical electroosmotic flow parameter, under the applied electric field of 1 V/mm.

For the forced convection application, a prescribed heat flux of 1000 W/m² at the silicon surface has been considered. A maximum temperature difference of 80 °C is used, to avoid boiling inside the microchannel. A convection heat transfer coefficient of 25 W/m² K is considered, corresponding to natural convection with air at the external ambient temperature. The thermal conductivity of glass is 0.8 W/m K, and using such values one reaches an effective heat transfer coefficient (h_{ef}) of 24.62 W/m² K and $Bi = 0.00041$. A parabolic velocity profile is expected for $Z = 0.5$, since it represents a situation where the EDL length is greater than the channel diameter, equivalent to a constant pressure gradient across the channel.

Table 2.2 Convergence of some of the first ten eigenvalues for a typical electroosmotic flow ($Z = 25$)

	$S_{\text{exc}} = 0$	$S_{\text{exc}} = 10$	$S_{\text{exc}} = 20$	$S_{\text{exc}} = 30$	$S_{\text{exc}} = 40$	$S_{\text{exc}} = 50$
μ_2	19.42276	19.42214	19.42211	19.42211	19.42210	19.42210
μ_4	57.65417	57.63324	57.63238	57.63228	57.63225	57.63224
μ_6	95.62972	95.47825	95.47373	95.47318	95.47306	95.47303
μ_8	134.07163	133.15624	133.14161	133.13997	133.13962	133.13952
μ_{10}	183.12336	170.77145	170.73302	170.72925	170.72847	170.72824

The value of $Z = 50$ represents a typical electroosmotic flow, and then a plug flow velocity profile is expected when the zeta potentials at the surfaces are the same.

As shown in Table 2.2, the required truncation orders to reach convergence in the eigenvalue problem solution have been inspected, when a typical electroosmotic velocity profile is used ($Z = 25$). It is clear that $S_{\text{exc}} = 30$ terms are enough to assure convergence to within six significant digits in the eigenvalues results, in the range of eigenvalue orders analyzed. S is the number of terms in the truncated series for the temperature field (for the computation of the dimensionless temperature), and S_{exc} extra terms are used in the eigenvalue problem representation to assure and verify convergence, thus $S_{\text{aux}} = S + S_{\text{exc}}$.

It is also of interest to analyze the convergence of the method (GITT) as directly applied to the original convection–diffusion problem. Table 2.3 shows the results for the bulk temperature convergence, with $Z = 25$, applied electric field of 1 V/mm and heat flux of 1500 W/m². Clearly, the eigenfunction expansion for the bulk temperatures presents an excellent convergence behavior, with five significant digits practically converged at truncation orders as low as $S = 15$. On the other hand, the Nusselt number formal expressions demonstrate a slower convergence behavior of up to three converged digits at the same truncation orders, as expected, though sufficiently accurate, with just a simple boundary filter as the one here used to accelerate convergence.

Figure 2.9 shows the variation of EDL against the Z parameter, for a 10^{-4} M solution, and Fig. 2.9 shows the velocity profile for the same conditions, illustrating

Table 2.3 Dimensionless bulk temperature convergence behavior

	X	$S = 5$	$S = 10$	$S = 15$	$S = 20$
$\tau = 2000$ ($t = 1.39$ s)	250	0.27029	0.27019	0.27020	0.27020
	500	0.37750	0.37737	0.37738	0.37738
	1000	0.42374	0.42361	0.42361	0.42361
	2000	0.42639	0.42626	0.42627	0.42627
$\tau = 6000$ ($t = 4.17$ s)	250	0.34837	0.34813	0.34814	0.34814
	500	0.52720	0.52682	0.52684	0.52684
	1000	0.66065	0.66030	0.66032	0.66032
	2000	0.69584	0.69557	0.69558	0.69558
$\tau = 12,000$ ($t = 8.347$ s)	250	0.35838	0.35832	0.35832	0.35832
	500	0.54767	0.54756	0.54756	0.54756
	1000	0.69972	0.69952	0.69952	0.69952
	2000	0.75248	0.75222	0.75222	0.75222

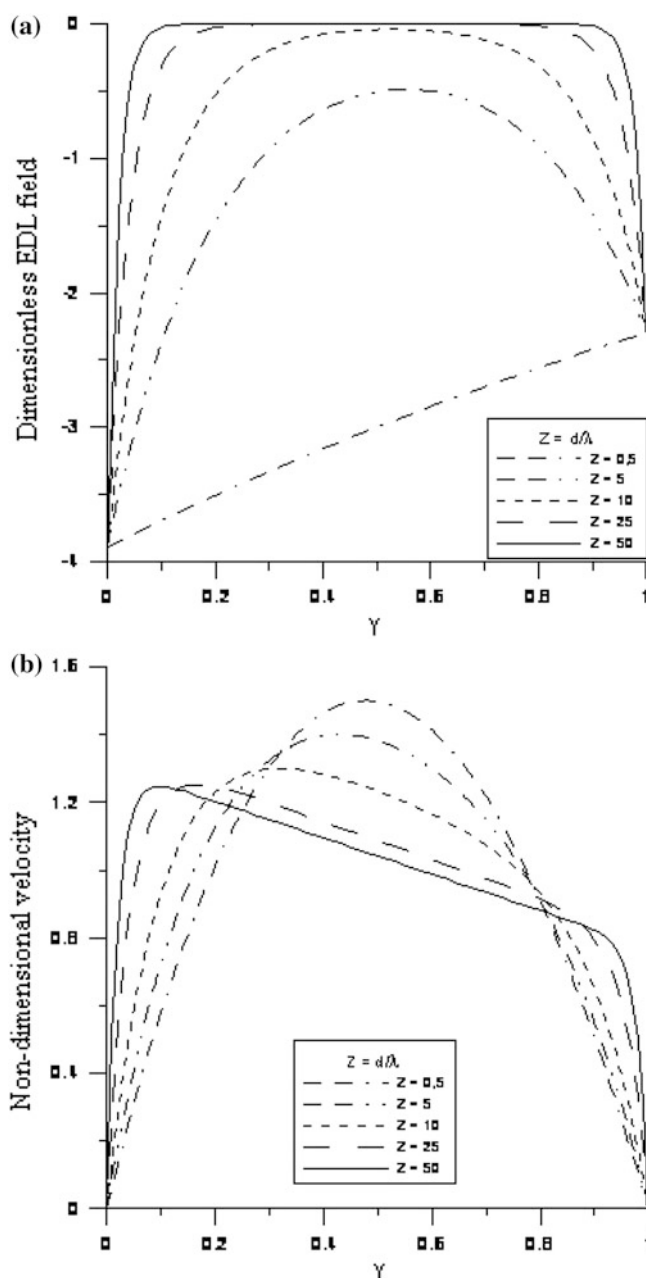


Fig. 2.9 (a) EDL field variation with Z parameter. (b) Velocity profile variation with Z parameter

the quite variable shapes of velocity profiles that are achievable in such physical situations. Figure 2.10a, b presents the dimensionless bulk temperature for different prescribed heat fluxes and electric fields, respectively, where one may readily observe the increase in bulk temperature levels within the channel when either the

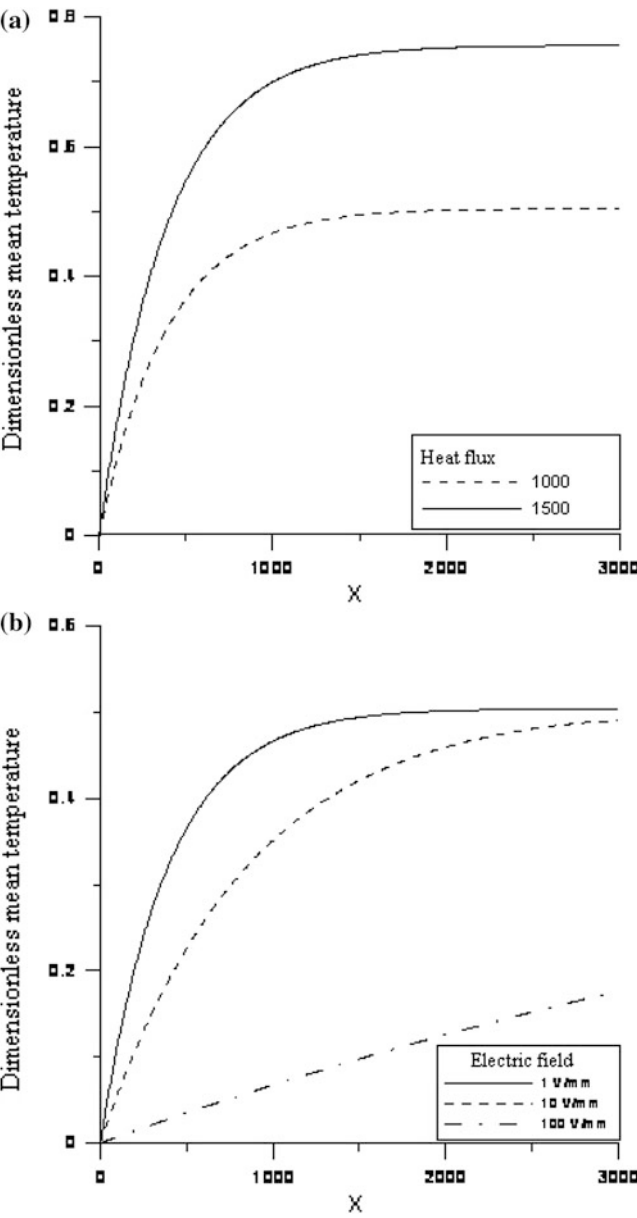


Fig. 2.10 (a) Bulk temperature variation with applied wall heat flux (1 V/mm). (b) Bulk temperature variation with applied electric field (1000 W/m²)

prescribed heat flux is increased or the applied electric field is reduced. Such information can be used so as to select the values that might be used in actual experimental runs, in order to avoid boiling of the fluid inside the channel. Also, Fig. 2.11a, b shows the local Nusselt numbers along both surfaces (silicon and

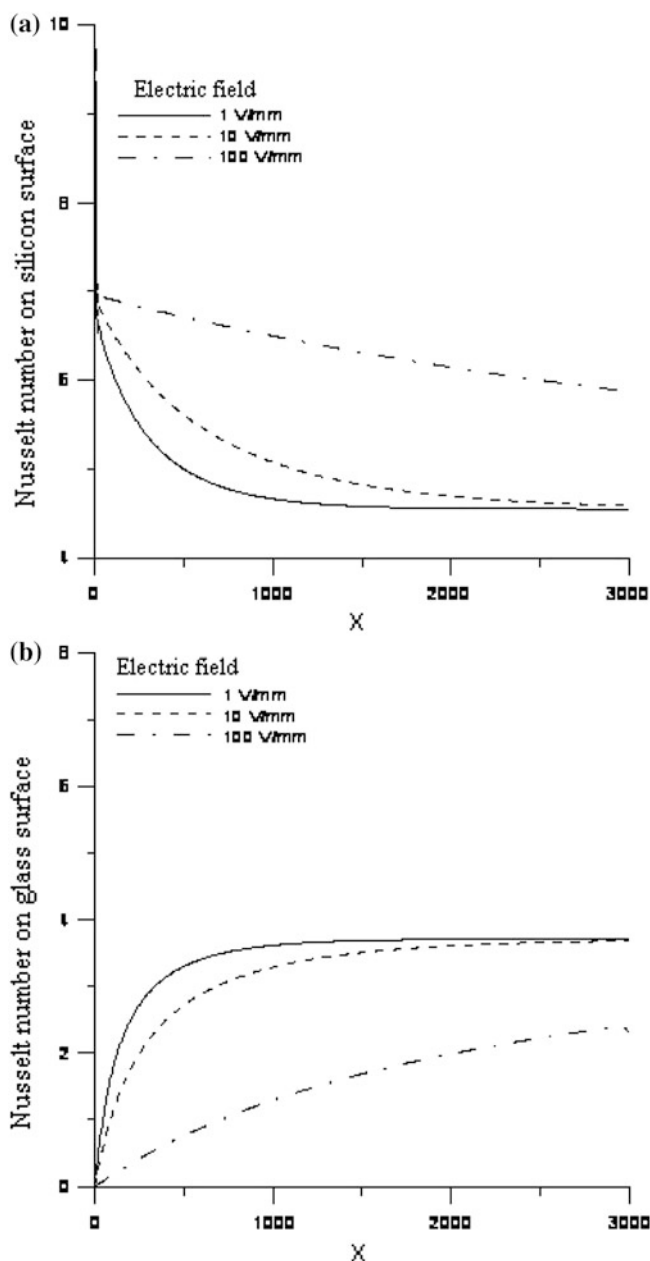


Fig. 2.11 (a) Local Nusselt number on silicon surface for different applied electric fields (b) Local Nusselt number on glass surface for different applied electric fields

glass), for different velocity profiles, as obtained by the variation in electric field, i.e., a greater electric field represents a larger average velocity and thus a larger Nusselt number at the heated surface.

2.4 Inverse Problem Analysis for Convection in Slip Flow

The accuracy of the analytic-based solutions for forced convection in microchannels with slip flow, shown in the previous sections, are dependent on the also accurate determination of the momentum and thermal accommodation coefficients, as required by the slip and temperature jump boundary conditions inherent to the slip flow model, that accounts for non-continuum effects at the fluid–surface interactions. Fundamental experimental work on rarefied gas dynamics has offered measurements of the tangential momentum accommodation coefficient, requiring, for instance, high vacuum and molecular beams impinging on carefully prepared substrates, such as reviewed in Agrawal and Prabhu (2008), but very few results are available for the actual conditions of the flow configuration within microchannels and their bounding walls (Arkilic et al. 2001; Hsieh et al. 2004). The experiments indicate that the tangential momentum accommodation coefficient generally assumes values between 0.2 and 1.0, with the lower limit being associated with exceptionally smooth surfaces and the upper limit with very rough or highly oxidized surfaces (Barber and Emerson 2006). Similar considerations are pertinent to the measurement of thermal accommodation coefficients (Rader et al. 2005), where an even more limited experimental database is available, and apparently no previous work seems to be available on the identification of this coefficient in actual heat and fluid flow conditions within specific pressure and temperature levels pertinent to MEMS applications, and in addition, for actual geometry and finishing of the microchannel walls.

Thus, we take advantage of the accuracy, robustness, and efficiency of the direct problem solution, to tackle the associated inverse heat transfer problem analysis (Naveira-Cotta et al. 2011a, b; Knupp et al. 2012a) toward the simultaneous estimation of momentum and thermal accommodation coefficients in microchannel flows with velocity slip and temperature jump (Naveira-Cotta et al. 2009, 2010a, b). A Bayesian inference approach is adopted in the solution of the identification problem, based on the Markov Chain Monte Carlo method (MCMC) and the Metropolis–Hastings algorithm (Kaipio and Somersalo 2004; Gamerman and Lopes 2006). Only simulated temperature measurements at the external faces of the channel walls, obtained for instance via infrared thermography with microscopic lens, are used in the inverse analysis in order to demonstrate the capabilities of the proposed approach. A sensitivity analysis allows for the inspection of the identification problem behavior with respect to the parameters to be estimated.

The approach here employed in the direct problem solution for forced convection in microchannels is borrowed from a previous work on diffusion in heterogeneous media, with arbitrarily space variable thermophysical properties

(Naveira-Cotta et al. 2009). In this sense, the dimensionless velocity fields are mathematically equivalent to space variable thermal capacitances, and the solution procedure is here briefly described.

For a general purpose automatic implementation, it is quite desirable to employ a flexible computational approach to handle eigenvalue problems with arbitrarily variable coefficients. Thus, the Generalized Integral Transform Technique (GITT) is here employed in the solution of the Sturm–Liouville problem via the proposition of a simpler auxiliary eigenvalue problem, as detailed in Sect. 1.3.4, and expanding the unknown eigenfunctions in terms of the chosen basis (Naveira-Cotta et al. 2011a, b). Also, the variable equation coefficients may themselves be expanded in terms of known eigenfunctions (Naveira-Cotta et al. 2009), so as to allow for a fully analytical implementation of the coefficient matrices in the transformed system.

In order to illustrate the direct–inverse problem analysis, we consider a two-dimensional situation consisting of a parallel-plate microchannel, with steady thermally developing laminar flow of a gas within the slip flow regime. The fluid is assumed to enter the channel with a fully developed velocity profile and a uniform temperature, exchanging heat by convection with the surroundings with an external heat transfer coefficient that is not known a priori in the inverse problem analysis. Thermophysical properties are assumed to be constant, while axial diffusion and viscous dissipation are neglected. Although more involved formulations could be handled by the present approach, the direct problem solution here illustrated suffices to demonstrate the applicability of the proposed direct–inverse problem analysis. In dimensionless form, the energy equation for the fluid within the microchannel, accounting for the symmetry of the heat transfer process, can be written as

$$W(Y) \frac{\partial \theta(Y, Z)}{\partial Z} = \frac{\partial^2 \theta(Y, Z)}{\partial Y^2}, \quad 0 < Y < 1, \quad Z > 0 \quad (2.80)$$

$$\theta(Y, 0) = 1, \quad 0 \leq Y \leq 1, \quad (2.81)$$

$$\frac{\partial \theta(Y, Z)}{\partial Y} \Big|_{Y=0} = 0, \quad \frac{\partial \theta(Y, Z)}{\partial Y} \Big|_{Y=1} = -\text{Bi}_{\text{eff}} \theta(1, Z), \quad Z > 0 \quad (2.82a, b)$$

The function $W(Y)$ is related to the fully developed velocity profile across the microchannel, while the parameter Bi_{eff} includes the temperature jump condition and the external convective heat transfer. Eventually, conjugated effects represented by heat conduction resistance across the wall, if relevant, can also be directly incorporated into this coefficient. The dimensionless groups are defined as

$$\begin{aligned} Y &= \frac{y}{y_1}; \quad Z = \frac{\alpha_f z}{u_{\text{av}} y_1^2}; \quad \theta(Y, Z) = \frac{T(y, z) - T_\infty}{T_0 - T_\infty}; \quad W(Y) = \frac{u(y)}{u_{\text{av}}}; \\ \text{Bi} &= \frac{h_e y_1}{k_f}; \quad \text{Kn} = \frac{\lambda}{2y_1}; \quad \text{Bi}_{\text{eff}} = \frac{\text{Bi}}{1 + 2\text{Kn}\beta_t \text{Bi}} \end{aligned} \quad (2.83)$$

and,

$$\beta_t = \frac{(2 - \alpha_t)}{\alpha_t} \frac{2\gamma}{(\gamma + 1)} \frac{1}{\text{Pr}} \quad (2.84)$$

is the wall temperature jump coefficient and α_t is the thermal accommodation coefficient, λ is the molecular mean free path, $\gamma = c_p/c_v$, while c_p is specific heat at constant pressure, c_v specific heat at constant volume, Pr is the Prandtl number, $2y_I$ is the distance between the plates, u_{av} is the average flow velocity, k_f is the thermal conductivity of the fluid, α_f is the thermal diffusivity of the fluid, T_0 is the inlet flow temperature, T_∞ is the external environment temperature, and h_e is the external heat transfer coefficient. Conjugation effects and viscous dissipation are disregarded in this particular example and the Knudsen number, Kn , is within the slip flow regime bounds. The dimensionless velocity profile is given as

$$W(Y) = \frac{6\text{Kn}\beta_v + 3(1 - Y^2)/2}{1 + 6\text{Kn}\beta_v} \quad (2.85)$$

where

$$\beta_v = \frac{(2 - \alpha_m)}{\alpha_m} \quad (2.86)$$

is the wall velocity slip coefficient and α_m is the tangential momentum accommodation coefficient.

The corresponding temperature jump is accounted for to provide the temperature distribution at the wall:

$$\theta_w(Z) = \frac{1}{1 + 2\text{KnBi}\beta_t} \theta(1, Z) \quad (2.87)$$

The vector of parameters to be estimated in this slip flow example is then given as

$$\mathbf{P}^T = [\beta_v, \beta_t, \text{Bi}_{\text{eff}}] \quad (2.88)$$

The solution of the dimensionless problem (2.80)–(2.82) is then a special case from the general solution given in (Naveira-Cotta et al. 2009), and written as

$$\theta(Y, Z) = \sum_{i=1}^{\infty} \bar{f}_i \tilde{\psi}_i(Y) e^{-\mu_i^2 Z}, \quad \text{with } \bar{f}_i = -\frac{\tilde{\psi}'_i(1)}{\mu_i^2} \quad (2.89)$$

where $\psi_i(Y)$ are eigenfunctions of the following Sturm–Liouville problem:

$$\frac{d^2\psi_i(Y)}{dY^2} + \mu_i^2 W(Y)\psi_i(Y) = 0, \quad 0 < Y < 1 \quad (2.90)$$

$$\frac{d\psi_i(Y)}{dY}\bigg|_{Y=0} = 0, \quad \frac{d\psi_i(Y)}{dY}\bigg|_{Y=1} = -\text{Bi}_{\text{eff}}\psi_i(1) \quad (2.91a, b)$$

with the corresponding normalization integral and normalized form of the eigenfunction

$$N_i = \int_0^1 W(Y)\psi_i^2(Y)dY; \quad \tilde{\psi}_i(Y) = \frac{\psi_i(Y)}{N_i^{1/2}} \quad (2.92a, b)$$

The solution of problem (2.90)–(2.91) was obtained in (Mikhailov and Cotta 2005; Cotta et al. 2004b) in terms of the confluent hypergeometric function, also known as Kummer function, as discussed in Sect. 2.2, readily available in the *Mathematica system* (Wolfram 2008). Here, a more convenient path for the corresponding inverse problem analysis was chosen, in light of the intensive computational task required by the parameters identification algorithm, so as to provide a solution in terms of simpler functions. Thus, following the ideas in Sect. 1.3.4, the solution of problem (7) is provided as an eigenfunction expansion from a simpler eigenvalue problem (Naveira-Cotta et al. 2009), that retains the same boundary conditions of the original problem, but avoids the variable coefficient corresponding to the dimensionless velocity field, in the form:

$$\frac{d^2\Omega_n(Y)}{dY^2} + \lambda_n^2\Omega_n(Y) = 0, \quad 0 < Y < 1 \quad (2.93)$$

$$\frac{d\Omega_n(Y)}{dY}\bigg|_{Y=0} = 0, \quad \frac{d\Omega_n(Y)}{dY}\bigg|_{Y=1} = -\text{Bi}_{\text{eff}}\Omega_n(1) \quad (2.94a, b)$$

which is readily solved as

$$\Omega_n(Y) = \cos(\lambda_n Y), \quad \lambda_n \tan(\lambda_n) = \text{Bi}_{\text{eff}}, \quad n = 1, 2, \dots \quad (2.95a, b)$$

Once the auxiliary eigenfunctions and eigenvalues have been obtained, we may express the desired eigenfunction of the original problem as an expansion of these simpler functions:

$$\psi_i(Y) = \sum_{n=1}^{\infty} \tilde{\Omega}_n(Y)\bar{\psi}_{i,n}, \quad \text{inverse} \quad (2.96)$$

$$\bar{\psi}_{i,n} = \int_0^1 \tilde{\Omega}_n(Y) \psi_i(Y) dY, \quad \text{transform} \quad (2.97)$$

The integral transformation of the original eigenvalue problem is then performed by employing the operator $\int_0^1 \tilde{\Omega}_n(Y)(\cdot) dY$, over Eq. (2.90), which results in the following algebraic eigenvalue problem for the original problem eigenvalues and corresponding eigenvectors:

$$(\mathbf{A} - \mu^2 \mathbf{B}) \bar{\boldsymbol{\psi}} = 0 \quad (2.98)$$

$$\bar{\boldsymbol{\psi}} = \{\bar{\psi}_{n,m}\}; \quad \mathbf{B} = \{B_{n,m}\}, \quad B_{n,m} = \int_0^1 W(Y) \tilde{\Omega}_n(Y) \tilde{\Omega}_m(Y) dY \quad (2.99\text{a-c})$$

$$A_{n,m} = \lambda_n^2 \delta_{n,m}, \quad \text{where } \delta_{n,m} = 1, \text{ for } n = m, \text{ or } \delta_{n,m} = 0, \text{ for } n \neq m \quad (2.100)$$

The algebraic problem (2.98)–(2.100) can be numerically solved to provide results for the eigenvalues μ^2 and eigenvectors $\bar{\boldsymbol{\psi}}$ from this matrix eigenvalue problem analysis (Naveira-Cotta et al. 2009), which will be combined within the inverse formula (2.96) to provide the desired eigenfunctions of the original eigenvalue problem.

The fluid temperature at the interface with the wall is then directly computed from the inverse formula as

$$\theta(1, Z) = \sum_{i=1}^{\infty} \tilde{f}_i \tilde{\psi}_i(1) e^{-\mu_i^2 Z} \quad (2.101)$$

A Bayesian estimator (Kaipio and Somersalo 2004) is basically concerned with the analysis of the *posterior probability density*, which is the conditional probability of the parameters given the measurements, while the likelihood is the conditional probability of the measurements given the parameters. If we assume the parameters and the measurement errors to be independent Gaussian random variables, with known means and covariance matrices, and that the measurement errors are additive, a closed form expression can be derived for the posterior probability density. In this case, the estimator that maximizes the posterior probability density can be recast in the form of a minimization problem involving the *maximum a posteriori objective function* (MAP). On the other hand, if different prior probability densities are assumed for the parameters, the posterior probability distribution may not allow for an analytical treatment. In this case, Markov Chain Monte Carlo (MCMC) methods are used to draw samples of all possible parameters, so that inference on the posterior probability becomes inference on the samples. This section makes use of Bayesian techniques for the estimation of parameters in microscale forced

convection problems, via the MCMC method (Kaipio and Somersalo 2004; Gamerman and Lopes 2006), as applied to the simultaneous identification of the momentum and thermal accommodation coefficients in slip flow modeling. The Metropolis–Hastings algorithm is employed for the sampling procedure (Gamerman and Lopes 2006), implemented in the *Mathematica* platform (Wolfram 2008).

Consider the vector of parameters appearing in the physical model formulation as

$$\mathbf{P}^T \equiv [P_1, P_2, \dots, P_{N_p}] \quad (2.102)$$

where N_p is the number of parameters. For the solution of the inverse problem of estimating $\mathbf{P}$, we assume available the measured temperature data given as

$$(\mathbf{Y} - \mathbf{T}) = (\vec{Y}_1 - \vec{T}_1, \vec{Y}_2 - \vec{T}_2, \dots, \vec{Y}_{N_x} - \vec{T}_{N_x}) \quad (2.103)$$

where $\vec{Y}_i$ contains the measured temperatures for each of the N_x sensors at time t_i , $i = 1, \dots, N_t$, that is,

$$(\vec{Y}_i - \vec{T}_i) = (Y_{i1} - T_{i1}, Y_{i2} - T_{i2}, \dots, Y_{iN_x} - T_{iN_x}) \quad \text{for } i = 1, \dots, N_t \quad (2.104)$$

so that we have $N_m = N_x N_t$ measurements in total. In the present steady-state estimation procedure, the sensors are assumed to be distributed along the channel wall length and the measurements are taken as an average in a time interval within the steady period ($N_t = 1$).

Bayes' theorem can then be stated as (Kaipio and Somersalo 2004; Gamerman and Lopes 2006):

$$p_{\text{posterior}}(\mathbf{P}) = p(\mathbf{P}|\mathbf{Y}) = \frac{p(\mathbf{P})p(\mathbf{Y}|\mathbf{P})}{p(\mathbf{Y})} \quad (2.105)$$

where $p_{\text{posterior}}(\mathbf{P})$ is the posterior probability density, that is, the conditional probability of the parameters $\mathbf{P}$ given the measurements $\mathbf{Y}$; $p(\mathbf{P})$ is the prior density, that is, a statistical model for the information about the unknown parameters prior to the measurements; $p(\mathbf{Y}|\mathbf{P})$ is the likelihood function, which gives the relative probability density of different measurement outcomes $\mathbf{Y}$ with a fixed $\mathbf{P}$, and $p(\mathbf{Y})$ is the marginal probability density of the measurements, which plays the role of a normalizing constant.

In this work we assume that the measurement errors are Gaussian random variables, with known (modeled) means and covariances, and that the measurement errors are additive and independent of the unknowns. With these hypotheses, the likelihood function can be expressed as (Kaipio and Somersalo 2004; Gamerman and Lopes 2006):

$$p(\mathbf{Y}|\mathbf{P}) = (2\pi)^{-M/2} |\mathbf{W}|^{-1/2} \exp \left\{ -\frac{1}{2} [\mathbf{Y} - \mathbf{T}(\mathbf{P})]^T \mathbf{W}^{-1} [\mathbf{Y} - \mathbf{T}(\mathbf{P})] \right\} \quad (2.106)$$

where $\mathbf{W}$ is the covariance matrix of the measurement errors.

The inference based on simulation techniques uses samples to extract information about the posterior distribution $p(\mathbf{P}|\mathbf{Y})$. As a sample is always a partial substitute of the information contained in a density, simulation-based methods are inherently approximate and should only be used when it is impossible to include the extraction of analytical information from the posteriori, as is the case in the present study. Unfortunately, for most problems of practical relevance it is complicated to generate the posteriori $p(\mathbf{P}|\mathbf{Y})$. Therefore, more sophisticated methods are required to obtain a sample of $p(\mathbf{P}|\mathbf{Y})$, for example, the simulation technique based on Markov chains (Gamerman and Lopes 2006). The numerical method most used to explore the space of states of the posteriori is the Monte Carlo approach. The Monte Carlo simulation is based on a large number of samples of the probability density function (in this case, the function of the posterior probability density $p(\mathbf{P}|\mathbf{Y})$). Several sampling strategies are proposed in the literature, including the Monte Carlo method with Markov Chain (MCMC), adopted in this work, where the basic idea is to simulate a “random walk” in the space of $p(\mathbf{P}|\mathbf{Y})$ that converges to a stationary distribution, which is the distribution of interest in the problem.

A Markov chain is a stochastic process $\{\mathbf{P}_0, \mathbf{P}_1, \dots\}$ such that the distribution of $\mathbf{P}_i$, given all previous values $\mathbf{P}_0, \dots, \mathbf{P}_{i-1}$, depends only on $\mathbf{P}_{i-1}$. That is, it interprets the fact that for a process satisfying the Markov property of Eq. (2.105), given the present, the past is irrelevant to predict its position in a future instant (Gamerman and Lopes 2006):

$$p(\mathbf{P}_i \in A | \mathbf{P}_0, \dots, \mathbf{P}_{i-1}) = p(\mathbf{P}_i \in A | \mathbf{P}_{i-1}) \quad (2.107)$$

The most commonly used algorithm in the Monte Carlo method with Markov Chain is the Metropolis–Hastings, here employed (Kaipio and Somersalo 2004; Gamerman and Lopes 2006). The Metropolis–Hastings algorithm uses the same idea of the rejection methods, i.e., a value is generated from an auxiliary distribution and accepted with a given probability. This correction mechanism ensures the convergence of the chain for the equilibrium distribution. That is, the algorithm now includes an additional step, where the transition mechanism depends on a proposal for a transition and a stage of assessing the equilibrium density, but this is represented by the global transition via the probability of acceptance.

The Metropolis–Hastings algorithm uses an auxiliary probability density function, $q(\mathbf{P}^*|\mathbf{P})$, from which it is easy to obtain sample values. Assuming that the chain is in a state $\mathbf{P}$, a new candidate value, $\mathbf{P}^*$, is generated from the auxiliary distribution $q(\mathbf{P}^*|\mathbf{P})$, given the current state of the chain $\mathbf{P}$. The new value $\mathbf{P}^*$ is accepted with probability given as Eq. (2.106) below, where the ratio that appears in this equation was called by Hastings (Gamerman and Lopes 2006) the ratio test, today called the ratio of Hastings “RH”:

$$\text{RH}(\mathbf{P}, \mathbf{P}^*) = \min \left[1, \frac{p(\mathbf{P}^*|\mathbf{Y})q(\mathbf{P}^*|\mathbf{P})}{p(\mathbf{P}|\mathbf{Y})q(\mathbf{P}|\mathbf{P}^*)} \right] \quad (2.108)$$

where $p(\mathbf{P}|\mathbf{Y})$ is the *a posteriori* distribution of interest. An important observation is that we only need to know $p(\mathbf{P}|\mathbf{Y})$ up to a constant, since we are working with ratios between densities and such normalization constant is canceled.

In practical terms, this means that the simulation of a sample of $p(\mathbf{P}|\mathbf{Y})$ using the Metropolis–Hastings algorithm can be outlined as follows (Gamerman and Lopes 2006):

1. Boot up the iterations counter of the chain $i = 0$ and assign an initial value $\mathbf{P}^{(0)}$.
2. Generate a candidate value $\mathbf{P}^*$ of the distribution $q(\mathbf{P}^*|\mathbf{P})$.
3. Calculate the probability of acceptance of the candidate value $\text{RH}(\mathbf{P}, \mathbf{P}^*)$ by Eq. (2.106).
4. Generate a random number u with uniform distribution, i.e., $u \sim U(0, 1)$,
5. If $u \leq \text{RH}$ then the new value is accepted and we let $\mathbf{P}^{(i+1)} = \mathbf{P}^*$. Otherwise, the new value is rejected and we let $\mathbf{P}^{(i+1)} = \mathbf{P}^{(i)}$.
6. Increase the counter i to $i + 1$ and return to step 2.

The transition core $q(\mathbf{P}^*|\mathbf{P})$ defines only a proposal for a movement that can be confirmed by $\text{RH}(\mathbf{P}, \mathbf{P}^*)$. For this reason it is usually called the proposal or density distribution. The success of the method depends on not so low acceptance rates and proposals that are easy to simulate. The method replaces a difficulty in generating $p(\mathbf{P}|\mathbf{Y})$ by several generations of the proposal $q(\mathbf{P}^*|\mathbf{P})$. In this study we have chosen to adopt symmetrical chains, i.e., $q(\mathbf{P}^*|\mathbf{P}) = q(\mathbf{P}|\mathbf{P}^*)$ for all $(\mathbf{P}^*, \mathbf{P})$. In this case, Eq. (2.109) reduces to the ratio of the posterior densities calculated at the previous and proposed chain positions, and does not depend on $q(\mathbf{P}^*|\mathbf{P})$.

Before addressing the estimation of the unknown parameters, it is desirable to perform a sensitivity analysis, when the behavior of the determinant of the matrix $\mathbf{J}^T \mathbf{J}$ (Ozisik and Orlande 2000) needs to be analyzed in order to inspect the influence of the number of parameters to be estimated in the solution of the inverse problem. The sensitivity matrix $\mathbf{J}$ is defined as

$$\mathbf{J}(\mathbf{P}) = \left[\frac{\partial \mathbf{T}^T(\mathbf{P})}{\partial \mathbf{P}} \right]^T = \begin{bmatrix} \frac{\partial T_1}{\partial P_1} & \frac{\partial T_1}{\partial P_2} & \frac{\partial T_1}{\partial P_3} & \cdots & \frac{\partial T_1}{\partial P_{N_p}} \\ \frac{\partial T_2}{\partial P_1} & \frac{\partial T_2}{\partial P_2} & \frac{\partial T_2}{\partial P_3} & \cdots & \frac{\partial T_2}{\partial P_{N_p}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial T_{N_m}}{\partial P_1} & \frac{\partial T_{N_m}}{\partial P_2} & \frac{\partial T_{N_m}}{\partial P_3} & \cdots & \frac{\partial T_{N_m}}{\partial P_{N_p}} \end{bmatrix} \quad (2.109)$$

Where N_m is the total number of measurements and N_p is total number of unknown parameters. The sensitivity coefficients $J_{ij} = \frac{\partial T_i}{\partial P_j}$ give the sensitivity of T_i (solution of the direct problem) with respect to changes in the parameter P_j . A small value of the magnitude of J_{ij} indicates that large changes in P_j yield small changes in T_i . It can be easily noticed that the estimation of the parameter P_j is extremely

difficult in such cases, because basically the same value for T_i would be obtained for a wide range of values of P_j . In fact, when the sensitivity coefficients are small, $|\mathbf{J}^T \mathbf{J}| \approx 0$ and the inverse problem is said to be ill-conditioned. It can also be shown that $|\mathbf{J}^T \mathbf{J}|$ is null if any column of $\mathbf{J}$ can be expressed as a linear combination of other columns (Ozisik and Orlande 2000). Therefore, it is desirable to have linearly independent sensitivity coefficients J_{ij} with large magnitudes, so that the parameter estimation problem is not very sensitive to measurement errors and accurate estimates of the parameters can be obtained. The comparison of the magnitude of the sensitivity coefficients, as well as the analysis of possible linear dependence, is more easily performed using the reduced sensitivity coefficients instead of the original ones. The reduced sensitivity coefficients are obtained by multiplying the original sensitivity coefficients, J_{ij} , by the parameters that they refer to. Therefore, they have units of the measured variables, which is used as a basis of comparison.

The direct problem solution was first validated by direct comparison with the benchmark results provided in Mikhailov and Cotta (2005), as illustrated in Cotta et al. (2009b), for the case of a parallel-plates under prescribed uniform wall temperature ($\text{Bi} \rightarrow \infty$). The constructed algorithm for the inverse analysis was then also validated for this same benchmark problem with $\text{Bi} \rightarrow \infty$, from a theoretical perspective, assuming the fluid temperature at the wall to be measurable. Then, the theory presented in Naveira-Cotta et al. (2010a, b), takes advantage of the accuracy, robustness, and efficiency of the direct problem solution of Sects. 1.3.1 and 1.3.4, to tackle the associated inverse heat transfer problem, toward the simultaneous estimation of momentum and thermal accommodation coefficients in microchannel flows with velocity slip and temperature jump. Here this approach is employed in a more challenging example, so as to explore the limits of the identification procedure just summarized.

Only wall temperature measurements, obtained from simulated data, are employed in this section, which may be obtained for instance through infrared thermography. A total of 1000 uniformly distributed points along the dimensionless channel length, $Z_f = 5$, are initially adopted. The simulated measurements are considered normally distributed with averages at the simulated values and 1 % standard deviation. They were obtained with 50 terms in the eigenfunction expansions, while the direct problem solution within the inverse problem procedure was handled with 20 terms only, in order to avoid the inverse crime. The test case data obtained from a hypothetical nitrogen flow within a silicon substrate is summarized in Table 2.4.

Table 2.4 Summary of the pertinent data

Parameter	Value
Kn	0.025
β_v	1.5
β_t	3.88
Bi_{eff}	0.8

The simulated wall temperature measurements (symbols) are plotted with the exact solution (solid line) in Fig. 2.12, where the representative experimental error adopted can be clearly noticed.

Figure 2.13 illustrates the reduced sensitivity coefficients for each of the three parameters β_v (dotted line), β_t (dashed line), Bi_{eff} (dot-dashed line). This figure indicates that the estimation of the effective Biot number and β_t , should not pose

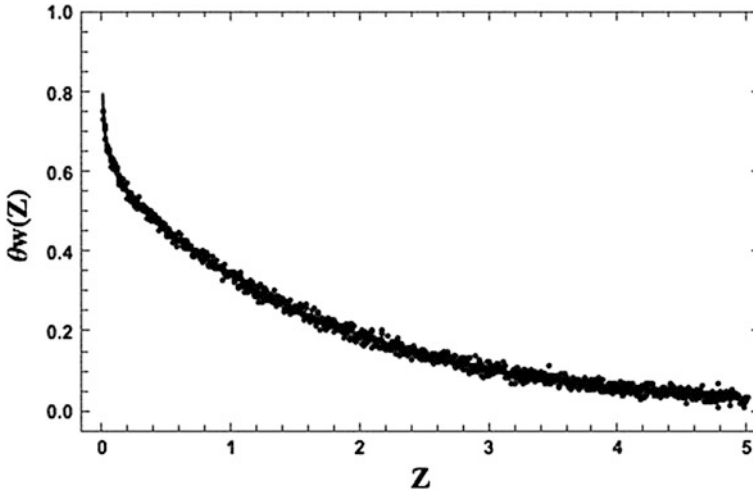


Fig. 2.12 Simulated wall temperature measurements with 1 % standard deviation

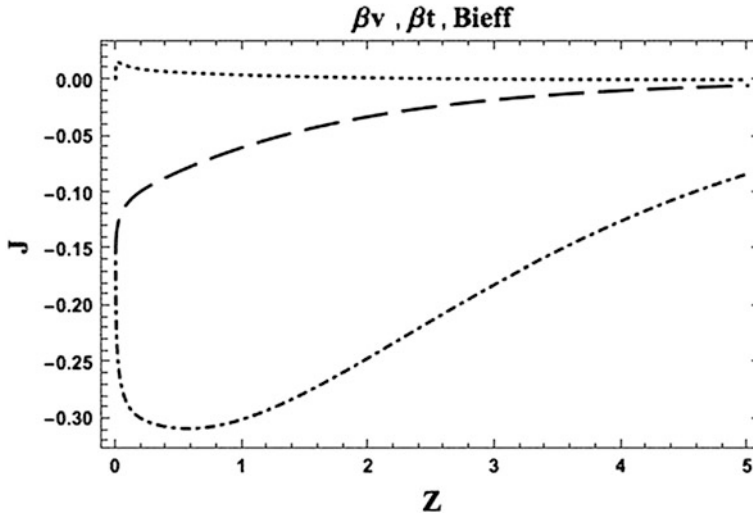


Fig. 2.13 Comparison of reduced sensitivity coefficients along the channel for the three parameters to be estimated

difficulties, since their sensitivity coefficients are large and linearly independent with respect to the others. On the other hand, the sensitivity coefficients with respect to the slip boundary condition coefficient, β_v , are much smaller, and some difficulties in its estimation can be anticipated, unless an informative prior is provided. In fact, an initial attempt of estimating the three parameters was performed by providing non-informative uniform probability distribution functions for the effective Biot number, a poor informative prior for β_t through a truncated normal distribution probability with very large standard deviation of 50 %, and an informative truncated normal prior for β_v with a standard deviation of 35 %, within the admissible minimum and maximum interval values for each parameter. It was then observed, especially for the parameter β_v , that a more informative prior would be required for achieving a fast convergence in the estimation procedure and a narrow interval of the estimated distribution. Fortunately, *a priori* information for both β_v and Bi can in principle be obtained for most experimental conditions, by utilizing pressure and mass flow rate measurements to approximate the slip coefficient and by employing classical correlations for estimating the external heat transfer coefficient, respectively. We have then proceeded to the analysis of the inverse problem by providing more informative probability distributions as prior for β_v , while maintaining the same probability distribution as prior for the other two parameter Bi_{eff} and β_t . The Gaussian priors for β_v and β_t were initially assumed with means at the exact values.

Figure 2.14a–c illustrates the evolution of the Markov chains (up to 50,000 states) for the estimation of the three parameters, β_v , β_t , and Bi_{eff} , respectively, involving a Gaussian prior for β_v with 20 % standard deviation. Also shown are dotted lines for the expected exact value and dashed lines for the admissible minimum and maximum limits for each parameter. One may clearly observe that the Markov chain for the parameter Bi_{eff} has a distinguished behavior of a very fast convergence, in comparison to the other two parameters, requiring less than 1,000 states. The slip boundary condition coefficient, β_v , appears to be the worst one to estimate in the present situation, as also indicated by the respective sensitivity analysis (Fig. 2.13).

Another form of analyzing these results can be provided by both sets of curves in Figs. 2.15a–c and 2.16a–c. In Fig. 2.15a–c it is shown the prior probability distribution functions, which can then be compared to the subsequent sampling frequency histograms of the posterior, neglecting the first 10,000 states in each chain, presented in Fig. 2.16a–c. One may clearly observe that the identification algorithm markedly narrows the *a posteriori* distributions, with respect to all three priors distributions, within the domain defined by the minimum and maximum values of the admissible intervals. For the effective Biot number, as shown in Fig. 2.16c, the posterior distribution is shown in the superposed zoomed image.

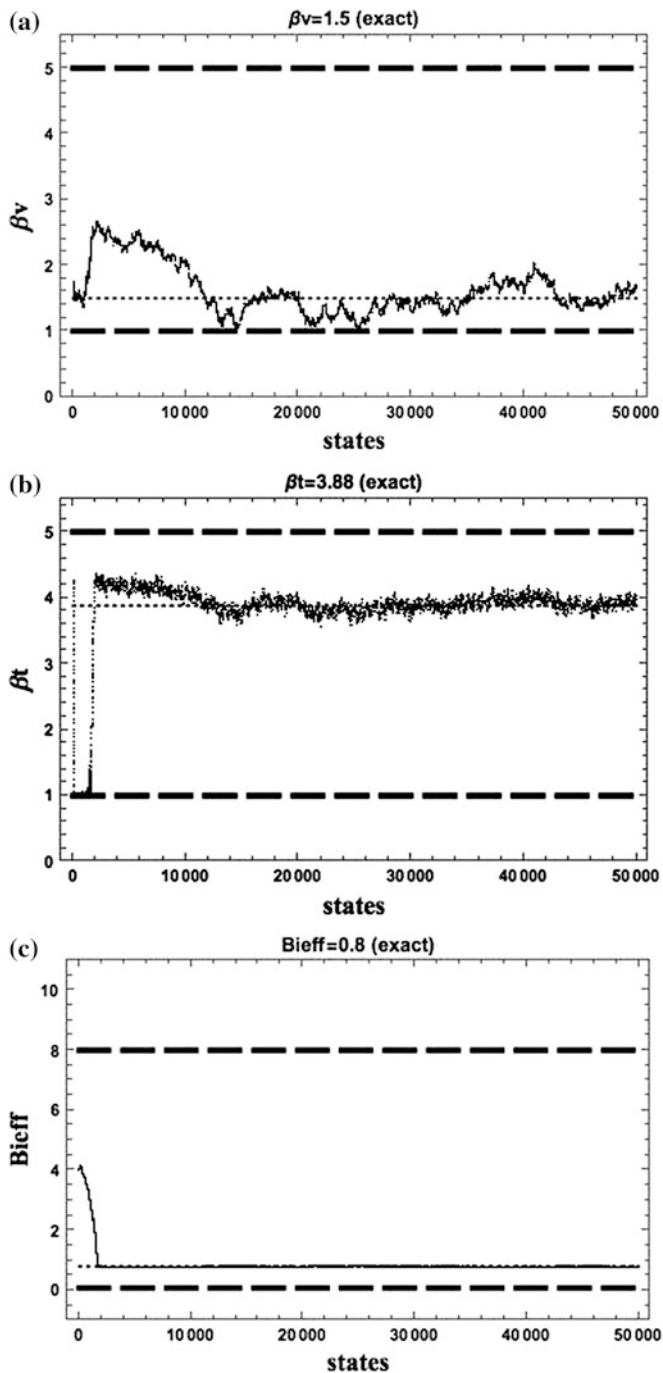


Fig. 2.14 Markov chains evolution for each of the three parameters in the slip flow test case, for truncated normal distribution with standard deviation of 20 % as prior for β_v : (a) β_v , (b) β_t , (c) Bi_{eff}

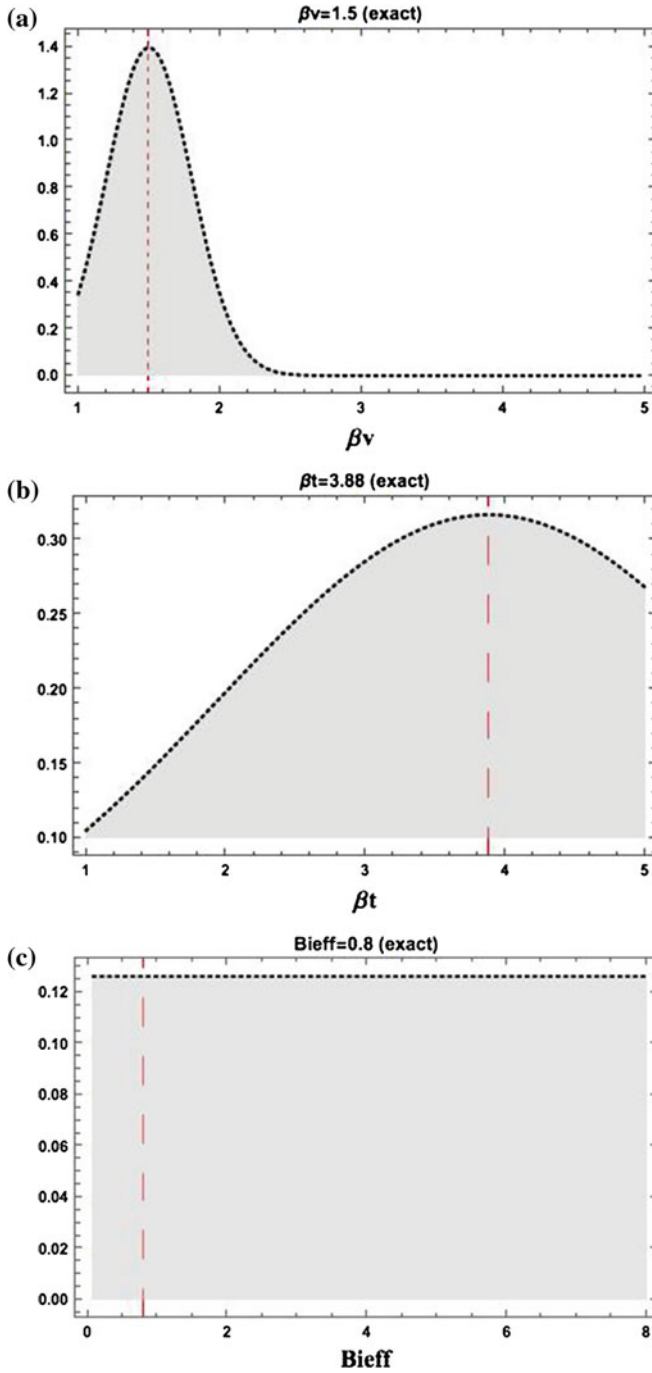


Fig. 2.15 Prior information for each of the three parameters: **(a)** truncated normal distribution with standard deviation of 20 % for β_v ; **(b)** truncated normal distribution with standard deviation of 50 % for β_t ; **(c)** uniform distribution for Bi_{eff}

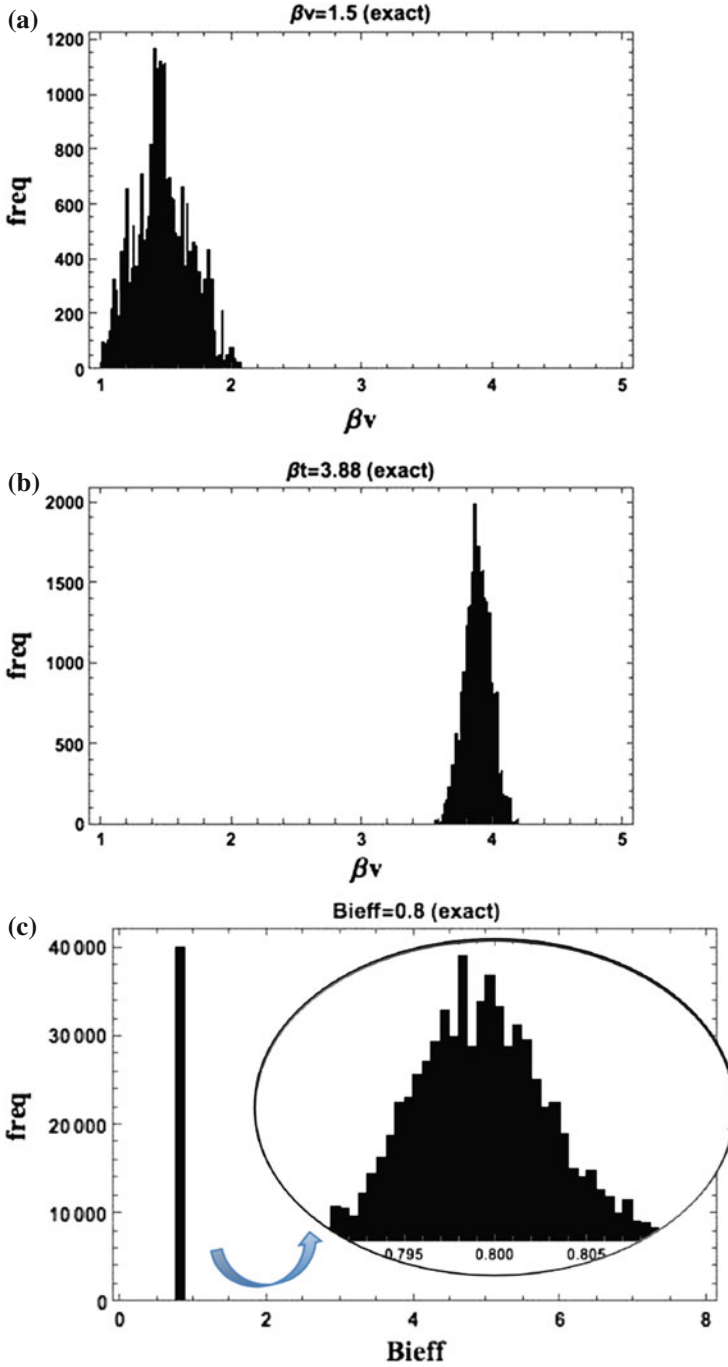


Fig. 2.16 Posterior information for each of the three parameters: (a) for β_v ; (b) for β_t ; (c) for Bi_{eff}

Table 2.5 summarizes the input data and illustrates not only the estimated values, after neglecting the first 10,000 states in each chain (total of 50,000 states), but also the minimum and maximum values of the 95 % confidence intervals for such estimated parameters. For all three parameters, including the least sensitive parameter, β_v , the exact value lies within the confidence intervals, though it presents the widest interval among the three estimated parameters.

Figure 2.17 provides a direct comparison of the velocity distribution across the channel, as computed from the exact (solid line) and estimated (long dashed line) values of the slip coefficient, and also the velocity profiles as obtained from the lower (dotted line) and upper (dashed-dotted line) bounds of the 95 % confidence interval. In this case some slight changes in the velocity distribution are observable to the graph scale. As expected from the reasonably accurate estimation of the slip coefficient shown in Table 2.5, the two curves (exact and estimated) are essentially coincident.

Table 2.5 Estimated parameters values with 50,000 states in Markov chains (first 10,000 states for the chains burn-in) and the corresponding 95 % confidence intervals, for truncated normal distribution with standard deviation of 20 % as prior for β_v

P	Exact	Search interval [min, max]	Initial guess	Estimated	Confidence interval [min, max]
β_v	1.5	[1.0, 5.0]	3	1.4731	[1.1014, 1.8649]
β_l	3.88	[1.0, 5.0]	3	3.8908	[3.6886, 4.0770]
Bi_{eff}	0.8	[0.008, 8.0]	4.04	0.7992	[0.7923, 0.8068]

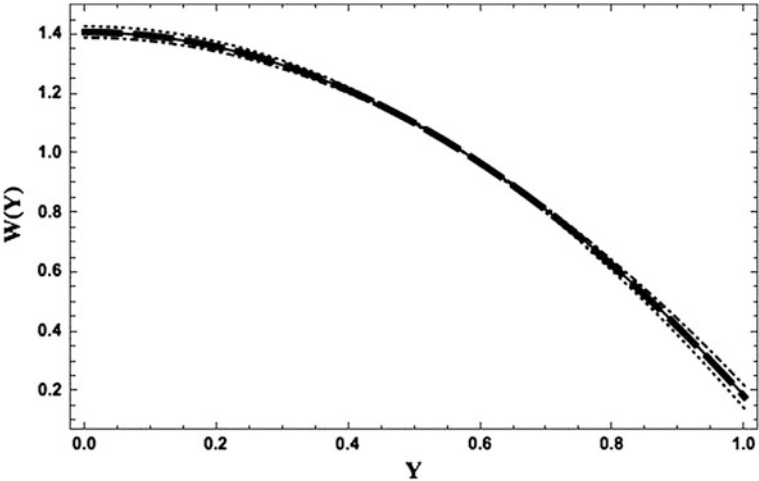


Fig. 2.17 Reconstructed velocity with estimated parameters (*long dashes*), with lower (*dotted*), upper (*dashed-dotted*) bounds of confidence interval, and exact (*solid*)

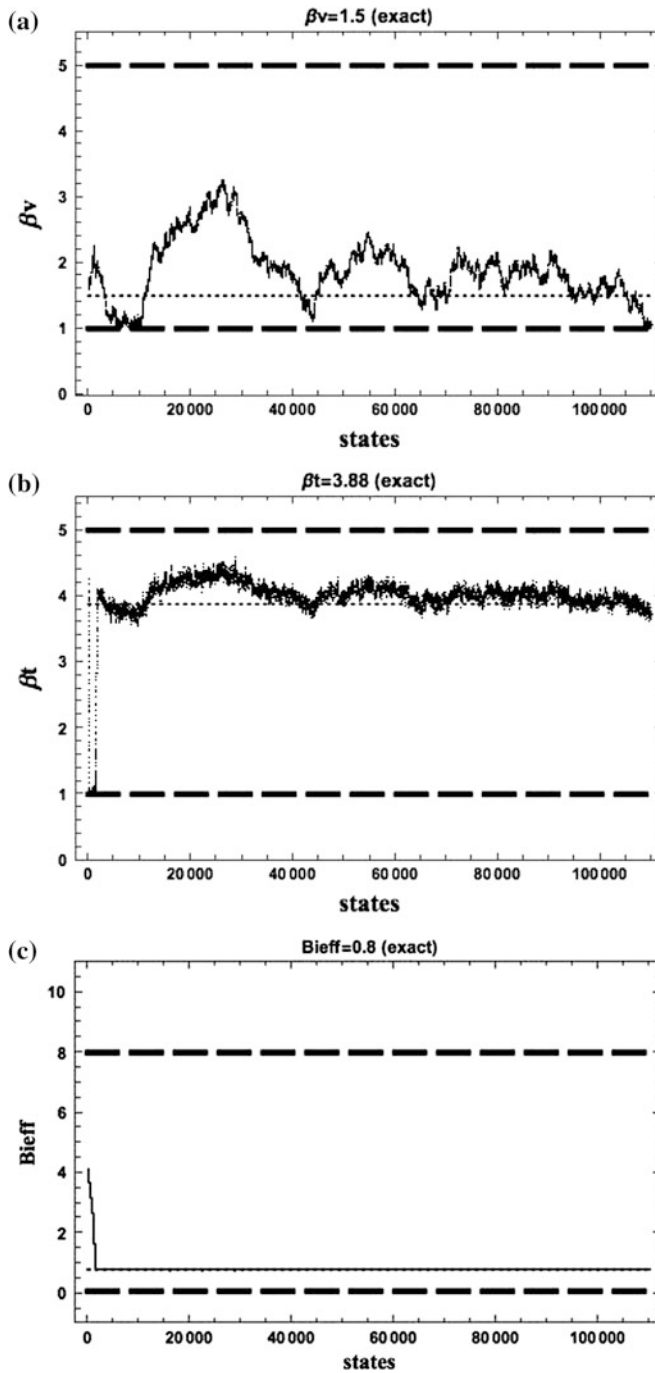


Fig. 2.18 Markov chains evolution for each of the three parameters in the slip flow test case, for truncated normal distribution with standard deviation of 35 % as prior for β_v : (a) β_v , (b) β_t , (c) Bi_{eff}

Table 2.6 Estimated parameters values with 110,000 states in Markov chains (neglecting first 40,000 states for the chains burn-in) and the corresponding 95 % confidence intervals for 35 % standard deviation in the priors distributions

P	Exact	Search interval [min, max]	Initial guess	Estimated	Confidence interval [min, max]
β_v	1.5	[1.0, 5.0]	1.65	1.7618	[1.1633, 2.2525]
β_t	3.88	[1.0, 5.0]	4.628	3.9915	[3.7628, 4.2058]
Bi_{eff}	0.8	[0.008, 8.0]	4.04	0.7960	[0.7880, 0.8042]

Then, we illustrate the effect of increasing the standard deviation on the average values informed as prior distribution for the coefficient β_v , now markedly increased to 35 %. From Fig. 2.18a–c for the three Markov chains, one may conclude that the β_v Markov chain convergence is the one most noticeably affected, in comparison to the previously presented case, where a narrower standard deviation of the priors distribution for the coefficient β_v have been employed. Clearly, an increased number of states on the β_v Markov chain (up to 110,000) is needed to achieve convergence, after the first 40,000 states (burn-in period), while the Markov chains for the other two parameters do not present a marked variation with respect to those presented in Fig. 2.14a–c. Also shown are dotted lines for the expected exact value and dashed lines for the admissible limits for each parameter.

Table 2.6 summarizes the input data and illustrates not only the estimated values, after neglecting the first 40,000 states in each chain, but also the minimum and maximum values of the 95 % confidence intervals for such estimated parameters. As observed from Table 2.6, the less informative prior similarly affects the width of the confidence interval for the parameter β_v , though the estimated value remains fairly reasonable. The estimation of the other two parameters is less affected.

2.5 Microchannels with Corrugated Walls

Microchannels with corrugated or rough walls along their length are fairly common, either as a result of inherent fabrication irregularities or as a strategy for heat or mass transfer enhancement in different contexts and applications. Simulation of heat and fluid flow in such configurations in general requires costly purely discrete solutions of the governing mass, momentum, and energy balances for irregular domains. The present section is aimed at illustrating how the Generalized Integral Transform Technique (GITT) can be applied in the hybrid numerical–analytical solution of the Navier–Stokes equations for flow within corrugated channels (Silva et al. 2004, 2005, 2007, 2011), as well as in the solution of the energy equation for both steady and transient regimen through an approximate analysis of the flow problem and considering axial diffusion and preheating/cooling of microchannels (Castellões and Cotta 2008; Castellões et al. 2010).

2.5.1 Flow Analysis via Full Navier–Stokes Equations

Laminar flow within a microchannel with longitudinally corrugated walls is here analyzed (Silva et al. 2004, 2005, 2007, 2011). The arbitrarily irregular boundaries can be originated by inherent irregularities in the microfabrication process of a straight channel or produced on purpose for heat and mass transfer enhancement goals. Figure 2.19 provides a general schematic drawing for the channel flow problem to be studied. Laminar incompressible flow of a Newtonian fluid is considered, for a two-dimensional geometry and steady state, governed by the continuity and Navier–Stokes equations, written in the primitive variables formulation and in dimensionless form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.110)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{\text{Re}} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (2.111)$$

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{\text{Re}} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (2.112)$$

In order to achieve automatic satisfaction of the continuity equation and eliminate the pressure field, the streamfunction only formulation is here preferred, as in previous solutions of the Navier–Stokes equations through GITT, for instance (Perez-Guerrero and Cotta 1992, 1995; Perez-Guerrero et al. 2000). The definition of streamfunction is employed according to:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \quad (2.113a, b)$$

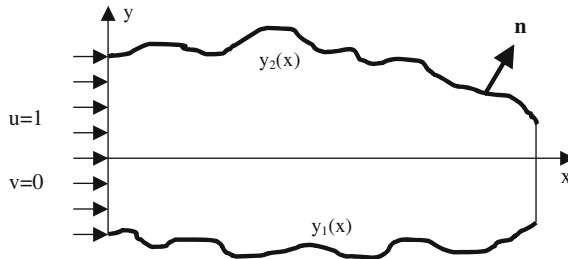


Fig. 2.19 Schematic representation of the fluid flow problem in a two-dimensional microchannel with longitudinally corrugated walls

The dimensionless groups employed in Eqs. (2.110)–(2.112) are given as

$$\begin{aligned} x &= x^*/b; \quad y = y^*/b; \quad y_1(x) = y_1^*(x^*)/b; \quad y_2(x) = y_2^*(x^*)/b; \\ u &= u^*/u_0; \quad v = v^*/u_0; \quad p = p^*/\rho u_0^2; \quad \text{Re} = bu_0/\nu \end{aligned} \quad (2.114)$$

where b represents half the distance between the walls at the duct inlet.

The flow equations are then rewritten in the streamfunction only formulation and becomes:

$$\frac{\partial \psi}{\partial y} \left(\frac{\partial^3 \psi}{\partial x^3} + \frac{\partial^3 \psi}{\partial x \partial y^2} \right) - \frac{\partial \psi}{\partial x} \left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3} \right) = \frac{1}{\text{Re}} \nabla^4 \psi \quad (2.115)$$

where $\psi(x, y)$ is the streamfunction which obeys the following no-slip and impermeable walls boundary conditions in the transversal coordinate y :

$$\psi(x, -y_1(x)) = k_1; \quad \frac{\partial \psi(x, -y_1(x))}{\partial \mathbf{n}} = 0; \quad \psi(x, y_2(x)) = k_2; \quad \frac{\partial \psi(x, y_2(x))}{\partial \mathbf{n}} = 0 \quad (2.116\text{a–d})$$

where in the application that follows a symmetric boundary shape will be considered, given as $y_1(x) = y_2(x) = 1 + f(x)$, with $f(x)$ being the function that represents the corrugation at the walls, while $\mathbf{n}$, k_1 , and k_2 are the normal vector to the wall surface, and the values of the streamfunction at the walls, respectively.

The boundary conditions at the channel entrance are given as

$$\psi(0, y) = k_1; \quad \left. \frac{\partial \psi(x, y)}{\partial x} \right|_{x=0} = 0 \quad (2.117\text{a, b})$$

Considering a fully developed flow condition at the channel exit, for a straight parallel-plate channel geometry, the exit boundary conditions are given as

$$\psi(x_\infty, y) = k_1 + \int_{-y_1}^y u_\infty(y') dy'; \quad \left. \frac{\partial \psi(x, y)}{\partial x} \right|_{x=x_\infty} = 0 \quad (2.118\text{a, b})$$

where the dimensionless fully developed velocity profile is written as

$$u_\infty(y) = \frac{3}{4} Q (1 - y^2) \quad (2.119)$$

where Q is given as

$$\psi(0, y_2(x)) = k_2 = Q + k_1 \quad (2.120)$$

For improved computational performance, it is convenient to define a filter to homogenize the boundary conditions in the transversal coordinate, y . A general analytical filter may be considered, following (Silva et al. 2004, 2011), as

$$\psi(x, y) = \phi(x, y) + F(x, y) \quad (2.121)$$

where $\phi(x, y)$ represents the unknown potential to be determined, while $F(x, y)$ is the proposed filter, which recovers the value of the streamfunction, $\psi(x, y)$, at the wall. Substituting Eq. (2.121) into Eq. (2.115) yields:

$$\begin{aligned} & \frac{\partial \phi}{\partial y} \left(\frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial x \partial y^2} \right) - \frac{\partial \phi}{\partial x} \left(\frac{\partial^3 \phi}{\partial x^2 \partial y} + \frac{\partial^3 \phi}{\partial y^3} \right) + \frac{\partial \phi}{\partial y} \left(\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial x \partial y^2} \right) - \frac{\partial \phi}{\partial x} \left(\frac{\partial^3 F}{\partial x^2 \partial y} + \frac{\partial^3 F}{\partial y^3} \right) \\ & + \frac{\partial F}{\partial y} \left(\frac{\partial^3 \phi}{\partial x^3} + \frac{\partial^3 \phi}{\partial x \partial y^2} \right) - \frac{\partial F}{\partial x} \left(\frac{\partial^3 \phi}{\partial x^2 \partial y} + \frac{\partial^3 \phi}{\partial y^3} \right) + \frac{\partial F}{\partial y} \left(\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial x \partial y^2} \right) - \frac{\partial F}{\partial x} \left(\frac{\partial^3 F}{\partial x^2 \partial y} + \frac{\partial^3 F}{\partial y^3} \right) \\ & = \frac{1}{\text{Re}} \left[\left(\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} \right) + \left(\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} \right) \right] \end{aligned} \quad (2.122)$$

and for the boundary conditions

$$\phi(x, -y_1(x)) = k_1 - F(x, -y_1(x)); \quad \frac{\partial \phi(x, -y_1(x))}{\partial \mathbf{n}} = 0 \quad (2.123a, b)$$

$$\phi(x, y_2(x)) = k_2 - F(x, y_2(x)); \quad \frac{\partial \phi(x, y_2(x))}{\partial \mathbf{n}} = 0 \quad (2.124a, b)$$

An analytical filter $F(x, y)$ can be readily constructed by considering that at each longitudinal position along the corrugated channel, a fully developed velocity profile, $u_{\infty}(y; x)$, is attained, thus completely neglecting the inertia terms at each cross section. This analytical solution offers both an approximate expression for the velocity field at very low Reynolds number and a straightforward filtering solution that shall be employed in the following development. For convenience in the filter definition, a coordinate transformation is introduced, given as

$$\eta = y - y_3(x); \quad y_3(x) = \frac{1}{2} [y_2(x) - y_1(x)]; \quad y_0(x) = \frac{1}{2} [y_2(x) + y_1(x)] \quad (2.125a-c)$$

Then, the filtering solution based on the fully developed flow solution is written as

$$F(\eta, x) = \frac{3}{4} Q \left[\left(\frac{\eta}{y_0} \right) - \frac{1}{3} \left(\frac{\eta}{y_0} \right)^3 \right] + \frac{Q}{2} + k_1 \quad (2.126)$$

Defining a normalized space variable, $\xi = \eta/y_0$, within the domain $[-1, 1]$, one may rewrite Eq. (2.126) as

$$F(\xi) = \frac{3}{4}Q \left[\xi - \frac{1}{3}\xi^3 \right] + \frac{Q}{2} + k_1 \quad (2.127)$$

In light of the homogeneous boundary conditions that result from the filtering proposed, the auxiliary eigenvalue problem for the integral transformation process is chosen in the transversal direction, y . The following integral transform pair is then proposed:

$$\text{Transform: } \bar{\phi}_i(x) = \int_{-y_1}^{y_2} \tilde{Y}_i(x, y) \phi(x, y) dy \quad (2.128)$$

$$\text{Inverse: } \phi(x, y) = \sum_{i=1}^{\infty} \tilde{Y}_i(x, y) \bar{\phi}_i(x) \quad (2.129)$$

where $\tilde{Y}_i(x, y)$ are the normalized eigenfunctions, and the adopted eigenvalue problem, in terms of the original transversal coordinate, is given as

$$\frac{d^4 Y_i(x, y)}{dy^4} = \beta_i^4 Y_i(x, y) \quad (2.130)$$

$$Y_i(x, -y_1) = 0; \quad \frac{dY_i(x, -y_1)}{dn} = 0 \quad (2.131a, b)$$

$$Y_i(x, y_2) = 0; \quad \frac{dY_i(x, y_2)}{dn} = 0 \quad (2.132a, b)$$

where the independent variable x becomes a parameter in the eigenvalue problem, in light of the x variable cross section. Employing the same coordinate transformation used for the filter, the eigenvalue problem formulation is rewritten as

$$\frac{d^4 Y_i(\xi)}{d\xi^4} = (y_0 \beta_i)^4 Y_i(\xi) = \lambda_i^4 Y_i(\xi) \quad (2.133)$$

$$Y_i(-1) = 0; \quad \frac{dY_i(-1)}{d\xi} = 0 \quad (2.134a, b)$$

$$Y_i(1) = 0; \quad \frac{dY_i(1)}{d\xi} = 0 \quad (2.135a, b)$$

which leads to the following eigenfunctions, $Y_i(\xi)$:

$$Y_i(\xi) = \begin{cases} \frac{\cos(\lambda_i \xi)}{\cos(\lambda_i)} - \frac{\cosh(\lambda_i \xi)}{\cosh(\lambda_i)}, & i = 1, 3, 5, \dots \\ \frac{\sin(\lambda_i \xi)}{\sin(\lambda_i)} - \frac{\sinh(\lambda_i \xi)}{\sinh(\lambda_i)}, & i = 2, 4, 6, \dots \end{cases} \quad (2.136a, b)$$

and the corresponding normalization integral is determined from:

$$N_i = y_0 \int_0^1 Y_i^2(\xi) d\xi, \quad i = 1, 2, 3, \dots \quad (2.137)$$

thus, $N_i(x) = 2y_0(x)$. Then the normalized eigenfunctions, $\tilde{Y}_i(x, y)$, are constructed as

$$\tilde{Y}_i(x, y) = \frac{Y_i(x, y)}{\sqrt{N_i}} \quad (2.138)$$

Applying the integral transformation to the filtered problem, employing the inverse formula in the various potential derivatives, one reaches the following transformed ordinary differential system:

$$\begin{aligned} \frac{d^4 \bar{\phi}_i}{dx^4} = & -\lambda_i^4 \bar{\phi}_i + L_i + \text{Re} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left[A_{ijk} \bar{\phi}_j \bar{\phi}_k + B_{ijk} \bar{\phi}_j \frac{d\bar{\phi}_k}{dx} + C_{ijk} \bar{\phi}_j \frac{d^2 \bar{\phi}_k}{dx^2} \right. \\ & + D_{ijk} \bar{\phi}_j \frac{d^3 \bar{\phi}_k}{dx^3} + E_{ijk} \frac{d\bar{\phi}_j}{dx} \bar{\phi}_k + F_{ijk} \frac{d\bar{\phi}_j}{dx} \frac{d\bar{\phi}_k}{dx} + G_{ijk} \frac{d\bar{\phi}_j}{dx} \frac{d^2 \bar{\phi}_k}{dx^2} \Big] \\ & + \sum_{j=1}^{\infty} \left[H_{ij} \bar{\phi}_j + I_{ij} \frac{d\bar{\phi}_j}{dx} + J_{ij} \frac{d^2 \bar{\phi}_j}{dx^2} + K_{ij} \frac{d^3 \bar{\phi}_j}{dx^3} \right] \end{aligned} \quad (2.139)$$

where the coefficients are defined in Silva et al. (2004, 2007, 2011) and the respective transformed boundary conditions are given as

$$\bar{\phi}_i(0) = 0, \quad \bar{\phi}_i(\infty) = 0 \quad (2.140a, b)$$

$$\frac{d\bar{\phi}_i(0)}{dx} = 0, \quad \frac{d\bar{\phi}_i(\infty)}{dx} = 0 \quad (2.140c, d)$$

In the discussion that follows, results are presented for two situations, corresponding to a sinusoidal wavy channel wall and a corrugated triangular-shaped wall constructed via Fourier transforms (Sias et al. 2005). Different values of Reynolds number were considered, $\text{Re} = 100, 300, 500$, and 700 , in a wide range of relatively large Reynolds numbers for microchannel applications, with corrugation

amplitudes of 3 and 5 % of half the distance between the channel walls and corrugation frequencies of π and 3π .

First, results are presented for a sinusoidally wavy wall, with bounding surfaces defined by the function $f(x) = \alpha \sin[\omega(x - 3)]$, as shown in Fig. 2.20. Tables 2.7, 2.8, 2.9, and 2.10 illustrate the convergence of the streamfunction values along the line $y = 0.5$, for the two extreme values of Re here analyzed, and for the two relative corrugation amplitudes.

For $\omega = \pi$, as can be seen from Tables 2.7 and 2.8, an excellent convergence rate of the streamfunction could be observed, reaching, for instance, five significant digits of convergence with truncation orders as low as 14 terms for $Re = 100$, increasing to 18–22 terms for the larger value of $Re = 700$. In this case, for a few

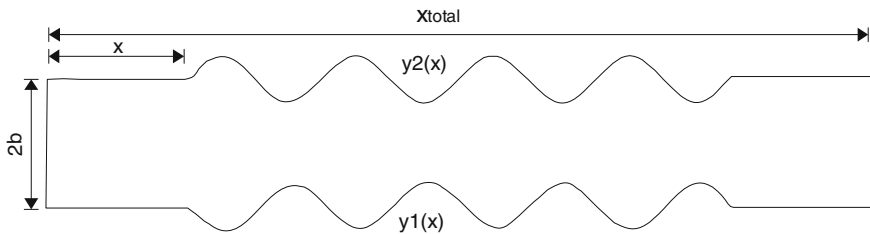


Fig. 2.20 Schematic representation of the channel with sinusoidal wavy walls

Table 2.7 Streamfunction convergence at $y = 0.5$ for $Re = 100$, $\alpha = 3\%$, and $\omega = \pi$

N	$x = 0$	$x = 3$	$x = 9$	$x = 15$	$x = 18$
6	0.68750	0.69504	0.68989	0.68267	0.68863
10	0.68750	0.69510	0.69025	0.68296	0.68867
14	0.68750	0.69511	0.69029	0.68299	0.68867
18	0.68750	0.69511	0.69029	0.68299	0.68867
22	0.68750	0.69511	0.69029	0.68299	0.68867
26	0.68750	0.69511	0.69029	0.68299	0.68867

Table 2.8 Streamfunction convergence at $y = 0.5$ for $Re = 700$, $\alpha = 5\%$, and $\omega = \pi$

N	$x = 0$	$x = 3$	$x = 9$	$x = 15$	$x = 18$
6	0.68750	0.69902	0.68609	0.67429	0.68829
10	0.68750	0.70089	0.68884	0.67648	0.69051
14	0.68750	0.70154	0.69061	0.67796	0.69130
18	0.68750	0.70156	0.69108	0.67839	0.69147
22	0.68750	0.70157	0.69126	0.67856	0.69153
26	0.68750	0.70157	0.69127	0.67858	0.69155

positions, convergence can be reached to four significant digits within the range of truncation orders explored in Table 2.8. For $\omega = 3\pi$, shown in Tables 2.9 and 2.10, the convergence behavior is also satisfactory, again yielding five significant digits of convergence in the case of $Re = 100$, while for the more critical situation of $Re = 700$, in some points three significant digits are fully converged to the maximum truncation order of just 26 terms here considered.

Figure 2.21a, b presents results for the product friction factor-Reynolds number ($f Re$), obtained for the various Reynolds numbers ($Re = 100, 300, 500, 700$) and for two situations in terms of the corrugation shape, namely, (a) $\alpha = 3\%$ and $\omega = \pi$; (b) $\alpha = 5\%$ and $\omega = 3\pi$. The overall behavior follows the findings of Wang and Chen (2002) and the benchmark results of Silva et al. (2011), with the marked increase on friction factor with the increase on both amplitude and frequency of the corrugations.

A brief convergence analysis of the longitudinal velocity component is also provided in Table 2.11, for $Re = 100$, $\alpha = 3\%$, and $y = 0.5$, with $\omega = \pi$ and 3π . Convergence was critically compared at two different sections in each case, one between the peaks of the corrugations, where the channel is narrower, and the other between the valleys of the corrugations, where the channel is wider. The two sections are roughly half way between the entrance and the exit of the channel. As expected, the convergence rates are not as fast as for the streamfunction (Perez-Guerrero and Cotta 1992, 1995; Perez-Guerrero et al. 2000). Nevertheless,

Table 2.9 Streamfunction convergence at $y = 0.5$ for $Re = 100$, $\alpha = 3\%$, and $\omega = 3\pi$

N	$x = 0$	$x = 3$	$x = 9$	$x = 15$	$x = 18$
6	0.68750	0.71216	0.69012	0.66000	0.68550
10	0.68750	0.71224	0.69003	0.65973	0.68537
14	0.68750	0.71227	0.69017	0.65976	0.68537
18	0.68750	0.71227	0.69019	0.65977	0.68537
22	0.68750	0.71227	0.69019	0.65977	0.68537
26	0.68750	0.71227	0.69019	0.65977	0.68537

Table 2.10 Streamfunction convergence at $y = 0.5$ for $Re = 700$, $\alpha = 5\%$, and $\omega = 3\pi$

N	$x = 0$	$x = 3$	$x = 9$	$x = 15$	$x = 18$
6	0.68750	0.66418	0.58295	0.56011	0.62729
10	0.68750	0.69444	0.63440	0.60499	0.66630
14	0.68750	0.69439	0.63575	0.60612	0.66632
18	0.68750	0.69442	0.63682	0.60705	0.66652
22	0.68750	0.69449	0.63754	0.60753	0.66640
26	0.68750	0.69456	0.63815	0.61072	0.67079

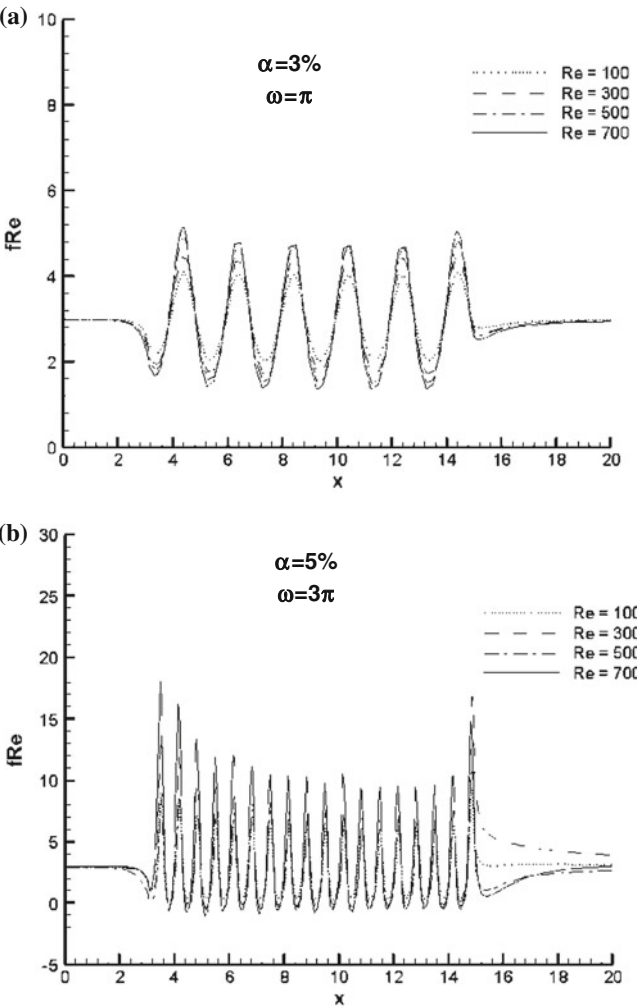


Fig. 2.21 Product friction factor-Reynolds number along the sinusoidally wavy channel for (a) $\alpha = 3\%$ and $\omega = \pi$; (b) $\alpha = 5\%$ and $\omega = 3\pi$

Table 2.11 Convergence of the longitudinal velocity component for $Re = 100$, $\alpha = 3\%$, and $y = 0.5$ (sinusoidal wavy wall)

N	$\omega = \pi$		$\omega = 3\pi$	
	$x_{conv} = 8.50$	$x_{div} = 9.50$	$x_{conv} = 8.14992$	$x_{div} = 9.84994$
6	1.15150	1.12331	1.14896	1.13601
10	1.15519	1.12137	1.15661	1.12803
14	1.15478	1.12135	1.15250	1.13025
18	1.15485	1.12134	1.15445	1.12985
22	1.15483	1.12135	1.15351	1.13001
26	1.15484	1.12134	1.15405	1.12992

for a corrugation amplitude of 3 %, four to five significant digits are achieved in the velocity value, for both frequencies analyzed, $\omega = \pi$ and 3π , within the maximum truncation order of just 26 terms here adopted.

Figure 2.22 provides an example of the streamfunction isolines as obtained from the present solution, taking only the case with more significant recirculations within the channel corrugation regions ($Re = 700$, $\alpha = 5\%$, and $\omega = 3\pi$).

Next, the case of a triangular corrugated wall is considered, as expressed by the following Fourier transforms representation of the triangular wave:

$$f(x) = \alpha \sum_{n=1}^5 \frac{8 \sin(n\pi/2)}{(n\pi)^2} \sin[n\omega(x - 3)] \quad (2.141)$$

Figure 2.23 provides an illustration of this geometry, and the corrugation frequencies were again taken as $\omega = \pi$ and 3π , while α represents the amplitude of the half wave.

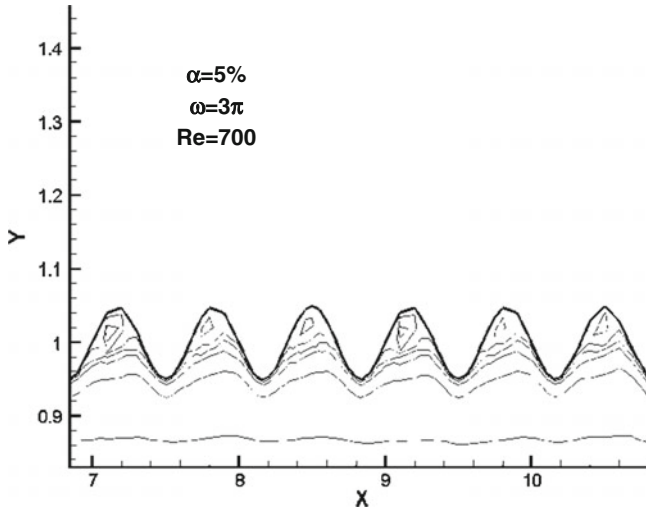


Fig. 2.22 Streamfunction isolines for $Re = 700$, $\alpha = 5\%$, and $\omega = 3\pi$ (sinusoidal wavy wall)

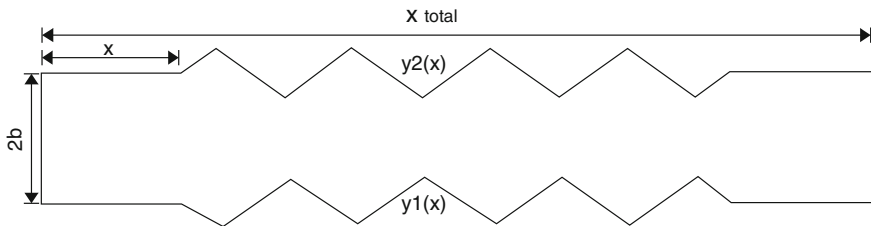


Fig. 2.23 Schematic representation of the microchannel with triangular-shaped corrugation

Table 2.12 Convergence of the streamfunction along $y = 0.5$ for $Re = 100$, $\alpha = 3\%$, and $\omega = \pi$ (triangular-shaped corrugation)

N	$x = 0$	$x = 3$	$x = 9$	$x = 15$	$x = 18$
6	0.68750	0.68975	0.68719	0.67810	0.68351
10	0.68750	0.69176	0.68874	0.68100	0.68536
14	0.68750	0.69177	0.68877	0.68102	0.68536
18	0.68750	0.69177	0.68877	0.68102	0.68536
22	0.68750	0.69177	0.68877	0.68102	0.68536
26	0.68750	0.69177	0.68877	0.68102	0.68536

Table 2.13 Convergence of the streamfunction along $y = 0.5$ for $Re = 100$, $\alpha = 3\%$, and $\omega = 3\pi$ (triangular-shaped corrugation)

N	$x = 0$	$x = 3$	$x = 9$	$x = 15$	$x = 18$
6	0.68750	0.66246	0.71067	0.66793	0.66666
10	0.68750	0.69596	0.62044	0.55732	0.60890
14	0.68750	0.69597	0.62040	0.55762	0.60882
18	0.68750	0.69597	0.62045	0.55751	0.60882
22	0.68750	0.69597	0.62048	0.55758	0.60882
26	0.68750	0.69597	0.62049	0.55757	0.60882

Tables 2.12 and 2.13 illustrate the convergence behavior of the streamfunction along the line $y = 0.5$, with $Re = 100$, $\alpha = 3\%$, and $\omega = \pi$ and 3π , respectively, for the triangular-shaped corrugated wall as reproduced by the Fourier transform above. In both situations one can observe the excellent convergence characteristics, with five fully converged significant digits within the range of parameters here considered.

Figure 2.24 provides the curves for the product friction factor-Reynolds number (fRe) as obtained for the channel with triangular-shaped corrugation, for $Re = 100$, $\omega = \pi$, $\alpha = 3$, and 5% . Again, one may notice an increase in friction factor, for increasing corrugation amplitude.

As in the previous case of a sinusoidal wall, Table 2.14 provides a brief illustration of the convergence behavior of the longitudinal velocity component in the case of a triangular-shaped corrugation. The velocity field is tabulated at $y = 0.5$ and for $Re = 100$, $\alpha = 3\%$, $\omega = \pi$, and $\omega = 3\pi$. Also in this case, an excellent convergence pattern is achieved, with some slight reduction in the convergence rates at the locations where the channel is more narrow for $\omega = 3\pi$.

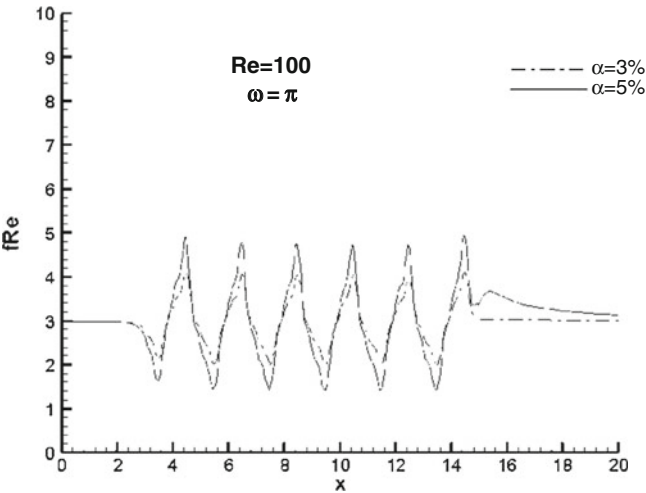


Fig. 2.24 Product friction factor-Reynolds number along the *triangular-shaped* corrugated channel for $Re = 100$, $\omega = \pi$, $\alpha = 3$, and 5%

Table 2.14 Convergence of the longitudinal velocity component for $Re = 100$, $\alpha = 3\%$, and $y = 0.5$ (triangular wavy wall)

N	$\omega = \pi$		$\omega = 3\pi$	
	$x_{conv} = 8.49990$	$x_{div} = 9.49990$	$x_{conv} = 8.15$	$x_{div} = 9.82499$
6	1.15088	1.12674	1.16363	1.11390
10	1.15342	1.12234	1.18285	1.15667
14	1.15252	1.12280	1.17748	1.16454
18	1.15283	1.12261	1.18073	1.16181
22	1.15268	1.12270	1.17897	1.16299
26	1.15277	1.12265	1.18008	1.16241

2.5.2 Heat Transfer Analysis with Approximate Velocity Profile

Transient laminar forced convection within microchannels formed by corrugated plates is now considered (Castellões and Cotta 2008; Castellões et al. 2010). Three regions along the channel are considered in the problem formulation, as described in Fig. 2.25. First, an adiabatic region with smooth walls, followed by the heat transfer section with prescribed temperatures at the corrugated walls, and the third one, following the corrugated region, is again made of smooth adiabatic walls. The problem formulation here presented and solved by integral transforms is fairly general, but the test case to be analyzed adopts the same sinusoidal walls geometry and

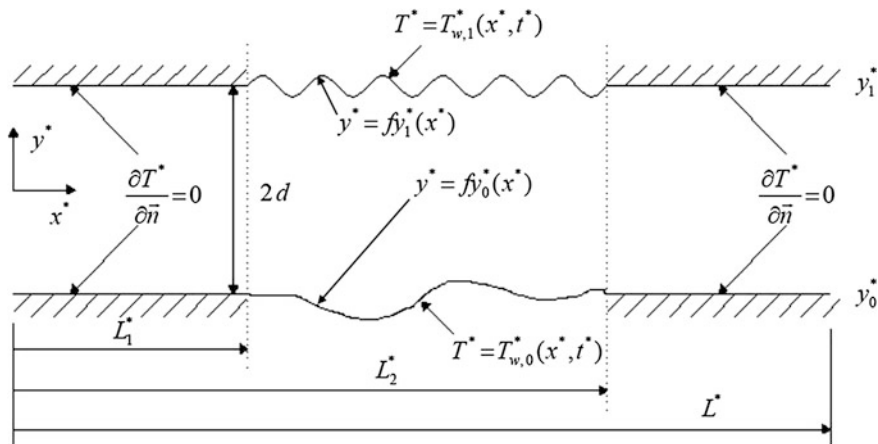


Fig. 2.25 Geometry and coordinates system for heat transfer in corrugated channel

boundary conditions as presented by Wang and Chen (2002). The walls boundaries are then described by the following functions along the longitudinal coordinate:

$$fy_0^*(x^*) = \begin{cases} y_0^*, & 0 < x^* < L_1^* \\ y_0^* + \alpha^* \sin(\pi x^* - L_1^*) \frac{1}{L_2^*}, & L_1^* < x^* < L_2^* \\ y_0^*, & L_2^* < x^* < L^* \end{cases} \quad (2.142)$$

$$fy_1^*(x^*) = \begin{cases} y_1^*, & 0 < x^* < L_1^* \\ y_1^* + \alpha^* \sin(\pi x^* - L_1^*) \frac{1}{L_2^*}, & L_1^* < x^* < L_2^* \\ y_1^*, & L_2^* < x^* < L^* \end{cases} \quad (2.143)$$

The two-dimensional steady flow is assumed to be laminar and incompressible, with constant thermophysical properties, while viscous dissipation and natural convection effects are neglected. Due to the possible low values of Péclet number, in light of the lower range of Reynolds numbers, axial diffusion along the fluid is not disregarded. Also, the flow is assumed to be fully developed at the first section entrance, but varies along the axial coordinate once the corrugated section is reached.

In obtaining the velocity field along the flow, the full Navier–Stokes equations should be employed, yielding the variable velocity components and pressure field along the transversal and longitudinal directions, as demonstrated in the previous section, where the Generalized Integral Transform Technique (GITT) has been employed in the hybrid numerical–analytical solution of this laminar flow (Silva et al. 2007, 2011). However, for sufficiently low Reynolds numbers and mild variations on the wall corrugations, an approximate solution for the velocity field has been previously proposed (Özişik et al. 1982), essentially by accounting for the variable cross section within the local mass balance equation, but neglecting the momentum equations influence on the velocity components development. These

explicit solutions for the velocity components are particularly handy, especially in design, optimization, and inverse analysis, and shall be here exploited to demonstrate the transient thermal problem solution. Thus, following this approach, the velocity components are analytically derived as (Castellões et al. 2010):

$$u^*(x^*, y^*) = \frac{-(y_0^* - y_1^*)}{(fy_0^*(x^*) - fy_1^*(x^*))^3} [-y^{*2} + y^* (fy_0^*(x^*) + fy_1^*(x^*)) - fy_0^*(x^*)fy_1^*(x^*)] \quad (2.144)$$

$$v^*(x^*, y^*) = \frac{-(y_0^* - y_1^*)}{(fy_0^*(x^*) - fy_1^*(x^*))^4} [6(y - fy_0^*(x^*))(y - fy_1^*(x^*)) \cdot (fy_1^{*'}(x^*)(y - fy_0^*(x^*)) - fy_0^{*'}(x^*)(y - fy_1^*(x^*)))] \quad (2.145)$$

It can be noticed that according to the approximate solution in Eqs. (2.45) and (2.46), at the entrance and exit of the corrugated section, as well as for a smooth heat transfer section, the flow becomes the classical parabolic fully developed velocity profile for parallel plates, given in the present coordinates system by

$$u^*(y^*) = \frac{6}{(y_0^* - y_1^*)^2} [-y^{*2} + y^* (y_0^* + y_1^*) - y_0^*y_1^*] \quad (2.146)$$

while the transversal velocity component vanishes. Once the velocity field is available, the transient energy balance is to be solved, given as

$$\frac{\partial T^*(x^*, y^*, t^*)}{\partial t^*} + u^*(x^*, y^*) \frac{\partial T^*}{\partial x^*} + v^*(x^*, y^*) \frac{\partial T^*}{\partial y^*} = \alpha \left(\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{\partial^2 T^*}{\partial y^{*2}} \right), \quad 0 < x^* < L^*, fy_0^*(x^*) < y^* < fy_1^*(x^*), t^* > 0 \quad (2.147)$$

$$T^*(x^*, y^*, 0) = T_0^*(x^*, y^*) \quad (2.148)$$

$$T^*(0, y^*, t^*) = T_e^*(y^*, t^*), \quad \left. \frac{\partial T^*}{\partial x^*} \right|_{x^*=L^*} = 0 \quad (2.149a, b)$$

$$\text{for } 0 < x^* < L_1^*, \begin{cases} \left. \frac{\partial T^*}{\partial y^*} \right|_{y^*=fy_0^*(x^*)} = 0 \\ \left. \frac{\partial T^*}{\partial y^*} \right|_{y^*=fy_1^*(x^*)} = 0 \end{cases}, \quad (2.150a, b)$$

$$\text{for } L_1^* \leq x^* \leq L_2^*, \begin{cases} T^*(x^*, y^* = fy_0^*(x^*), t^*) = T_{w,0}^* \\ T^*(x^*, y^* = fy_1^*(x^*), t^*) = T_{w,1}^* \end{cases} \quad (2.150c, d)$$

$$\text{for } L_2^* < x^* < L^*, \begin{cases} \left. \frac{\partial T^*}{\partial y^*} \right|_{y^* = fy_0^*(x^*)} = 0 \\ \left. \frac{\partial T^*}{\partial y^*} \right|_{y^* = fy_1^*(x^*)} = 0 \end{cases} \quad (2.150\text{e, f})$$

The following dimensionless groups are defined:

$$\begin{aligned} \zeta &= \frac{\alpha}{u_m d} x^* = \frac{1}{\text{Pe}} \frac{x^*}{d}, \quad y = \frac{y^*}{d}, \quad u(\zeta, y) = \frac{u^*(x^*, y^*)}{u_m}, \quad v(\zeta, y) = \frac{v^*(x^*, y^*)}{u_m} \\ t &= \frac{\alpha}{d^2} t^*, \quad T_A(\zeta, y, t) = \frac{T^*(x^*, y^*, t^*) - T_{w,0}^*}{\Delta T_c}, \quad \text{Pe} = \frac{u_m d}{\alpha}, \quad d = \frac{y_1^* - y_0^*}{2} \end{aligned} \quad (2.151)$$

Then, the dimensionless form of the temperature problem is written as

$$\begin{aligned} \frac{\partial T_A}{\partial t} + u(\zeta, y) \frac{\partial T_A}{\partial \zeta} + \text{Pe } v(\zeta, y) \frac{\partial T_A}{\partial y} &= \frac{1}{\text{Pe}^2} \frac{\partial^2 T_A}{\partial \zeta^2} + \frac{\partial^2 T_A}{\partial y^2}, \\ 0 < \zeta < L, fy_0(\zeta) < y < fy_1(\zeta), t > 0 \end{aligned} \quad (2.152)$$

$$T_A(\zeta, y, 0) = T_{A,0}(\zeta, y) \quad (2.153)$$

$$T_A(0, y, t) = T_{A,e}(y, t), \quad \left. \frac{\partial T_A}{\partial \zeta} \right|_{\zeta=L} = 0 \quad (2.154\text{a, b})$$

$$\text{for } 0 < \zeta < L_1, \begin{cases} \left. \frac{\partial T_A}{\partial y} \right|_{y=fy_0(\zeta)} = 0 \\ \left. \frac{\partial T_A}{\partial y} \right|_{y=fy_1(\zeta)} = 0 \end{cases}, \quad (2.155\text{a, b})$$

$$\text{for } L_1 \leq \zeta \leq L_2, \begin{cases} T_A(\zeta, y = fy_0(\zeta), t) = 0 \\ T_A(\zeta, y = fy_1(\zeta), t) = T_{A,w,1} \end{cases} \quad (2.155\text{c, d})$$

$$\text{for } L_2 < \zeta < L, \begin{cases} \left. \frac{\partial T_A}{\partial y} \right|_{y=fy_0(\zeta)} = 0 \\ \left. \frac{\partial T_A}{\partial y} \right|_{y=fy_1(\zeta)} = 0 \end{cases} \quad (2.155\text{e, f})$$

It can be observed from Eq. (2.155c,d) that the formulation does not impose axial symmetry to the thermal problem, but if symmetry prevails, Eq. (2.155d) shall be a homogeneous one.

In light of the discontinuity on the boundary conditions at the junction of the regions, it is adequate to propose a domain decomposition to handle the three mathematical problems separately, coupled at the cross sections between each pair of regions. The three problems formulation should therefore include the continuity conditions of temperature and heat flux at the fluid interfaces between the regions. Thus, the problem formulation for the first adiabatic region becomes

$$\frac{\partial T_1}{\partial t} + u(\xi, y) \frac{\partial T_1}{\partial \xi} + \text{Pe } v(\xi, y) \frac{\partial T_1}{\partial y} = \frac{1}{\text{Pe}^2} \frac{\partial^2 T_1}{\partial \xi^2} + \frac{\partial^2 T_1}{\partial y^2}, \quad (2.156)$$

$$0 < \xi < L_1, f_{y0}(\xi) < y < f_{y1}(\xi), t > 0$$

$$T_1(\xi, y, 0) = T_{A,0}(\xi, y) \quad (2.157)$$

$$T_1(0, y, t) = T_{A,e}(y, t), \left. \frac{\partial T_1}{\partial \xi} \right|_{\xi=L_1} = \left. \frac{\partial T_{A,2}}{\partial \xi} \right|_{\xi=L_1}, \left. \frac{\partial T_1}{\partial y} \right|_{y=f_{y0}(\xi)} = 0, \left. \frac{\partial T_1}{\partial y} \right|_{y=f_{y1}(\xi)} = 0 \quad (2.158\text{a-d})$$

For the heat exchanging section, we have

$$\frac{\partial T_{A,2}}{\partial t} + u(\xi, y) \frac{\partial T_{A,2}}{\partial \xi} + \text{Pe } v(\xi, y) \frac{\partial T_{A,2}}{\partial y} = \frac{1}{\text{Pe}^2} \frac{\partial^2 T_{A,2}}{\partial \xi^2} + \frac{\partial^2 T_{A,2}}{\partial y^2}, \quad (2.159)$$

$$L_1 < \xi < L_2, f_{y0}(\xi) < y < f_{y1}(\xi), t > 0$$

$$T_{A,2}(\xi, y, 0) = T_{A,0}(\xi, y) \quad (2.160)$$

$$T_{A,2}(\xi = L_1, y, t) = T_1(\xi = L_1, y, t), T_{A,2}(\xi = L_2, y, t) = T_3(\xi = L_2, y, t) \quad (2.161\text{a, b})$$

$$T_{A,2}(\xi, y = f_{y0}(\xi), t) = 0, T_{A,2}(\xi, y = f_{y1}(\xi), t) = T_{A,w,1} \quad (2.162\text{a, b})$$

and finally for the exiting section, also adiabatic:

$$\frac{\partial T_3}{\partial t} + u(\xi, y) \frac{\partial T_3}{\partial \xi} + \text{Pe } v(\xi, y) \frac{\partial T_3}{\partial y} = \frac{1}{\text{Pe}^2} \frac{\partial^2 T_3}{\partial \xi^2} + \frac{\partial^2 T_3}{\partial y^2}, \quad (2.163)$$

$$L_2 < \xi < L, f_{y0}(\xi) < y < f_{y1}(\xi), t > 0$$

$$T_3(\xi, y, 0) = T_{A,0}(\xi, y) \quad (2.164)$$

$$\left. \frac{\partial T_3}{\partial \xi} \right|_{\xi=L_2} = \left. \frac{\partial T_{A,2}}{\partial \xi} \right|_{\xi=L_2}, \left. \frac{\partial T_3}{\partial \xi} \right|_{\xi=L_1} = 0, \left. \frac{\partial T_3}{\partial y} \right|_{y=f_{y0}(\xi)} = 0, \left. \frac{\partial T_3}{\partial y} \right|_{y=f_{y1}(\xi)} = 0 \quad (2.165\text{a-d})$$

The temperature problem for the heated section remains nonhomogeneous in Eq. (2.162b). In order to homogenize the problem in the transversal direction y , a filtering solution, $\phi(\xi, y)$, is employed for the potential $T_{A,2}(\xi, y, t)$, in the form:

$$T_{A,2}(\xi, y, t) = T_2(\xi, y, t) + \phi(\xi, y) \quad (2.166)$$

A simple and sufficiently general form for the filter $\phi(\xi, y)$ is obtained by satisfying the diffusion operator in the transversal direction, which yields

$$\phi(\xi, y) = T_{A,w,1} \frac{fy_0(\xi) - y}{fy_0(\xi) - fy_1(\xi)} \quad (2.167)$$

Then, the homogeneous filtered problem for the corrugated section is given as

$$\frac{\partial T_2}{\partial t} + u(\xi, y) \frac{\partial T_2}{\partial \xi} + \text{Pe } v(\xi, y) \frac{\partial T_2}{\partial y} = \frac{1}{\text{Pe}^2} \frac{\partial^2 T_2}{\partial \xi^2} + \frac{\partial^2 T_2}{\partial y^2} + g(\xi, y), \quad (2.168)$$

$$L_1 < \xi < L_2, fy_0(\xi) < y < fy_1(\xi), t > 0$$

$$T_2(\xi, y, t = 0) = T_{A,0}(\xi, y) - \phi(\xi, y) \quad (2.169)$$

$$\begin{aligned} T_2(\xi = L_1, y, t) &= T_1(\xi = L_1, y, t) - \phi(\xi = L_1, y), \\ T_2(\xi = L_2, y, t) &= T_3(\xi = L_2, y, t) - \phi(\xi = L_2, y) \end{aligned} \quad (2.170a, b)$$

$$T_2(\xi, y = fy_0(\xi), t) = 0, \quad T_2(\xi, y = fy_1(\xi), t) = 0 \quad (2.171a, b)$$

where

$$g(\xi, y) = \frac{1}{\text{Pe}^2} \frac{\partial^2 \phi}{\partial \xi^2} - \text{Pe } v(\xi, y) \frac{\partial \phi}{\partial \xi} - u(\xi, y) \frac{\partial \phi}{\partial y} \quad (2.172)$$

Next, the decomposed domain is described with three different coordinates systems, which match at the two interfaces, as shown in Fig. 2.26. The normalized longitudinal coordinates are then computed in terms of the original dimensionless coordinate as

$$x_1 = \frac{1}{L_1} \xi, \quad x_2 = \frac{1}{(L_2 - L_1)} (L_2 - \xi), \quad x_3 = \frac{1}{(L - L_2)} (\xi - L_2) \quad (2.173)$$

Thus, in terms of the redefined coordinates systems the dimensionless problem formulation is given, for each of the three regions, as

$$\frac{\partial T_1}{\partial t} + u(x_1, y) \frac{\partial T_1}{\partial x_1} \frac{1}{L_1} + \text{Pe } v(x_1, y) \frac{\partial T_1}{\partial y} = \frac{1}{\text{Pe}^2} \frac{1}{L_1^2} \frac{\partial^2 T_1}{\partial x_1^2} + \frac{\partial^2 T_1}{\partial y^2}, \quad (2.174)$$

$$0 < x_1 < 1, \quad y_0 < y < y_1, \quad t > 0$$

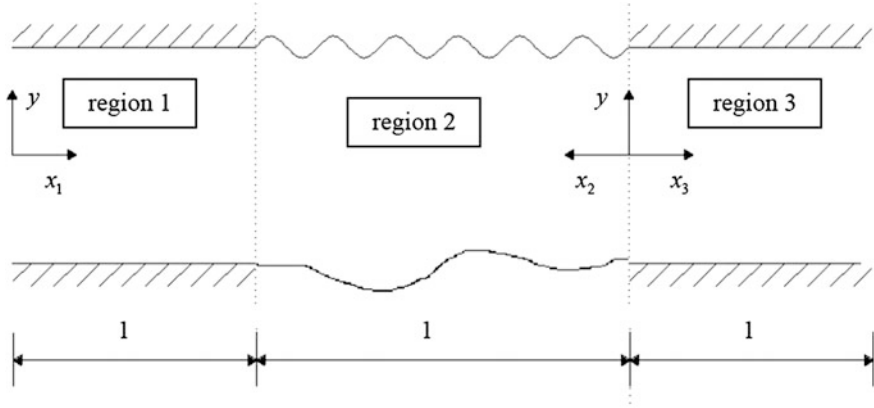


Fig. 2.26 Coordinates systems for the decomposed domain in heat transfer analysis of corrugated channel

$$T_1(x_1, y, t = 0) = T_{1,0}(x_1, y) \quad (2.175)$$

$$T_1(x_1 = 0, y, t) = T_{A,e}(y, t), \quad \left. \frac{\partial T_1}{\partial x_1} \right|_{x_1=1} = - \left. \frac{\partial T_2}{\partial x_2} \right|_{x_2=1} \frac{L_1}{L_2 - L_1} \quad (2.176a, b)$$

$$\left. \frac{\partial T_1}{\partial y} \right|_{y=y_0} = 0, \quad \left. \frac{\partial T_1}{\partial y} \right|_{y=y_1} = 0 \quad (2.177a, b)$$

$$\begin{aligned} \frac{\partial T_2}{\partial t} + u(x_2, y) \frac{\partial T_2}{\partial x_2} \left(\frac{-1}{(L_2 - L_1)} \right) + \text{Pe} v(x_2, y) \frac{\partial T_2}{\partial y} &= \frac{1}{\text{Pe}^2} \frac{1}{(L_2 - L_1)^2} \frac{\partial^2 T_2}{\partial x_2^2} \\ &+ \frac{\partial^2 T_2}{\partial y^2} + g(x_1, y), \quad 0 < x_2 < 1, f y_0(x_2) < y < f y_1(x_2), t > 0 \end{aligned} \quad (2.178)$$

$$T_2(x_2, y, t = 0) = T_{2,0}(x_2, y) - \phi(x_2, y) \quad (2.179)$$

$$\begin{aligned} T_2(x_2 = 0, y, t) &= T_3(x_3 = 0, y, t) - \phi(x_2 = 0, y), \\ T_2(x_2 = 1, y, t) &= T_1(x_1 = 1, y, t) - \phi(x_2 = 1, y) \end{aligned} \quad (2.180a, b)$$

$$T_2(x_2, y = f y_0(x_2), t) = 0, \quad T_2(x_2, y = f y_1(x_2), t) = 0 \quad (2.181a, b)$$

$$\phi(x_2, y) = T_{A,w,1} \frac{f y_0(x_2) - y}{f y_0(x_2) - f y_1(x_2)}, \quad (2.182a)$$

$$g(x_2, y) = \frac{1}{\text{Pe}^2(L_2 - L_1)^2} \frac{\partial^2 \phi}{\partial x_2^2} - \text{Pe } v(x_2, y) \frac{\partial \phi}{\partial y} + u(x_2, y) \frac{\partial \phi}{\partial x_2} \frac{1}{(L_2 - L_1)} \quad (2.182b)$$

$$\begin{aligned} \frac{\partial T_3}{\partial t} + u(x_3, y) \frac{\partial T_3}{\partial x_3} \frac{1}{(L - L_2)} + \text{Pe } v(x_3, y) \frac{\partial T_3}{\partial y} &= \frac{1}{\text{Pe}^2} \frac{1}{(L - L_2)^2} \frac{\partial^2 T_3}{\partial x_3^2} + \frac{\partial^2 T_3}{\partial y^2}, \\ 0 < x_3 < 1, \quad y_0 < y < y_1, \quad t > 0 \end{aligned} \quad (2.183)$$

$$T_3(x_3, y, t = 0) = T_{3,0}(x_3, y) \quad (2.184)$$

$$\left. \frac{\partial T_3}{\partial x_3} \right|_{x_3=0} = - \left. \frac{\partial T_2}{\partial x_2} \right|_{x_2=0} \frac{L - L_2}{L_2 - L_1}, \quad \left. \frac{\partial T_3}{\partial x_3} \right|_{x_3=1} = 0 \quad (2.185a, b)$$

$$\left. \frac{\partial T_3}{\partial y} \right|_{y=y_0} = 0, \quad \left. \frac{\partial T_3}{\partial y} \right|_{y=y_1} = 0 \quad (2.186a, b)$$

One may observe that after the coordinates systems redefinition, the problems are coupled only at the interfaces and all the domains limits are the same, and therefore the dependent variables may be algebraically interpreted as applied to one single domain in the longitudinal coordinate ($0 < x < 1$).

Following the GITT formalism, the auxiliary problems are now defined to construct the eigenfunction expansions in each region. For regions 1 and 3 it is adopted the same eigenvalue problem, with second kind boundary conditions, given as

$$\frac{d^2 \psi_i}{dy^2} + \mu_i^2 \psi_i(y) = 0, \quad y_0 < y < y_1, \quad (2.187)$$

$$\left. \frac{d\psi_i}{dy} \right|_{y=y_0} = 0, \quad \left. \frac{d\psi_i}{dy} \right|_{y=y_1} = 0 \quad (2.188a, b)$$

whose solution in terms of eigenfunctions, norms, and eigenvalues is readily found as

$$\psi_i(y) = \begin{cases} 1, & i = 0 \\ \cos(\mu_i(y - y_0)), & i = 1, 2, 3, \dots \end{cases} \quad (2.189)$$

$$N_{\psi_i} = \int_{y_0}^{y_1} \psi_i^2(y) dy = \begin{cases} y_1 - y_0, & i = 0 \\ \frac{y_1 - y_0}{2}, & i = 1, 2, 3, \dots \end{cases} \quad (2.190)$$

$$\mu_i = \begin{cases} 0, & i = 0 \\ \frac{i\pi}{y_1 - y_0}, & i = 1, 2, 3, \dots \end{cases} \quad (2.191)$$

For region 2, the auxiliary problem has to account for the irregular walls, which is incorporated into the eigenfunctions and eigenvalues via the functions of x that describe the transversal domain bounding walls, as shown below:

$$\frac{\partial^2 \Gamma_m}{\partial y^2} + \beta_m^2(x) \Gamma_m(x, y) = 0, \quad f y_0(x) < y < f y_1(x) \quad (2.192)$$

$$\Gamma_m(y = f y_0(x)) = 0, \quad \Gamma_m(y = f y_1(x)) = 0 \quad (2.193a, b)$$

Thus, the x -dependent eigenfunctions, norms, and eigenvalues are given as

$$\Gamma_m(x, y) = \sin(\beta_m(x) (y - f y_0(x))) \quad (2.194)$$

$$N_{\Gamma_m}(x) = \int_{f y_0(x)}^{f y_1(x)} \Gamma_m^2(x, y) dy = \frac{f y_1(x) - f y_0(x)}{2} \quad (2.195)$$

$$\beta_m(x) = \frac{m\pi}{f y_1(x) - f y_0(x)}, \quad m = 1, 2, 3, \dots \quad (2.196)$$

Once the eigenvalue problems have been defined and solved, the integral transform pairs (transform-inverse) are constructed as

$$\text{Transform: } \bar{T}_{1,i}(x, t) = \int_{y_0}^{y_1} T_1(x, y, t) \psi_i(y) dy \quad (2.197)$$

$$\text{Inverse: } T_1(x, y, t) = \sum_{i=0}^{\infty} \frac{1}{N_{\psi_i}} \bar{T}_{1,i}(x, t) \psi_i(y) \quad (2.198)$$

$$\text{Transform: } \bar{T}_{2,m}(x, t) = \int_{f y_0(x)}^{f y_1(x)} T_2(x, y, t) \Gamma_m(x, y) dy \quad (2.199)$$

$$\text{Inverse: } T_2(x, y, t) = \sum_{m=1}^{\infty} \frac{1}{N_{\Gamma_m}(x)} \bar{T}_{2,m}(x, t) \Gamma_m(x, y) \quad (2.200)$$

$$\text{Transform: } \bar{T}_{3,i}(x, t) = \int_{y_0}^{y_1} T_3(x, y, t) \psi_i(y) dy \quad (2.201)$$

$$\text{Inverse: } T_3(x, y, t) = \sum_{i=0}^{\infty} \frac{1}{N_{\psi_i}} \bar{T}_{3,i}(x, t) \psi_i(y) \quad (2.202)$$

One may start with the integral transformation of Eqs. (2.174)–(2.177) for region 1, operating with $\int_{y_0}^{y_1} \psi_i(y)(\cdot) dy$, and after substitution of the inverse formula (2.198) and some manipulation, the transformed system becomes

$$\frac{\partial \bar{T}_{1,i}}{\partial t} + \frac{1}{L_1} \sum_{j=0}^{\infty} A_{ij} \frac{\partial \bar{T}_{1,j}}{\partial x} = \frac{1}{\text{Pe}^2 L_1^2} \frac{\partial^2 \bar{T}_{1,i}}{\partial x^2} - \mu_i^2 \bar{T}_{1,i}(x, t), \quad 0 < x < 1, \quad t > 0 \quad (2.203)$$

$$\bar{T}_{1,i}(x, t = 0) = \bar{T}_{1,0,i}(x) \quad (2.204)$$

$$\bar{T}_{1,i}(x = 0, t) = \bar{T}_{e,i}(t), \quad (2.205a)$$

$$\left. \frac{\partial \bar{T}_{1,i}}{\partial x} \right|_{x=1} = -\frac{L_1}{L_2 - L_1} \sum_{n=1}^{\infty} \frac{1}{N_{\Gamma_n(x=1)}} B_{\text{in}}(x=1) \left. \frac{\partial \bar{T}_{2,n}}{\partial x} \right|_{x=1} \quad (2.205b)$$

Similarly, Eqs. (2.178)–(2.182) are operated on with $\int_{f y_0(x)}^{f y_1(x)} \Gamma_m(x, y)(\cdot) dy$ to yield the transformed version for the temperature problem in region 2

$$\begin{aligned} \frac{\partial \bar{T}_{2,m}}{\partial t} = & \frac{1}{\text{Pe}^2 (L_2 - L_1)^2} \frac{\partial^2 \bar{T}_{2,m}}{\partial x^2} \\ & + \sum_{n=1}^{\infty} \frac{\partial \bar{T}_{2,n}}{\partial x} \left[\frac{1}{\text{Pe}^2 (L_2 - L_1)^2} \left(2 \frac{d}{dx} \left(\frac{1}{N_{\Gamma_n(x)}} \right) N_{\Gamma_n(x)} \delta_{mn} + \frac{2}{N_{\Gamma_n(x)}} F_{mn}(x) \right) \right. \\ & + \frac{1}{L_2 - L_1} \left(\frac{1}{N_{\Gamma_n(x)}} D_{mn}(x) \right) \left. \right] + \sum_{n=1}^{\infty} \bar{T}_{2,n}(x, t) \left[\frac{1}{\text{Pe}^2 (L_2 - L_1)^2} \right. \\ & \times \left(\frac{d^2}{dx^2} \left(\frac{1}{N_{\Gamma_n(x)}} \right) N_{\Gamma_n(x)} \delta_{mn} + 2 \frac{d}{dx} \left(\frac{1}{N_{\Gamma_n(x)}} \right) F_{mn}(x) + \frac{1}{N_{\Gamma_n(x)}} G_{mn}(x) \right) \\ & + \frac{1}{L_2 - L_1} \left(\frac{1}{N_{\Gamma_n(x)}} C_{mn}(x) + \frac{d}{dx} \left(\frac{1}{N_{\Gamma_n(x)}} \right) D_{mn}(x) \right) \\ & \left. - \text{Pe} \left(\frac{1}{N_{\Gamma_n(x)}} E_{mn}(x) \right) - \beta_m^2(x) \delta_{mn} \right] + \bar{g}_m(x), \quad 0 < x < 1, \quad t > 0 \end{aligned} \quad (2.206)$$

$$\bar{T}_{2,m}(x, t = 0) = \bar{T}_{2,0,m}(x) - \bar{\phi}_m(x) \quad (2.207)$$

$$\bar{T}_{2,m}(x=0,t) = \sum_{j=0}^{\infty} \left(\frac{1}{N_{\psi_j}} B_{mj}(x=0) \bar{T}_{3,j}(x=0,t) \right) - \bar{\phi}_m(x=0) \quad (2.208)$$

$$\bar{T}_{2,m}(x=1,t) = \sum_{j=0}^{\infty} \left(\frac{1}{N_{\psi_j}} B_{mj}(x=1) \bar{T}_{1,j}(x=1,t) \right) - \bar{\phi}_m(x=1) \quad (2.209)$$

and finally, for region 3, Eqs. (2.183)–(2.186) are operated on with $\int_{y_0}^{y_1} \psi_i(y) (\cdot) dy$, to furnish

$$\frac{\partial \bar{T}_{3,i}}{\partial t} + \frac{1}{L-L_2} \sum_{j=0}^{\infty} A_{ij} \frac{\partial \bar{T}_{3,j}}{\partial x} = \frac{1}{\text{Pe}^2 (L-L_2)^2} \frac{\partial^2 \bar{T}_{3,i}}{\partial x^2} - \mu_i^2 \bar{T}_{3,i}(x,t), \quad (2.210)$$

$$0 < x < 1, \quad t > 0,$$

$$\bar{T}_{3,i}(x, t=0) = \bar{T}_{3,0,i}(x) \quad (2.211)$$

$$\left. \frac{\partial \bar{T}_{3,i}}{\partial x} \right|_{x=0} = -\frac{L-L_2}{L_2-L_1} \sum_{n=1}^{\infty} \frac{1}{N_{\Gamma_m(x=0)}} B_{in}(x=0) \left. \frac{\partial \bar{T}_{2,n}}{\partial x} \right|_{x=0}, \quad \left. \frac{\partial \bar{T}_{3,i}}{\partial x} \right|_{x=1} = 0 \quad (2.212a, b)$$

The coefficients that appear on the transformed system are analytically obtained from the following integrations:

$$A_{ij} = \frac{1}{N_{\psi_j}} \int_{y_0}^{y_1} u(y) \psi_i(y) \psi_j(y) dy, \quad B_{ni}(x) = \int_{y_0}^{y_1} \psi_i(y) \Gamma_n(x, y) dy \quad (2.213a, b)$$

$$C_{mn}(x) = \int_{f_{y_0}(x)}^{f_{y_1}(x)} u(x, y) \Gamma_m(x, y) \frac{\partial \Gamma_n}{\partial x} dy, \quad D_{mn}(x) = \int_{f_{y_0}(x)}^{f_{y_1}(x)} u(x, y) \Gamma_m(x, y) \Gamma_n(x, y) dy \quad (2.214a, b)$$

$$E_{mn}(x) = \int_{f_{y_0}(x)}^{f_{y_1}(x)} v(x, y) \Gamma_m(x, y) \frac{\partial \Gamma_n}{\partial y} dy, \quad F_{mn}(x) = \int_{f_{y_0}(x)}^{f_{y_1}(x)} \Gamma_m(x, y) \frac{\partial \Gamma_n}{\partial x} dy \quad (2.215a, b)$$

$$G_{mn}(x) = \int_{f_{y0}(x)}^{f_{y1}(x)} \Gamma_m(x, y) \frac{\partial^2 \Gamma_n}{\partial x^2} dy, \quad \bar{g}_m(x) = \int_{f_{y0}(x)}^{f_{y1}(x)} g(x, y) \Gamma_m(x, y) dy \quad (2.216a, b)$$

$$\bar{\phi}_m(x) = \int_{f_{y0}(x)}^{f_{y1}(x)} \phi(x, y) \Gamma_m(x, y) dy \quad (2.217)$$

Equations (2.203)–(2.217) form an infinite system of partial differential equations having as independent variables the unified dimensionless longitudinal coordinate, x , and dimensionless time, t , and as dependent variables the transformed temperatures in each region. For computational purposes, the system is truncated to a finite order, truncating the eigenfunction expansions for each field in a sufficiently large number of terms for each region (N_1 , N_2 and N_3). Due to the x -variable nature of the system coefficients, Eqs. (2.213)–(2.217), the PDE system has to be numerically solved, for instance employing the Method of Lines with local error control as implemented on the function `NDSolve` of the *Mathematica* symbolic-numerical computation platform (Wolfram 2008).

For the thermal problem, results are reported in terms of the bulk temperature and the local Nusselt numbers at the two channel walls (yielding the same results for the symmetric situation), as defined below:

$$T_m(\xi, t) = \frac{\int_{f_{y0}(\xi)}^{f_{y1}(\xi)} u(\xi, y) T_A(\xi, y, t) dy}{\int_{f_{y0}(\xi)}^{f_{y1}(\xi)} u(\xi, y) dy} \quad (2.218)$$

$$\text{Nu}_1(\xi, t) = \frac{-4}{T_m(\xi, t)} (\nabla T_A \cdot \mathbf{n}_{y1}), \quad \text{Nu}_0(\xi, t) = \frac{4}{T_m(\xi, t)} (\nabla T_A \cdot \mathbf{n}_{y0}) \quad (2.219a, b)$$

The constructed computer code was first verified against the transient convection results within smooth parallel-plates channel of Sect. 2.2 (Castellões et al. 2007), with one single region, i.e., only the heat transfer section. A thorough convergence analysis was undertaken on the temperature results so as to provide confidence on the numerical results reported. The truncation orders of $N_1 = 4$, $N_2 = 8$, and $N_3 = 4$ were sufficient to provide three to four significant digits in the temperature field results, in a wide range of the dimensionless longitudinal coordinate and for Péclet numbers $\text{Pe} = 1, 10$, and 30 . It was also concluded that the order of the expansions is required to gradually increase as Pe increases. The analysis for smooth parallel plates was also validated by comparing the results to a previously reported analytical solution (Tan and Normandia 1975), which accounts for the upstream adiabatic region, under steady laminar forced convection for low Péclet numbers (Castellões et al. 2010).

Figure 2.27 illustrates the influence of axial heat diffusion in the fluid, showing the steady-state dimensionless bulk temperatures along the channel for $Pe = 1, 10$, and 30 , and especially across the interface between the upstream adiabatic region and the heat transfer section, represented by a dashed line. The precooling (or heating) effect provided by the presence of the upstream region is noticeable for the lower values of Péclet number. This effect is quite relevant along the transient state as well, as illustrated in Fig. 2.28 for $Pe = 30$. As a result, heat transfer enhancement effect is in fact observed, resulting in higher values of the Nusselt number in this region for decreasing Péclet number. As can be deduced, significant errors may result if experimental results are employed to estimate average Nusselt numbers that assume the bulk temperature at the heat transfer section inlet as the uniform temperature at the upstream region inlet.

Results now illustrate the verification of the approximate velocity field given by Eqs. (2.144)–(2.146) in describing the flow behavior in the corrugated channel. This simplified approach is expected to provide reasonable results for lower values of the Reynolds number and for smoother corrugations. Thus, numerical results for the full Navier–Stokes formulation were employed in the verification of the present approximate analytical solutions, as obtained from the GITT hybrid numerical–analytical solution of the previous section, as reported in (Silva 2003). Figure 2.29a depicts the longitudinal velocity component at different axial positions along the corrugated section, for the selected values of $Re = 10$ and $\alpha = 0.1$, and Fig. 2.29b presents the velocity component for a less smooth corrugation, with $\alpha = 0.2$.

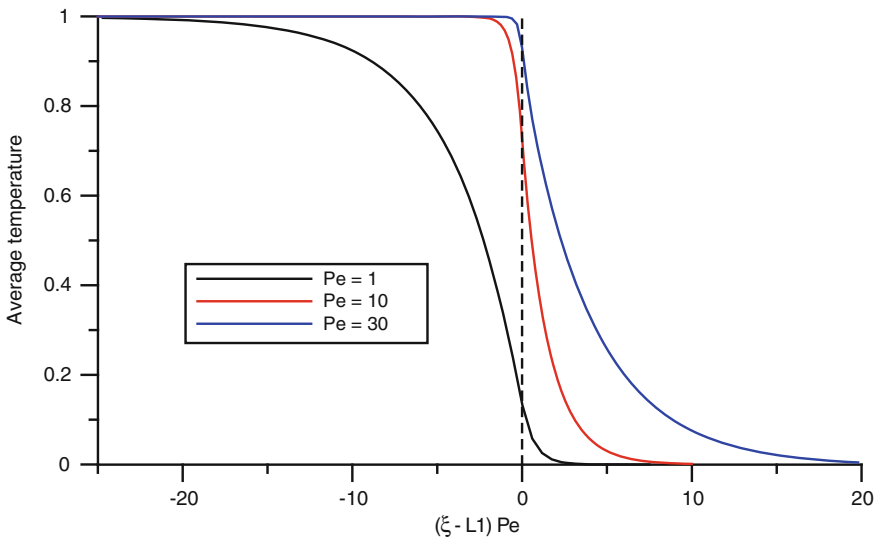


Fig. 2.27 Comparison of dimensionless bulk temperatures in smooth parallel-plates channel along both the upstream adiabatic region and heat transfer section at steady state for different values of Péclet number (*dashed line* is the interface between the two regions)

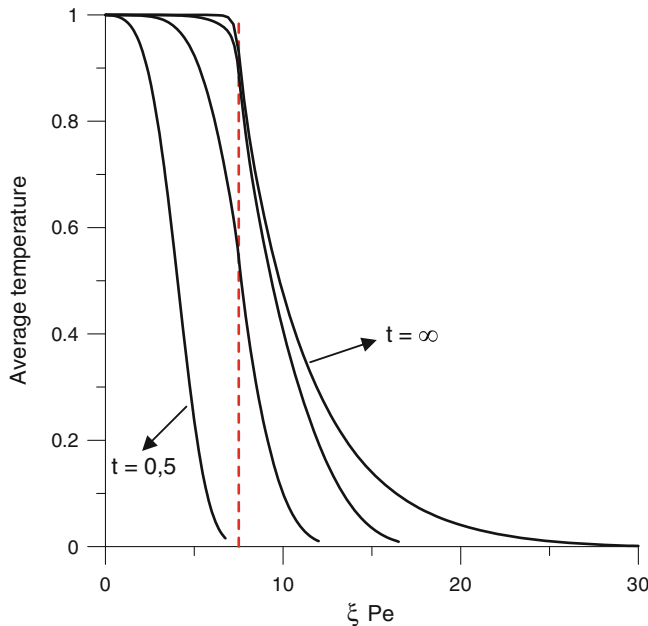


Fig. 2.28 Transient behavior of the dimensionless bulk temperature in smooth parallel-plates channel along both the upstream adiabatic region and the heat transfer section for $Pe = 30$ and $t = 0.5, 1.0, 1.5$, and steady state (dashed line is the interface between the two regions)

One may observe that the increase in the wall corrugation amplitude yields a less accurate velocity component calculated from the approximation through Eqs. (2.144)–(2.146), especially in regions closer to the inlet of the corrugated region. Also, increasing the Reynolds number results in some loss of accuracy in the simplified solution, but its relative merits should be highlighted, as observed in Fig. 2.30 which demonstrates that the results are still reasonably accurate for the case of $Re = 100$ and $\alpha = 0.1$, and apparently the increase in corrugation amplitude of Fig. 2.29b was more significant in deviating the approximate solution from the converged GITT results for the full Navier–Stokes formulation than the increase in Reynolds number of Fig. 2.30. The comparisons of the longitudinal velocity component are emphasized since the related convective term is the most important one in the heat transfer problem.

Proceeding to the transient thermal analysis, Fig. 2.31 illustrates the time evolution of the bulk temperature for $Pe = 10$, where the dashed line denotes the interface between the adiabatic region 1 and the corrugated section, region 2. One may observe that due to the low Reynolds number employed (and consequently low Péclet number), the axial diffusion along the fluid has a sensible effect on the bulk temperature evolution within the access region (region 1) along the transient period. The steady-state results are, however, closer to the situation of an unheated inlet section, which would be obtained by the model that neglects axial diffusion of heat

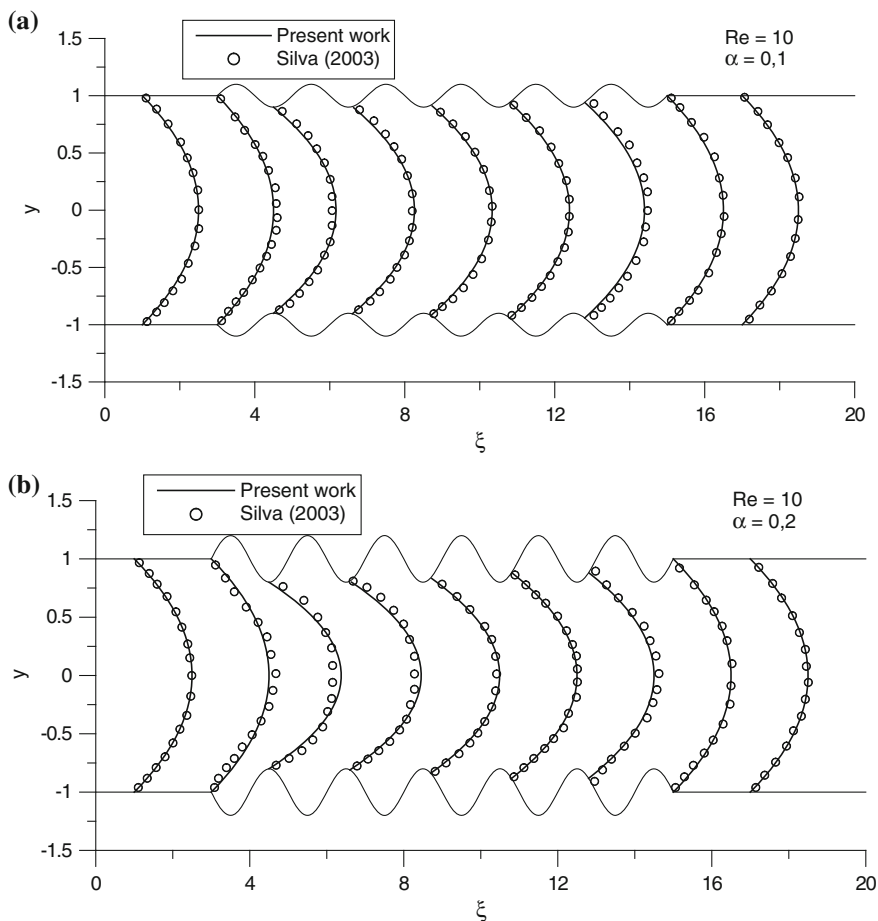


Fig. 2.29 Comparison of longitudinal velocity component for corrugated channel between approximate results (Castellões et al. 2010) and GITT solution (Silva 2003; Silva et al. 2011), for (a) $Re = 10$ and $\alpha = 0.1$ and (b) $Re = 10$ and $\alpha = 0.2$

within the fluid. Hence, for microchannel applications that involve low Péclet numbers, the behavior of the thermal wave front can be markedly affected by the presence of an adiabatic inlet section. It is also observed that the bulk temperature behavior presents a fluctuating shape due to the presence of the wall corrugations.

Figure 2.32 illustrates the effect of the Péclet number on the bulk temperatures for the steady-state situation. It can be clearly observed a more significant pre-heating effect in region 1 due to the lower value of Pe , but also the more pronounced effects on the bulk temperature fluctuations due to the wall corrugations in the case of a smaller axial diffusion of heat, when the transversal effect plays a major role in the governing equation.

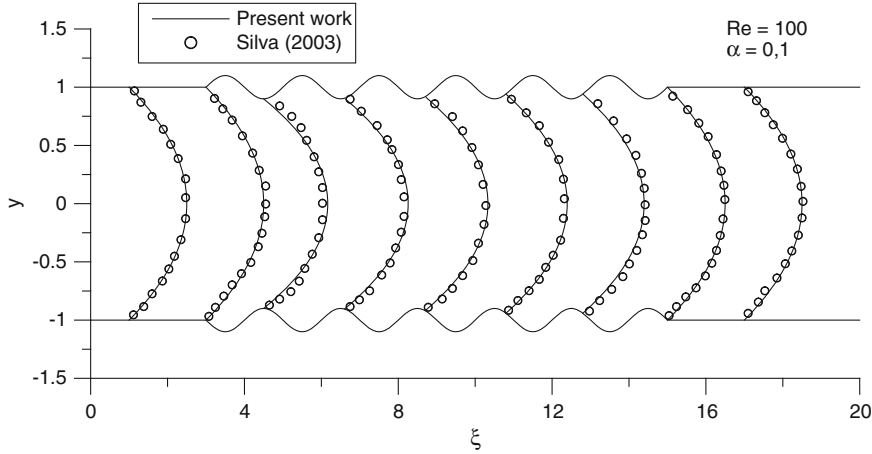


Fig. 2.30 Comparison of longitudinal velocity component for corrugated channel between approximate results (Castellões et al. 2010) and GITT solution (Silva 2003; Silva et al. 2011), for $Re = 100$ and $\alpha = 0.1$

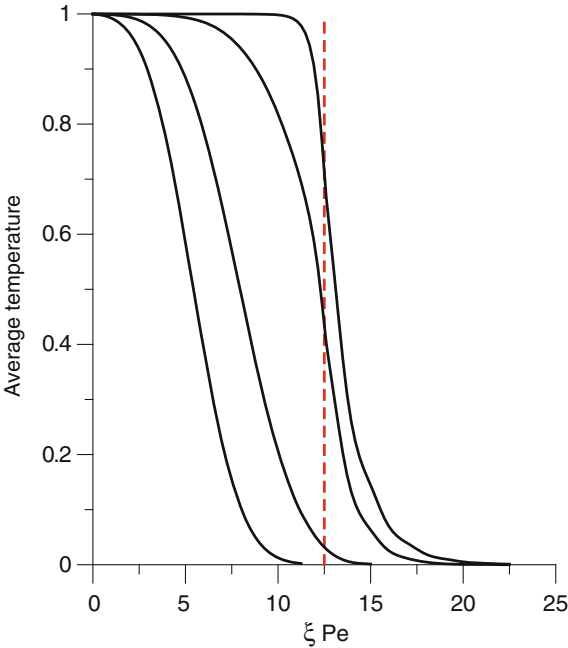


Fig. 2.31 Transient behavior of bulk temperature on regions 1 and 2 for $Pe = 10$, $\alpha = 0.1$, and $t = 2, 3, 5$, and steady state (Castellões et al. 2010)

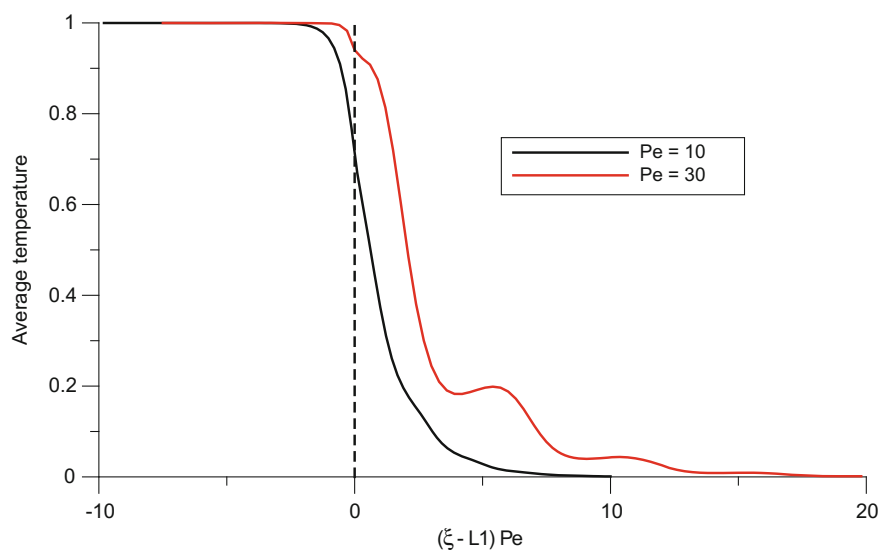
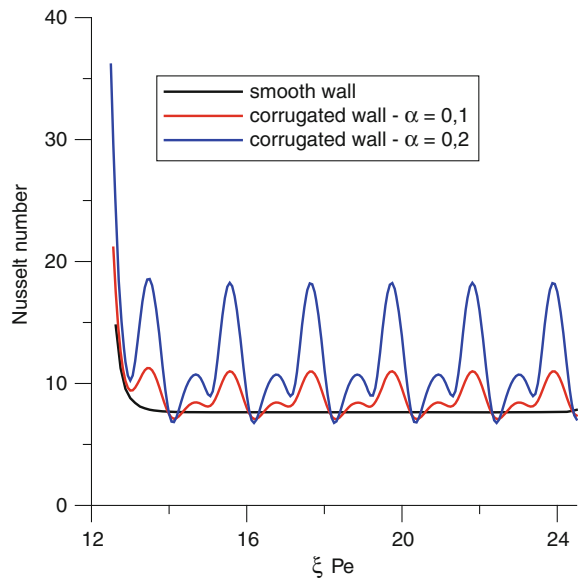


Fig. 2.32 Influence of Péclet number on the bulk temperature behavior along regions 1 and 2 in steady state, for $Pe = 10$ and 30 , and $\alpha = 0.1$ (Castellões et al. 2010)

In order to investigate the heat transfer enhancement, Fig. 2.33 depicts the local Nusselt number for $Pe = 10$ for both $\alpha = 0.1$ and 0.2 . As reference, the smooth parallel-plate channel case is also plotted. It is observed that even with the lower corrugation amplitude value some noticeable heat transfer enhancement is already

Fig. 2.33 Local Nusselt numbers at steady state for smooth and corrugated channels with $Pe = 10$ and $\alpha = 0$ (smooth parallel-plate channel), 0.1 , and 0.2 (Castellões et al. 2010)



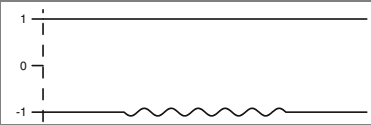
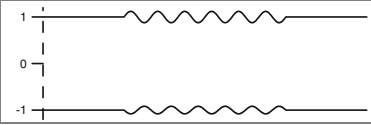
evident, and marked increases in the local heat transfer coefficient are achieved for the higher corrugation amplitude value for this value of Pe .

Aimed at investigating the corrugation effects of one wall on the heat transfer rate at the opposite wall, a few results are now reported for nonsymmetrical cases (Castellões 2010), as summarized in Table 2.15.

Besides the three nonsymmetrical cases given in Table 2.15, in order to provide comparative results, two additional symmetrical cases are also included in the analysis, namely the case with no corrugation (smooth walls) and the reference case proposed by Wang and Chen (2002), which is symmetrical with the lower and upper plates corrugation amplitude given as $\alpha = 0.1$.

Figure 2.34 depicts the local Nusselt number at the lower plate, which has the same corrugation amplitude for Cases 0, 1, 2, and the base case. It is observed that for lower corrugation amplitudes of the upper plate, higher amplitude oscillations of the Nusselt number are observed at the lower plate. For instance, one may observe that Case 0, which employs a smooth upper plate ($\alpha_1 = 0$), and Case 1, which employs $\alpha_1 = 0.05$, yields higher amplitude oscillations for the lower plate Nusselt number in comparison to the reference case ($\alpha_1 = 0.1$), while Case 3, which employs a higher corrugation amplitude at the upper plate ($\alpha_1 = 0.15$), yields a lower amplitude oscillation for the local Nusselt number estimated at the lower plate. The local Nusselt number at the upper plate for these different configurations is shown in Fig. 2.35. It is observed that keeping the corrugation amplitude of the lower plate constant and varying the corrugation amplitude of the upper plate yields significant variation in both amplitude and phase lag of the local Nusselt number at the upper plate. One may notice that a nontrivial behavior is observed, for instance Case 0, which employs a smooth upper wall ($\alpha_1 = 0$), yields significant oscillation on the Nusselt number at this same wall due to corrugation effects of the opposite

Table 2.15 Summary of the nonsymmetrical wall corrugation cases

Case	Corrugation amplitude at the lower wall, α_0	Corrugation amplitude at the upper wall, α_1	Schematic drawing
0	0.1	0	
1	0.1	0.05	
2	0.1	0.15	

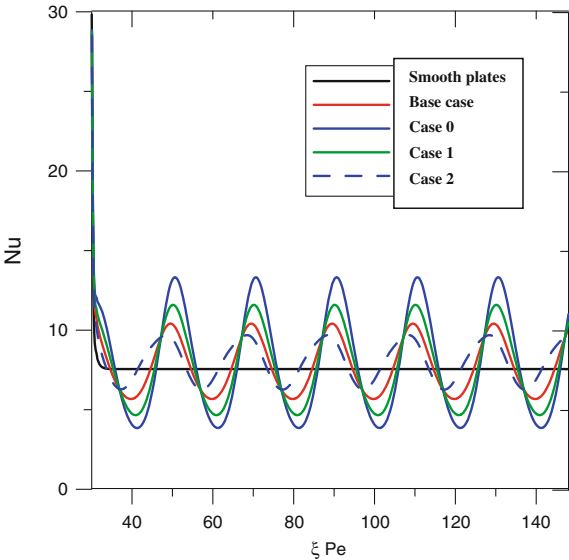


Fig. 2.34 Local Nusselt numbers at the lower wall at steady state for nonsymmetrical corrugated channels with $Pe = 10$ (Castellões 2010)

Fig. 2.35 Local Nusselt numbers at the upper wall at steady state for nonsymmetrical corrugated channels with $Pe = 10$ (Castellões 2010)

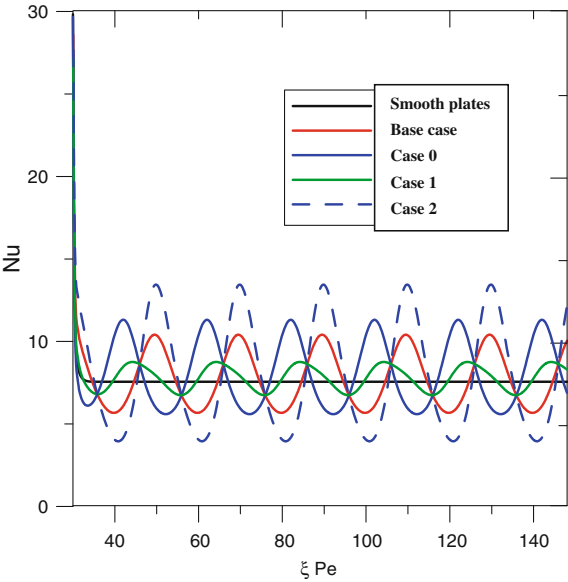


plate. This Nusselt number amplitude oscillation is even higher than observed for Case 1 and the base case, which employ corrugated upper walls. Nonetheless, the case which yields the highest amplitude oscillation on the calculated Nusselt number at the upper wall is Case 2, which in fact corresponds to the case with highest corrugation amplitude at the upper wall ($\alpha_1 = 0.15$).

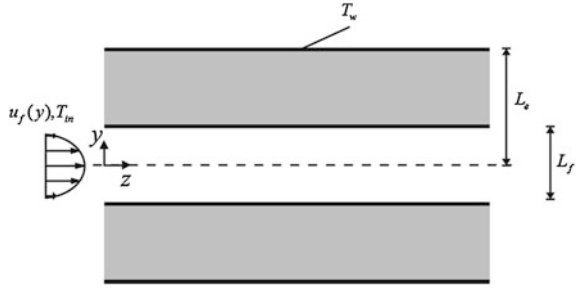
2.6 Conjugated Heat Transfer in Microchannels

Participation of the microchannel or microsystem substrate in the overall heat transfer process can be a major effect in the prediction of heat transfer coefficients in such situations, as briefly discussed in Sect. 1.2. Conjugated heat transfer in microchannels has been studied, for instance, both analytically and experimentally by Nunes et al. (2010), and its relevance was clearly demonstrated. In that work, since the substrate was formed by metallic walls, an improved lumped-differential formulation was developed and employed for the heat conduction along the walls, which could then be incorporated as boundary conditions in the convective heat transfer formulation for the fluid temperature. This section deals with a more general formulation for conjugated heat transfer in microchannels, by introducing the strategy of single domain reformulation in combination with the integral transform method, as described in Sect. 1.3.3. This approach was introduced in the solution of conjugated heat transfer in the simpler geometry of a parallel-plates channel (Knupp et al. 2012a, b), and then progressively extended to include other effects of relevance at the microscale, such as axial diffusion, preheating, or cooling ahead of the heat transfer section, rectangular cross sections, and transient state (Knupp et al. 2013a, b, 2014a, 2015a), besides, more recently, velocity and temperature jump conditions (Knupp et al. 2015c). The presentation that follows is initially based on a parallel-plates channel configuration (Knupp et al. 2012a), illustrating the analysis of first-order slip flow modeling (Knupp et al. 2015c), and then the extension to a three-dimensional configuration represented by a rectangular microchannel with its corresponding substrate (Knupp et al. 2014a).

2.6.1 *Parallel-Plates Channel*

A simple geometry of a parallel-plates channel is first illustrated for the analysis of conjugated heat transfer in microchannels, so as to allow for an exact analytical solution for comparison purposes. The schematic drawing presented in Fig. 2.36 illustrates the problem geometry and coordinates system, which involves incompressible flow in a parallel-plates channel, undergoing heat transfer due to a prescribed temperature at the external face of the channel wall (T_w), different from the inlet fluid temperature (T_{in}). Axial heat diffusion is disregarded in both fluid stream and channel walls, and thus the channel wall participates on the heat transfer process

Fig. 2.36 Schematic drawing of the conjugated problem in a parallel-plates channel



through transversal heat conduction only. The flow is considered hydrodynamically developed with velocity profile $u_f(y)$. The corresponding Knudsen number may fall either within the region of validity of the classical continuum governing equations and boundary conditions (Knupp et al. 2012b, 2013b) or within the region of validity of the first-order slip flow modeling (Knupp et al. 2015c).

Considering the more general situation, the dimensionless mathematical formulation of this conjugated problem with a temperature jump interface condition is given as

Fluid heat convection equation:

$$U_f(Y) \frac{\partial \theta_f(Y, Z)}{\partial Z} = \frac{\partial^2 \theta_f}{\partial Y^2}, \quad 0 < Y < Y_i, \quad Z > 0 \quad (2.220)$$

$$\theta_f(Y, 0) = 1, \quad 0 \leq Y \leq Y_i \quad (2.221)$$

$$\left. \frac{\partial \theta_f}{\partial Y} \right|_{Y=0} = 0, \quad Z > 0 \quad (2.222)$$

$$2Kn\beta_t \left. \frac{\partial \theta_f}{\partial Y} \right|_{Y=Y_i} + \theta_f(Y_i, Z) = \theta_s(Y_i, Z), \quad Z > 0 \quad (2.223)$$

Solid heat conduction equation:

$$\frac{\partial^2 \theta_s}{\partial Y^2} = 0, \quad Y_i < Y < 1, \quad Z > 0 \quad (2.224)$$

$$\left. \frac{k_s}{k_f} \frac{\partial \theta_s}{\partial Y} \right|_{Y=Y_i} = \left. \frac{\partial \theta_f}{\partial Y} \right|_{Y=Y_i}, \quad Z > 0 \quad (2.225)$$

$$\theta_s(1, Z) = 0, \quad Z > 0 \quad (2.226)$$

where Y_i indicates the dimensionless interface position between the fluid stream and the wall internal surface. The other corresponding dimensionless groups are given as

$$\begin{aligned}
Z = \frac{z/L_e}{\text{RePr}} = \frac{z}{L_e \text{Pe}}; \quad Y = \frac{y}{L_e}; \quad U = \frac{u}{u_{\text{av}}}; \quad \theta = \frac{T - T_w}{T_{\text{in}} - T_w}; \\
\text{Re} = \frac{u_{\text{av}} L_e}{\nu_f}; \quad \text{Pr} = \frac{\nu_f}{\alpha_f}; \quad \text{Pe} = \text{RePr} = \frac{u_{\text{av}} L_e}{\alpha_f}; \quad \text{Kn} = \frac{\lambda}{2L_f}
\end{aligned} \tag{2.227}$$

and

$$\beta_t = \frac{(2 - \alpha_t)}{\alpha_t} \frac{2\gamma}{(\gamma + 1)} \frac{1}{\text{Pr}} \tag{2.228}$$

is the wall temperature jump coefficient, α_t is the thermal accommodation coefficient, λ is the molecular mean free path, and $\gamma = c_p/c_v$. The dimensionless velocity profile, $U_f(Y)$, appearing in Eq. (2.220) in the slip flow regime situation is given by Mikhailov and Cotta (2005)

$$U_f(Y) = \frac{6\text{Kn}\beta_v + 3(1 - Y^2)/2}{1 + 6\text{Kn}\beta_v} \tag{2.229}$$

where

$$\beta_v = \frac{2 - \alpha_m}{\alpha_m} \tag{2.230}$$

is the wall velocity slip coefficient and α_m is the tangential momentum accommodation coefficient.

The solution through single domain formulation and integral transforms is achieved by proposing a fictitious thin layer between the fluid stream and the channel wall, in such a way that this layer can be chosen with dimensionless thickness and thermal conductivity, L_{fic} and K_{fic} , respectively, so as to impose the desired temperature jump condition that results in $\theta_f(Y_i, Z) \neq \theta_s(Y_i, Z)$.

A one-dimensional steady-state heat conduction formulation is imposed to the fictitious layer, which results in a linear temperature distribution in the space coordinate, yielding the following relations for the interface conditions:

$$\begin{aligned}
-K_{\text{fic}} \frac{\partial \theta_{\text{fic}}}{\partial Y} = -\frac{\partial \theta_f}{\partial Y} \Big|_{Y=Y_i}, \quad \theta_{\text{fic}}(Y_i, Z) = \theta_f(Y_i, Z), \\
\theta_{\text{fic}}(Y_i + L_{\text{fic}}, Z) = \theta_s(Y_i + L_{\text{fic}}, Z)
\end{aligned} \tag{2.231a-c}$$

and, therefore,

$$\theta_s(Y_i + L_{\text{fic}}, Z) - \theta_f(Y_i, Z) = -\frac{L_{\text{fic}}}{K_{\text{fic}}} \frac{\partial \theta_f}{\partial Y} \Big|_{Y=Y_i} \tag{2.232}$$

However, from the temperature jump condition given by Eq. (2.223), the temperature difference between the fluid at the interface and the channel wall internal surface must be equal to

$$\theta_s(Y_i + L_{\text{fic}}, Z) - \theta_f(Y_i, Z) = -2\text{Kn}\beta_t \left. \frac{\partial \theta_f}{\partial Y} \right|_{Y=Y_i} \quad (2.233)$$

Therefore, the simulation of a temperature jump condition through the single domain formulation strategy, using the fictitious material layer, can be obtained with the relation

$$\frac{L_{\text{fic}}}{K_{\text{fic}}} = 2\text{Kn}\beta_t \quad (2.234)$$

Hence, the fictitious layer introduced can be set with arbitrary values for the dimensionless thickness and thermal conductivity in such a way that the ratio given by Eq. (2.234) is attained, in order to achieve the desired temperature jump condition. The conjugated problem including the temperature jump condition can then be written as a single domain formulation with space variable coefficients

$$U(Y) \frac{\partial \theta^*(Y, Z)}{\partial Z} = \frac{\partial}{\partial Y} \left(K(Y) \frac{\partial \theta^*}{\partial Y} \right), \quad 0 < Y < 1 + L_{\text{fic}}, \quad Z > 0 \quad (2.235)$$

$$\theta^*(Y, 0) = 1, \quad 0 \leq Y \leq 1 + L_{\text{fic}} \quad (2.236)$$

$$\left. \frac{\partial \theta^*}{\partial Y} \right|_{Y=0} = 0, \quad \theta^*(1 + L_{\text{fic}}, Z) = 0, \quad Z > 0 \quad (2.237a, b)$$

where

$$U(Y) = \begin{cases} U_f(Y), & \text{if } 0 < Y < Y_i \\ 0, & \text{if } Y_i < Y < 1 + L_{\text{fic}} \end{cases} \quad (2.238)$$

$$K(Y) = \begin{cases} 1, & \text{if } 0 < Y < Y_i \\ K_{\text{fic}}, & \text{if } Y_i < Y < Y_i + L_{\text{fic}} \\ k_s/k_f, & \text{if } Y_i + L_{\text{fic}} < Y < 1 + L_{\text{fic}} \end{cases} \quad (2.239)$$

Once the problem given by Eqs. (2.235)–(2.239) is solved, the final temperature distribution can be readily obtained by disregarding the fictitious layer from the solution

$$\theta(Y, Z) = \begin{cases} \theta^*(Y, Z), & \text{if } 0 \leq Y \leq Y_i \\ \theta^*(Y + L_{\text{fic}}, Z), & \text{if } Y > Y_i \end{cases} \quad (2.240)$$

So, whereas θ^* is defined for $0 \leq Y \leq 1 + L_{\text{fic}}$, θ is defined for $0 \leq Y \leq 1$, the original problem domain before the introduction of the fictitious layer.

Following the single domain formulation procedure described in Sect. 1.3.3, the problem given by Eqs. (2.235)–(2.239) can be solved through the proposition of the eigenvalue problem directly obtained from the application of separation of variables, so that all the information concerning the transition of the two subdomains are represented within the eigenvalue problem, by means of the space variable coefficients $K(Y)$ and $U(Y)$. Therefore, the following eigenvalue problem is employed:

$$\frac{d}{dY} \left(K(Y) \frac{d\psi_i(Y)}{dY} \right) + \mu_i^2 U(Y) \psi_i(Y) = 0 \quad (2.241)$$

$$\left. \frac{d\psi_i}{dY} \right|_{Y=0} = 0 \quad (2.242)$$

$$\psi_i(1 + L_{\text{fic}}) = 0 \quad (2.243)$$

The eigenvalue problem given by Eqs. (2.241)–(2.243) is solved by the GITT itself in order to provide a hybrid numerical–analytical solution, as described in Sect. 1.3.4, and also detailed in (Knupp et al. 2012b) for this particular one-dimensional problem. Once this solution is made available, problem (2.235)–(2.239) becomes exactly transformable, and the analytical solution can be written as

$$\theta^*(Y, Z) = \sum_{i=1}^N \bar{\theta}_{0,i}^* \exp(-\mu_i^2 Z) \tilde{\psi}_i(Y) \quad (2.244)$$

where the inlet condition is transformed as

$$\bar{\theta}_{0,i}^* = \int_0^1 U(Y) \tilde{\psi}_i(Y) \theta^*(Y, 0) dY \quad (2.245)$$

The conjugated problem here considered, either with the classical temperature continuity at the fluid–solid interface or with the temperature jump condition, also allows for exact solution through the Classical Integral Transform approach, as detailed in Knupp et al. (2012a, 2015c), respectively. This exact solution is used to provide benchmark results in the discussion that follows.

Some results are also reported for the local Nusselt number, calculated as (Knupp et al. 2015a–c)

$$\text{Nu}_Z = \frac{h_Z D_h}{k_f} = - \frac{\frac{2L_f}{L_e} \frac{\partial \theta_f(Y, Z)}{\partial Y} \Big|_{Y=Y_i}}{\theta_{f,b}(Z) - \theta_s(Y_i, Z)} \quad (2.246)$$

where $D_h = 2L_f$ is the hydraulic diameter. The dimensionless fluid bulk temperature is given as

$$\theta_{f,b}(Z) = \frac{\int_0^{Y_i} U_f(Y) \theta_f(Y, Z) dY}{\int_0^{Y_i} U_f(Y) dY} \quad (2.247)$$

In order to avoid the direct evaluation of the derivative $\partial\theta/\partial Y|_{Y=Y_i}$ when using the single domain formulation, the following alternative integral balance formula (Cotta and Mikhailov 1997) has been used:

$$\left. \frac{\partial\theta}{\partial Y} \right|_{Y=Y_i} = \int_0^{Y_i} U(Y) \frac{\partial\theta}{\partial Z} dY \quad (2.248)$$

In the example chosen to demonstrate the methodology, the dimensionless thermal conductivity has been calculated motivated by an application with a microchannel made of PMMA (polymethyl methacrylate), with $k_s = 0.2 \text{ W/m}^\circ\text{C}$, and with air as the working fluid ($k_f = 0.0271 \text{ W/m}^\circ\text{C}$), so that $k_s/k_f = 7.38$. Also, for this example, it has been considered $\beta_v = 1.5$, $\beta_r = 2.0$, and $\text{Kn} = 0.025$ (if not explicitly stated otherwise). For the hybrid solution with the single domain formulation and integral transforms, it has been employed the dimensionless length value of $L_{\text{fic}} = 0.05$ in all situations, and the dimensionless thermal conductivity has been obtained from Eq. (2.234) in order to achieve the desired temperature jump condition.

The convergence behavior is presented in Table 2.16, showing the temperature at three different transversal positions, $Y = 0$ (channel centreline), $Y = 0.25$, and $Y = 0.5$ (wall), at the axial length of $Z = 0.05$, for different truncation orders in the eigenvalue problem solution, from $M = 25$ up to $M = 300$, besides the exact solution. All results are presented with $N = 5$ terms in the final temperature expansion which is enough to yield full convergence to the five digits shown. For this example with conjugation effects, it is observed that a convergence of at least three significant digits is obtained for $M < 300$, yielding at least two digits of agreement with the exact solution. In the worst situation, occurring at the wall, the relative error remains at most around 1 % for $M = 300$. Figure 2.37 depicts the transversal temperature profiles at three different axial lengths, $Z = 0.05$, 0.5 , and 0.2 , where one can observe a good agreement of the proposed solution with the exact solution throughout, for both the fluid stream and the channel wall regions.

The convergence behavior analysis for the fully developed Nusselt number is illustrated in Table 2.17, which shows the calculated fully developed Nusselt numbers for different truncation orders in the eigenvalue problem solution, from $M = 25$ to $M = 300$, besides the exact fully developed Nusselt number. Here a convergence of two significant digits is observed for $M < 300$, with two significant digits of agreement with the exact value, presenting a relative error of 0.6 % at $M = 300$.

Table 2.16 Convergence behavior of the temperature distribution with respect to the eigenvalue problem truncation order (Knupp et al. 2015c)

M	$\theta(0, 0.05)$	Relative err. (%)	$\theta(0.25, 0.05)$	Relative err. (%)	$\theta(0.5, 0.05)$	Relative err. (%)
50	0.89526	0.36	0.72414	1.02	0.29571	5.86
75	0.89589	0.29	0.72592	0.78	0.30023	4.42
100	0.89674	0.20	0.72758	0.55	0.30403	3.21
125	0.89705	0.16	0.72843	0.43	0.30641	2.46
150	0.89730	0.13	0.72886	0.37	0.30724	2.19
175	0.89750	0.11	0.72938	0.30	0.30868	1.73
200	0.89760	0.10	0.72957	0.28	0.30906	1.61
225	0.89772	0.09	0.72985	0.24	0.30979	1.38
250	0.89779	0.08	0.73001	0.22	0.31021	1.25
275	0.89785	0.07	0.73015	0.20	0.31049	1.16
300	0.89791	0.06	0.73030	0.18	0.31092	1.02
Exact	0.89849	—	0.73159	—	0.31412	—

Fig. 2.37 Transversal temperature profiles (exact solution and single domain formulation with GITT) (Knupp et al. 2015c)

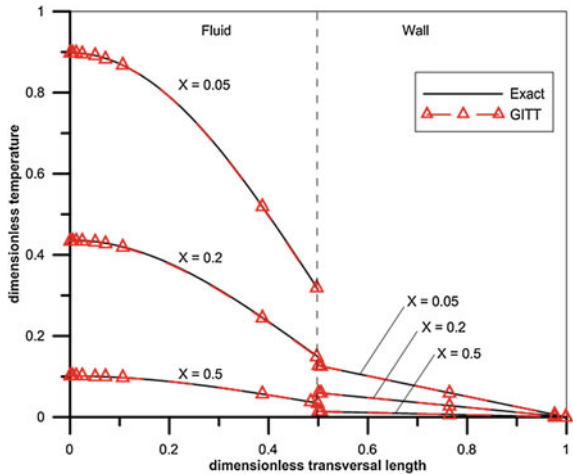
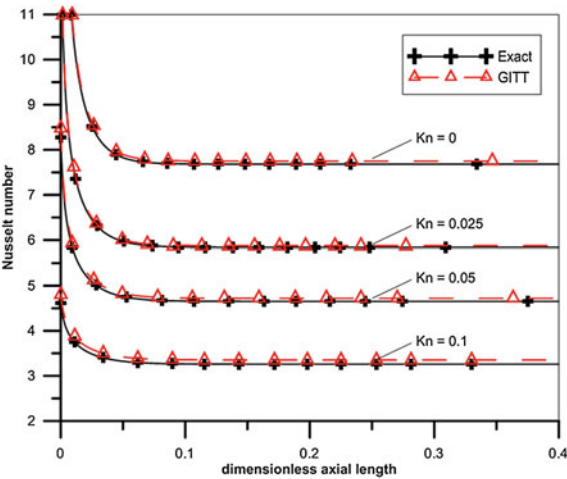


Figure 2.38 depicts the local Nusselt number (exact solution and single domain formulation with GITT) for different Knudsen numbers, $\text{Kn} \rightarrow 0$ (no-slip and no temperature jump), $\text{Kn} = 0.025, 0.05$, and 0.1 . Despite the fairly good agreement of the proposed hybrid solution with the exact one for all cases, further improvement on convergence behavior and agreement with the exact solution can be obtained for instance by fixing $K_{\text{fic}} = 1$ (the same dimensionless thermal conductivity of the fluid) in all cases, with the thickness L_{fic} calculated accordingly, in order to yield a simpler space variable coefficient $K(Y)$, with less abrupt variations.

Table 2.17 Convergence behavior of the fully developed Nusselt number with respect to the eigenvalue problem truncation order in the example with conjugation effects (Knupp et al. 2015c)

M	Nu	Relative error (%)
50	6.0480	3.58
75	5.9895	2.58
100	5.9491	1.89
125	5.9241	1.46
150	5.9132	1.27
175	5.8992	1.03
200	5.8938	0.939
225	5.8864	0.813
250	5.8818	0.733
275	5.8783	0.674
300	5.8741	0.603
Exact	5.8390	—

Fig. 2.38 Local Nusselt numbers (exact solution and single domain formulation with GITT), for different Knudsen numbers (Knupp et al. 2015c)



2.6.2 Rectangular Channel

It is here considered the incompressible internal laminar flow of a Newtonian fluid inside a rectangular microchannel (Knupp et al. 2014a–c), where the external wall faces are subjected to a prescribed temperature T_w . The channel walls participate in the heat transfer process both in the transversal and longitudinal directions. The fluid flows with a fully developed velocity profile in laminar regime, $u(x,y)$, without slip, and thermally developing from an uniform inlet temperature, T_{in} . A schematic drawing of the geometry and coordinates is given in Fig. 2.39a, b.

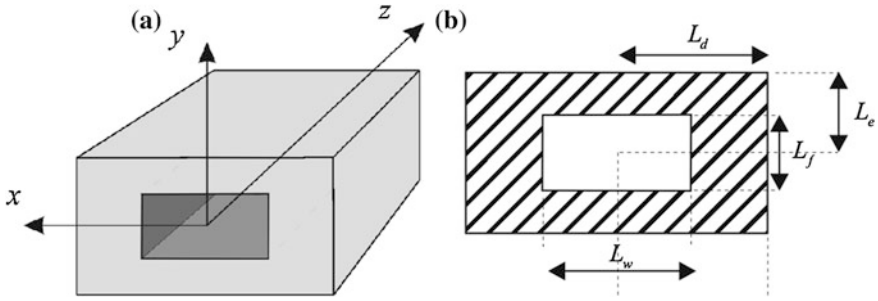


Fig. 2.39 Schematic representation of the three-dimensional conjugated problem in a rectangular microchannel: (a) coordinates system; (b) cross section

For a sufficiently long microchannel, the steady-state conjugated problem is thus formulated as a single domain, as described in Sect. 1.3.3, in the form:

$$u(x, y)w_f \frac{\partial T(x, y, z)}{\partial z} = \frac{\partial}{\partial x} \left(k(x, y) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k(x, y) \frac{\partial T}{\partial y} \right) + k(x, y) \frac{\partial^2 T}{\partial z^2},$$

$$0 < x < L_d, \quad 0 < y < L_e, \quad 0 < z < z_\infty \quad (2.249)$$

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = 0; \quad T(L_d, y, z) = T_w \quad (2.250a, b)$$

$$\left. \frac{\partial T}{\partial y} \right|_{y=0} = 0; \quad T(x, L_e, z) = T_w \quad (2.251a, b)$$

$$T(x, y, 0) = T_{in}; \quad \left. \frac{\partial T}{\partial z} \right|_{z=z_\infty} = 0 \quad (2.252a, b)$$

where the following space variable functions are defined to represent the abrupt transition at the interfaces between the fluid and solid regions:

$$u(x, y) = \begin{cases} u_f(x, y), & \text{for } 0 \leq y \leq L_f/2 \text{ and } 0 \leq x \leq L_w/2 \\ 0 & \text{elsewhere} \end{cases} \quad (2.253)$$

$$k(x, y) = \begin{cases} k_f, & \text{for } 0 \leq y \leq L_f/2 \text{ and } 0 \leq x \leq L_w/2 \\ k_s, & \text{elsewhere} \end{cases} \quad (2.254)$$

where w_f is the fluid thermal capacitance, $\rho_f c_{pf}$, k_s is the thermal conductivity of the wall material, k_f is the fluid thermal conductivity, and $u_f(x, y)$ is the fully developed velocity profile. Now consider the definition of the following dimensionless groups:

$$\begin{aligned} Z = \frac{z/D_h}{\text{RePr}} = \frac{z}{D_h \text{Pe}}; \quad Y = \frac{y}{L_e}; \quad X = \frac{x}{L_d}; \quad U_f = \frac{u}{u_{av}}; \quad \theta = \frac{T - T_w}{T_{in} - T_w}; \\ Y_i = \frac{L_f}{2L_e}; \quad X_i = \frac{L_w}{2L_d}; \quad K = \frac{k}{k_f}; \quad \text{Re} = \frac{u_{av} D_h}{\nu}; \quad \text{Pr} = \frac{\nu}{\alpha}; \\ \text{Pe} = \text{Re Pr} = \frac{u_{av} D_h}{\alpha}; \quad \alpha = \frac{k_f}{w_f}; \quad \sigma_y = \frac{L_e}{\frac{L_f L_w}{L_f + L_w}}; \quad \sigma_x = \frac{L_d}{\frac{L_f L_w}{L_f + L_w}} \end{aligned} \quad (2.255)$$

where u_{av} is the average flow velocity. The problem given by Eqs. (2.249)–(2.254) may then be rewritten in dimensionless form, already with homogeneous boundary conditions in the X and Y directions, as

$$\begin{aligned} U(X, Y) \frac{\partial \theta(X, Y, Z)}{\partial Z} = \frac{K(X, Y)}{\text{Pe}^2} \frac{\partial^2 \theta}{\partial Z^2} + \frac{4}{\sigma_y^2} \frac{\partial}{\partial Y} \left(K(X, Y) \frac{\partial \theta}{\partial Y} \right) \\ + \frac{4}{\sigma_x^2} \frac{\partial}{\partial X} \left(K(X, Y) \frac{\partial \theta}{\partial X} \right), \quad 0 < X < 1, \quad 0 < Y < 1, \quad 0 < Z < Z_\infty \end{aligned} \quad (2.256)$$

$$\left. \frac{\partial \theta}{\partial X} \right|_{X=0} = 0; \quad \theta(1, Y, Z) = 0 \quad (2.257a, b)$$

$$\left. \frac{\partial \theta}{\partial Y} \right|_{Y=0} = 0; \quad \theta(X, 1, Z) = 0 \quad (2.258a, b)$$

$$\theta(X, Y, 0) = 1; \quad \left. \frac{\partial \theta}{\partial Z} \right|_{Z=Z_\infty} = 0 \quad (2.259a, b)$$

The dimensionless space variable coefficients are given as

$$U(X, Y) = \begin{cases} U_f(X, Y), & \text{for } 0 \leq Y \leq Y_i \text{ and } 0 \leq X \leq X_i \\ 0, & \text{elsewhere} \end{cases} \quad (2.260)$$

$$K(X, Y) = \begin{cases} 1, & \text{for } 0 \leq Y \leq Y_i \text{ and } 0 \leq X \leq X_i \\ k_s/k_f, & \text{elsewhere} \end{cases} \quad (2.261)$$

The dimensionless fully developed velocity profile is written as (Cotta 1993)

$$U_f(X, Y) = A^* \sum_{k=1}^{\infty} B_k F_k(Y) G_k(X) \quad (2.262)$$

where

$$A^* = \frac{48}{\pi^3} \left[1 - \frac{192}{\pi^5} \sum_{i=1}^{\infty} \frac{1}{Y_i} \tanh(k\pi Y_i/2X_i) \right]^{-1}, \quad B_k = \frac{(-1)^{(k-1)/2}}{k^3} \quad (2.263a, b)$$

$$F_k(Y) = 1 - \cosh(a_k Y) / \cosh(a_k Y_i), \quad G_k(X) = \cos(a_k X), \quad a_k = \frac{k\pi}{2X_i} \quad (2.264a-c)$$

As proposed in Knupp et al. (2013a, b) for a two-dimensional problem, a pseudo-transient term is incorporated to the above formulation as an alternative to numerically solving the transformed boundary value problem

$$\begin{aligned} \frac{\partial \theta(X, Y, Z, t)}{\partial t} = & -U(X, Y) \frac{\partial \theta}{\partial Z} + \frac{K(X, Y)}{\text{Pe}^2} \frac{\partial^2 \theta}{\partial Z^2} + \frac{4}{\sigma_y^2} \frac{\partial}{\partial Y} \left(K(X, Y) \frac{\partial \theta}{\partial Y} \right) \\ & + \frac{4}{\sigma_x^2} \frac{\partial}{\partial X} \left(K(X, Y) \frac{\partial \theta}{\partial X} \right), \quad 0 < X < 1, \quad 0 < Y < 1, \quad 0 < Z < Z_{\infty} \end{aligned} \quad (2.265)$$

$$\left. \frac{\partial \theta}{\partial X} \right|_{X=0} = 0; \quad \theta(1, Y, Z, t) = 0 \quad (2.266a, b)$$

$$\left. \frac{\partial \theta}{\partial Y} \right|_{Y=0} = 0; \quad \theta(X, 1, Z, t) = 0 \quad (2.267a, b)$$

$$\theta(X, Y, 0, t) = 1; \quad \left. \frac{\partial \theta}{\partial Z} \right|_{Z=Z_{\infty}} = 0 \quad (2.268a, b)$$

$$\theta(X, Y, Z, 0) = \theta_0 \quad (2.269)$$

where the dimensionless initial condition θ_0 is an arbitrary function, preferably an estimate of the steady-state solution so as to accelerate convergence of the pseudo-transient solution. Thus, for the integral transform solution of problem (2.265)–(2.269), via the partial transformation scheme, the following double integral transformation is proposed:

$$\text{transform : } \bar{\theta}_i(Z, t) = \int_0^1 \int_0^1 \tilde{\psi}_i(X, Y) \theta(X, Y, Z, t) dX dY \quad (2.270)$$

$$\text{inverse : } \theta(X, Y, Z, t) = \sum_{i=1}^{\infty} \tilde{\psi}_i(X, Y) \bar{\theta}_i(Z, t) \quad (2.271)$$

where the eigenfunctions, $\tilde{\psi}_i(X, Y)$, and corresponding eigenvalues, μ_i , are obtained from the following two-dimensional eigenvalue problem:

$$\frac{4}{\sigma_x^2} \frac{\partial}{\partial X} \left(K(X, Y) \frac{\partial \psi_i(X, Y)}{\partial X} \right) + \frac{4}{\sigma_y^2} \frac{\partial}{\partial Y} \left(K(X, Y) \frac{\partial \psi_i(X, Y)}{\partial Y} \right) + \mu_i^2 \psi_i = 0 \quad (2.272)$$

$$\left. \frac{\partial \psi_i}{\partial X} \right|_{X=0} = 0; \quad \psi_i(1, Y) = 0 \quad (2.273a, b)$$

$$\left. \frac{\partial \psi_i}{\partial Y} \right|_{Y=0} = 0; \quad \psi_i(X, 1) = 0 \quad (2.274a, b)$$

with the normalized eigenfunctions given as

$$\tilde{\psi}_i(X, Y) = \frac{\psi_i(X, Y)}{\sqrt{N_i}} \quad (2.275)$$

$$N_i = \int_0^1 \int_0^1 \psi_i^2(X, Y) dX dY \quad (2.276)$$

The solution of problem (2.272)–(2.274) requires application of the Generalized Integral Transform Technique (GITT) itself (Cotta 1993), following the procedure detailed in Sect. 1.3.4, and here summarized for the specific application under consideration. For the moment, considering that the eigenfunctions $\tilde{\psi}_i(X, Y)$ and eigenvalues μ_i are known, the integral operator $\int_0^1 \int_0^1 \tilde{\psi}_i(X, Y)(\cdot) dX dY$ is applied to problem (2.265)–(2.269), yielding the following one-dimensional partial differential system:

$$\frac{\partial \bar{\theta}_i(Z, t)}{\partial t} + \mu_i^2 \bar{\theta}_i = g_i(Z, t, \bar{\theta}), \quad i = 1, 2, \dots \quad (2.277)$$

where

$$g_i(Z, t, \bar{\theta}) = - \sum_{n=1}^{\infty} \frac{\partial \bar{\theta}_n}{\partial Z} \int_0^1 \int_0^1 U(X, Y) \tilde{\psi}_i \tilde{\psi}_n dX dY \quad (2.278)$$

$$+ \sum_{n=1}^{\infty} \frac{\partial^2 \bar{\theta}_n}{\partial Z^2} \int_0^1 \int_0^1 \frac{K(X, Y)}{\text{Pe}^2} \tilde{\psi}_i \tilde{\psi}_n dX dY$$

$$\bar{\theta} = \{\bar{\theta}_1, \bar{\theta}_2, \dots\} \quad (2.279)$$

with the transformed boundary and initial conditions

$$\bar{\theta}_i(0, t) = \int_0^1 \int_0^1 \tilde{\psi}_i dX dY; \quad \left. \frac{\partial \bar{\theta}_i}{\partial Z} \right|_{Z=Z_{\infty}} = 0 \quad (2.280a, b)$$

$$\bar{\theta}_i(Z, 0) = \int_0^1 \int_0^1 \tilde{\psi}_i \bar{\theta}_0 dX dY \quad (2.281)$$

The partial differential system given by Eqs. (2.277)–(2.281), once truncated to a sufficiently large finite-order N to satisfy the requested accuracy, can be numerically solved for the transformed potentials, $\bar{\theta}_i(Z, t)$. The representation of the dimensionless temperature field, θ , is then obtained at any position (X, Y, Z) and pseudo-time t through the inverse formula given by Eq. (2.271). It should be recalled that t in this pseudo-transient formulation should be large enough so that the steady state be reached.

Back to the solution of the space variable coefficients eigenvalue problem defined in Eqs. (2.272)–(2.274), following the formalism presented in Sect. 1.3.4, a simpler eigenvalue problem is considered for implementing its GITT solution. This auxiliary problem is here taken simply as a constant coefficients case

$$\frac{\partial^2 \Omega_i(X, Y)}{\partial X^2} + \frac{\partial^2 \Omega_i(X, Y)}{\partial Y^2} + \lambda_i^2 \Omega_i(X, Y) = 0 \quad (2.282)$$

$$\left. \frac{\partial \Omega_i}{\partial X} \right|_{X=0} = 0; \quad \Omega_i(1, Y) = 0 \quad (2.283a, b)$$

$$\left. \frac{\partial \Omega_i}{\partial Y} \right|_{Y=0} = 0; \quad \Omega_i(X, 1) = 0 \quad (2.284a, b)$$

Applying separation of variables to problem (2.282)–(2.284) above, it is then proposed that the auxiliary eigenfunction be expressed by the product of two one-dimensional eigenfunctions as

$$\Omega(X, Y) = \zeta(X)\xi(Y) \quad (2.285)$$

where

$$\frac{d^2\zeta_m(X)}{dX^2} + \lambda_{X,m}^2\zeta_m = 0 \quad (2.286)$$

$$\left.\frac{d\zeta_m}{dX}\right|_{X=0} = 0; \quad \zeta_m(1) = 0 \quad (2.287a, b)$$

and

$$\frac{d^2\xi_n(Y)}{dY^2} + \lambda_{Y,n}^2\xi_n = 0 \quad (2.288)$$

$$\left.\frac{d\xi_n}{dY}\right|_{Y=0} = 0; \quad \xi_n(1) = 0 \quad (2.289a, b)$$

The eigenvalue problems (2.286)–(2.289) have straightforward analytical solutions and the auxiliary eigenfunctions and eigenvalues are then written as

$$\Omega_{m,n}(X, Y) = \zeta_m(X)\xi_n(Y) \quad (2.290)$$

$$\lambda_{m,n}^2 = \lambda_{X,m}^2 + \lambda_{Y,n}^2 \quad (2.291)$$

Thus, the following integral transform pair is proposed for the determination of the original eigenfunctions, $\psi(X, Y)$:

$$\text{Transform: } \bar{\psi}_{m,n} = \int_0^1 \int_0^1 \tilde{\Omega}_{m,n}(X, Y)\psi(X, Y)dXdY \quad (2.292)$$

$$\text{Inverse: } \psi(X, Y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{\Omega}_{m,n}(X, Y)\bar{\psi}_{m,n} \quad (2.293)$$

where

$$\tilde{\Omega}_{m,n}(X, Y) = \frac{\Omega_{m,n}(X, Y)}{\sqrt{N_{m,n}}} \quad (2.294)$$

$$N_{m,n} = \int_0^1 \int_0^1 \Omega_{m,n}^2(X, Y) dX dY \quad (2.295)$$

In order to more efficiently compute the eigenfunction expansion represented by the inverse formula in Eq. (2.293), the double summation is rewritten as a single summation, according to an appropriate reordering rule for the most relevant terms in the final numerical result of the multiple sum (Cotta and Mikhailov 1997; Correa et al. 1997). Since the full solution is not known a priori, one fairly simple and effective reordering scheme is based on the sum of the squared eigenvalues, as given by Eqs. (2.249)–(2.254). Then, the inversion formula is rewritten as

$$\psi(X, Y) = \sum_{i=1}^{\infty} \tilde{\Omega}_i(X, Y) \bar{\psi}_i \quad (2.296)$$

where for each index i is associated a pair (m, n) corresponding to the indices in the eigenvalues for each coordinate direction, $\lambda_{X,m}^2 + \lambda_{Y,n}^2$, once the result is reorganized in ascending order. The eigenvalue problem (2.272)–(2.274) is then integral transformed through the operator $\int_0^1 \int_0^1 \tilde{\Omega}_i(X, Y)(\cdot) dX dY$, resulting in the following algebraic eigensystem analysis:

$$(\mathbf{A} + \mathbf{C})\{\bar{\psi}\} = \mu^2 \mathbf{B}\{\bar{\psi}\} \quad (2.297)$$

$$\begin{aligned} A_{ij} = & \int_0^1 \left\{ \left[1 - \frac{4}{\sigma_x^2} K(1, Y) \right] \tilde{\Omega}_i(1, Y) \frac{\partial \tilde{\Omega}_j}{\partial X} \Big|_{X=1} - \int_0^1 \left[1 - \frac{4}{\sigma_x^2} K(X, Y) \right] \frac{\partial \tilde{\Omega}_i}{\partial X} \frac{\partial \tilde{\Omega}_j}{\partial X} dX \right\} dY \\ & + \int_0^1 \left\{ \left[1 - \frac{4}{\sigma_y^2} K(X, 1) \right] \tilde{\Omega}_i(X, 1) \frac{\partial \tilde{\Omega}_j}{\partial Y} \Big|_{Y=1} - \int_0^1 \left[1 - \frac{4}{\sigma_y^2} K(X, Y) \right] \frac{\partial \tilde{\Omega}_i}{\partial Y} \frac{\partial \tilde{\Omega}_j}{\partial Y} dY \right\} dX \end{aligned} \quad (2.298)$$

$$C_{ij} = \lambda_i^2 \delta_{ij}, \quad B_{ij} = \int_0^1 \int_0^1 U(X, Y) \tilde{\Omega}_i(X, Y) \tilde{\Omega}_j(X, Y) dX dY, \quad (2.299a, b)$$

The algebraic eigenvalue problem defined by Eqs. (2.297)–(2.299) can then be numerically handled by well-established algorithms and subroutines for matrix eigenvalue problems, once system (2.297) is truncated to a sufficiently large finite-order M , providing the eigenvalues, μ_i , and corresponding eigenvectors, $\{\bar{\psi}_i\}$, which are combined through the inverse formula with reordering scheme given by Eq. (2.296), resulting in an explicit representation of the desired original eigenfunctions, $\psi_i(X, Y)$, thus completing the hybrid numerical–analytical solution.

The analysis is started by verifying the convergence of the numerical solution of the eigenvalue problem (2.272)–(2.274) and comparing it to previously published results for purely convective heat transfer in the square channel case (Cotta 1993; Knupp et al. 2014a), which can be reproduced with $L_e = L_d$ and $L_f = L_w$, and by letting $X_i \rightarrow 1$ and $Y_i \rightarrow 1$, which corresponds to considering the very thin wall case without participation. From the results for the first ten eigenvalues of the two-dimensional problem, presented in Table 2.18, one can observe that for a truncation order in the algebraic eigensystem of $M = 60$, after employing the reordering scheme, at least five significant digits of agreement with the reference solution (Cotta 1993) are achieved. It should be mentioned that the reference results (Cotta 1993) were obtained with 12 terms in expansion for each coordinate direction, i.e., a total of 144 terms in the double summation without reordering. It is also observable that even at fairly low truncation orders ($M = 30$), at least four significant digits in the first ten eigenvalues have been reproduced.

Next, the analysis of the conjugated problem is performed, considering the case of a square rectangular channel with participating walls with $X_i = Y_i = 0.5$ and $k_s/k_f = 0.25$ (Knupp et al. 2014a), which corresponds to water flow inside acrylic (PMMA) microchannels. Figure 2.40a, b illustrates the space variable coefficients, $U(X, Y)$ and $K(X, Y)$, respectively, for this case.

Table 2.19 presents the convergence of the first ten eigenvalues of the conjugated problem with respect to the truncation order M of the algebraic eigensystem, where it can be observed a convergence to at least four significant digits within this range of truncation order, as also pointed out in (Knupp et al. 2014a).

Also, in Fig. 2.41 it is presented the convergence characteristics of the 15th eigenfunction of the two-dimensional problem with space variable coefficients, $\psi_{15}(X, Y)$, as a function of the coordinate Y and at $X = 0$, for different truncation orders M , where full convergence to the graphic scale is observed at the largest truncation orders of $M = 50$ and 60.

Table 2.18 Verification of the GITT solution for the two-dimensional eigenvalue problem in the pure convection case for a square channel ($X_i \rightarrow 1, Y_i \rightarrow 1$)

Order/M	20	30	40	50	60	Cotta (1993) ^a
1	2.97753	2.97753	2.97753	2.97753	2.97753	2.9775
2	17.7709	17.7708	17.7708	17.7708	17.7708	17.7708
3	18.4230	18.4229	18.4229	18.4228	18.4228	18.4228
4	39.4731	39.4726	39.4725	39.4725	39.4725	39.4724
5	48.3331	48.3312	48.3307	48.3305	48.3305	48.3304
6	48.5449	48.5432	48.5427	48.5426	48.5425	48.5425
7	74.1169	74.0761	74.0754	74.0752	74.0751	74.0751
8	80.1736	80.1128	80.1121	80.112	80.1119	80.1117
9	93.7606	93.5279	93.5234	93.5225	93.5221	93.5219
10	93.8132	93.5851	93.5807	93.5797	93.5794	93.5791

^aCotta (1993, p. 337)

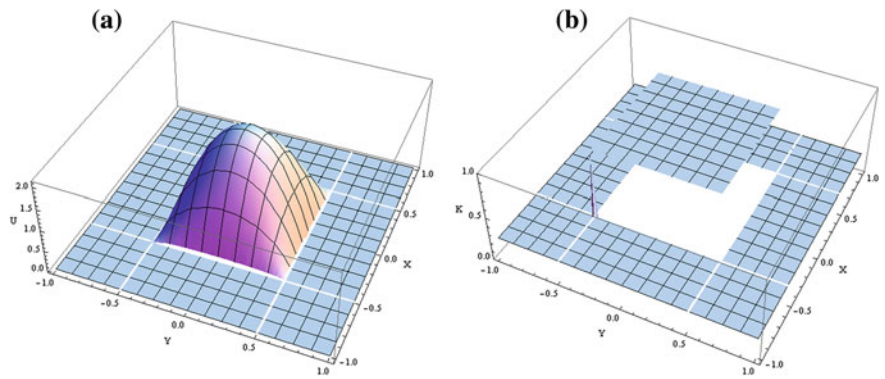


Fig. 2.40 Space variable coefficients in the single domain formulation: (a) $U(X, Y)$, (b) $K(X, Y)$ (Knupp et al. 2014a)

Table 2.19 Convergence of the first ten eigenvalues for the two-dimensional eigenvalue problem in the conjugated problem with a square channel ($X_i = Y_i = 0.5$)

Order/M	20	30	40	50	60
1	1.39134	1.38445	1.38423	1.38053	1.38041
2	4.78343	4.75225	4.75111	4.73512	4.73457
3	5.01807	4.98355	4.98342	4.96632	4.96613
4	6.14949	6.11103	6.11017	6.09191	6.09142
5	6.84602	6.82480	6.82418	6.81570	6.81535
6	6.91248	6.88612	6.8855	6.87494	6.87466
7	7.90935	7.86898	7.86790	7.85054	7.84986
8	8.24535	8.23550	8.23237	8.22749	8.22632
9	9.48937	9.46766	9.46472	9.45748	9.45653
10	10.1862	10.1161	10.1062	10.0821	10.0784

Figure 2.42 illustrates the transversal profiles of the temperature field, along the coordinate Y , from the channel centerline ($Y = 0$) up to the wall external surface ($Y = 1$), at $X = 0$, for various longitudinal positions along the flow, where one may clearly distinguish the different behaviors of the temperature field in the fluid ($Y < 0.5$) and wall ($Y > 0.5$) regions. The present solution is then verified by comparing numerical results of the temperature field against the finite elements solution from the commercial code COMSOL Multiphysics. It can then be noticed from Fig. 2.42 that the GITT and COMSOL solutions are fully coincident to the graphic scale.

A convergence analysis of the temperature field is then provided in Table 2.20, for the longitudinal position $Z = 0.5$ where it can be observed that with up to $N = 70$ terms in the eigenfunction expansion, one can achieve a consistent convergence of at least four significant digits in the dimensionless temperature field. The numerical

Fig. 2.41 Convergence of the eigenfunction $\psi_{15}(X, Y)$ for different truncation orders, M

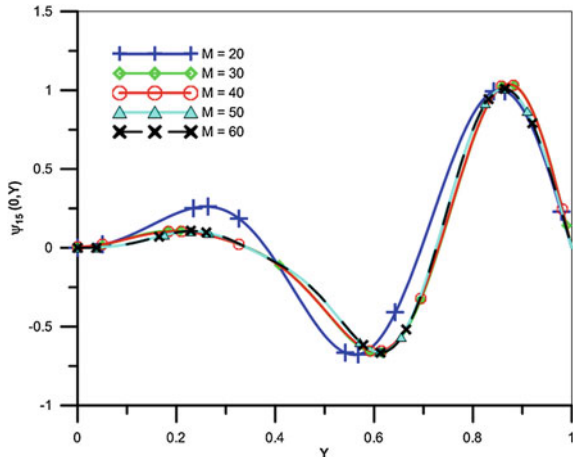


Fig. 2.42 Temperature profiles along the transversal coordinate (Y), at $X = 0$, from the channel centerline up to the wall external surface, for different longitudinal positions along the channel (aspect ratio = 1)

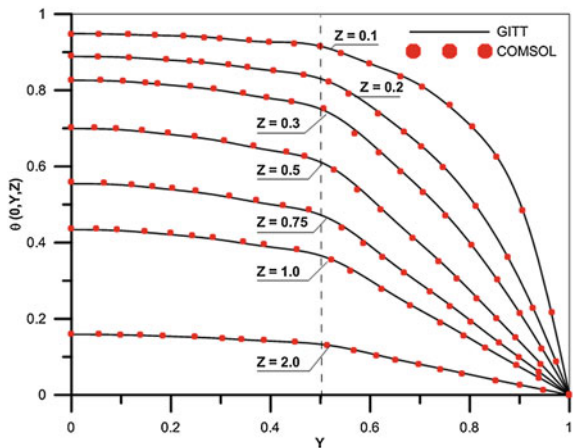


Table 2.20 Convergence of the temperature field eigenfunction expansion ($X = 0$)

Truncation order N	$Z = 0.5$			
	$Y = 0.0$	$Y = 0.3$	$Y = 0.6$	$Y = 0.9$
30	0.6991	0.6666	0.5169	0.1400
40	0.6994	0.6667	0.5169	0.1402
50	0.6994	0.6667	0.5169	0.1403
60	0.6994	0.6667	0.5169	0.1402
70	0.6993	0.6667	0.5169	0.1402
Numerical ^a	0.70	0.67	0.51	0.14

^aNumerical solution (COMSOL Multiphysics)

results from the COMSOL Multiphysics simulation are also shown in the last line of the column, and an agreement to the first two significant digits can be observed, since the third and fourth digits of the finite elements solution are not yet fully converged due to the meshing size employed in this example, despite the fact that the “extra fine” meshing option has been adopted in the automatic mesh generation of the COMSOL code. For further accuracy, it would be necessary a few trials to improve the meshing or to adopt an adaptive meshing scheme.

Figure 2.43 illustrates the evolution of the fluid bulk temperature along the flow for rectangular channels of different aspect ratio, $L_w/L_f = 1, 2.5, 5,$ and 10 , in comparison with the solution for the limiting case of a parallel-plates channel, $L_w/L_f \rightarrow \infty$. In this graph one may observe the progressive tendency of the bulk fluid temperature curves to approach the limiting case of a parallel-plates channel as the aspect ratio is increased, as expected, and notice that the bulk temperature progresses in a shorter dimensionless duct length toward the prescribed external wall temperature, as the aspect ratio is increased from the square channel to the parallel-plates case.

Additional verification of the GITT hybrid solution in comparison against the COMSOL numerical solution is also provided by considering a different aspect ratio of 2.5 (with $X_i = 0.7143$ and $Y_i = 0.5$). Figure 2.44a presents the transversal temperature distributions along the coordinate Y , from the center of the channel ($Y = 0$) up to the wall external surface ($Y = 1$), at $X = 0$, whereas Fig. 2.44b provides the temperature distributions along the coordinate X , also from the center of the channel ($X = 0$) up to the wall external surface ($X = 1$), at $Y = 0$, for different longitudinal positions along the channel length.

The location of the solid–fluid interface is drawn in these graphs represented with vertical dashed line to aid in the identification of the transition region. Also,

Fig. 2.43 Evolution of the bulk fluid temperature along the channel length for different aspect ration and comparison with the limiting case of a parallel-plates channel conjugated problem solution

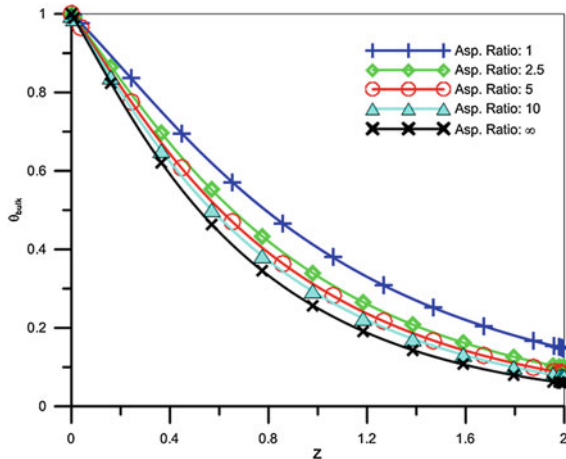
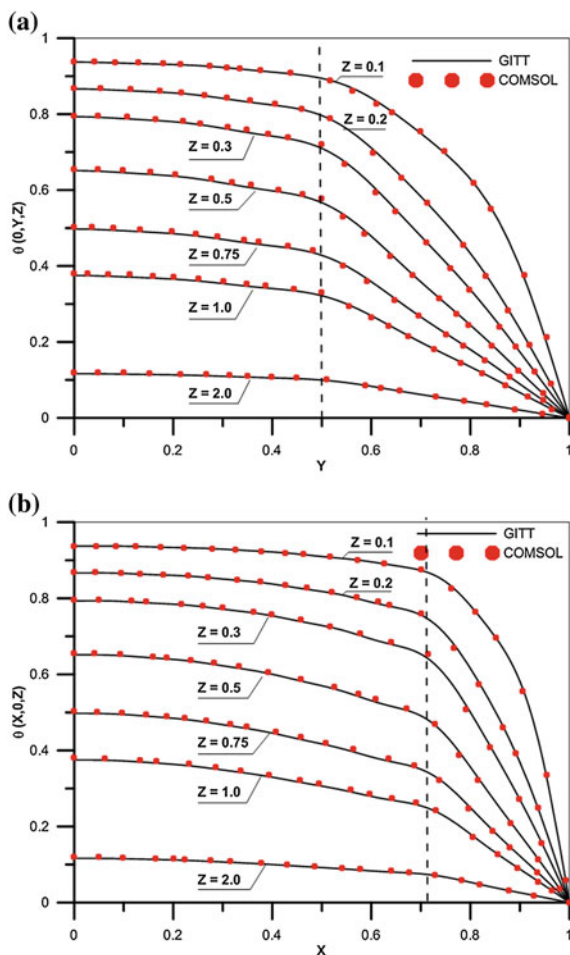


Fig. 2.44 Temperature distribution for the aspect ratio case of 2.5 along (a) transversal coordinate Y , at $X = 0$; and (b) along transversal coordinate X , at $Y = 0$



the approximate results from the COMSOL code are plotted in red dots over the solid black lines for the GITT solution. The comparison between the integral transforms and finite element solutions are once again in excellent agreement to the graphic scale.

2.7 Microchannels with Arbitrary Cross Section

This section focus on the heat and fluid flow analysis for microchannels with arbitrarily shaped cross sections, complementing in scope the ideas in Sect. 2.5, that deals with irregular boundaries along the channel length, and Sect. 2.6, which analyzed the conjugated heat transfer problem in regular geometries. The integral

transform solution of heat and fluid flow in irregular channels has been dealt with before for laminar flows (Aparecido and Cotta 1987, 1990, 1992; Barbuto and Cotta 1997; Quaresma et al. 2006; Monteiro et al. 2010), through direct integral transformation over the irregular domain, based on eigenvalue problems written for the same irregular region. Here, an alternative solution path is employed to deal with arbitrarily shaped channels, through the single domain reformulation, both for the fully developed flow and for the conjugated heat transfer analysis.

2.7.1 Flow Problem

Steady-state laminar incompressible fully developed flow of a Newtonian fluid is here considered, within a channel with arbitrary cross section which remains uniform along the flow direction (Knupp et al. 2015). The momentum equation in the longitudinal direction (z), formulated as a single region model, is given as

$$\frac{\partial}{\partial x} \left(v(x, y) \frac{\partial u(x, y)}{\partial x} \right) + \frac{\partial}{\partial y} \left(v(x, y) \frac{\partial u(x, y)}{\partial y} \right) - \frac{C(x, y)}{\rho(x, y)} = 0 \quad (2.300)$$

$$u(0, y) = u(L_x, y) = u(x, 0) = u(x, L_y) = 0 \quad (2.301a-d)$$

with

$$C(x, y) = \begin{cases} \frac{\partial p}{\partial z} = \Delta p / L_z, & \text{in the fluid stream} \\ 0, & \text{in the solid region} \end{cases} \quad (2.302)$$

where the space variable kinematic viscosity and density, $v(x, y)$ and $\rho(x, y)$, are given, respectively, by

$$v(x, y) = \begin{cases} v_f, & \text{in the fluid stream} \\ v_s \rightarrow \infty, & \text{in the solid region} \end{cases} \quad (2.303)$$

$$\rho(x, y) = \begin{cases} \rho_f, & \text{in the fluid stream} \\ \rho_s, & \text{in the solid region} \end{cases} \quad (2.304)$$

where v_f and ρ_f represent the fluid kinematic viscosity and density, and v_s and ρ_s represent the equivalent solid properties. For the kinematic viscosity corresponding to the solid region, v_s , the value can be arbitrarily chosen, as long as it is high enough so that the calculated velocities in the solid region will result to be zero, as physically required.

Proceeding with the GITT formalism for the solution of the problem given by Eqs. (2.300)–(2.304), the following transform/inverse pair is defined:

$$\text{Transform: } \bar{u}_i = \int_0^{L_y} \int_0^{L_x} \tilde{\chi}_i(x, y) u(x, y) dx dy \quad (2.305)$$

$$\text{Inverse: } u(x, y) = \sum_{i=1}^{\infty} \tilde{\chi}_i(x, y) \bar{u}_i \quad (2.306)$$

with

$$\tilde{\chi}_i(x, y) = \frac{\chi_i(x, y)}{\sqrt{N_i}} \quad (2.307)$$

$$N_i = \int_0^{L_x} \int_0^{L_y} \chi_i^2(x, y) dx dy \quad (2.308)$$

where the normalized eigenfunctions $\tilde{\chi}_i(x, y)$ and corresponding eigenvalues η_i are calculated from the solution of a properly chosen eigenvalue problem. As discussed in Sect. 1.3.3, the integral transformation of the single domain reformulation consists mainly in adopting the eigenvalue problem obtained by directly applying separation of variables to the single domain model, as formulated in problem (2.300)–(2.304), so that all the information concerning the transition of the two original domains may be represented within the eigenfunctions. Hence, the following eigenvalue problem is considered:

$$\frac{\partial}{\partial x} \left(v(x, y) \frac{\partial \chi_i(x, y)}{\partial x} \right) + \frac{\partial}{\partial y} \left(v(x, y) \frac{\partial \chi_i(x, y)}{\partial y} \right) + \eta_i^2 \chi_i = 0 \quad (2.309)$$

$$\chi(0, y) = \chi(L_x, y) = \chi(x, 0) = \chi(x, L_y) = 0 \quad (2.310a-d)$$

Even though it is not possible to obtain an explicit analytic solution for the eigenvalue problem given by Eqs. (2.309)–(2.310), the GITT itself can be employed in order to provide a hybrid numerical–analytical solution, by transforming the original problem into a standard algebraic eigenvalue problem, as detailed in Sect. 1.3.4 (Cotta 1993). Once this solution is made available, problem (2.300)–(2.304) becomes fully transformable when operated with $\int_0^1 \int_0^1 \tilde{\chi}_i(x, y) (\cdot) dx dy$, yielding a linear decoupled algebraic problem for the transformed velocities, with analytical solution given as

$$\bar{u}_i = \frac{1}{\eta_i^2} \int_0^1 \int_0^1 \tilde{\chi}_i(x, y) \frac{C(x, y)}{\rho(x, y)} dx dy, \quad i = 1, 2, \dots \quad (2.311)$$

The transformed potentials $\bar{u}_i$ can then be readily substituted into the inverse formula given by Eq. (2.306), yielding an analytic representation for the velocity field $u(x, y)$.

One may notice that the bounding domain for the analysis of the flow problem is not required to be the same as the actual physical domain, including the whole solid region, since the objective of proposing a bounding domain for the flow problem is merely to simplify the solution of the original irregular region problem, by performing the integral transformation over a regular domain.

2.7.2 Heat Transfer Problem

The conjugated heat transfer problem involving thermally developing flow is now handled for the irregular channel and the surrounding solid walls. The transient single domain formulation for the energy balance, grouping the convection and axial diffusion terms as a source function, is then written as

$$w(x, y) \frac{\partial T(x, y, z, t)}{\partial t} = \frac{\partial}{\partial x} \left(k(x, y) \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left(k(x, y) \frac{\partial T}{\partial y} \right) + P(\mathbf{x}, t, T) \quad (2.312)$$

with

$$P(\mathbf{x}, t, T) = -u(x, y)w(x, y) \frac{\partial T}{\partial z} + k(x, y) \frac{\partial^2 T}{\partial z^2} \quad (2.313)$$

with boundary conditions, already made homogeneous through filtering in both the x and y directions, given as

$$\left[\alpha + \beta k(x, y) \frac{\partial}{\partial \mathbf{n}} \right] T(x, y, z, t) = 0, \quad x, y \in S_{xy}, \quad 0 < z < L_z, \quad t > 0 \quad (2.314)$$

where S_{xy} refers to the outside wall surface, where the boundary conditions can be of first or third kind, depending on the chosen values of the coefficients α and β . The initial condition is written in a general form as

$$T(x, y, z, 0) = f(x, y, z), \quad 0 < x < L_x, \quad 0 < y < L_y, \quad 0 < z < L_z \quad (2.315)$$

while the boundary conditions in the longitudinal direction are taken in the present case as

$$T(x, y, 0, t) = T_{\text{in}}(x, y), \quad \left. \frac{\partial T}{\partial z} \right|_{z=L_z} = 0, \quad 0 < x < L_x, \quad 0 < y < L_y, \quad t > 0 \quad (2.316a, b)$$

The space variable coefficients are given as

$$u(x, y) = \begin{cases} u_f(x, y), & \text{fluid stream} \\ u \rightarrow 0, & \text{solid region} \end{cases} \quad (2.317)$$

$$w(x, y) = \begin{cases} w_f, & \text{fluid stream} \\ w_s, & \text{solid region} \end{cases} \quad (2.318)$$

$$k(x, y) = \begin{cases} k_f, & \text{fluid stream} \\ k_s, & \text{solid region} \end{cases} \quad (2.319)$$

It should be recalled that the terms with partial derivatives with respect to the coordinate z were gathered into the source term $P(\mathbf{x}, t, T)$, since the partial transformation scheme shall be adopted in this case, aimed at eliminating, by integral transformation, only the coordinates x and y , in which the diffusive effects are predominant, and thus not applying the integral transformation over the coordinate z .

Hence, proceeding with the GITT formalism for partial transformation in order to solve the problem given by Eqs. (2.312)–(2.319), the following integral transform pair is employed:

$$\text{Transform: } \bar{T}_i(z, t) = \int_0^{L_y} \int_0^{L_x} w(x, y) \tilde{\psi}_i(x, y) T(x, y, z, t) dx dy \quad (2.320)$$

$$\text{Inverse: } T(x, y, z, t) = \sum_{i=1}^{\infty} \tilde{\psi}_i(x, y) \bar{T}_i(z, t) \quad (2.321)$$

where the eigenfunctions $\tilde{\psi}_i(x, y)$ and corresponding eigenvalues μ_i are obtained from the solution of the following eigenvalue problem:

$$\frac{\partial}{\partial x} \left(k(x, y) \frac{\partial \psi_i(x, y)}{\partial x} \right) + \frac{\partial}{\partial y} \left(k(x, y) \frac{\partial \psi_i(x, y)}{\partial y} \right) + \mu_i^2 w(x, y) \psi_i = 0 \quad (2.322)$$

$$\left[\alpha + \beta k(x, y) \frac{\partial}{\partial \mathbf{n}} \right] \psi_i(x, y) = 0, \quad x, y \in S_{xy} \quad (2.323)$$

with normalized eigenfunctions and norms written as

$$\tilde{\psi}_i(x, y) = \frac{\psi_i(x, y)}{\sqrt{N_i}}, \quad N_i = \int_0^{L_y} \int_0^{L_x} w(x, y) \psi_i^2(x, y) dx dy, \quad (2.324a, b)$$

Once again the GITT itself must be employed in the solution of the eigenvalue problem (2.322)–(2.323), by transforming the original problem into an algebraic

eigensystem, which is readily solved, and then yielding analytical representations for the eigenfunctions (Cotta 1993). After the eigenfunctions $\tilde{\psi}_i(x, y)$ are made available, problem (2.312)–(2.319) is integral transformed through the operator $\int_0^{L_y} \int_0^{L_x} \tilde{\psi}_i(x, y)(\cdot) dx dy$, yielding the one-dimensional transformed partial differential system

$$\frac{\partial \bar{T}_i(z, t)}{\partial t} + \mu_i^2 \bar{T}_i = \bar{g}_i(z, t, \bar{\mathbf{T}}), \quad i = 1, 2, \dots \quad (2.325)$$

$$\begin{aligned} \bar{g}_i(z, t, \bar{\mathbf{T}}) = & - \sum_{j=1}^{\infty} \frac{\partial \bar{T}_j}{\partial z} \int_0^{L_y} \int_0^{L_x} w(x, y) u(x, y) \tilde{\psi}_i(x, y) \tilde{\psi}_j(x, y) dx dy \\ & - \sum_{j=1}^{\infty} \frac{\partial^2 \bar{T}_j}{\partial z^2} \int_0^{L_y} \int_0^{L_x} k(x, y) \tilde{\psi}_i(x, y) \tilde{\psi}_j(x, y) dx dy \end{aligned} \quad (2.326)$$

where

$$\bar{\mathbf{T}} = \{\bar{T}_1, \bar{T}_2, \dots\} \quad (2.327)$$

with the transformed boundary and initial conditions

$$\bar{T}_i(0, t) = \bar{T}_{in} \equiv \int_0^{L_y} \int_0^{L_x} w(x, y) \tilde{\psi}_i(x, y) T_{in}(x, y) dx dy, \quad \left. \frac{\partial \bar{T}_i}{\partial z} \right|_{z=L_z} = 0 \quad (2.328a, b)$$

$$\bar{T}_i(z, 0) = \bar{f}_i(z) \equiv \int_0^{L_y} \int_0^{L_x} w(x, y) \tilde{\psi}_i(x, y) f(x, y, z) dx dy \quad (2.329)$$

The transformed system given by Eqs. (2.325)–(2.329), upon truncation to a user-prescribed finite-order N , is numerically solved for the transformed potentials $\bar{T}_i(z, t)$, with $i = 1, 2, \dots, N$. Then, the inversion formula given by Eq. (2.321), can be readily employed to recover the sought temperature field $T(x, y, z, t)$.

2.7.3 Test Cases

The analysis is started with the comparison of the solution of the flow problem via the single domain formulation and integral transforms with an exact analytical solution. Aimed at choosing a problem for which exact analytical solution is readily available, a

simple case is first investigated, consisting of a Hagen–Poiseuille flow. For the definition of a single region model defined in a rectangular domain, consider Fig. 2.45, which depicts the circular channel with radius $R = 1$ inside a square with side $2R$. In Fig. 2.45, the gray hatched region is considered to be the channel walls, whereas the blank region is considered to be the fluid stream region. This cross section is supposed to remain uniform along the longitudinal direction. In the same figure, it is also represented the lines r_1 and r_2 , which are going to be used later for plotting the velocity profiles for comparison purposes.

As an illustration, the input data has been taken as $\Delta p/L_z = -8$, $v_f = 1$, and $\rho_f = 1$, yielding unitary average velocity inside the circular channel. The exact analytical solution for the fully developed velocity field inside the circular duct is given as

$$u(r) = 2(1 - r^2) \quad (2.330)$$

which is going to be used as a benchmark for verification purposes.

For the solution via the single domain/integral transforms approach, besides the pressure difference and the fluid properties, already defined above, the following values have been assigned to the parameters related to the solid region, $v_s = 10^2$ and $\rho_s = \rho_f$. Tables 2.21 and 2.22 illustrate the convergence behavior of the velocity field, presenting the calculated velocity u at some different positions along directions r_1 and r_2 (defined in Fig. 2.45), respectively, by varying the truncation order in the velocity eigenfunction expansion, $N = 50, 65, 80, 95, 110$, with a fixed number of terms in the solution of the eigenvalue problem ($M = 120$). It is

Fig. 2.45 Schematic representation of the channel cross section (*circular tube*) employed in the single domain formulation and integral transforms solution verification

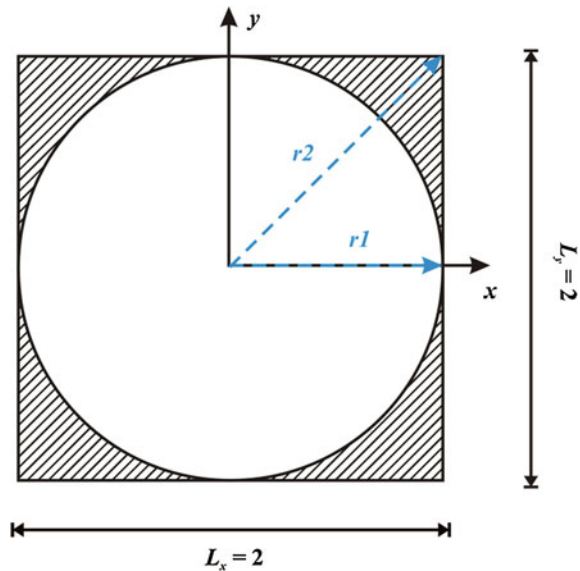


Table 2.21 Convergence behavior of the velocity field (along r_1) with respect to the truncation order of the velocity field expansion (N), with fixed $M = 120$ terms in the eigenvalue problem solution (flow in circular tube)

N	u			
	$r_1 = 0.2$	$r_1 = 0.4$	$r_1 = 0.6$	$r_1 = 0.8$
$N = 50$	1.9200	1.6730	1.2652	0.6994
$N = 65$	1.9188	1.6732	1.2642	0.6986
$N = 80$	1.9198	1.6726	1.2644	0.6984
$N = 95$	1.9198	1.6736	1.2652	0.6996
$N = 110$	1.9192	1.6734	1.2652	0.6996
Exact	1.92	1.68	1.28	0.72

Table 2.22 Convergence behavior of the velocity field (along r_2) with respect to the truncation order of the velocity field expansion (N), with fixed $M = 120$ terms in the eigenvalue problem solution (flow in circular tube)

N	u			
	$r_2 = 0.2$	$r_2 = 0.4$	$r_2 = 0.6$	$r_2 = 0.8$
$N = 50$	1.9182	1.6724	1.2786	0.7338
$N = 65$	1.9176	1.6714	1.2784	0.7330
$N = 80$	1.9184	1.6722	1.2778	0.7326
$N = 95$	1.9180	1.6718	1.2780	0.7326
$N = 110$	1.9178	1.6722	1.2784	0.7328
Exact	1.92	1.68	1.28	0.72

observed a convergence of at least three significant digits with the increase in the truncation order, and an agreement of at least two significant digits with the exact velocities at the selected positions, calculated from Eq. (2.330). It is inspected that in the worst case the relative error remains below 3 %, which occurs near the fluid–solid interface.

Aimed at giving an overall picture of the approximate solution for the flow problem obtained via the single domain approach, in Fig. 2.46a, b this approximate solution is critically compared to the exact analytical velocity profiles along the two directions r_1 and r_2 (defined in Fig. 2.45), respectively, where an excellent agreement between the approximate and exact results is observed throughout, even for the region near the interface between the fluid and solid regions. The figures also clearly show that the solid region recovers zero velocity, as it should be, and the no-slip condition at the circular contour corresponding to the channel walls is satisfied.

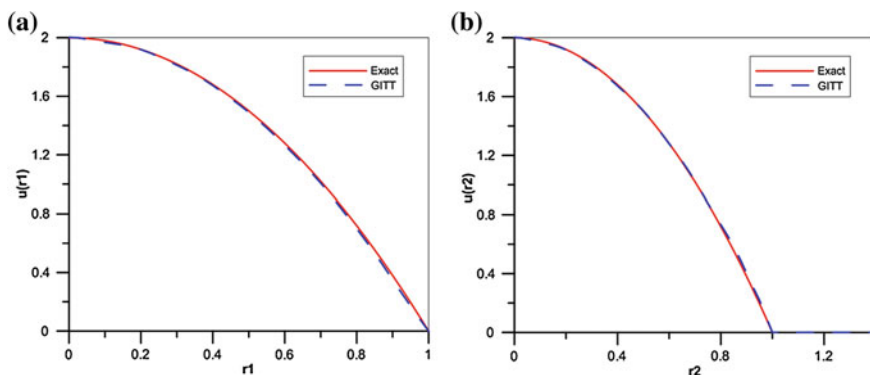


Fig. 2.46 Comparison between exact and single domain formulation/integral transforms solution (flow in circular tube): (a) velocity profile along r_1 ; (b) velocity profile along r_2

Now the flexibility and robustness of the single domain/integral transforms approach is illustrated through the investigation of the flow in a channel with the cross-sectional shape presented in Fig. 2.47, which also defines the single region model in a rectangular domain, where the gray hatched region is considered to be the region of solid walls, whereas the blank region is considered to be the region of the fluid stream, where the fully developed velocity field is to be determined. Also for this case, it has been prescribed unitary dimensionless average velocity inside the channel. Since there is no exact solution available for the velocity in the proposed channel, the problem was also modeled on the commercial CFD solver Comsol Multiphysics, in order to provide a numerical solution through the finite element method for comparison purposes.

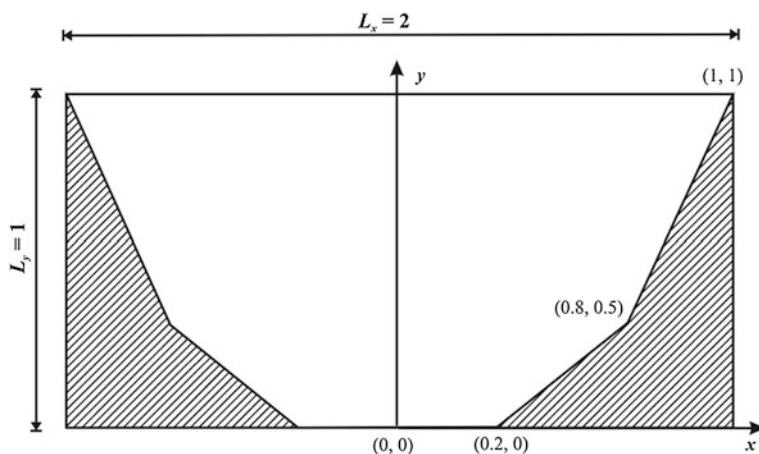


Fig. 2.47 Schematic representation of the channel cross section employed in the single domain formulation

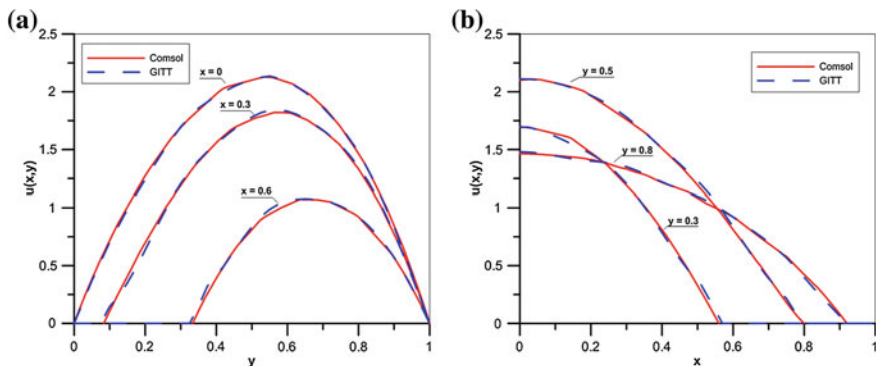


Fig. 2.48 Comparison between finite elements solution (COMSOL) and GITT with single domain formulation for flow in irregular channel—(a) velocity profiles along y ; (b) velocity profiles along x

In Fig. 2.48a, b, the GITT solution of the flow problem is critically compared to the velocity profiles obtained through the Comsol Multiphysics software along y (for $x = 0, 0.3, 0.6$) and along x (for $y = 0.3, 0.5, 0.8$), respectively, and an excellent agreement between the hybrid and numerical solutions is observed throughout. Again, it becomes evident that the single domain formulation/integral transforms solution recovers the expected null velocity fields at the solid region, also respecting the no-slip boundary condition, while allowing for a more straightforward integral transformation of the complex configuration, by enveloping the irregular channel within a simpler regular domain with the help of the space variable coefficients.

The conjugated heat transfer problem is now analyzed, for the proposed geometry inspired by a laser-ablated microchannel on a polymeric substrate, as presented in Fig. 2.49, considering a dimensionless average velocity $U_{\text{avg}} = 2$ inside the channel. A particular dimensionless numerical case with $T_{\text{in}} = 0$ and prescribed uniform temperature at the external walls, $T = 1$ ($\alpha = 1$ and $\beta = 0$), with dimensionless solid thermal conductivity, $k_s = 0.25$, and fluid thermal conductivity, $k_f = 1$, typical of the case of water flow inside acrylic channels, is more closely analyzed to report a few representative results.

Figure 2.50a provides the temperature distributions along the x coordinate at $y = 1$, from the centerline of the channel up to the external wall surface, while Fig. 2.50b shows the temperature distributions along the vertical y coordinate at the centerline $x = 0$, all the way from the bottom to the top base of the polymeric structure. Again, the agreement of the hybrid GITT solution with the purely numerical solution of the CFD solver is remarkable, all along the thermally developing region, as can be seen from the various curves for different values of the longitudinal coordinate, z .

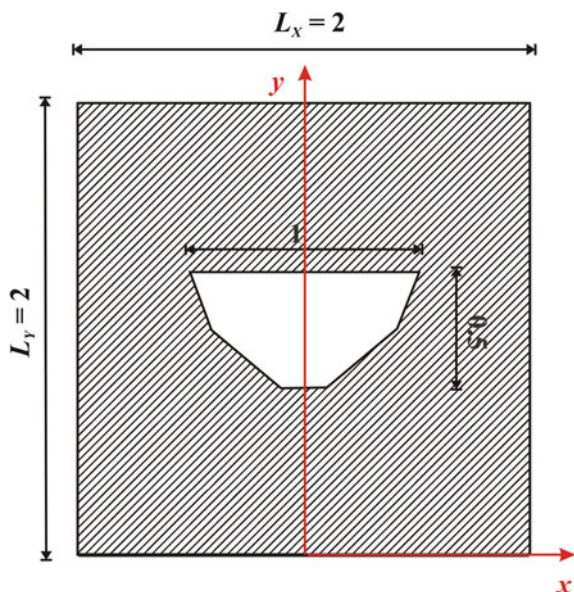


Fig. 2.49 Schematic representation of microchannel on polymeric substrate for analysis of conjugated heat transfer in irregular channel and surrounding substrate

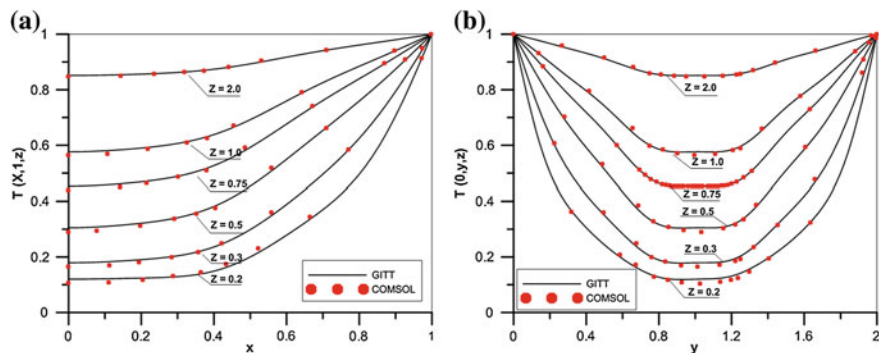


Fig. 2.50 Comparison of GITT and numerical (COMSOL Multiphysics) solutions of conjugated heat transfer in complex configuration. (a) Temperature distributions along x ; (b) temperature distributions along y

A comparison of the GITT and the COMSOL solutions is also provided along the actual transient behavior of the conjugated problem, by showing in Fig. 2.51 the time evolution of the channel centerline temperature ($x = 0$, $y = 1$), for different values of the longitudinal coordinate, z . Again, the curves from the hybrid and numerical solutions are practically coincident to the graphical scale, throughout both the time and longitudinal coordinate domains.

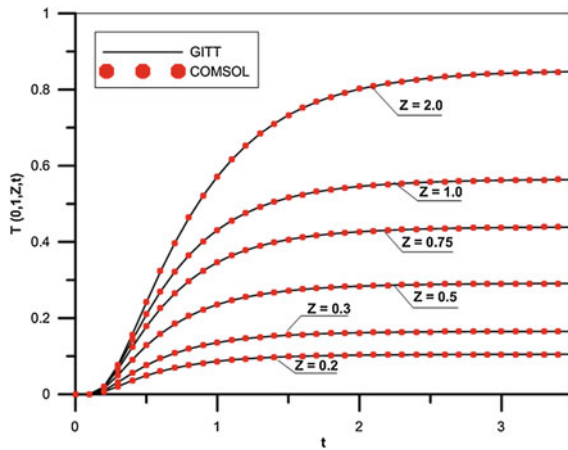


Fig. 2.51 Comparisons between GITT and COMSOL solutions of the channel centerline temperature ($x = 0, y = 1$) as a function of time (t), and for different longitudinal positions (z)

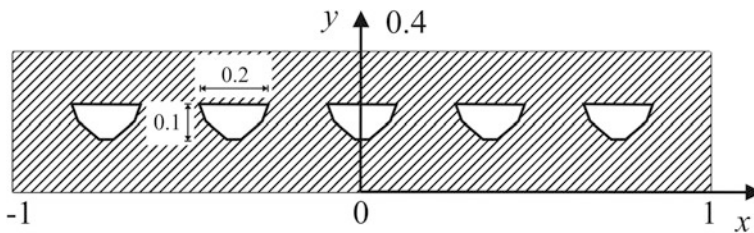


Fig. 2.52 Schematic representation of the multi-stream heat exchanger cross section

In order to further illustrate the robustness and flexibility of the single domain/integral transforms approach, an additional example is provided by considering a multi-stream heat exchanger with five microchannels, as shown in Fig. 2.52. The wall and fluid temperature profiles are plotted in Fig. 2.53a, b, comparing the GITT and COMSOL calculated temperatures for different longitudinal positions (z) across the transversal directions: (a) along x , at $y = 0.2$, and (b) along y at $x = 0$. It may be observed that there is excellent adherence to the graphical scale between the two sets of temperature results. Clearly, the transitions between the solid and fluid regions are accurately accounted for by the single domain formulation and the corresponding eigenfunctions. It should be pointed out that this problem employs the same entrance and boundary conditions previously used, i.e., $T_{in} = 0$ and $T = 1$ ($\alpha = 1$ and $\beta = 0$), and dimensionless solid thermal conductivity, $k_s = 0.25$, and fluid thermal conductivity, $k_f = 1$. Thus, the multi-stream heat exchanger problem can be adequately modeled as a single

domain by just properly defining the space variable coefficients so as to capture the five microchannels geometry and their transitions within the substrate.

Besides the remarkable adherence between the hybrid and numerical solutions presented, a few more results are presented in Tables 2.23 and 2.24 in order to

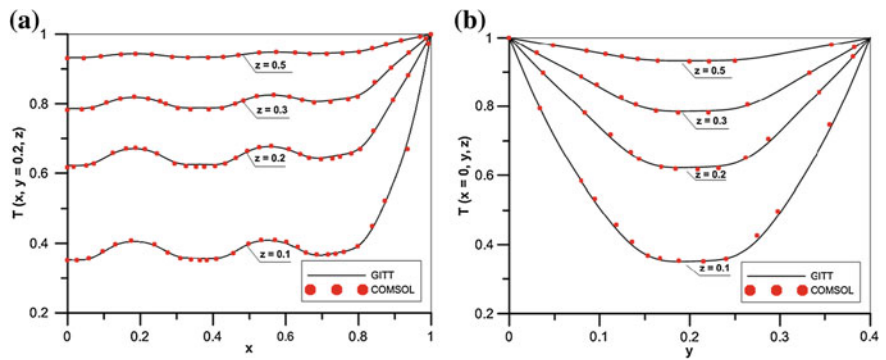


Fig. 2.53 Comparisons between GITT and COMSOL solutions for the fluid and wall temperature profiles for the multi-stream heat exchanger: (a) along x , at $y = 0.2$, and (b) along y at $x = 0$, for different longitudinal positions (z)

Table 2.23 Convergence behavior of the steady-state temperature profile along x with respect to the truncation order of the expansion (N), with fixed $M = 120$ terms in the eigenvalue problem solution (multi-stream heat exchanger), at (a) $z = 0.1$; (b) $z = 0.2$

(a)				
N	$T(x, y = 0.2, z = 0.1)$			
	$x = 0.0$	$x = 0.3$	$x = 0.6$	$x = 0.9$
$N = 75$	0.3537	0.3642	0.3979	0.5828
$N = 85$	0.3533	0.3640	0.3975	0.5824
$N = 95$	0.3532	0.3646	0.3988	0.5825
$N = 105$	0.3527	0.3661	0.3986	0.5825
$N = 115$	0.3519	0.3631	0.3998	0.5824
COMSOL	0.35	0.36	0.40	0.58
(b)				
N	$T(x, y = 0.2, z = 0.2)$			
	$x = 0.0$	$x = 0.3$	$x = 0.6$	$x = 0.9$
$N = 75$	0.6238	0.6314	0.6686	0.8202
$N = 85$	0.6236	0.6312	0.6681	0.8192
$N = 95$	0.6235	0.6315	0.6687	0.8192
$N = 105$	0.6232	0.6325	0.6686	0.8193
$N = 115$	0.6224	0.6298	0.6697	0.8193
COMSOL	0.62	0.63	0.67	0.82

Table 2.24 Convergence behavior of the steady-state temperature profile along y with respect to the truncation order of the expansion (N), with fixed $M = 120$ terms in the eigenvalue problem solution (multi-stream heat exchanger), at (a) $z = 0.1$; (b) $z = 0.2$

(a)			
N	$T(x = 0, y, z = 0.1)$		
	$y = 0.1$	$y = 0.2$	$y = 0.3$
$N = 75$	0.5070	0.3537	0.4913
$N = 85$	0.5066	0.3533	0.4909
$N = 95$	0.5069	0.3532	0.4916
$N = 105$	0.5064	0.3527	0.4930
$N = 115$	0.5063	0.3519	0.4927
COMSOL	0.51	0.35	0.50
(b)			
N	$T(x = 0, y, z = 0.2)$		
	$y = 0.1$	$y = 0.2$	$y = 0.3$
$N = 75$	0.7403	0.6238	0.7265
$N = 85$	0.7403	0.6236	0.7267
$N = 95$	0.7403	0.6235	0.7270
$N = 105$	0.7400	0.6232	0.7278
$N = 115$	0.7399	0.6224	0.7276
COMSOL	0.74	0.62	0.73

illustrate the convergence behavior of the GITT solution with respect to the number of terms employed in the temperature field expansion, where it can be observed a full convergence of at least two significant digits, in the worst case, within the truncation orders considered, and the adherence to the COMSOL solution can be confirmed in tabular form.

Analytical Heat and Fluid Flow in Microchannels and
Microsystems

Cotta, R.M.; Knupp, D.C.; Naveira-Cotta, C.P.

2016, XII, 164 p., Hardcover

ISBN: 978-3-319-23311-6