Preface

The book we offer to the reader was conceived as a comprehensive course of homotopical topology, starting with the most elementary notions, such as paths, homotopies, and products of spaces, and ending with the most advanced topics, such as the Adams spectral sequence and K-theory. The history of homotopical, or algebraic, topology is short but full of sharp turns and breathtaking events, and this book seeks to follow this history as it unfolded.

It is fair to say that homotopical topology began with *Analysis Situs* by Henri Poincaré (1895). Poincaré showed that global analytic properties of functions, vector fields, and differential forms are greatly influenced by homotopic properties of the relevant domains of definition. Poincaré's methods were developed, in the half-century that followed *Analysis Situs*, by a constellation of great topologists which included such figures as James Alexander, Heinz Hopf, Andrey Kolmogorov, Hassler Whitney, and Lev Pontryagin. Gradually, it became clear that the homotopical properties of domains which were singled out by Poincaré could be best understood in the form of groups and rings associated, in a homotopy invariant way, with a topological space which has to be "good" from the geometric point of view, like smooth manifolds or triangulations. Thus, *Analysis Situs* became algebraic topology.

Still the algebra used by the algebraic topology of that epoch was very elementary: It did not go far beyond the classification of finitely generated Abelian groups (with some exceptions, however, such as Van Kampen's theorem about fundamental groups). More advanced algebra (which, actually, was not developed by algebraists of that time and had to be started from scratch by topologists under the name "homological algebra," aka "abstract nonsense") invaded topology, which made algebraic topology truly algebraic. This happened in the late 1940s and early 1950s. The leading item of this new algebra appeared in the form of a "spectral sequence." The true role of a spectral sequence in topology was discovered mainly by Jean-Pierre Serre (who was greatly influenced by older representatives of the French school of mathematics, mostly by Henri Cartan, Armand Borel, and Jean Leray).

The impact of spectral sequences on algebraic topology was tremendous: Many major problems of topology, both solved and unsolved, became exercises for students.

The progress of the new algebraic topology was very impressive but short-lived: As early as in the late 1950s, the results became less and less interesting, and the proofs became more and more involved. The last big achievement of the algebraic topology which was started by Serre was the Adams spectral sequence, which, in a sense, absorbed all major notions and methods of contemporary algebraic topology. Using his spectral sequence, J. Frank Adams was able to prove the famous Frobenius conjecture (the dimension of a real division algebra must be 1, 2, 4, or 8); it was also used by René Thom in his seminal work, becoming the starting point of the so-called cobordism theory.

Reviving ailing algebraic topology required strong means, and such means were found in the newly developed K-theory. Created by J. Frank Adams, Michael Atiyah, Raoul Bott, and Friedrich Hirzebruch, K-theory (which may be regarded as a branch of the broader "algebraic K-theory") had applications which were unthinkable from the viewpoint of "classical" algebraic topology. It is sufficient to say that the Frobenius conjecture was reduced, via K-theory, to the following question: For which positive integers n is $3^n - 1$ divisible by 2^n? (Answer: for $n = 1, 2$, and 4.)

Developing K-theory was more or less completed in the mid-1960s. Certainly, it was not the end of algebraic topology. Very important results were obtained later; some of them, belonging to Sergei Novikov, Victor Buchstaber, Alexander Mishchenko, James Becker, and Daniel Gottlieb, are discussed in the last chapter of this book. Many excellent mathematicians continue to work in algebraic topology. Still, one can say that, from the students' point of view, algebraic topology can now be seen as a completed domain, and it is possible to study it from the beginning to the end. (We can add that this is not only possible, but also highly advisable: Algebraic topology provides a necessary background for geometry, analysis, mathematical physics, etc.) This book is intended to help the reader achieve this goal.

The book consists of an introduction and six chapters. The introduction introduces the most often used topological spaces (from spheres to the Cayley projective plane) and major operations over topological spaces (products, bouquets, suspensions, etc.). The chapter titles are as follows: "Homotopy"; "Homology"; "Spectral Sequences of Fibrations"; "Cohomology Operations"; "The Adams Spectral Sequence"; and "K-theory and Other Extraordinary Theories." Chapters are divided into parts called "Lectures," which are numerated throughout the book from Lecture 1 to Lecture 44. Lectures are divided into sections numerated with Arabic numbers, and some sections are divided into subsections labeled with capital letters of the Roman alphabet. For example, Lecture 13 consists of Sects. 13.1, 13.2, 13.3, . . . ,13.11, and Sect. 13.8 consists of the subsections $A, B, \dots, E$ (which are referred to, in further parts of the book, as Sects. 13.8.A, 13.8.B, and so on).

To present this huge material in one volume of moderate size, we had to be very selective in presenting details of proofs. Many proofs in this book are algebraic, and they often involve routine verifications of independence of the result of a

construction of some arbitrary choices within this construction, of exactness of this or that sequence, of group or ring axioms for this or that addition or multiplication. These verifications are necessary, but they often repeat each other, and if included in the book, they will only lead to excessively increasing the volume and irritating the reader, who will probably skip them. On the other hand, we did not want to follow some authors of books who skip details of proofs which are inconvenient, for this or that reason, for an honest presentation. We did our best to avoid this pattern: If a part of a proof is left to the reader as an exercise, then we are sure that this exercise should not be difficult for somebody who has consciously absorbed the preceding material.

As should be clear from the preceding sentences, the book contains many exercises; actually, there are approximately 500 of them. They are numerated within each lecture. They may be serve the usual purposes of exercises: Instructors can use them for homework and tests, while readers can solve them to check their understanding or to get some additional information. But at least some of them must be regarded as a necessary part of the course; in some (not very numerous) cases we will make references to exercises from preceding sections or lectures. We hope that the reader will appreciate this style. The most visible consequence of this approach to exercises is that they are not concentrated in one special section (which is common for many textbooks) but rather scattered throughout every lecture.

Being a part of geometry, homotopic topology requires, for its understanding, a lot of graphic material. Our book contains more than 100 drawings ("figures"), which are supposed to clarify definitions, theorems, or proofs. But the book also contains a chain of drawings that are pieces of art rather than rigorous mathematical figures. These pictures were drawn by A. Fomenko; some of them were displayed at various exhibitions. All of them are supposed to present not the rigorous mathematical meaning, but rather the spirit and emotional contents of notions and results of homotopical topology. They are located in the appropriate places in the book. A short explanation for these pictures can be found at the end of the book.

We owe our gratitude to many people. The first, mimeographed, version of the beginning of this book (which roughly corresponded to Chap. 1 and a considerable part of Chap. 2) was written in collaboration with Victor Gutenmacher; we are deeply grateful to him for his help. The idea of formally publishing this book was suggested to us by Sergei Novikov. We are grateful to him for this suggestion. Some improvements to the book were suggested by several students of the University of California; we are grateful to all of them, especially to Colin Hagemeyer. The whole idea of publishing this book under the auspices of Springer belonged to Boris Khesin and Anton Zorich; we thank them heartily. And the last but, maybe, the most important thanks go to the brilliant team of editors at Springer, especially to Eugene Ha and Jay Popham. It is the result of their work that the book looks as attractive as it does.

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