Chapter 2
Homology

Lecture 12 Main Definitions and Constructions

Besides the homotopy groups $\pi_n(X)$, there are other series of groups corresponding in a homotopy invariant way to a topological space X; the most notable are homology and cohomology groups, $H_n(X)$ and $H^n(X)$. Compared with homotopy groups, they have an important flaw—their accurate definition requires substantial algebraic work—and important advantages: Their computation is much easier, we will calculate them more or less immediately for the majority of topological spaces known to us, and also they are geometrically better visualizable [there are no counterintuitive phenomena like $\pi_3(S^2) \cong \mathbb{Z}$]. The information of a simply connected topological space contained in homology groups is comparable with that contained in homotopy groups.

The main geometric idea of homology is as follows. Spheroids are replaced by *cycles*; an n-dimensional cycle is, roughly, an n-dimensional surface, maybe a sphere, but it may be something different, say, a torus. The relation of being homotopic is replaced by a relation of being *homological*: Two n-dimensional cycles are homological if they cobound a piece of surface of dimension $n+1$. How do we define cycles and those pieces of surfaces which they bound, the so-called chains? One can try to present them as continuous maps of some standard objects, spheres and something else (k-dimensional manifolds?). But this leads to severe difficulties, especially in dimensions > 2. It is easier to define cycles and chains as the union of standard "bricks." The role of these bricks is assumed by "singular simplices."

Notice that the construction of homology (and cohomology) groups does not require a fixation of a base point.

A. Fomenko, D. Fuchs, *Homotopical Topology*, Graduate Texts in Mathematics 273,
DOI 10.1007/978-3-319-23488-5_2

12.1 Singular Simplices, Chains, and Homology

Let $A_0, A_1, \ldots, A_q$ be points of the space $\mathbb{R}^n$, $n \geq q$, not contained in one $(q-1)$-dimensional plane. The convex hull of these points is called the *Euclidean simplex* with *vertices* $A_0, A_1, \ldots, A_q$ (this notion is known to us from Lecture 5; see Sect. 5.8). The convex hulls of (nonempty) subsets of the set of vertices are called *faces* of the simplex; they are also Euclidean simplices. Euclidean simplices of the same dimension are essentially the same, and this motivates us to choose one *standard* Euclidean simplex. The usual choice of the standard simplex is the simplex Δ^n in $\mathbb{R}^{n+1}$ with the ends of coordinate vectors taken for vertices. Thus,

$$\Delta^n = \left\{(t_0, t_1, \ldots, t_n) \in \mathbb{R}^{q+1} \mid t_0 \geq 0, t_1 \geq 0, \ldots, t_n \geq 0, \sum\nolimits_{i=0}^{n} t_i = 1\right\}.$$

Let X be an arbitrary topological space. We define an n-dimensional *singular simplex* of X simply as a continuous map of Δ^n into X. An n-dimensional *singular chain* of X is a formal finite linear combination of n-dimensional singular simplices with integral coefficients: $\sum_i k_i f_i$, $f_i \colon \Delta^n \to X$. The set of all n-dimensional singular chains of X is denoted as $C_n(X)$. The usual addition of linear combinations makes $C_n(X)$ an Abelian group; thus, $C_n(X)$ is the free Abelian group generated by the set of all n-dimensional singular simplices of X.

Next we describe the *boundary homomorphism* $\partial = \partial_n \colon C_n(X) \to C_{n-1}(X)$. Since the group $C_n(X)$ is free, it is sufficient to define ∂ for the generators, that is, for singular simplices. For a singular simplex f we put

$$\partial f = \sum_{i=0}^{n} (-1)^i \Gamma_i f,$$

where $\Gamma_i f$ is the ith face of f, which is defined as the restriction of f to the ith face $\Gamma_i \Delta^n$,

$$\Gamma_i \Delta^n = \{(t_0, t_1, \ldots, t_n) \in \Delta^n \mid t_i = 0\}$$

[we identify $\Gamma_i \Delta^n$ with Δ^{n-1} using the correspondence

$$(t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots .t_n) \leftrightarrow (t_0, \ldots, t_{i-1}, t_{i+1}, \ldots .t_n)].$$

Theorem. *The composition*

$$C_{n+1}(X) \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$$

is trivial; in other words, $\mathrm{Im}(\partial_{n+1}) \subset \mathrm{Ker}(\partial_n)$.

Proof. A direct verification is based on the equality

$$\Gamma_i \Gamma_j f = \begin{cases} \Gamma_{j-1} \Gamma_i f, & \text{if } j > i, \\ \Gamma_j \Gamma_{i+1} f, & \text{if } j \leq i. \end{cases}$$

To make our upcoming life slightly easier, we assume that $C_n(X) = 0$ for $n < 0$ and extend the definition of ∂ accordingly. The theorem is not affected.

Main Definition. The quotient group

$$H_n(X) = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$$

is called the *nth homology group* of X. In particular, $H_0(X) = C_0(X)/\operatorname{Im} \partial_1$ and $H_n(X) = 0$ for $n < 0$.

There are also common notations $\operatorname{Ker} \partial_n = Z_n(X)$ and $\operatorname{Im} \partial_{n+1} = B_n(X)$. Thus, $H_n(X) = Z_n(X)/B_n(X)$. Elements of the groups $Z_n(X)$ and $B_n(X)$ are called, respectively, *cycles* and *boundaries*. (Thus, every boundary is a cycle, but the converse is, generally, false.) If the difference of two cycles is a boundary, then these cycles are called *homologous*. Thus, the homology group is the group of classes of homologous cycles (which may be called *homology classes*).

If the group $H_n(X)$ is finitely generated, then its rank is called the nth *Betti number* of X.

12.2 Chain Complexes, Map, and Homotopies

A *chain complex*, or simply a complex, is an (infinite in both directions) sequence of groups and homomorphisms

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all n.

The group $H_n = \operatorname{Ker} \partial_n / \operatorname{Im} \partial_{n+1}$ is called the nth homology group of the complex.

EXERCISE 1. Let

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

be a complex. Put $\widetilde{C}_n = C_n \oplus C_{n+1}$ and define $\widetilde{\partial}_n : \widetilde{C}_n \to \widetilde{C}_{n-1}$ by the formula $\widetilde{\partial}_n(c, c') = (\partial_n c, \partial_{n+1} c' + (-1)^n c),\ c \in C_n, c' \in C_{n+1}$. Prove that

$$\dots \xrightarrow{\widetilde{\partial}_{n+2}} \widetilde{C}_{n+1} \xrightarrow{\widetilde{\partial}_{n+1}} \widetilde{C}_n \xrightarrow{\widetilde{\partial}_n} \widetilde{C}_{n-1} \xrightarrow{\widetilde{\partial}_{n-1}} \dots$$

is a complex and that the homology of this complex is trivial ($\widetilde{H}_n = 0$ for all n).

Our main example of a chain complex, so far, is the *singular complex* of a space X: $C_n = C_n(X)$. This complex is *positive*, which means that $C_n = 0$ for $n < 0$.

Mostly, we will consider positive complexes, but there will be exceptions, and the first exception appears immediately: The *augmented* or *reduced* singular complex of a space X,

$$\dots \xrightarrow{\partial_{n+2}} \widetilde{C}_{n+1}(X) \xrightarrow{\partial_{n+1}} \widetilde{C}_n(X) \xrightarrow{\partial_n} \widetilde{C}_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots,$$

is defined by the formula

$$\widetilde{C}_n(X) = \begin{cases} C_n(X), & \text{if } n \neq -1, \\ \mathbb{Z}, & \text{if } n = -1, \end{cases}$$

and ∂_n are all as before, except $\partial_0 \colon C_0(X) \to \mathbb{Z}$, more commonly denoted as ϵ and called an *augmentation*, which takes every zero-dimensional singular simplex of X into $1 \in \mathbb{Z}$. Thus, the reduced complex of X looks like

$$\dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow \dots .$$

[Thus, for a zero-dimensional chain $c = \sum k_i f_i$, $\epsilon(c) = \sum k_i$; the number $\epsilon(c)$ is sometimes called the *index* of the zero-dimensional chain c; it may be denoted as $\operatorname{ind}(c)$.] A natural question arises: Why is this complex called *reduced*? It looks bigger than the unreduced complex. The answer is in the following proposition.

Proposition 1. *The homology $\widetilde{H}_n(X)$ of the reduced singular complex* (called the *reduced homology* of X) *is related to the usual homology as follows. If X is not empty, then*

$$H_n(X) = \begin{cases} \widetilde{H}_n(X), & \text{if } n \neq 0, \\ \widetilde{H}_0(X) \oplus \mathbb{Z}, & \text{if } n = 0; \end{cases}$$

if X is empty, then the only nonzero reduced homology group of X is $\widetilde{H}_{-1}(X) = \mathbb{Z}$.

Proof. Obvious.

Back to algebra. If $\mathcal{C} = \{C_n, \partial_n\}$ and $\mathcal{C}' = \{C'_n, \partial'_n\}$ are two chain complexes, then a *chain map*, or a *homomorphism* $\varphi \colon \mathcal{C} \to \mathcal{C}'$, is defined as a sequence of group homomorphisms $\varphi_n \colon C_n \to C'_n$ which make the diagram

$$\begin{array}{ccccccccc}
\dots & \xrightarrow{\partial_{n+2}} & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \xrightarrow{\partial_{n-1}} & \dots \\
 & & \downarrow{\scriptstyle \varphi_{n+1}} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \\
\dots & \xrightarrow{\partial'_{n+2}} & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \xrightarrow{\partial'_{n-1}} & \dots
\end{array}$$

commutative.

From this commutativity, $\varphi_n(\operatorname{Ker}\partial_n) \subset \operatorname{Ker}(\partial'_n)$ and $\varphi_n(\operatorname{Im}\partial_{n+1}) \subset \operatorname{Im}(\partial'_{n+1})$, so there arise homomorphisms $\varphi_* = \varphi_{*n} \colon H_n(\mathcal{C}) \to H_n(\mathcal{C}')$ with obvious properties, like $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$. For our main example, a continuous map $h \colon X \to Y$ naturally induces homomorphisms $h_\# = h_{\# n}: C_n(X) \to C_n(Y)$, $h_\#\left(\sum_i k_i f_i\right) = \sum_i k_i(h \circ f_i)$ and also $h_{\# n}: \widetilde{C}_n(X) \to \widetilde{C}_n(Y)$ (with $h_{\#,-1} = \mathrm{id}$) which comprise homomorphisms between both unreduced and reduced singular complexes. Thus, there arise maps $f_*: H_n(X) \to H_n(Y)$ and $\widetilde{H}_n(X) \to \widetilde{H}_n(Y)$ (with the same obvious properties).

Again back to algebra. Let $\mathcal{C} = \{C_n, \partial_n\}$ and $\mathcal{C}' = \{C'_n, \partial'_n\}$ be two chain complexes and $\varphi = \{\varphi_n\}, \psi = \{\psi_n\}: \mathcal{C} \to \mathcal{C}'$ be two chain maps. A *chain homotopy* between φ and ψ is a sequence $D = \{D_n: C_n \to C'_{n+1}\}$ satisfying the identities

$$D_{n-1} \circ \partial_n + \partial'_{n+1} \circ D_n = \psi_n - \varphi_n.$$

For the reader's convenience (or inconvenience?) we show all the maps involved in this definition in one diagram (which, certainly, is not commutative):

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, shift right, "\psi_{n+1}"'] \arrow[d, shift left, "\varphi_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, shift right, "\psi_n"'] \arrow[d, shift left, "\varphi_n"] \arrow[dl, "D_n"] & C_{n-1} \arrow[r] \arrow[d, shift right, "\psi_{n+1}"'] \arrow[d, shift left, "\varphi_{n+1}"] \arrow[dl, "D_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

If chain maps φ, ψ can be connected by a chain homotopy, they are called (*chain*) *homotopic*.

Proposition 2. *If chain maps $\varphi, \psi: \mathcal{C} \to \mathcal{C}'$ are homotopic, then the induced homology maps $\varphi_*, \psi_*: H_n(\mathcal{C}) \to H_n(\mathcal{C}')$ are equal.*

Proof. Let $D = \{D_n\}$ be a homotopy between φ and ψ. If $c \in \operatorname{Ker} \partial_n \subset C_n$, then

$$\psi_n(c) - \varphi_n(c) = D_{n-1} \circ \partial_n(c) + \partial'_{n+1} \circ D_n(c) = \partial'_{n+1}(D_n(c)) \in \operatorname{Im} \partial'_{n+1};$$

that is, $\varphi_n(c)$ and $\psi_n(c)$ are homologous for every cycle $c \in C_n$. Thus, $\varphi_{*n} = \psi_{*n}$.

EXERCISE 2. A complex $(\mathcal{C})$ is called *contractible* if the identity map $\mathrm{id}: \mathcal{C} \to \mathcal{C}$ is homotopic to the zero map $0: \mathcal{C} \to \mathcal{C}$. A complex $(\mathcal{C})$ is called acyclic if $H_n(\mathcal{C}) = 0$ for all n.

(Warmup) Prove that a contractible complex is acyclic.
(a) Prove that the complex $\{\widetilde{C}_n, \widetilde{\partial}_n\}$ from Exercise 1 is not only acyclic but also contractible.
(b) Prove that the complex

$$\cdots \leftarrow 0 \leftarrow 0 \leftarrow \mathbb{Z}_2 \xleftarrow{\text{onto}} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \leftarrow 0 \leftarrow 0 \ldots$$

is acyclic but not contractible.
(c) Let $(\mathcal{C}) = \{C_n, \partial_n\}$ be a positive ($C_n = 0$ for $n < 0$) free (all C_n are free Abelian groups) complex. Prove that if $(\mathcal{C})$ is acyclic, then it is contractible.

Finally, we will establish a connection between chain homotopies considered here with homotopies between continuous maps. (This connection is actually a justification for the term "chain homotopy.") Namely, we will show how a homotopy between continuous maps $f, g: X \to Y$ determines a chain homotopy between the maps $f_\#, g_\#$ of singular complexes.

We begin with a geometric construction which presents a covering of a cylinder $\Delta^n \times I$ by $n+1$ Euclidean simplices (in the language of Sect. 5.8, it is a *triangulation* of $\Delta^n \times I$). Recall that $\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum t_i = 1\}$. The vertices of Δ^n are $v_i = (0, \dots, 0, 1, 0, \dots, 0)$ with $1 = t_i$. For $0 \leq i \leq n$, put

$$A_i = \{((t_0, \dots, t_n), t) \in \Delta^n \times I \mid t_0 + \dots + t_{i-1} \leq t \leq t_0 + \dots t_i\}$$

(where the empty sum is regarded as 0). It is easy to see that A_i is the convex hull of $(v_0, 1), \dots, (v_i, 1), (v_i, 0), \dots, (v_n, 0)$, that is, the Euclidean simplex with the vertices $(v_0, 1), \dots, (v_i, 1), (v_i, 0), \dots, (v_n, 0)$. Indeed, all these points belong to A_i, and if $y = ((t_0, \dots, t_n), t)$, then $y = t_0(v_0, 1) + \dots + t_{i-1}(v_{i-1}, 1) + t_i'(v_i, 1) + t_i''(v_i, 0) + t_{i+1}(v_{i+1}, 0) + \dots + t_n(v_n, 0)$, where $t_i' = t - (t_0 + \dots + t_{i-1})$ and $t_i'' = t_i - t_i' = (t_0 + \dots + t_i) - t$, so if $y \in A_i$, then the sum of the coefficients is 1 and all of them are between 0 and 1.

For $n = 1$ and 2, this triangulation is shown in Fig. 59 (familiar to the reader from elementary geometry textbooks).

Let $\alpha_i = \alpha_i(\Delta^n): \Delta^{n+1} \to \Delta^n \times I$ be the affine homeomorphism of Δ^{n+1} onto A_i preserving the order of vertices. These α_is are singular simplices of $\Delta^n \times I$. Consider the faces $\Gamma_j \alpha_i$ $(0 \leq i \leq n, 0 \leq j \leq n+1)$. First, $\Gamma_i \alpha_i = \Gamma_i \alpha_{i-1}$ $(1 \leq i \leq n)$; in addition to that, $\Gamma_0 \alpha_0 = \mathrm{id}_{\Delta^n} \times 0$, $\Gamma_{n+1} \alpha_n = \mathrm{id}_{\Delta^n} \times 1$. Second,

$$\Gamma_j \alpha_i(\Delta^n) = \begin{cases} \alpha_{i-1}(\Gamma_j \Delta^n), & \text{if } j < i, \\ \alpha_i(\Gamma_{j-1} \Delta_n), & \text{if } j > i+1. \end{cases}$$

Next, let us calculate the boundary of $\alpha(\Delta^n) = \sum_i (-1)^i \alpha_i(\Delta^n)$.

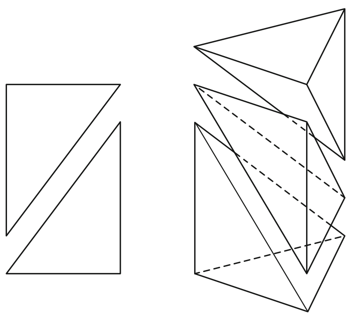


Fig. 59 Triangulations of cylinders over simplices

$$\begin{aligned}\partial\alpha(\Delta^n) &= \textstyle\sum_{j=0}^{n+1}\sum_{i=0}^{n}(-1)^{i+j}\Gamma_j\alpha_i(\Delta^n) = \mathrm{id}_{\Delta^n}\times 0 + \\ &\quad \left[\textstyle\sum_{j=2}^{n+1}\sum_{i=0}^{j-2}+\sum_{j=0}^{n-1}\sum_{i=j+1}^{n}\right](-1)^{i+j}\Gamma_j\alpha_i(\Delta^n) - \mathrm{id}_{\Delta^n}\times 1 \\ &= \mathrm{id}_{\Delta^n}\times 0 + \textstyle\sum_{i=0}^{n-1}\sum_{j=0}^{n}(-1)^{i+j+1}\alpha_i(\Gamma_j\Delta^n) - \mathrm{id}_{\Delta^n}\times 1 \\ &= \mathrm{id}_{\Delta^n}\times 0 - \mathrm{id}_{\Delta^n}\times 1 - \alpha(\partial\Delta^n).\end{aligned}$$

Now let $f, g: X \to Y$ be two continuous maps and let $H: X\times I \to Y$ be a homotopy connecting f with g. For an n-dimensional singular simplex $b: \Delta^n \to X$, define an $(n+1)$-dimensional singular chain B of Y as $(H\circ(b\times I))_\#\alpha(\Delta^n)$; the correspondence $b \mapsto B$ is extended to a homomorphism $C_n(X) \to C_{n+1}(Y)$, which we take for D_n. The previous computations show that for any chain $c \in C_n(X)$,

$$\partial D_n(c) = f_\#(c) - g_\#(c) - D_{n-1}(\partial c),$$

which means that $\{D_n\}$ is a chain homotopy between $f_\#$ and $g_\#$ (see Fig. 60).

We arrive at the following result.

Theorem. *If continuous maps $f, g: X \to Y$ are homotopic, then the chain maps $f_\#, g_\#$ are chain homotopic.*

Corollary 1. *If continuous maps $f, g: X \to Y$ are homotopic, then for all n the induced homology homomorphisms $f_*, g_*: H_n(X) \to H_n(Y)$ coincide.*

Corollary 2. *A homotopy equivalence $f: X \to Y$ induces for all n isomorphisms $f_*: H_n(x) \xrightarrow{\cong} H_n(Y)$. In particular, homotopy equivalent spaces have isomorphic homology groups.*

(Question: And what about weak homotopy equivalence? The answer is in Lecture 14.)

EXERCISE 3. Prove the last three statements for reduced homology.

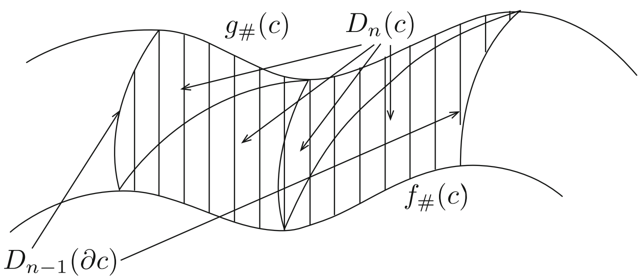


Fig. 60 From a homotopy to a chain homotopy

12.3 First Calculations

The groups of singular chains are usually huge and difficult to deal with; they are not fit for systematic calculations of homology groups. There are some efficient indirect methods of homology calculations which will be presented in the nearest future. Still, some direct calculations are possible and, actually, necessary for developing those indirect methods.

A: Homology of the One-Point Space

Let pt denote the one-point space. Then in every dimension $n \geq 0$ there is only one singular simplex $f_n\colon \Delta^n \to \mathrm{pt}$. In particular, $\Gamma_i f_n = f_{n-1}$ for all i, and $\partial f_n = f_{n-1} - f_{n-1} + f_{n-1} - \cdots + (-1)^n f_{n-1}$, which is 0 if n is odd and f_{n-1} if n is even and positive. Thus, the (unreduced) singular complex of pt has the form

$$\dots \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0 \to 0 \to \dots,$$

and

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, \text{ if } n = 0, \\ 0, \ \text{ if } n \neq 0. \end{cases}$$

Add to this that $\widetilde{H}_0(\mathrm{pt}) = 0$; this shows that $\widetilde{H}_n(\mathrm{pt}) = 0$ for all n.

A space whose homology is the same as that of pt is called *acyclic*.

Corollary (of homotopy invariance of homology). *Contractible spaces are acyclic.*

The converse is not true; fans of the function $\sin \dfrac{1}{x}$ will appreciate an example in Fig. 61. There are more interesting examples, say, the Poincaré sphere with one point deleted.

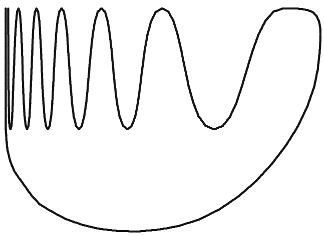


Fig. 61 A noncontractible acyclic space

B: Zero-Dimensional Homology

Theorem. *If X is path connected, then $H_0(X) = \mathbb{Z}$.*

Proof. Zero-dimensional singular simplices of X are just points of X; one-dimensional simplices are paths, and the boundary of a path joining x_0 with x_1 is $x_1 - x_0$. If X is connected, then every zero-dimensional chain $\sum_i k_i f_i$ (which is always a cycle) is homological to $\left(\sum_i k_i\right) f_0$, where f_0 is an arbitrarily fixed zero-dimensional singular simplex; indeed, if s_i is a path joining f_0 with f_i, then $\partial \sum_i k_i s_i = \sum_i k_i(f_i - f_0) = \sum_i k_i f_i - \left(\sum_i k_i\right) f_0$. We see that if $\sum_i k_i = 0$, then the chain is homological to zero. The converse is also true: The sum of the coefficients of the boundary of a one-dimensional singular simplex, and hence of the boundary of every zero-dimensional singular chain, is zero. We see that the map $\epsilon : C_0(X) = Z_0(x) \to \mathbb{Z}$ establishes an isomorphism $H_0(X) \to \mathbb{Z}$.

Equivalent statement (for a path connected X): $\widetilde{H}_0(X) = 0$.

EXERCISE 4. Prove that if $f: X \to Y$ is a continuous map between two path connected spaces, then $f_*: H_0(X) \to H_0(Y)$ is an isomorphism.

C: Homology and Components

Standard simplices are connected. Hence, every singular simplex of a space belongs to one of the path components of this space. This shows that $C_n(X) = \bigoplus_\alpha C_n(X_\alpha)$, where the X_αs are path components of X, and also $Z_n(X) = \bigoplus_\alpha Z_n(X_\alpha)$, $B_n(X) = \bigoplus_\alpha B_n(X_\alpha)$, $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$. In particular, the two previous computations imply the following. (1) For an arbitrary X, $H_0(X)$ is a free Abelian group generated by the path components of X; (2) If the space X is discrete, then $H_n(X) = 0$ for any $n \neq 0$.

12.4 Relative Homology

Let (X, A) be a topological pair; that is, A is a subset of a space X. Then $C_n(A) \subset C_n(X)$. The group $C_n(X, A) = C_n(X)/C_n(A)$ is called the groups of (*relative*) *singular chains of the pair* (X, A) or *of X modulo A*. Obviously, $C_n(X, A)$ is a free Abelian group generated by singular simplices $f: \Delta^n \to X$ such that $f(\Delta^n) \not\subset A$. Since $\partial(C_n(A)) \subset C_{n-1}(A)$, there arise a quotient homomorphism $\partial: C_n(X, A) \to C_{n-1}(X, A)$ and a complex

$$\dots \xrightarrow{\partial} C_{n+1}(X, A) \xrightarrow{\partial} C_n(X, A) \xrightarrow{\partial} C_{n-1}(X, A) \xrightarrow{\partial} \dots.$$

The homology groups of this complex are denoted $H_n(X,A)$ and are called *relative homology groups*. One can say that $H_n(X,A)$ is the quotient $Z_n(X,A)/B_n(X,A)$ of the group of *relative cycles* over the group of *relative boundaries*. Here a relative cycle is a singular chain of X whose boundary lies in A, and a relative boundary is a chain of X which becomes a boundary after adding a chain from A. (Obviously, relative boundaries are relative cycles.)

EXERCISE 5. Compute $H_0(X,A)$ in the case when X and A are both connected and in the general case.

EXERCISE 6. Construct for an arbitrary space X and an arbitrary point $x_0 \in X$ a natural isomorphism $\widetilde{H}_n(X) = H_n(X, x_0)$.

The boundary of a relative cycle is an absolute (that is, usual) cycle in A; the correspondence $c \mapsto \partial c$ determines (for every n) a *boundary homomorphism*

$$\partial_*: H_n(X,A) \to H_{n-1}(A)$$

(indeed, if $c - c'$ is a relative boundary, then $\partial c - \partial c'$ is an absolute boundary in A). The homomorphism ∂_* is included in a *homology sequence of a pair* (similar to a homotopy sequence of a pair; see Sect. 8.7; but it looks simpler than the homotopy sequence, since it involves only Abelian groups):

$$\dots \xrightarrow{\partial_*} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial_*} H_{n-1}(A) \xrightarrow{i_*} \dots,$$

where i_* is induced by the inclusion map $i: A \to X$ and j_* is induced by the projection $C_n(X) \to C_n(X)/C_n(A) = C_n(X,A)$.

Theorem. *The homology sequence of a pair is exact.*

We prefer to have this theorem in a "more general" algebraic form. Let $\mathcal{C} = \{C_n, \partial_n\}$ be a complex and let $\mathcal{C}' = \{C'_n, \partial'_n\}$ be a subcomplex which means $C'_n \subset C_n$, $\partial_n(C'_n) \subset C'_{n-1}$ for all n and $\partial'_n(c) = \partial_n(c)$ for all $c \in C'_n$. There arise a quotient complex $\mathcal{C}'' = \mathcal{C}/\mathcal{C}' = \{C''_n = C_n/C'_n, \partial''_n\}$ with a naturally defined ∂''_n, and also inclusion and projection homomorphisms $\iota: \mathcal{C}' \to \mathcal{C}$ and $\pi: \mathcal{C} \to \mathcal{C}''$. There also arise "connecting homomorphisms"

$$\partial_*: H_n(\mathcal{C}'') \to H_{n-1}(\mathcal{C}').$$

Namely, let $\gamma'' \in H_n(\mathcal{C}'')$ be an arbitrary homology class and let $c'' \in \operatorname{Ker} \partial''_n \subset C''_n = C_n/C'_n$ be a representative of γ''. Let $c \in C_n$ be a representative of (the coset) c''. The equality $\partial''_n c'' = 0$ means precisely that $c' = \partial_n c \in C'_{n-1}$. Moreover, $\partial'_{n-1} c' = \partial_{n-1} c' = \partial_{n-1} \circ \partial_n c = 0$. Thus, $c' \in \operatorname{Ker} \partial'_{n-1}$ and hence belongs to the homology class in $\gamma' \in H_{n-1}(\mathcal{C}')$; we take this class for $\partial_*(\gamma'')$.

EXERCISE 7. Prove that the correspondence $\gamma'' \mapsto \gamma'$ provides a well-defined homomorphism $\partial_*: H_n(\mathcal{C}'') \to H_{n-1}(\mathcal{C}')$. In particular, γ' does not depend on the choice of c'' in γ'' and of c in c''. Moreover, one needs to check that ∂_* is a homomorphism.

Algebraic Theorem. *The sequence*

$$\dots \xrightarrow{\partial_*} H_n(\mathcal{C}') \xrightarrow{\iota_*} H_n(\mathcal{C}) \xrightarrow{\pi_*} H_n(\mathcal{C}'') \xrightarrow{\partial_*} H_{n-1}(\mathcal{C}') \xrightarrow{\iota_*} \dots$$

is exact.

EXERCISE 8. Prove the algebraic theorem. (The proof has some resemblance to the proof of exactness of the homotopy sequence of a pair in Sect. 8.7.)

The algebraic theorem implies the theorem above; it will be used many more times in this book, including exercises later in this section.

Notice that a map $f:(X,A) \to (Y,B)$ between topological pairs (that is, a map $f:X \to Y$ such that $f(A) \subset B$) induces homomorphisms $f_*:H_n(X,A) \to H_n(Y,B)$ and a homomorphism of the homology sequence of the pair (X,A) into the homology sequence of the pair (Y,B), that is, a "commutative ladder"

$$\begin{array}{ccccccccc}
\dots \to & H_n(A) & \to & H_n(X) & \to & H_n(X,A) & \to & H_{n-1}(A) & \to \dots \\
& \Big\downarrow{\scriptstyle (f|_A)_*} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle f_*} & & \Big\downarrow{\scriptstyle (f|_A)_*} & \\
\dots \to & H_n(B) & \to & H_n(Y) & \to & H_n(Y,B) & \to & H_{n-1}(B) & \to \dots
\end{array}$$

with exact rows. Add to that $H_n(X) = H_n(X,\emptyset)$ (in this sense relative homology is a generalization of absolute homology) and that the mysterious homomorphism $j_*:H_n(X) \to H_n(X,A)$ is actually induced by the map $j = \mathrm{id}:(X,\emptyset) \to (X,A)$.

EXERCISE 9. Construct the *homology sequence of a triple*,

$$\dots \to H_n(A,B) \to H_n(X,B) \to H_n(X,A) \to H_{n-1}(A,B) \to \dots$$

$(B \subset A \subset X)$ and prove its properties, including the exactness. (Compare to Exercise 10 in Sect. 8.7.) (In the case when A is not empty, a combination of this exercise with Exercise 5 gives rise to a *reduced* homology sequence of a pair, with the absolute groups H replaced by $\widetilde{H}$).

The exactness of homology sequences of pairs and triples (combined with the five-lemma; see Sect. 8.8) has a standard set of corollaries. Among them, there is a homotopy invariance of relative homology: If $f:X \to Y$ is a homotopy equivalence, $f(A) \subset B$, and the map $A \to B$ arising is also a homotopy equivalence, then $f_*:H_n(X,A) \to H_n(Y,B)$ is an isomorphism for all n.

(We have to disappoint a reader who expects an exact "homology sequence of a fibration" relating homology groups of the total space, the base, and the fiber of a fibration. The relations between homology and fibrations are more complicated, and we will thoroughly study them in the subsequent chapters of this book.)

12.5 Relative Homology as Absolute

The results here provide the main technical tool to effectively compute homology.

Theorem. *Let (X, A) be a topological pair.*

(1) *The inclusion $X \to X \cup CA$, where $X \cup CA$ is obtained from X by attaching the cone over A, induces for every n an isomorphism*

$$H_n(X, A) \cong H_n(X \cup CA, CA) = H_n(X \cup CA, v) = \widetilde{H}_n(X \cup CA),$$

where v is the vertex of the cone CA.

(2) *If (X, A) is a Borsuk pair* (see Sect. 5.6), *for example, a CW pair* (see again Sect. 5.6), *then*

$$p_*: H_n(X, A) \to H_n(X/A, a) = \widetilde{H}_n(X/A)$$

[where $p: X \to X/A$ is the projection and $a = p(A)$] is an isomorphism for all n.

COMMENTS. 1. Part (2) follows from part (1) because of the homotopy equivalence $X \cup CA \sim X/A$ for Borsuk pairs (see Sect. 5.6 again). Thus, we need to prove only part (1).

2. In Sect. 9.10, we showed how relative homotopy groups can be presented as absolute homotopy groups of a certain space. Here we do the same for homology groups, and it is obvious that for homology the construction is much simpler than for homotopy. This may be regarded as a first illustration of a reason why homology groups are way easier to compute than homotopy groups.

The proof of the theorem is based on the so-called refinement lemma, whose proof is based on the so-called transformator lemma. Both lemmas (especially, the first) have considerable independent value. We arrange the proof in the following order. First, we state the refinement lemma. Then we state and prove the transformator lemma. Then we prove the refinement lemma. And after that we prove our theorem.

Let X be a topological space and let $\mathcal{U} = \{U_\alpha\}$ be an open covering of X. We say that a singular simplex $f: \Delta^n \to X$ is subordinated to the covering $\mathcal{U}$ if $f(\Delta^n)$ is contained in U_α for some α. Let $C_n^{\mathcal{U}}(X)$ be a subgroup of $C_n(X)$ generated by singular simplices subordinated to $\mathcal{U}$. It is obvious that $\partial(C_n^{\mathcal{U}}(X)) \subset C_{n-1}^{\mathcal{U}}(X)$: If a singular simplex is subordinated to $\mathcal{U}$, then all its faces are subordinated to $\mathcal{U}$. Thus, the groups $C_n^{\mathcal{U}}(X)$ form a subcomplex of the singular complex of X.

Refinement Lemma. *The inclusion of the complex $\{C_n^{\mathcal{U}}(X)\}$ into the complex $\{C_n(X)\}$ induces a homology isomorphism. In other words, (1) every singular cycle of X is homologous to a cycle composed of singular simplices subordinated to $\mathcal{U}$ and (2) if two such cycles are homologous in X, then their difference equals a boundary of a chain composed of singular simplices subordinated to $\mathcal{U}$.*

To prove this lemma, we need "transformators."

Definition. A *transformator* τ is a rule which assigns to every topological space X and every integer n a homomorphism $\tau_n^X : C_n(X) \to C_n(X)$ such that

(1) $\tau_0^X = \text{id}$ for every X.
(2) $\partial_n \circ \tau_n^X = \tau_{n-1}^X \circ \partial_n$ for every X and every n.
(3) If $h: X \to Y$ is a continuous map, then $h_\# \circ \tau_n^X = \tau_n^Y \circ h_\#$ for every n.

Example 1 (Barycentric Transformator). The barycentric subdivision of the standard simplex Δ^n (see Fig. 21 in Sect. 5.8) consists of $(n + 1)!$ n-dimensional Euclidean simplices corresponding to chains $\delta^0 \subset \delta^1 \subset \cdots \subset \delta^n$ of faces of dimensions $0, 1, \ldots, n$; the vertices of the simplex corresponding to this chain are centers of $\delta^0, \delta^1, \ldots, \delta^n$. In other words, simplices of the subdivision correspond to permutations $\sigma \in S_{n+1}$: The simplex $\beta_\sigma \Delta^n$ corresponding to a permutation σ of $0, 1, \ldots, n$ has vertices

$$u_k^\sigma = \frac{v_{\sigma(0)} + v_{\sigma(1)} + \cdots + v_{\sigma(k)}}{k+1},\ k = 0, 1, \ldots, n,$$

where $v_0, v_1, \ldots, v_n$ are the vertices of Δ^n in their natural order. The correspondence $v_i \mapsto u_i^\sigma$ is extended to an affine map $\beta_\sigma : \Delta^n \to \Delta^n$, which may be regarded as an n-dimensional singular simplex of Δ^n. Put $\beta(\Delta^n) = \sum_{\sigma \in S_{n+1}} \text{sgn}(\sigma)\beta_\sigma$. A direct computation shows that $\partial(\beta(\Delta^n)) = \sum_{i=0}^n (-1)^i \beta(\Gamma_i \Delta^n)$ (the faces inside Δ^n are cancelled; there remain only simplices of barycentric subdivisions of faces of Δ^n, and they appear in $\partial(\beta(\Delta^n))$ with proper signs).

EXERCISE 10. Reconstruct the details of this direct computation.

Now to the transformator. For a chain $c = \sum_i k_i f_i \in C_n(X)$, we put $\beta_n^X(c) = \sum_i k_i (f_i)_\#(\beta(\Delta^n))$. This is a transformator: Properties (1) and (3) are immediately clear, and property (2) follows from the formula for $\partial(\beta(\Delta^n))$.

Example 2 (Backward Transformator). Let $\omega : \Delta^n \to \Delta^n$ be the affine homeomorphism reversing the order of vertices ($\omega(v_i) = v_{n-1}$). For $c = \sum_i k_i f_i \in C_n(X)$, put $\omega_n^X(c) = \sum_i k_i (-1)^{\frac{n(n+1)}{2}} (f_i \circ \omega)$. It is immediately clear that $\{\omega_n^X\}$ satisfies conditions (1) and (3) from the definition of a transformator, and a direct computation shows that condition (2) is also satisfied.

EXERCISE 11. Reconstruct the details of this direct computation.

We will use the backward transformator later, in Lecture 16.

Transformator Lemma. *Let $\tau = \{\tau_n^X\}$ be a transformator. Then for every X the chain map $\tau^X = \{\tau_n^X : C_n(X) \to C_n(X)\}$ is homotopic to the identity. Thus, $(\tau^X)_{*n} : H_n(X) \to H_n(X)$ is $\text{id}_{H_n(X)}$.*

Moreover, a homotopy $D_n^X : C_n(X) \to C_{n+1}(X)$ between τ^X and id *can be defined in such a way that $f_{\#,n+1} \circ D_n^X = D_n^Y \circ f_{\#n}$ for every continuous map $f : X \to Y$.*

Proof of Transformator Lemma. We put $D_0^X = 0$ for all X. Let $n > 0$. Assume that for all X and $m < n$ we have already defined homomorphisms $D_m^X : C_m(X) \to$

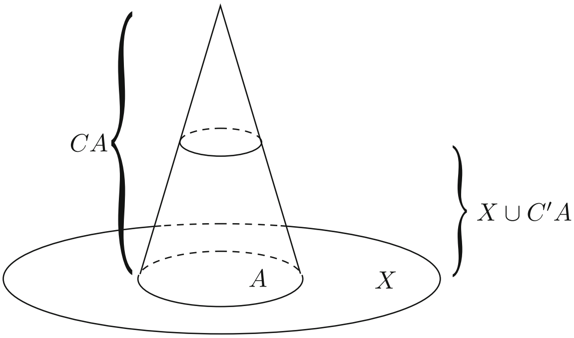


Fig. 62 The two-set covering of $X \cup CA$

$C_{m+1}(X)$ which satisfy all the conditions required (including the condition $\partial_{m+1} \circ D_m^X + D_{m-1}^X \circ \partial_m = \tau_m^X - \mathrm{id}$). The construction of D_n^X we begin with is $D_n^{\Delta^n}(\mathrm{id})$. The desired property is

$$\partial D_n^{\Delta^n}(\mathrm{id}) = \tau_n^{\Delta^n}(\mathrm{id}) - \mathrm{id} - D_{n-1}^{\Delta^n}(\partial\, \mathrm{id}).$$

But $\partial \circ D_{n-1}^{\Delta^n}(\partial\, \mathrm{id}) = \tau_{n-1}^{\Delta^n}(\partial\, \mathrm{id}) - \partial\, \mathrm{id} - D_{n-2}^{\Delta^n}(\partial\partial\, \mathrm{id}) = \partial(\tau_n^{\Delta^n}(\mathrm{id}) - \mathrm{id})$, which shows that $\partial(\tau_n^{\Delta^n}(\mathrm{id}) - \mathrm{id} - D_{n-1}^{\Delta^n}(\partial\, \mathrm{id})) = 0$. Since $H_n(\Delta^n) = 0$ (Δ^n is connected), the cycle $\tau_n^{\Delta^n}(\mathrm{id}) - \mathrm{id} - D_{n-1}^{\Delta^n}(\partial\, \mathrm{id}) \in C_n(\Delta^n)$ is a boundary of some chain in $C_{n+1}(\Delta^n)$; we choose such a chain and take it for $D_n^{\Delta^n}(\mathrm{id})$. After that, for an arbitrary X and arbitrary $c = \sum_i k_i f_i \in C_n(X)$, we put $D_n^X(c) = \sum_i k_i (f_i)_\#(D_n^{\Delta^n}(\mathrm{id})$. This D_n^X obviously satisfies the conditions in the "moreover" part of the lemma.

Proof of the Refinement Lemma. We use the barycentric transformator β. We need to prove that (1) every cycle from $C_n(X)$ is homologous to a cycle in $C_n^{\mathcal{U}}(X)$ and (2) if a cycle from $C_n^{\mathcal{U}}(X)$ is a boundary of some chain from $C_{n+1}(X)$, then it is a boundary of some chain from $C_{n+1}^{\mathcal{U}}(X)$. This follows from the following three facts. (A) For every chain $c \in C_n(X)$ the chain $(\beta_n^X)^N(c)$ with a sufficiently big N is contained in $C_n^{\mathcal{U}}(X)$ (it is obvious). (B) A cycle c is homologous to $\beta(c)$, and hence to $\beta^N(c)$ (the transformator lemma). (C) If a cycle c belongs to $C_n^{\mathcal{U}}(X)$, then the difference $c - \beta(c)$, and hence the difference $c - \beta^N(c)$, is a boundary of a chain from $C_{n+1}^{\mathcal{U}}(X)$ (the "moreover" part of the transformator lemma).

Proof of Theorem. We need to prove only part (1). Consider the covering $\mathcal{U}$ of $C \cup CA$ by two open sets: CA (without the base) and $X \cup C'A$, where $C'A$ is the lower half of the cone (without the upper base): See Fig. 62.

It follows from the relative version of the transformator lemma (which, on one side, can be proved precisely as the absolute version, and, on the other side, follows from the absolute version and the five-lemma) that the homology of the pair $(X \cup CA, CA)$ can be computed with the chain groups

$$C_n^{\mathcal{U}}(X \cup CA, CA) = C_n^{\mathcal{U}}(X \cup CA)/C_n^{\mathcal{U}}(CA);$$

the covering of the cone CA, induced by the covering $\mathcal{U}$, we denote again by $\mathcal{U}$. But obviously

$$C_n^{\mathcal{U}}(X \cup CA)/C_n^{\mathcal{U}}(CA) = C_n(X \cup C'A)/C_n(C'A) = C_n(X \cup C'A, C'A).$$

Thus,

$$\begin{aligned}\widetilde{H}_n(X \cup CA) = H_n(X \cup CA, \mathrm{pt}) &= H_n(X \cup CA, CA)\\ &= H_n(X \cup C'A, C'A) = H_n(X, A)\end{aligned}$$

(the last equality follows from the homotopy invariance of homology).

12.6 Generalizations of the Refinement Lemma: Sufficient Sets of Singular Simplices

The refinement lemma says that for computing homology groups of spaces and pairs it is possible to consider only singular simplices satisfying some additional condition. This additional condition (for the refinement lemma this is the condition of being subordinated to an open covering) may be different.

Definition. A set $\mathcal{S}$ of singular simplices is called sufficient if all faces of a singular simplex from $\mathcal{S}$ also belong to $\mathcal{S}$, so the groups $C_n^{\mathcal{S}}(X) \subset C_n(X)$ form a subcomplex of the singular complex of X, and if the inclusion map of this subcomplex induces a homology isomorphism. In other words, for every n, every cycle from $C_n(X)$ is homologous to some cycle belonging to $C_n^{\mathcal{S}}(X)$, and if a cycle belonging to $C_n^{\mathcal{S}}(X)$ equals the boundary of some chain from $C_{n+1}(X)$, then it is also a boundary of a chain in $C_{n+1}^{\mathcal{S}}(X)$. The usual procedure of proving sufficiency of some set $\mathcal{S}$ of singular simplices is to find some way of "approximating" singular simplices with all faces in $\mathcal{S}$ by chains in $C_n^{\mathcal{S}}(X)$ with the same boundary. We will not prove any general result of this kind but will list several sufficient sets in the form of exercises (the statement in the last of these exercises will actually be proved quite soon).

EXERCISE 12. If X is a smooth manifold (say, a smooth surface of some dimension in some Euclidean space), then smooth singular simplices form a sufficient set.

EXERCISE 13. If X is a domain in an Euclidean space, then affine singular simplices form a sufficient set.

EXERCISE 14. If X is a triangulated space, then affine isomorphisms of standard simplices onto the simplices of the triangulation form a sufficient set.

12.7 More Applications of the Refinement Lemma

We will give here in the form of exercises two additional properties of homology groups. In the next lecture we will prove similar statements in the CW context.

EXERCISE 15. Let (X,A) be a topological pair, and let $B \subset A$. The inclusion map $(X-B, A-B) \to (X,A)$ induces a homomorphism

$$H_n(X-B, A-B) \to H_n(X,A)$$

called an *excision homomorphism*. Prove that if $\overline{B} \subset \operatorname{Int} A$, then the excision homomorphism is an isomorphism. (This statement is called the excision theorem, or, within a certain axiomatic approach to homology theory, the excision axiom. The conditions on X, A, B which imply the excision isomorphism may be different.)

EXERCISE 16. Let $X = A \cup B$, $A \cap B = C$. We suppose that the excision homomorphisms $H_n(B,C) \to H_n(X,A)$ and $H_n(A,C) \to H_n(X,B)$ are isomorphisms. Then the homomorphisms

$$H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\text{exc.}^{-1}} H_n(B,C) \xrightarrow{\partial_*} H_{n-1}(C)$$
$$H_n(X) \xrightarrow{j_*} H_n(X,B) \xrightarrow{\text{exc.}^{-1}} H_n(A,C) \xrightarrow{\partial_*} H_{n-1}(C)$$

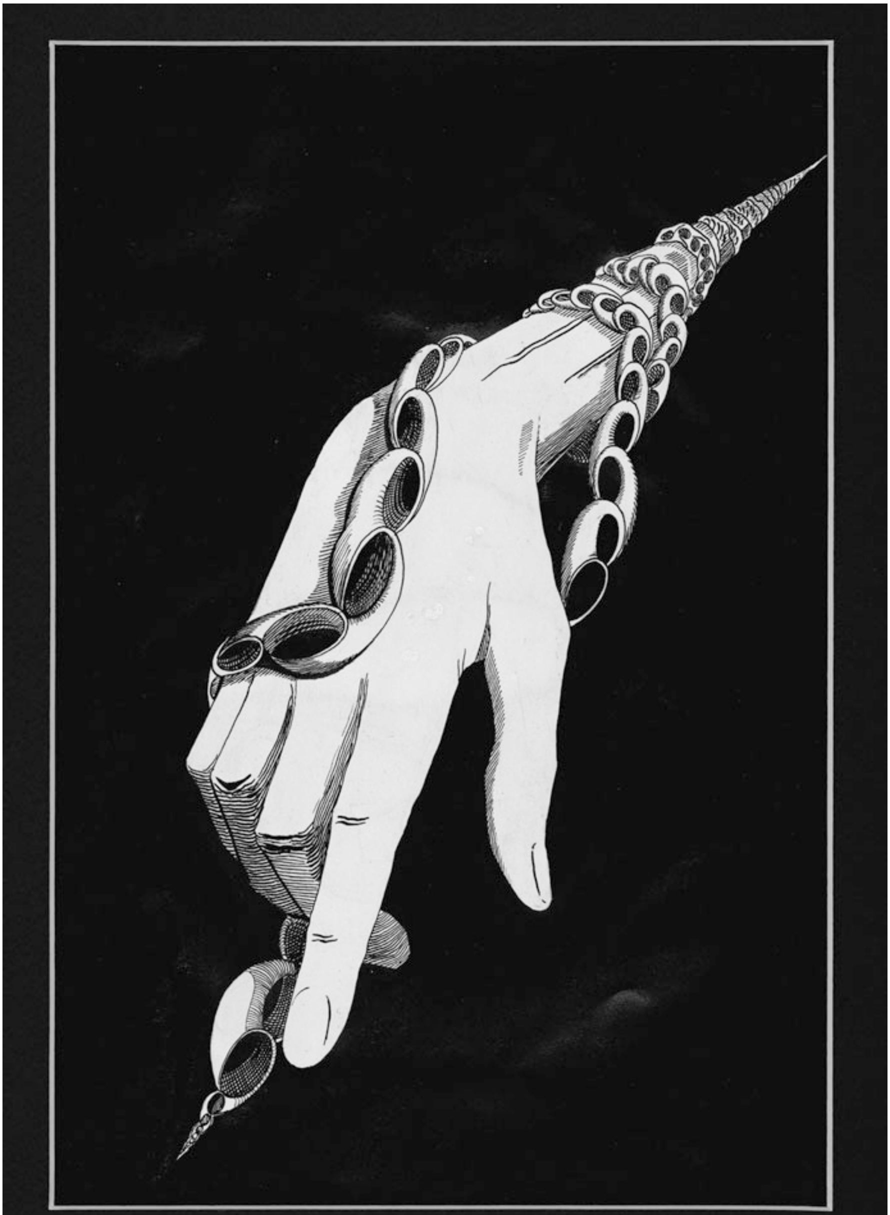
are the same, and we denote them as γ_n. The sequence

$$\dots \to H_n(C) \xrightarrow{\alpha_n} H_n(A) \oplus H_n(B) \xrightarrow{\beta_n} H_n(X) \xrightarrow{\gamma_n} H_{n-1}(C) \to \dots,$$

where α_n is the *difference* of the homomorphisms induced by the inclusions $C \to A$ and $C \to B$ and β_n is the *sum* of the homomorphisms induced by the inclusions $A \to X$ and $B \to X$, is called the *Mayer–Vietoris homology sequence* or the *homology sequence of the triad* $(X; A, B)$. Prove that this sequence is exact.

Lecture 13 Homology of CW Complexes

In this lecture, we will see that it is possible to compute the homology groups of CW complexes via a complex way narrower than the singular complex. We have to begin with the homology of spheres and bouquets of spheres.



13.1 Homology of Spheres: Suspension Isomorphism

Theorem 1. *If $n > 0$, then*

$$H_m(S^n) = \begin{cases} \mathbb{Z}, \text{ if } m = 0, n, \\ 0, \text{ if } m \neq 0, n. \end{cases}$$

The homology of the (two-point) sphere S^0 looks different: $H_0(S^0) = \mathbb{Z} \oplus \mathbb{Z}$, $H_m(S^0) = 0$, if $m \neq 0$. To make the statement better looking, we may consider the reduced homology.

Theorem $\widetilde{1}$. *For all n,*

$$\widetilde{H}_m(S^n) = \begin{cases} \mathbb{Z}, \text{ if } m = n, \\ 0, \text{ if } m \neq n. \end{cases}$$

Proof of Theorem 1 Consider a portion of the reduced homology sequence of the pair (D^n, S^{n-1}):

$$\begin{array}{ccccccc} \widetilde{H}_m(D^n) & \to & H_m(D^n, S^{n-1}) & \to & \widetilde{H}_{m-1}(S^{n-1}) & \to & \widetilde{H}_{m-1}(S^{n-1}) \\ \| & & \| & & & & \| \\ 0 & & \widetilde{H}_m(S^n) & & & & 0 \end{array}$$

[the equalities come from Sect. 12.3.A and Sect. 12.5 (part (2) of the theorem)]. From the exactness of the sequence, we have $\widetilde{H}_m(S^n) = \widetilde{H}_{m-1}(S^{n-1})$, which completes the proof, since for $n = 0$ the statement is known to us.

The isomorphism $\widetilde{H}_m(S^n) = \widetilde{H}_{m-1}(S^{n-1})$ constructed in the proof is generalized as the following suspension isomorphism.

Theorem 2. *For any topological space X and any n,*

$$\widetilde{H}_n(\Sigma X) = \widetilde{H}_{n-1}(X).$$

Proof. It follows from the reduced homology sequence of the pair (CX, X), the contractibility of CX, the equality $\Sigma X = CX/X$, and the (obvious) fact that (CX, X) is a Borsuk pair.

Remark. From the point of view of the Eckmann–Hilton duality (Lecture 4), this isomorphism is dual to $\pi_n(X) = \pi_{n-1}(\Omega X)$. Freudenthal's theorem (Lecture 10) is dual to a relation between the homology groups of X and ΩX which will be studied in Chap. 3.

EXERCISE 1 (A more precise version of Theorem 2). Let $f\colon \Delta^{n-1} \to X$ be a singular simplex of X. The composition

$$\Delta^n = C\Delta^{n-1} \xrightarrow{Cf} CX \xrightarrow{\text{proj.}} \Sigma X$$

is a singular simplex of ΣX, which we denote as Σf. Prove that the maps

$$\Sigma\colon C_{n-1}(X) \to C_n(\Sigma X),\ \sum_i k_i f_i \mapsto \sum_i k_i(\Sigma f_i)$$

commute with ∂ and induce the isomorphism $\widetilde{H}_{n-1}(X) \xrightarrow{\Sigma} \widetilde{H}_n(\Sigma X)$.

EXERCISE 2. Using Exercise 1, construct singular cycles representing the homology of spheres.

EXERCISE 3. Prove that a generator of a group $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$ is represented by a one-simplex relative cycle $f\colon \Delta^n \to D^n$, where f is a homeomorphism.

EXERCISE 4. Construct a relative version of the isomorphism Σ of Exercise 1 and prove that it commutes with maps f_* and ∂_*.

13.2 Homology of Bouquets of Spheres and Other Bouquets

Theorem 1. *Let A be an arbitrary set and let S^n_α, $\alpha \in A$, be copies of the standard n-dimensional sphere. Then*

$$\widetilde{H}_n\left(\bigvee_{\alpha\in A} S^n_\alpha\right) = \begin{cases} \bigoplus_{\alpha\in A}\mathbb{Z}\alpha, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

Here $\bigoplus_{\alpha\in A}\mathbb{Z}\alpha$ is the free Abelian group generated by the set A, that is, the sum of groups $\mathbb{Z}$ corresponding to the spheres of the bouquet.

Proof. This follows from Theorem 2 of Sect. 13.1, since $\bigvee_{\alpha\in A} S^n_\alpha$ is homotopy equivalent to the suspension of $\bigvee_{\alpha\in A} S^{n-1}_\alpha$ (and even is homeomorphic to this suspension if the latter is understood in the base point version), and for the bouquet of the zero-dimensional spheres the statement is true. Also, this follows from the next theorem.

Theorem 2. *If (X_α, x_α) are base point spaces which are Borsuk pairs, then for any m,*

$$\widetilde{H}_m\left(\bigvee_{\alpha\in A} X_\alpha\right) = \bigoplus_{\alpha\in A}\widetilde{H}_m(X_\alpha).$$

Proof. A bouquet is the quotient space of a disjoint union under the union of the base points.

EXERCISE 5. Construct the previous isomorphism at the level of cycles, establish its relative version, and prove the compatibility with f_* and ∂_*.

13.3 Maps of Spheres into Spheres and of Bouquets of Spheres into Bouquets of Spheres

Recall that a continuous map of S^n into S^n has a *degree*, an integer which characterizes its homotopy class (Sect. 10.3). A continuous map

$$g\colon \bigvee_{\alpha\in A} S^n_\alpha \to \bigvee_{\beta\in B} S^n_\beta$$

(where S^n_α, S^n_β are copies of the sphere S^n) has a whole *matrix of degrees* $\{d_{\alpha\beta} \mid \alpha \in A, \beta \in B\}$, where $d_{\alpha\beta}$ is the degree of the map

$$S^n \xrightarrow{i_\alpha} \bigvee S^n_\alpha \xrightarrow{g} \bigvee S^n_\beta \xrightarrow{p_\beta} S^n,$$

where i_α is the identity map of S^n onto S^n_α and p_β is the identity map of S^n_β of S^n and the constant map on the other spheres of the bouquet.

EXERCISE 6. Do the degrees $d_{\alpha\beta}$ determine the homotopy class of the map g?

Theorem. *The matrix of the map*

$$\begin{array}{ccc} H_n\left(\bigvee_{\alpha\in A} S^n_\alpha\right) & \xrightarrow{g_*} & H_n\left(\bigvee_{\beta\in B} S^n_\beta\right) \\ \| & & \| \\ \bigoplus_{\alpha\in A}\mathbb{Z}\alpha & & \bigoplus_{\beta\in B}\mathbb{Z}\beta \end{array}$$

coincides with $\{d_{\alpha\beta}\}$. *In particular, the map*

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{f_*} & H_n(S^n) \\ \| & & \| \\ \mathbb{Z} & & \mathbb{Z} \end{array}$$

induced by the map $f\colon S^n \to S^n$ *of degree* d *is the multiplication by* d.

Proof. Since Σ preserves the degrees, both for maps $S^n \to S^n$ and homomorphisms $H_n(S^n) \to H_n(S^n)$, our statement for some dimension n and some matrix $\{d_{\alpha\beta}\}$ implies our statement for dimension $n+1$ and the same matrix. On the other side, in dimension 0 everything is known (obvious). However, this does not resolve our problem: The trouble is that a base point–preserving map $S^0 \to S^0$ can have only degree 0 or 1. Thus, this suspension argumentation proves our theorem only for maps $g\colon \bigvee_\alpha S^n_\alpha \to \bigvee_\beta S^n_\beta$ which are n-fold suspensions of maps $\bigvee_\alpha S^0_\alpha \to \bigvee_\beta S^0_\beta$. Still, there are such maps, in particular, i_α and p_β. Thus, $(i_\alpha)_*\colon \mathbb{Z} \to \bigoplus_\alpha \mathbb{Z}\alpha$ takes a $c \in \mathbb{Z}$ into $c\alpha$ and $(p_\beta)_*\colon \bigoplus_\beta \mathbb{Z}\beta \to \mathbb{Z}$ takes $\sum c_\beta\beta$ into c_β. We want to prove that g_* takes $\sum_\alpha c_\alpha\alpha$ into $\sum_{\alpha,\beta} d_{\alpha\beta}c_\alpha\beta$, which (because of the computation of $(i_\alpha)_*$ and $(p_\beta)_*$ above) is the same as proving that $(p_\beta \circ g \circ i_\alpha)_*\colon \mathbb{Z} \to \mathbb{Z}$ is the multiplication by $d_{\alpha\beta}$. In other words, all we need is to prove that a map $S^n \to S^n$ of degree d

induces a homomorphism $H_n(S^n) \to H_n(S^n)$ which is the multiplication by d. Let us prove this (for $d = 1$, it is obvious).

Let $B = S_1^n \vee \cdots \vee S_d^n$, and let $r\colon S^n \to B$ be a map whose composition with each $p_k\colon B \to S^n (k = 1, \dots, d)$ has degree 1 (obviously, such a map exists). Let $s\colon B \to S^n$ map every sphere of the bouquet onto S^n by the identity map. Then $s \circ r$ is a map of degree d. Since $\deg(p_k \circ r) = 1$, the homomorphism $r_*\colon \mathbb{Z} \to \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (d summands) takes a $c \in \mathbb{Z}$ into $(c, \dots, c)$. Since $\deg(s \circ i_k) = 1$, the homomorphism $s_*\colon \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \to \mathbb{Z}$ takes $(c_1, \dots, c_d)$ into $c_1 + \cdots + c_d$. Hence, $(s \circ r)_*(c) = c + \cdots + c = dc$, which is what we needed to prove.

13.4 Cellular Complex

Let X be a CW complex and let $X^n = \operatorname{sk}_n X$ $(n = 0, 1, 2, \dots)$ be its skeletons. Let $\{e_\alpha^n \mid \alpha \in A_n\}$ be the set of all n-dimensional cells of X.

Pre-lemma. *The space X^n/X^{n-1} is homeomorphic to the bouquet $\bigvee_{\alpha \in A_n} S_\alpha^n$; if characteristic maps $f_\alpha\colon (D^n, S^{n-1}) \to (X^n, X^{n-1})$ are fixed, then there arises a canonical homeomorphism between X^n/X^{n-1} and $\bigvee_{\alpha \in A_n} S_\alpha^n$.*

Indeed, the maps f_α compose a continuous map $\coprod_\alpha (D_\alpha^n, S_\alpha^{n-1}) \to (X^n, X^{n-1})$, and it is obvious [follows from the properties of characteristic maps and Axiom (W)] that the map $(\coprod_\alpha D_\alpha^n)/(\coprod_\alpha S_\alpha^{n-1}) = \bigvee_\alpha S_\alpha^n \to X^n/X^{n-1}$ is a homeomorphism.

Lemma.

$$H_m(X^n, X^{n-1}) \cong \begin{cases} \text{free Abelian group generated by} & \\ \qquad\qquad n\text{--dimensional cells of } X, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

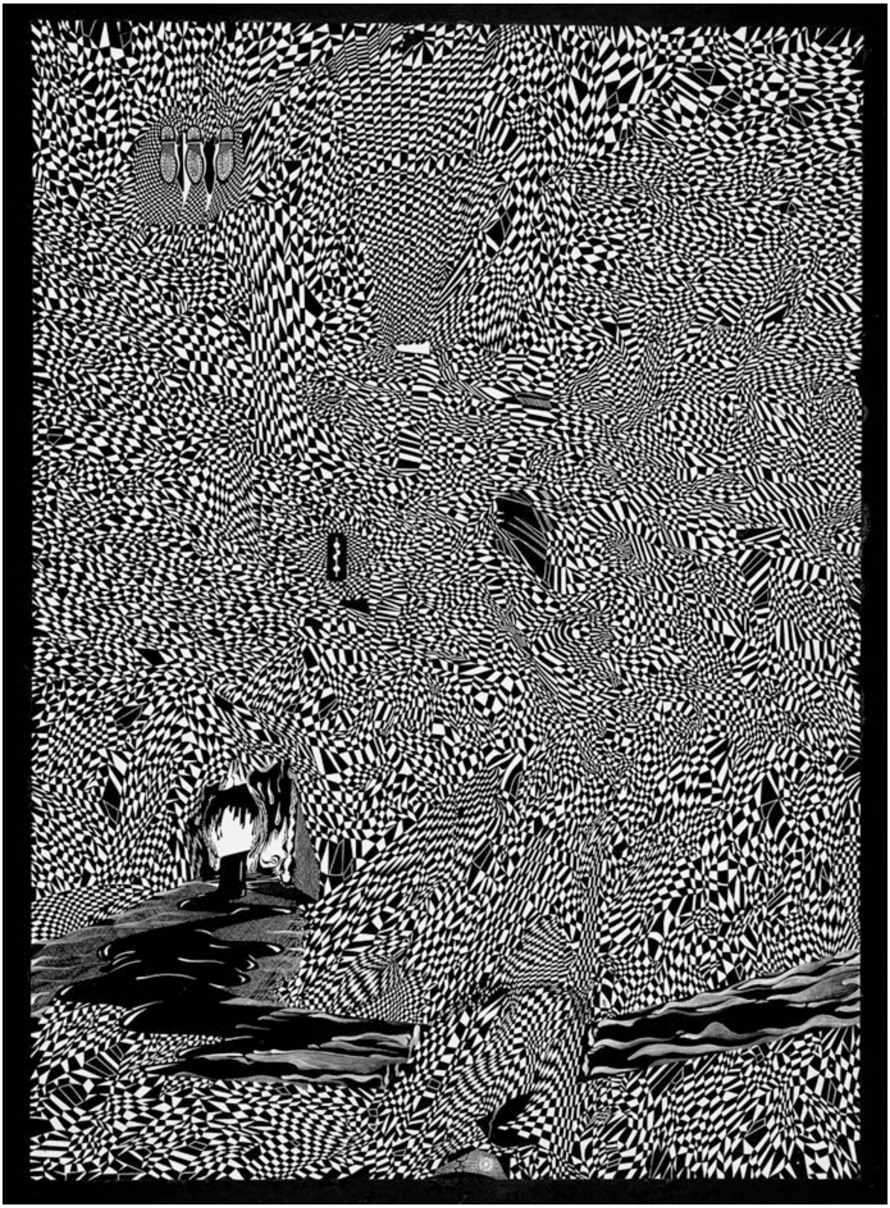
Proof. $H_m(X^n, X^{n-1}) = \widetilde{H}_m(X^n/X^{n-1}) = \widetilde{H}_m(\bigvee_{\alpha \in A_n} S_\alpha^n)$.

The group $\mathcal{C}_n(X) = H_n(X^n, X^{n-1})$ is called the *groups of cellular chains* of X. The *cellular differential* or *cellular boundary operator* $\partial = \partial_n\colon \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$ is defined as the connecting homomorphism

$$\begin{array}{ccc} H_n(X^n, X^{n-1}) & \xrightarrow{\partial_*} & H_{n-1}(X^{n-1}, X^{n-2}) \\ \| & & \| \\ \mathcal{C}_n(X) & & \mathcal{C}_{n-1}(X) \end{array}$$

from the homology sequence of the triple (X^n, X^{n-1}, X^{n-2}) (see Exercise 7 from Sect. 12.4).

AN OBVIOUS FACT: $\partial_{n-1} \circ \partial_n\colon \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X)$ is zero (follows from the equality $\partial \circ \partial = 0$ in the singular complex).



We obtain a complex

$$\dots \xrightarrow{\partial} \mathcal{C}_n(X) \xrightarrow{\partial} \dots \xrightarrow{\partial} \mathcal{C}_2(X) \xrightarrow{\partial} \mathcal{C}_1(X) \xrightarrow{\partial} \mathcal{C}_0(X) \xrightarrow{\partial} 0 \dots,$$

which is called the *cellular complex of* X. If we add the term $\mathcal{C}^{-1}(X) = \mathbb{Z}$ and augmentation $\partial_0 = \epsilon \colon \mathcal{C}_0(X) = H_0(X^0) \to \mathbb{Z}$, and then replace the notation $\mathcal{C}$ by $\widetilde{\mathcal{C}}$, we will get a definition of a reduced or augmented cellular complex.

There are two important things concerning cellular complexes. First, it is far from being as big as the singular complex; for example, for finite CW complexes the cellular chain groups are finitely generated. Moreover, not only the cellular chain groups, but also the cellular boundary operators have an explicit description that is easy to deal with. Second, we will prove that the homology of the cellular complex is the same as the homology of the singular complex. We will show how these results can be applied to calculating the homology of many classical CW complexes.

We will begin with the second part of this program.

13.5 Cellular Homology

Theorem. *For an arbitrary CW complex X, the homology of the cellular complex* $\{\mathcal{C}_n(X), \partial\}$ *coincides with the singular homology* $H_n(X)$.

Proof The proof consists of three steps.

Step 1. $H_n(X) = H_n(X^{n+1})$. Let $m > n$. From the exactness of homology sequence of the pair (X^{m+1}, X^m),

$$\begin{array}{ccccccc} H_{n+1}(X^{m+1}, X^m) & \to & H_n(X^m) & \to & H_n(X^{m+1}) & \to & H_n(X^{m+1}, X^m) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

we see that all homomorphisms

$$H_n(X^{n+1}) \to H_n(X^{n+2}) \to H_n(X^{n+3}) \to \dots$$

induced by the inclusion maps are isomorphisms. If X is finite dimensional, this settles our statement. In the general case, consider the map $H_n(X^{n+1}) \to H_n(X)$. Every $\alpha \in H_n(X)$ is represented by a finite sum of singular simplices, and every singular simplex is covered by a finite number of cells. This implies that α is represented by a cycle contained in some X^N, that is, belongs to the image of the map $H_n(X^N) \to H_n(X)$ (and we can assume that $N > n$). Since $H_n(X^{n+1}) \to H_n(X^N)$ is an isomorphism, α also belongs to the image of the map $H_n(X^{n+1}) \to H_n(X)$, so the latter is onto. Now let $\beta \in H_n(X^{n+1})$ be annihilated by the map $H_n(X^{n+1}) \to H_n(X)$. Then a cycle representing β is the boundary of some singular chain of X. But, as before, this chain must be contained in some X^N. Hence, β is also annihilated by some map $H_n(X^{n+1}) \to H_n(X^N)$, which is an isomorphism. Thus, $\beta = 0$ and our map $H_n(X^{n+1}) \to H_n(X)$ is one-to-one.

Step 2. $H_n(X^{n+1}) = H_n(X^{n+1}, X^{n-2})$. Let $m < n-1$. From the exactness of the homology sequence of the triple (X^{n+1}, X^m, X^{m-1}),

$$\begin{tikzcd}
H_n(X^m, X^{m-1}) \arrow[r] \arrow[d, equal] & H_n(X^n, X^{m-1}) \arrow[r] & H_n(X^n, X^m) \arrow[r] & H_{n-1}(X^m, X^{m-1}) \arrow[d, equal] \\
0 & & & 0
\end{tikzcd}$$

we see that all homomorphisms

$$\begin{tikzcd}
H_n(X^{n+1}, X^{n-2}) & H_n(X^{n+1}, X^{n-3}) \arrow[l] & \cdots \arrow[l] & H_n(X^{n+1}, X^{-1}) \arrow[l] \arrow[d, equal] \\
& & & H_n(X^{n-1})
\end{tikzcd}$$

are isomorphisms. This proves our statement.

Step 3. $H_n(X^{n+1}, X^{n-2}) = \dfrac{\operatorname{Ker}(\partial_n\colon \mathcal{C}_n(X) \to \mathcal{C}_{n-1}(X))}{\operatorname{Im}(\partial_{n+1}\colon \mathcal{C}_{n+1}(X) \to \mathcal{C}_n(X))}$. Consider the diagram

$$\begin{tikzcd}
\mathcal{C}_{n+1}(X) \arrow[d, equal] & H_n(X^{n-1}, X^{n-2}) = 0 \arrow[d] & & \\
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_*"] \arrow[dr, "\partial_{n+1}"'] & H_n(X^n, X^{n-2}) \arrow[r, "\alpha"] \arrow[d, "\beta"] & H_n(X^{n+1}, X^{n-2}) \arrow[r] & H_n(X^{n+1}, X^n) \arrow[d, equal] \\
& H_n(X^n, X^{n-1}) = \mathcal{C}_n(X) \arrow[d, "\partial_n"] & & 0 \\
& H_{n-1}(X^{n-1}, X^{n-2}) = \mathcal{C}_{n-1}(X) & &
\end{tikzcd}$$

where the row is a fragment of the homology sequence of the triple (X^{n+1}, X^n, X^{n-2}) and the column is a fragment of the homology sequence of the triple (X^n, X^{n-1}, X^{n-2}); in particular, both are exact. There are two zeroes in the diagram, and they show that α is an epimorphism, and β is a monomorphism. From this (and again the exactness of the sequences) we obtain

$$\begin{aligned}
H_n(X^{n+1}, X^{n-2}) &= H_n(X^n, X^{n-2})/Ker\alpha = H_n(X^n, X^{n-2})/\operatorname{Im}\partial_* \\
&= \beta(H_n(X^n, X^{n-2}))/\beta(\operatorname{Im}\partial_*) = \operatorname{Im}\beta/\operatorname{Im}(\beta \circ \partial_*) \\
&= \operatorname{Ker}\partial_n/\operatorname{Im}\partial_{n+1}.
\end{aligned}$$

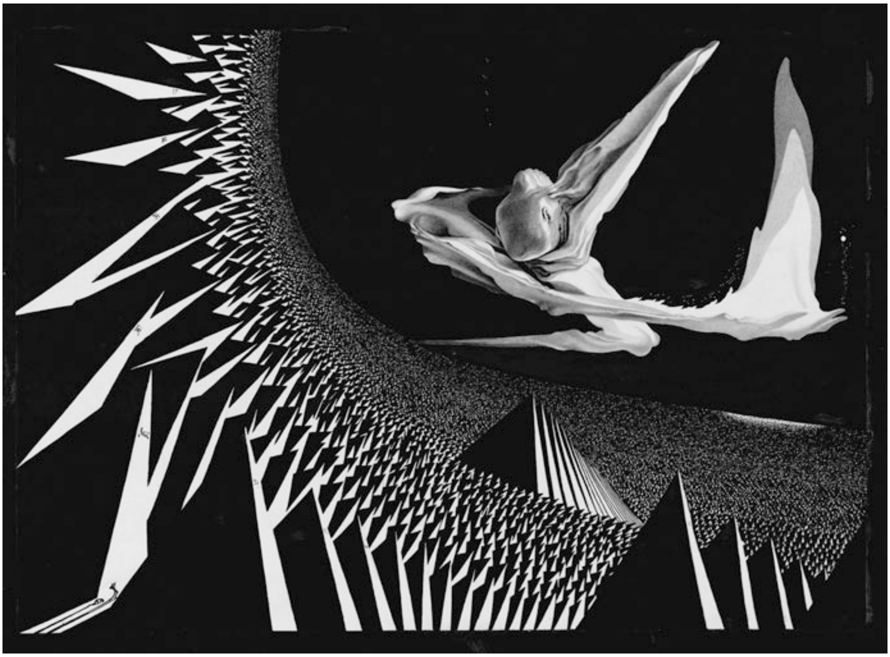
This completes step 3, and the combination of the three steps gives the isomorphism we need.

13.6 A Closer Look at the Cellular Complex

We already know that for a CW complex X, the group $\mathcal{C}_n(X)$ is isomorphic to a free Abelian group generated by n-dimensional cells of X. But the isomorphism is not genuinely canonical: It depends on a choice of characteristic maps of cells, which is not convenient because usually characteristic maps are not fixed—we know only that they exist. Actually, what we need to fix for every cell is not a characteristic map, but an *orientation*. A characteristic map of an n-dimensional cell establishes an isomorphism between two groups isomorphic to $\mathbb{Z}$: $H_n(D^n, S^n - 1) = \widetilde{H}_n(S^n)$ and $H_n(X^{n-1} \cup e, X^{n-1}) = \widetilde{H}_n((X^{n-1} \cup e)/X^{n-1})$ or $\widetilde{H}_n(X^n/(X^n - e))$ (which is the same group). One can say that the orientation of e is a choice of a generator in $\widetilde{H}_n((X^{n-1} \cup e)/X^{n-1}) \cong \mathbb{Z}$. Geometrically this indeed is an orientation: Say, if $n = 1$, then a choice of orientation is a choice of a direction of an arrow on e. In other words, characteristic maps f and $f \circ r$ always determine opposite orientations. (Zero-dimensional cells have canonical orientations.)

Thus, chains in $\mathcal{C}_n(X)$ can be presented as finite integral linear combinations of oriented n-dimensional cells, $\sum k_i e_i$. An orientation change for e_i results in a sign change for k_i.

There also exists a good description of the boundary homomorphism $\partial_{n+1}\colon \mathcal{C}_{n+1}(X) \to \mathcal{C}_n(X)$. Let e and f be cells of X of dimensions $n+1$ and n. In the homology sequence of the triple $(X^n \cup e, X^n, X^n - f)$, there is a homomorphism



$$\mathbb{Z} \cong H_{n+1}(X^n \cup e, X^n) \xrightarrow{\partial_*} H_n(X^n, X^n - f) \cong \mathbb{Z}.$$

The choice of the isomorphisms with $\mathbb{Z}$ corresponds to the orientations of the cells e, f. Every homomorphism $\mathbb{Z} \to \mathbb{Z}$ is a multiplication by some integer. This integer is called the *incidence number* of the *oriented cells* e and f and is denoted as $[e : f]$ (certainly, if $\overline{e}$ and f are disjoint, then $[e : f] = 0$). The orientation change for any of the cells e and f results in the sign change for $[e : f]$.

Theorem. *Let e be an oriented $(n+1)$-dimensional cell of X regarded as an element of $\mathcal{C}_{n+1}(X)$. Then*

$$\partial_{n+1}(e) = \sum_f [e : f]f,$$

where the sum is taken over all n-dimensional cells of X with fixed orientations. [This sum is always finite: The intersection $\overline{e} \cap f$ may be nonempty for only finitely many n-dimensional cells f—this is Axiom (C).]

EXERCISE 7. Prove this. *Recommendation:* It may be useful to consider the commutative diagram

$$\begin{tikzcd}
H_{n+1}(X^n \cup e, X^n) \arrow[r, "\partial_*"] \arrow[d] & H_n(X^n, X^n - f) \\
H_{n+1}(X^{n+1}, X^n) \arrow[r, "\partial_{n+1}"] & H_n(X^n, X^{n-1}) \arrow[u]
\end{tikzcd},$$

where the vertical maps are induced by the inclusion maps between pairs.

(A clarification is needed and possible in the case when $n = 0$. An oriented one-dimensional cell e is a path joining two zero-dimensional cells, f_0 and f_1. Then $\partial e = f_1 - f_0$; in particular, if $f_0 = f_1$, then $\partial e = 0$.)

The description of the boundary map in the preceding theorem motivates a better understanding of the incidence numbers. They can be described as degrees of maps $S^n \to S^n$. Namely, if $\varphi: S^n \to X^n$ is an attaching map for e (determined by a certain characteristic map for e) and $\psi: X^n/(X^n - f) = \overline{f}/f \to S^n$ is a homeomorphism determined by a certain characteristic map for F, then $[e : f]$ is nothing but the degree of the map

$$S^n \xrightarrow{\varphi} X^n \xrightarrow{\text{proj.}} X^n/(X^n - f) \xrightarrow{\psi} S^n.$$

The description of the degree of a map $S^n \to S^n$ given in Sect. 10.3 may be used as a geometric description of incidence numbers. Namely, take a regular value $x \in f$ of the attaching map $\varphi: S^n \to X^n$ [rather of the map $\varphi: \varphi^{-1}(f) \to f$] and compute the "algebraic number" of inverse images of x (that is, the number of inverse images where φ preserves the orientation minus the number of inverse images where φ reverses the orientation); this is $[e : f]$.

Having this in mind, we can give our theorem an aggressively tautological form: The boundary of a cell is the sum of cells which appear in the boundary of this cell with coefficients equal to the multiplicity of their appearance in this boundary.

13.7 First Applications

Theorem 1. *If the number of n-dimensional cells of a CW complex X is N, then the group $H_n(X)$ is generated by at most N generators; in particular, the nth Betti number $B_n(X)$ does not exceed N. For example, if X does not have n-dimensional cells at all, then $H_n(X) = 0$; in particular, if X is finite dimensional, then $H_n(X) = 0$ for all $n >$* $\dim X$. (Compare with homotopy groups!)

It follows directly from previous results.

Algebraic Lemma (Euler–Poincaré). *Let*

$$\dots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots$$

be a complex with the "total group" $\bigoplus_n C_n$ finitely generated. Let c_n be the rank of the group C_n and h_n be the rank of the homology group H_n. Then

$$\sum_n (-1)^n c_n = \sum_n (-1)^n h_n.$$

EXERCISE 8. Prove this.

Corollary. *Let X be a finite CW complex, and let c_n be the number of n-dimensional cells of X. Then*

$$\sum_n (-1)^n c_n = \sum_n (-1)^n B_n(X).$$

Thus, the number $\sum_n (-1)^n c_n$ does not depend on the CW structure; it is determined by the topology (actually, be the homotopy type) of X. This number is called the *Euler characteristic* of X and is traditionally denoted by $\chi(X)$.

Historical Remark. This number is attributed to Euler because of the Euler polyhedron theorem, which states that for every convex polyhedron in space, the numbers V, E, and F of vertices, edges, and faces are connected by the relation $V - E + F = 2$. Certainly, this is a computation of the Euler characteristic of the surface of the polyhedron, that is, of the sphere. It is worth mentioning that Euler was not the first to prove this theorem: It was proved, a century before Euler, by Descartes.

Now let us revisit the excision theorem and the Mayer–Vietoris sequence (Exercises 13 and 14 of Lecture 12).

Theorem 2 (Excision Theorem). *Let X be a CW complex and let A, B be CW subcomplexes of X such that $A \cup B = X$. Then (for every n)*

$$H_n(X, A) = H_n(B, A \cap B).$$

Indeed, X/A and $B/(A \cap B)$ are the same as CW complexes.

Theorem 3 (Mayer–Vietoris Sequence). *Let X be a CW complex and let A, B be CW subcomplexes of X such that $A \cup B = X$. Then there exists an exact sequence*

$$\cdots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \dots$$

(see the description of maps in Exercise 14 of Lecture 12).

Proof. Let $Y = (A \times 0) \cup ((A \cap B) \times I) \cup (B \times 1) \subset X \times I$ and let $C \subset Y$ be $(A \cap B) \times I$. Then Y/C and $\Sigma(A \cap B)$ (actually with the vertices merged; this slightly affects the case of dimension 0) are the same CW complexes (see schematic picture in Fig. 63).

Notice, in addition, that $C = A \coprod B$ and $Y \sim X$. The last homotopy equivalence is established by the obvious map $f\colon Y \to X$ (the restriction of the projection $X \times I \to X$) and a map $g\colon X \to Y$ which is defined in the following way. The homotopy $h_t\colon A \cap B \to Y$, $h_t(x) = (x, 1-t)$ is extended, by Borsuk's theorem, to a homotopy $H_t\colon A \to Y$ of the map $A \to Y$, $x \to (x, 1)$. Then maps $H_1\colon A \to Y$ and $B \to Y$, $x \mapsto (x, 0)$ agree on $A \cap B$ and hence compose a map $X \to Y$; this is g; the relations $f \circ g \sim \mathrm{id}$, $g \circ f \sim \mathrm{id}$ are obvious. Thus, $H_n(Y) = H_n(X), H_n(C) = H_n(A) \oplus H_n(B)$, and $H_n(Y, C) = H_{n-1}(A \cap B)$ (with small corrections in dimension 0), and the homology sequence of the pair (Y, C) is the Mayer–Vietoris sequence of the triad $(X; A, B)$.

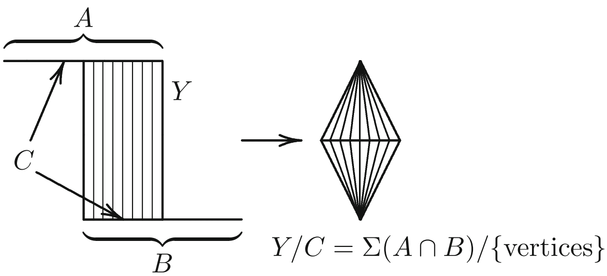


Fig. 63 To the proof of the Mayer–Vietoris theorem

13.8 Some Calculations

A: Spheres

We already know the homology of spheres, but let us calculate them again for practice in the technique based on cellular complexes. The sphere S^n has a CW structure with two cells, of dimensions 0 and n. Thus (if $n > 0$), $\mathcal{C}_0(S^n) = \mathcal{C}_n(S^n) = \mathbb{Z}$, and all other cellular chain groups are trivial. The differential ∂ has to be 0 (if $n > 1$, then this follows from the "dimension argumentations"; for $n = 1$, we use the remark after Exercise 7); hence,

$$H_i(S^n) = \mathcal{C}_i(S^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, n, \\ 0, & \text{if } i \neq 0, n. \end{cases}$$

EXERCISE 9. Prove this using another CW decomposition of S^n described in Sect. 5.4.

B: Projective Spaces

The cases of complex, quaternionic, and Cayley projective spaces are not more difficult than the cases of spheres: For the CW structures described in Sect. 5.4, there are no cells of adjacent dimensions, the differential ∂ is trivial, and the homology groups coincide with the cellular chain groups. Thus,

$$H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 2, 4, \dots, [2n, \text{if } n \text{ is finite}], \\ 0 & \text{for all other } i; \end{cases}$$

$$H_i(\mathbb{H}P^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 4, 8, \dots [, 4n, \text{if } n \text{ is finite}], \\ 0 & \text{for all other } i; \end{cases}$$

$$H_i(\mathbb{C}\mathbf{a}P^2) = \begin{cases} \mathbb{Z}, & \text{if } i = 0, 8, 16, \\ 0 & \text{for all other } i. \end{cases}$$

The real case is more complicated, since $\mathbb{R}P^n$ has cells $e^0, e^1, e^2, \dots$ [, e^n if n is finite].

Lemma. $[e^{i+1} : e^i] = \begin{cases} \pm 2, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases}$

Proof. The attaching map $f\colon S^i \to \mathbb{R}P^i$ is the standard twofold covering. The inverse image of (actually, any) point of $\mathbb{R}P^i$ consists of two points, and the restrictions of f to neighborhoods of these points are related by the antipodal map $S^i \to S^i$.

This antipodal map preserves the orientation if i is odd and reverses the orientation if i is even. Thus, the contributions of these two points in $[e^{i+1}, e^i]$ have the same sign if i is odd and have different signs if i is even. This implies the formula of the lemma.

Thus, the cellular complex of $\mathbb{R}P^n$ is as shown below.

$$\begin{array}{lccccccccccccc}
\text{(if } n \text{ is odd)} & & 0 & & 2 & & 0 \\
\text{(if } n \text{ is even)} & & 2 & & 0 & & 2 & & 0 & & 2 & & 0 \\
& \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \\
& \mathcal{C}_n & & \mathcal{C}_{n-1} & & \mathcal{C}_{n-2} & & & & \mathcal{C}_2 & & \mathcal{C}_1 & & \mathcal{C}_0
\end{array}$$

Since $\mathrm{Im}(\xrightarrow{0}) = 0$, $\mathrm{Ker}(\xrightarrow{0}) = \mathbb{Z}$, $\mathrm{Im}(\xrightarrow{2}) = 2\mathbb{Z}$, and $\mathrm{Ker}(\xrightarrow{2}) = 0$, the factorization yields

$$H_i(\mathbb{R}P^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 0 \text{ or } i = n \text{ and } n \text{ is odd,} \\ \mathbb{Z}_2, & \text{if } i \text{ is odd and } i < n, \\ 0 & \text{in all other cases.} \end{cases}$$

EXERCISE 10. Find the Euler characteristics of all finite-dimensional projective spaces.

C: Grassmann Manifolds

Again, in the complex and quaternion cases, there are no cells of adjacent dimensions, so the ith homology group is a free Abelian group of rank (= Betti number) equal to the number of i-dimensional cells. The Betti numbers are as follows. For i odd, $B_i(G(n,k)) = 0$; for i even, this is the number of Young diagrams of $\frac{i}{2}$ cells contained in the $k \times (n-k)$ rectangle. For quaternionic Grassmann manifolds everything is doubled: $B_i(\mathbb{H}G(n,k)) = B_{i/2}(\mathbb{C}G(n,k))$; in particular, $B_i(\mathbb{H}G(n,k)) = 0$ if i is not divisible by 4.

In the real case the situation is more complicated.

EXERCISE 11. Let Δ and Δ' be two Young diagrams with i and $i-1$ cells contained in the $k \times (n-k)$ rectangle. Prove that if $\Delta' \not\subset \Delta$, then $[e(\Delta) : e(\Delta')] = 0$. If $\Delta' \subset \Delta$ and the difference $\Delta - \Delta'$ consists of one cell with the coordinates (s,t), then

$$[e(\Delta) : e(\Delta')] = \begin{cases} \pm 2, & \text{if } s + t \text{ is even,} \\ 0, & \text{if } s + t \text{ iodd.} \end{cases}$$

Use this for computation of the homology of $G(n,k)$ with reasonably small n, k. Also, compute $H_{k(n-k)}(G(n,k))$.

EXERCISE 12. Find incidence numbers for the case of the manifold $G_+(n,k)$. In particular, find $H_{k(n-k)}(G_+(n,k))$.

D: Flag Manifolds

Again, the complex and quaternionic cases are relatively easy. The reader can try to investigate the real case.

E: Classical Surfaces

Classical surfaces with holes are homotopy equivalent to bouquets of circles, so we will consider classical surfaces without holes. The cellular complex for such a surface has the form

$$\begin{array}{ccccc} \mathbb{Z} & \xrightarrow{\partial_2} & \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} & \xrightarrow{\partial_1} & \mathbb{Z} \\ \mathcal{C}_2 & & \mathcal{C}_1 & & \mathcal{C}_0, \end{array}$$

where the number of the summands $\mathbb{Z}$ in $\mathcal{C}_1$ is $2g$, $2g+1$, or $2g+2$ if our surface is a sphere with g handles, a projective plane with g handles, or a Klein bottle with g handles, respectively. The differential ∂_1 is zero (every one-dimensional cell has equal endpoints). To find ∂_2, we consider the construction of the classical surface from a polygon (Sect. 1.10). Each of the $2g$ one-dimensional cells arising from the handles is obtained by attaching differently oriented sides of the polygon, so the incidence numbers of the two-dimensional cell with each of these 1-cells is 0. On the other hand, the other one-dimensional cells (if there are any) are obtained by attaching coherently oriented sides, and the incidence number with these cells is 2. Thus,

$$\partial_2(1) = \begin{cases} (0, \ldots, 0) & \text{for a sphere with } g \text{ handles,} \\ (0, \ldots, 0, 2) & \text{for a projective plane with } g \text{ handles,} \\ (0, \ldots, 0, 2, 2) & \text{for the Klein bottle with } g \text{ handles.} \end{cases}$$

This leads to the results for homology:

$$H_0(X) = \mathbb{Z} \text{ always,}$$

$$H_1(X) = \begin{cases} \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2g}, & \text{if X is a sphere with } g \text{ handles,} \\ \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2g} \oplus \mathbb{Z}_2, & \text{if } X \text{ is a projective plane with } g \text{ handles,} \\ \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{2g+1} \oplus \mathbb{Z}_2, & \text{if } X \text{ is a Klein bottle with } g \text{ handles,} \end{cases}$$

$$H_2(X) = \begin{cases} \mathbb{Z}, & \text{if } X \text{ is a sphere with handles,} \\ 0 & \text{in all other cases} \end{cases}$$

EXERCISE 13. Find the Euler characteristics of classical surfaces.

13.9 Chain Maps of Cellular Complexes

Let $h: X \to Y$ be a cellular map of a CW complex into a CW complex. Then $h(X^n) \subset Y^n$ for all n, and hence h induces a map $H_n(X^n, X^{n-1}) \to H_n(Y^n, Y^{n-1})$, that is, $\mathcal{C}_n(X) \to \mathcal{C}_n(Y)$, which we denote as $h_\#$ or $h_{\#n}$. Such maps induce a homomorphism between cellular complexes of X and Y, and the induced homology map is just $h_*: H_n(X) \to H_n(Y)$. To prove this, we need to consider every step of the proof of the theorem in Sect. 13.5, and to consider maps between the diagrams in these steps for X into similar diagram for Y. The commutativity of (three-dimensional) diagrams arising will imply our statement.

We can add that if $c = \sum_i k_i e_i \in \mathcal{C}_n(X)$, where e_i are n-dimensional cells of X, then $h_\#(c) = \sum_i k_i \big(\sum_j d_h(e_i, f_j) f_j \big)$, where the f_j are n-dimensional cells of Y and the number $d_h(e, f)$ is defined with the help of characteristic maps φ and ψ of e and f as the degree of the map

$$S^n = D^n/S^{n-1} \xrightarrow{\varphi} X^n/X^{n-1} \xrightarrow{h} Y^n/Y^{n-1} \xrightarrow{\text{proj.}} Y^n/(Y^n - f) \xrightarrow{\psi^{-1}} D^n/S^{n-1} = S^n.$$

Using the description of the degree of a map $S^n \to S^n$ in Sect. 10.3, we can say that $d_h(e, f)$ is the algebraic number of inverse images of a regular value $x \in f$ of the map $h: e \cap h^{-1}(f) \to f$.

Certainly, this construction works only for cellular maps, but it is not a big deal, since every continuous map is homotopic to a cellular map. (Not a big deal? We will cast a doubt on this statement in Lecture 16.) Thus, one can say that the cellular theory can be used as a substitute for the singular theory. But without the singular theory (which is topologically invariant from the very beginning) we would have had to prove that homeomorphic CW complexes have isomorphic homology groups.

13.10 Classical Complex

A cellular complex appears especially attractive when a CW structure is actually a triangulation (see Sect. 5.8). We consider a triangulated space X with an additional structure (a substitute for fixing characteristic maps): We suppose that the set of vertices of X is *ordered*, or, at least, vertices of every simplex are ordered in such a way that the ordering of vertices of a face of a simplex is always compatible with the ordering of vertices of this simplex. We refer to such triangulations as ordered triangulations. (For example, the barycentric subdivision of any triangulation is naturally ordered: Vertices of simplices of a barycentric subdivisions are centers of faces of simplices of the given triangulation and these are ordered by the dimensions of the faces.)

For a simplex with the vertices ordered, there is a canonical affine homeomorphism of the standard simplex onto this simplex; this homeomorphism can be regarded as a singular simplex of an ordered triangulated space X. We obtain a set of special singular simplices of X, and it is clear that faces of "special singular simplices" are also special. By this reason, linear combinations of special singular simplices form a subcomplex of the singular complex, and it is also clear that this subcomplex is precisely the cellular complex of the triangulation.

Historically, the complex described above is the first chain complex of a (orderly triangulated) topological space ever considered. It can be described very directly: Chains are integral linear combinations of simplices (remember the ordering!), and the boundary is given by the very familiar formula $\partial\big(\sum_i k_i s_i\big) = \sum_i k_i\big(\sum_j (-1)^j \Gamma_j s_i\big)$, where the s_i are simplices of our triangulation and the $\Gamma_j s_i$ are their faces. Obviously, the inclusion of the classical complex into the singular complex induces the isomorphism of the homology groups [to show this, the only thing we need to add to what we already know is that n-dimensional simplices regarded as singular simplices are relative cycles of (X^n, X^{n-1}), and their homology classes form the usual basis in $\mathcal{C}_n(X) = H_n(X^n, X^{n-1})$].

For the classical chain groups, the notation $\mathcal{C}_n^{\text{class}}(X)$ is often used.

Historical Remark. The classical definition of homology created the necessity of proving a topological invariance theorem: Homeomorphic triangulated spaces have isomorphic homology groups. The initial proof, due to J. Alexander, was long and complicated (hundreds of pages in old topology textbooks). There was an attempt to deduce the topological invariance of classical homology from the so-called *Hauptvermutung* (German for *main conjecture*) of combinatorial topology: Any two triangulations of a topological space have simplicially equivalent subdivisions. But the *Hauptvermutung* turns out to be false: The first counterexample was found by J. Milnor in 1961, and many other counterexamples were constructed later, in particular for simply connected smooth manifolds. The whole problem of topological invariance disappeared mysteriously when singular homology was defined. The first definition of singular homology was given by O. Veblen in the late 1920s but became broadly known some 10 years later.

EXERCISE 14. Using the classical complex, find the Betti numbers of the skeletons of the standard simplex. (Make your computations as explicit as possible.)

EXERCISE 15. (An algebraic lemma) Let $\{C_n, \partial_n\}, \{C'_n, \partial'_n\}$ be two positive $C_n = C'_n = 0$ (for $n < 0$) complexes of *free* Abelian groups, and let f be a homomorphism of the first complex into the second one. Prove that if f_{*n} is an isomorphism for all, then f is a homotopy equivalence. Deduce that the classical complex is homotopy equivalent to the singular complex.

EXERCISE 16. Prove that the cellular complex of a CW complex is homotopy equivalent to its singular complex. (There are several different ways of proving that, so we refrain from giving any hint.)

13.11 The Singular Complex as a CW Complex

We finish this lecture with a construction which may seem amusing to some readers but actually is quite useful (we will use it in the beginning of the next lecture). Let X be a topological space, and let $\mathrm{Sing}_n(X)$ be the set of all n-dimensional singular simplices of X. Consider a (monstrous, we agree) topological space $Y = \coprod_{n=0}^{\infty} \coprod_{\alpha\in\mathrm{Sing}_n(X)} \Delta^n_\alpha$ (where Δ^n_α is a copy of the standard simplex Δ^n) and make, for every n and every α, the identification $\Gamma_i\Delta^n_\alpha = \Delta^{n-1}_{\Gamma_i\alpha}$ (both are copies of Δ^{n-1} contained in Y). We denote the resulting space as $\mathcal{S}ing(X)$. This space has a natural CW structure (images of $\mathrm{Int}\,\Delta^n_\alpha \subset Y$ are cells of $\mathcal{S}ing(X)$ and the maps $\Delta^n \xrightarrow{=} \Delta^n_\alpha \xrightarrow{\subset} Y \xrightarrow{\text{proj.}} \mathcal{S}ing(X)$ can be taken for characteristic maps. [Notice that although the cells of $\mathcal{S}ing(X)$ look like simplices, its CW structure is not a triangulation: The intersection of closed simplices is not a face.] There is also a natural map $\mathcal{S}ing(X) \to X$, which induces the identity homomorphism in homology [just take $\alpha\colon \Delta^n \to X$ on $\Delta^n_\alpha \subset \mathcal{S}ing(X)$].

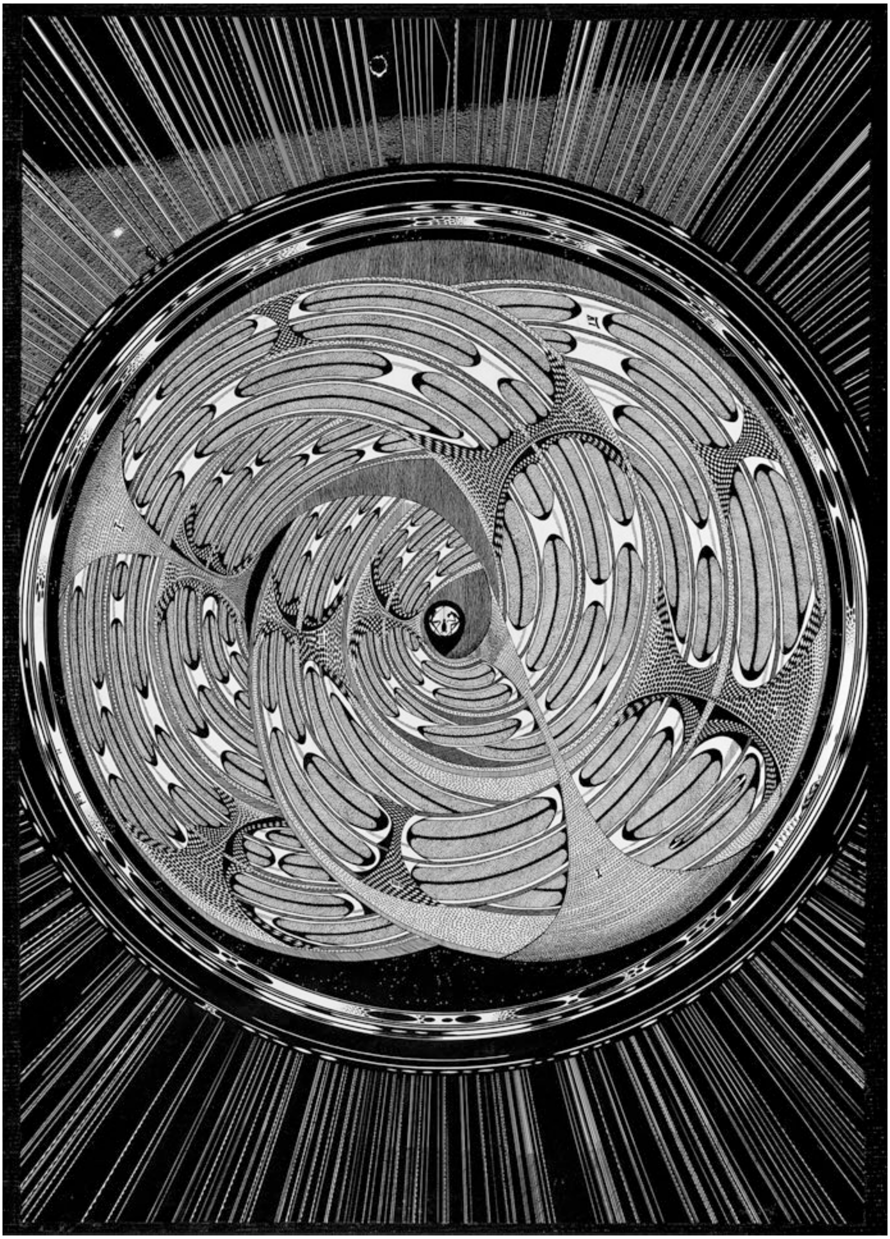
It is immediately obvious that the cellular complex of $\mathcal{S}ing(X)$ is the same as the singular complex of X; in particular, $H_n(\mathcal{S}ing(X)) = H_n(X)$ for all X. Actually, the spaces $\mathcal{S}ing(X)$ and X are weakly homotopy equivalent (and homotopy equivalent if X is a CW complex). We will see that later.

Let us add that the $\mathcal{S}ing$ construction is natural in the sense that a continuous map $X \to Y$ gives rise to a cellular map $\mathcal{S}ing(X) \to \mathcal{S}ing(Y)$ with the same induced map in homology. Also, if $A \subset X$, then $\mathcal{S}ing(A) \subset \mathcal{S}ing(X)$ and there arises a continuous map

$$(\mathcal{S}ing(X), \mathcal{S}ing(A)) \to (X, A)$$

which induces isomorphisms

$$H_n(\mathcal{S}ing(X), \mathcal{S}ing(A)) \to H_n(X, A).$$



Lecture 14 Homology and Homotopy Groups

The connection between homology and homotopy groups is seen always from the preliminary description of homology in the beginning of Lecture 12: Spheroids are cycles and homotopical spheroids are homological cycles. This suggests that there must be a natural map from homotopy groups into homology groups. This map, called the *Hurewicz homomorphism*, is the main subject of this lecture. We will see that the connection between homotopy and homology groups is deeper than it may seem at the beginning, but we also will show examples which should serve as a warning to a reader who expects too much of this connection.

14.1 Homology and Weak Homotopy Equivalences

Theorem. *If $f: X \to Y$ is a weak homotopy equivalence, then $f_*: H_n(X) \to H_n(Y)$ is an isomorphism for all n.*

Proof. Since both weak homotopy equivalences and homology homomorphisms are homotopy equivalent, we can replace the map f by the inclusion map $X \to Cyl(f)$ of X into the mapping cylinder of f (see Sects. 2.3 and 3.3). Because of this, we can assume that the given map f is an inclusion, so we have a pair (Y, X). Also, we have a pair $(\mathcal{S}ing(Y), (\mathcal{S}ing(X))$ and a continuous map $h: (\mathcal{S}ing(Y), \mathcal{S}ing(X)) \to (Y, X)$ which induces isomorphisms

$$h_*: H_n(\mathcal{S}ing(Y), \mathcal{S}ing(X)) \to H_n(Y, X)$$

(see Sect. 13.11).

On the other hand, since f is a weak homotopy equivalence, the map $f_*: \pi(\mathcal{S}ing(Y), X) \to \pi(\mathcal{S}ing(Y), Y)$ is a bijection, which means that the map $h: \mathcal{S}ing(Y) \to Y$ is homotopic to a map whose image is contained in X. Hence, the map $h_*: H_n(\mathcal{S}ing(Y), \mathcal{S}ing(X)) \to H_n(Y, X)$ is zero, which shows that $H_n(Y, X) = 0$ for all n. By exactness of the homology sequence of the pair (Y, X), this shows that all the homomorphisms $f_*: H_n(X) \to H_n(Y)$ are isomorphisms.

Recall that according to another result from Sect. 11.4, a map is a weak homotopy equivalence if and only if it induces an isomorphism in homotopy groups. Because of this, our theorem assumes the following memorable form.

Corollary. *If a continuous map induces an isomorphism between homotopy groups, then it also induces an isomorphism between homology groups.*

This will be further developed in the last section of this lecture.

To finish this section, we will formulate some exercises which will show that some statements looking similar to the preceding theorem and corollary are actually false.

EXERCISE 1. Prove that the spaces S^2 and $S^3 \times \mathbb{C}P^\infty$ have isomorphic homotopy groups but nonisomorphic homology groups. Same for the spaces $S^m \times \mathbb{R}P^n$ and $S^n \times \mathbb{R}P^m$ with $m \neq n, m \neq 1, n \neq 1$. (Compare with Exercises 5 and 6 in Lecture 11.)

EXERCISE 2. Prove that the spaces $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups but nonisomorphic homotopy groups.

EXERCISE 3. Prove that the Hopf map $S^3 \to S^2$ induces a trivial homomorphism in reduced homology groups but a nontrivial homomorphism in homotopy groups.

EXERCISE 4. Prove that the projection map $S^1 \times S^1 \to (S^1 \times S^1)/(S^1 \vee S^1) = S^2$ induces a trivial homomorphism in homotopy groups but a nontrivial homomorphism in reduced homology groups.

14.2 The Hurewicz Homomorphism

Let X be a topological space with a base point X_0. Let s_n be the canonical generator of the group $H_n(S^n) = \mathbb{Z}, n = 1, 2, \ldots$. For a $\varphi \in \pi_n(X, x_0)$ put

$$h(\varphi) = f_*(s_n) \in H_n(X),$$

where $f\colon S^n \to X$ is a spheroid of the class φ [obviously, $h(\varphi)$ does not depend on the choice of the spheroid f]. The function $\varphi \mapsto h(\varphi)$ is a homomorphism

$$h\colon \pi_n(X, x_0) \to H_n(X).$$

Indeed, let the spheroid f be the sum of spheroids $f', f''\colon S^n \to X$, that is, f is the composition

$$S^n \xrightarrow{\mu} S^n \vee S^n \xrightarrow{f' \vee f''} X$$

(see Fig. 37). Then $\mu_*(s) = s' + s''$ where $s', s'' \in H_n(S^n \vee S^n)$ are generators corresponding to the two spheres of the bouquet, and $f_*(s) = (f' \vee f'')_*(s' + s'') = f'_*(s) + f''_*(s)$.

This homomorphism is called the Hurewicz homomorphism; it is natural with respect to continuous maps (taking a base point into a base point).

EXERCISE 5. Prove that the diagram

$$\begin{tikzcd} \pi_n(X, x_0) \arrow[rr, "u_\#"] \arrow[dr, "h"'] & & \pi_n(X, x_1) \arrow[dl, "h"] \\ & H_n(X) & \end{tikzcd}$$

is commutative for any path u joining x_0 with x_1.

Theorem (Hurewicz). *Let* $\pi_0(X, x_0) = \cdots = \pi_{n-1}(X, x_0) = 0$*, where* $n \geq 2$*. Then* $H_1(X) = \cdots = H_{n-1}(X, x_0) = 0$ *and* $h\colon \pi_n(X, x_0) \to H_n(X, x_0)$ *is an isomorphism.*

Proof. By the theorem of Sect. 11.6, there exists a CW complex weakly homotopy equivalent to X. Since a weak homotopy equivalence induces isomorphisms both in homotopy groups and in homology groups (the first by Sect. 13.11, the second by Sect. 14.1), we can assume that X itself is a CW complex. Then Sect. 5.9 allows us to make an additional assumption that X has one vertex and no cells of dimensions $1, \ldots, n-1$. This already shows that $H_1(X) = \cdots = H_{n-1}(X) = 0$ (Theorem 1 in Sect. 13.7), and $H_n(X) = \mathcal{C}_n(X)/\operatorname{Im}\partial_{n-1}$ is not different from $\pi_n(X)$ according to the theorem in Sect. 11.3.

Corollary (The Inverse Hurewicz Theorem). *If* X *is simply connected and* $H_2(X) = \cdots = H_{n-1}(X) = 0$ $(n \geq 2)$*, then* $\pi_2(X) = \cdots = \pi_{n-1}(X) = 0$ *and* $h\colon \pi_n(X) \to H_n(X)$ *is an isomorphism.*

Together these theorems mean that the first nontrivial homotopy and homology groups of a simply connected space occur in the same dimension and are isomorphic.

EXERCISE 6. Prove that a simply connected CW complex with the same homology groups as S^n is homotopy equivalent to S^n. [*Hint:* Apply Whitehead's theorem to a spheroid $S^n \to X$ representing a generator of the group $\pi_n(X) \cong \mathbb{Z}$.] Do the same for the bouquet of spheres of the same dimensions.

Remark. Thus, we see that the triviality of the homotopy groups, as well as the triviality of the homology groups, implies the homotopy triviality (contractibility) of a simply connected CW complex. At the same time, we have the examples which show that neither the triviality of induced homotopy groups homomorphisms nor the triviality of induced homology homomorphisms secures homotopy triviality of a continuous map. It turns out that even these two trivialities together do not imply the homotopy triviality of a continuous map.

EXERCISE 7. Prove that the composition

$$S^1 \times S^1 \times S^1 \xrightarrow{\text{proj.}} (S^1 \times S^1 \times S^1)/\operatorname{sk}_2(S^1 \times S^1 \times S^1) = S^3 \xrightarrow{\text{Hopf}} S^2$$

induces a trivial map of both homotopy and homology groups but is not homotopic to a constant map.

EXERCISE 8. Do the same for the map

$$S^{2n-2} \times S^3 \xrightarrow{\text{proj.}} (S^{2n-2} \times S^3)/(S^{2n-2} \vee S^3) = S^{2n+1} \xrightarrow{\text{Hopf}} \mathbb{C}P^n.$$

14.3 The Case $n = 1$

Theorem (Poincaré). *For an arbitrary path connected space X, the Hurewicz homomorphism $h: \pi_1(X) \to H_1(X)$ is an epimorphism whose kernel is the commutator subgroup $[\pi_1(X), \pi_1(X)]$ of the group $\pi_1(X)$. Thus,*

$$H_1(X) \cong \pi_1(X)/[\pi_1(X), \pi_1(X)].$$

(Recall that the commutator subgroup $[G, G]$ of a group G is its subgroup generated by commutators $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$ for all $g_1, g_2 \in G$. The commutator subgroup is always normal. The group $G/[G, G]$ is obtained from g by *Abelianization*, that is, by imposing additional relations: Any two generators commute with each other.)

Proof of Theorem is a copy of the proof of the theorem in Sect. 14.2: We can assume that X is a CW complex with only one vertex, and for such spaces, it is sufficient to compare the procedures of computing the groups π_1 and H_1; see Sects. 7.6 and 13.5.

EXERCISE 9. Show that a loop $f: S^1 \to X$ determines an element of the kernel of the map $h: \pi_1(X) \to H_1(X)$ ("homologous to zero") if and only if it can be extended to the map into X of the disk (with the boundary S^1) with handles. Moreover, the minimal number of these handles is equal to the minimal number of commutators in $\pi_1(X)$ whose product is $[f]$.

EXERCISE 10. The space X_{Ab} is called an *Abelianization*, or *Quillenization*, of a path connected space X if the fundamental group of X_{Ab} is Abelian and there exists a continuous map $X \to X_{\mathrm{Ab}}$ inducing an isomorphism $H_n(X) \to H_n(X_{\mathrm{Ab}})$ for every n. Prove that X possesses an Abelianization if and only if

$$[\pi_1(X), \pi_1(X)] = [\pi_1(X), [\pi_1(X), \pi_1(X)]],$$

that is, if every element of $[\pi_1(X), \pi_1(X)]$ can be presented as a product of commutators of elements of $\pi_1(X)$ with elements of $[\pi_1(X), \pi_1(X)]$.

Remark. Our definition of an Abelianization is a simplified version of a more common definition in which the space X_{Ab} is assumed simple (see Sect. 8.2), or even an H-space (see Exercise 2 in Sect. 8.2) or even a loop space (see Lecture 4). This enhanced definition of an Abelianization plays an important technical role in one of the versions of constructing an algebraic K-functor. The problem of the existence of an Abelianization in this sense is much more complicated, and there are no general theorems about it. But there are several remarkable examples of the Abelianization, two of which we will mention. The first was discovered in 1971 by M. Barratt, D. Kahn, and S. Priddy: The Abelianization of the space $X = K(S_\infty, 1)$, where $S_\infty = \cup_n S_n$ is the group of finite permutations of the set $\mathbb{Z}_{>0}$, is $X_{\mathrm{Ab}} = (\Omega^\infty S^\infty)_0 = \cup_n (\Omega^n S^n)_0$ (the subscript 0 indicates that we consider only one component of the set). Another example belongs to G. Segal (1973) and states that if $X = K(B(\infty), 1)$ where $B(\infty)$ is the infinite braid group and hence X is the set of (unordered) countable subsets of the plane consisting, for some N (depending on the subset), of points $(n+1, 0), (n+2, 0), \dots$ and n more points different from each other and from the points listed above, then X_{Ab} is $\Omega^2 S^3$. In both cases, the space X has a complicated fundamental group and trivial higher homotopy groups, and the space X_{Ab} has a simple fundamental group ($\mathbb{Z}_2$ in the first case and $\mathbb{Z}$ in the second case) and complicated, so far unknown, homotopy groups. For further details, see Barratt and Priddy [20], Segal [74], and Fuchs [37].

EXERCISE 11. Prove that any two-dimensional homology class of an arbitrary space X can be represented by a sphere with handles; that is, for every $\alpha \in H_2(X)$, there exist a sphere with handles S and a continuous map $f: S \to X$ such that the map $f_*: H_2(S) \to H_2(X)$ takes the canonical generator of $H_2(S) = \mathbb{Z}$ into α.

14.4 The Relative Hurewicz Theorem

The relative Hurewicz homomorphism $h: \pi_n(X, A) \to H_n(X, A)$ is defined similarly to the absolute one. If $f: (D^n, S^{n-1}) \to (X, A)$ is a relative spheroid representing the class $\varphi \in \pi_n(X, A)$, then $h(\varphi)$ is the image of the canonical generator if the group $H_n(D^n, S^{n-1}) = \mathbb{Z}$ with respect to the homomorphism $f_*: H_n(D^n, S^{n-1}) \to H_n(X, A)$.

Theorem. *Let (X, A) be a topological pair such that the space X is path connected and A is simply connected. Let $n \geq 3$.*

(1) *Suppose that $\pi_2(X, A) = \cdots = \pi_{n-1}(X, A) = 0$. Then $H_1(X, A) = H_2(X, A) = \cdots = H_{n-1}(X, A) = 0$ and $h: \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.*
(2) *Suppose that $H_2(X, A) = \cdots = H_{n-1}(X, A) = 0$. Then $\pi_2(X, A) = \cdots = \pi_{n-1}(X, A) = 0$ and $h: \pi_n(X, A) \to H_n(X, A)$ is an isomorphism.*

Proof The proof can be obtained from the proof of the theorem in Sect. 14.2 by modifications characteristic for a transition from the absolute case to a relative case.

We begin by constructing a cellular approximation of the pair (X, A). For this purpose, we first find a cellular approximation (B, g) of A (see Sect. 11.6). Then we attach additional cells to B and successively expand the map $i \circ g: B \to X$ (where i is the inclusion map of A into X) to the new cells in such a way that B is expanded to a CW complex Y and $i \circ g$ is expanded to a weak homotopy equivalence $f: Y \to X$ (this is a replica of the construction in the proof of the theorem in Sect. 11.6). Since $f|_B = g$, the maps f and g compose a map $(Y, B) \to (X, A)$. We already know that f and g induce isomorphisms in both homotopy and homology groups, and the five-lemma implies that the map between the pairs induces isomorphisms for relative homotopy and homology groups. After this, we can assume that the pair (X, A) in the theorem is actually a CW pair.

According to Exercise 22 in Sect. 5.9, there exists a CW pair (X', A') homotopy equivalent to (X, A) and such that A' contains all cells of X' of dimension less than n. We can assume that the pair (X, A) itself has these properties. Then the relative version of the theorem in Sect. 11.3 (see Exercise 2, or, even better, Exercise 4 in Sect. 11.3) describes the group $\pi_n(X, A)$, and this description is not different from the description of $H_n(X, A)$.

EXERCISE 12. If A is not simply connected, then part (1) of the theorem remains true with the following modification: $H_n(X, A)$ is isomorphic to $\pi_n(X, A)$ *factorized over the natural action of* $\pi_1(A)$.

14.5 Whitehead's Theorem

(Not to be confused with a different theorem of the same Whitehead, in Sect. 11.5.)

Theorem. *Let X and Y be simply connected spaces, and let $f : X \to Y$ be a continuous map such that $f_* : \pi_2(X) \to \pi_2(Y)$ is an epimorphism.*

(1) *If the homomorphism $f_* : \pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m < n$ and an epimorphism for $m = n$, then the same is true for $f_* : H_m(X) \to H_m(Y)$.*
(2) *The same with π and H swapped.*

Proof. We may assume that f is an embedding, so (Y, X) is a topological pair. The exactness of homotopy and homology sequences of this pair yields a translation of conditions and claims of the theorem into the language of relative homotopy and homology groups. Namely, the condition "$f_* : \pi_2(X) \to \pi_2(Y)$ is an epimorphism" means precisely that $\pi_2(Y, X) = 0$; the condition "$f_* : \pi_m(X) \to \pi_m(Y)$ is an isomorphism for $m < n$ and an epimorphism for $m = n$" means that $\pi_m(Y, X) = 0$ for $m \leq n$; the same for homology groups. Thus, the theorem is equivalent to the relative Hurewicz theorem in Sect. 14.4.

Corollary. *If a continuous map $f : X \to Y$ between simply connected topological spaces induces an epimorphism $f_* : \pi_2(X) \to \pi_2(Y)$ and isomorphisms $f_* : H_m(X) \to H_m(Y)$ for all m, then f is a weak homotopy equivalence (a homotopy equivalence, if X and Y are CW complexes).*

Lecture 15 Homology with Coefficients and Cohomology

One can apply to the singular or cellular complex of a topological space the standard algebraic operations $- \otimes G$ and $\mathrm{Hom}(-, G)$. In this way, we obtain new complexes which also have homologies; these homologies are called *homology* and *cohomology* of the space *with coefficients (values) in G*. Certainly, the transition to these homology and cohomology may be regarded as a purely algebraic operation, but the experience shows that a too frankly algebraic presentation of this subject may scare a geometrically oriented reader off. To avoid hurting the feelings of such a reader, we will refer to tensor products, Homs, and other such things only when it is absolutely necessary. Still, we will have numerous such necessities.

15.1 Definitions

Let G be an Abelian group. A singular n-dimensional chain of a space X with coefficients in G is a formal linear combination of the form $\sum_i g_i f_i$ where $g_i \in G$ and $f_i: \Delta^n \to X$ are singular simplices. The group of n-dimensional singular chains of X with coefficients in G is denoted as $C_n(X;G)$; obviously, $C_n(X;G) = C_n(X) \otimes G$. Our previous group of chains, $C_n(X)$, is, in this notation, $C_n(X;\mathbb{Z})$. A singular n-dimensional *cochain* of X with coefficients (values) in G is defined as a function on the set of all n-dimensional singular simplices of X with values in G (no conditions like continuity are imposed). The group of n-dimensional cochains of X with coefficients in G is denoted as $C^n(X;G)$; obviously, $C^n(X;G) = \operatorname{Hom}(C_n(X), G)$. The value of a cochain c on a chain a is denoted as $\langle c, a\rangle$; thus, $\left\langle c, \sum_i g_i f_i\right\rangle = \sum_i c(f_i) g_i$. A generalization: if a bilinear multiplication (pairing) $G_1 \times G_2 \to G_3$ is given, then for $c \in C^n(X;G_1)$ and $a \in C_n(X;G_2)$ there arises the "value" $\langle c, a\rangle \in G_3$.

Boundary and coboundary operators

$$\begin{aligned} \partial = \partial_n &: C_n(X;G) \to C_{n-1}(X;G),\\ \delta = \delta^n &: C^n(X;G) \to C^{n+1}(X;G) \end{aligned}$$

are defined by the formulas

$$\partial \sum_i g_i f_i = \sum_i g_i \sum_{j=0}^{n} (-1)^j \Gamma_j f_i, \quad (\delta c)(f) = \sum_{j=0}^{n} (-1)^j c(\Gamma_j f).$$

Obviously, for every $c \in C^n(X;G)$ and $a \in C_{n+1}(X;G)$,

$$\langle c, \partial a\rangle = \langle \delta c, a\rangle.$$

A simple computation shows that $\partial\partial = 0$ and $\delta\delta = 0$ (the second follows from the first and the formula for $\langle -, -\rangle$ above), and we set

$$H_n(X;G) = \frac{\operatorname{Ker}[\partial_n: C_n(X;G) \to C_{n-1}(X;G)]}{\operatorname{Im}[\partial_{n+1}: C_{n+1}(X;G) \to C_n(X;G)]},$$

$$H^n(X;G) = \frac{\operatorname{Ker}[\delta^n: C^n(X;G) \to C^{n+1}(X;G)]}{\operatorname{Im}[\delta^{n-1}: C^{n-1}(X;G) \to C^n(X;G)]}.$$

The related terminology is *homology, cohomology, cycles, cocycles, boundaries. coboundaries, homological cycles, cohomological cocycles.*

Chain and cochain complexes may be augmented by maps

$$\begin{gathered} \epsilon: C_0(X;G) \to G, \ \epsilon^*: G \to C^0(X;G)\\ \epsilon \textstyle\sum_i g_i f_i = \sum_i g_i \text{ and } [\epsilon^*(g)](f) = g. \end{gathered}$$

The *reduced* homology and cohomology, $\widetilde{H}_n(X;G), \widetilde{H}^n(X;G)$, are the same as unreduced ones with obvious exceptions: $H_0(X;G) = \widetilde{H}_0(X;G) \oplus G$, $H^0(X;G) = \widetilde{H}^0(X;G) \oplus G$, if X is nonempty, and $\widetilde{H}_{-1}(X;G) = G = \widetilde{H}^{-1}(X;G)$ if X is empty.

15.2 Transfer of the Known Results

All major results of Lectures 12 and 13 and some results of Lecture 14 can be transferred to the new context without serious changes, either in statements or in proofs (for the proofs, we have an option to deduce new results from the old results using simple algebraic means; we will not do this, at least now).

A continuous map $h: X \to Y$ induces homology and cohomology homomorphisms, the latter of which acts in the "opposite direction":

$$h_*: H_n(X;G) \to H_n(Y;G), \ h^*: H^n(Y;G) \to H^n(X;G)$$

[the cochain map $h^{\#}: C^n(Y,G) \to C^n(X;G)$ is defined by the formula $[h^{\#}(c)](f) = c(h \circ f)$, where f is a singular simplex of X].

Homology with coefficients and cohomology are homotopy invariant: If $g \sim h$, then $g_* = h_*$ and $g^* = h^*$; in particular, homology with coefficients and cohomology of homotopy equivalent spaces are the same.

For a disjoint union $X = X_1 \sqcup \cdots \sqcup X_N$,

$$H_n(X;G) = \bigoplus\nolimits_i H_n(X_i;G), \ H^n(X;G) = \bigoplus\nolimits_i H^n(X;G).$$

For *infinite* disjoint unions, a difference appears between homology and cohomology: $H_n(X;G)$ is the *direct sum* of the groups $H_n(X_i;G)$, while $H^n(X;G)$ is the *direct product* of the groups $H^n(X_i;G)$.

For the one-point space pt,

$$\begin{aligned} H_0(\mathrm{pt};G) = G = H^0(\mathrm{pt};G), \\ H_n(\mathrm{pt};G) = 0 = H^n(\mathrm{pt};G) \text{ for } n \neq 0, \\ \widetilde{H}_n(\mathrm{pt};G) = 0 = \widetilde{H}^n(\mathrm{pt};G) \text{ for all } n. \end{aligned}$$

Relative homology with coefficients is defined precisely as usual (integral) relative homology, while in the definition of relative cohomology there arises a small (and expectable) new feature: The group $C^n(X,A;G)$ is a subgroup, not a quotient group, of $C^n(X;G)$; it consists of cochains from $C^n(X;G)$ which have zero restriction to $C_n(A) \subset C_n(X)$ (or, equivalently, assume zero value at every singular simplex in A).

The homology sequence of a pair (X,A) with coefficients in G looks the same as in the integral case (just insert "$;G$" where necessary). The cohomology sequence has all the arrows reversed:

$$\cdots \to H^{n-1}(A;G) \xrightarrow{\delta^*} H^n(X,A;G) \to H^n(X;G) \to H^n(A;G) \to \ldots.$$

The homomorphism $\delta^*: H^{n-1}(A;G) \to H^n(X;A)$ is defined in the following (expectable) way. For a class $\gamma \in H^(A;G)$, choose a representing cocycle $c \in C^{n-1}(A;G)$. Then expand the function c [on $(n-1)$-dimensional singular simplices of A] to all $(n-1)$-dimensional singular simplices of X (for example, set it equal to 0 on simplices not contained in A) and take the coboundary of the chain $c' \in C^{n-1}(X;G)$ arising. Then $\delta c'$ is zero on $C^n(A;G)$ (since c is a cocycle). Thus, $\delta c' \in C^n(X,A)$. It is a (relative) cocycle (since $\delta\delta = 0$), and its cohomology class $\beta \in H^n(X,A;G)$ does not depend on the arbitrary choices of the construction (c in γ and the extension c' of c; it is similar to Exercise 7 in Lecture 12). The function $\gamma \mapsto \beta$ is δ^*.

Both homology with coefficients and cohomology sequences of a pair are exact. There are also exact reduced homology with coefficients and cohomology sequences of pairs (no reducing for relative homology and cohomology groups) and exact homology with coefficients and cohomology sequences of triples.

For a Borsuk pair (X,A), there are isomorphisms

$$H_n(X,A;G) = \widetilde{H}_n(X/A;G), \; H^n(X,A;G) = \widetilde{H}^n(X/A;G)$$

established by the projection $X \to X/A$. For an arbitrary pair there are similar isomorphisms with X/A replaced by $X \cup CA$. Under the same assumptions as in Sect. 12.7, there are excision isomorphisms $H_n(X-B, A-B;G) = H_n(X,A;G)$ and $H^n(X-B,A-B;G) = H^n(X,A;G)$ and exact Mayer–Vietoris sequences; the cohomology Mayer–Vietoris sequences assume the form

$$\begin{aligned}\cdots \to H^{n-1}(A\cap B;G) &\to H^n(X;G)\\ &\to H^n(A;G)\oplus H^n(B;G) \to H^n(A\cap B;G) \to \ldots.\end{aligned}$$

For a CW complex, homology with coefficients and cohomology can be calculated through the cellular complex. Namely, for a CW complex X, $\mathcal{C}_n(X;G)$ is the group of linear combinations $\sum_i g_i e_i$, where e_i are oriented n-dimensional cells (an orientation change for a cell e_i results in a replacement of g_i by $-g_i$). Furthermore, $\mathcal{C}^n(X;G)$ is the group of G-valued functions on the set of oriented n-dimensional cells of X, where the orientation change for e_i leads to a sign change for the value at e_i. The boundary and coboundary operations act by the formulas

$$\partial\left(\sum_i g_i e_i\right) = \sum_i g_i \sum_f [e_i : f] f, \; [\delta c](e) = \sum_f [e:f]c(f),$$

where the inner summation on the right-hand side of the first formula is spread to all $(n-1)$-dimensional cells f of X and the summation in the second formula is spread to all n-dimensional cells of X.

Let us now show the results of calculating homology with coefficients and cohomology for the most important CW complexes. For spheres,

$$\widetilde{H}_m(S^n; G) = \widetilde{H}^m(S^n; G) = \begin{cases} G, & \text{if } m = n, \\ 0, & \text{if } m \neq n \end{cases}$$

(this fact certainly can be obtained with the cellular complexes, but the reader who wants to reconstruct all the proofs will have to do it at an earlier stage, as in Sect. 13.1). For complex, quaternion, and Cayley projective spaces, as well as for complex and quaternion Grassmann manifolds and flag manifolds, the homology with coefficients and cohomology are not different from the corresponding cellular chains and cochains. For example,

$$H_m(\mathbb{C}P^n; G) = H^m(\mathbb{C}P^n; G) = \begin{cases} G, \text{ if } m = 0, 2, 4, \dots [, 2n, \\ \qquad\qquad\qquad\qquad \text{if } n \text{ is finite}], \\ 0 \quad \text{for all other } m. \end{cases}$$

In the real case, the computation may be more complicated (compare Sect. 13.8), but it becomes much simpler if $G = \mathbb{Z}_2$, since in this case all the boundary and coboundary operators (in cellular complexes) are zero and homology with coefficients and cohomology again do not differ from the corresponding cellular chain and cochain groups. For example,

$$H_m(\mathbb{R}P^n; \mathbb{Z}_2) = H^m(\mathbb{R}P^n; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & \text{for } 0 \leq m \leq n, \\ 0 & \text{for all other } m. \end{cases}$$

Notice in addition that for a classical surface X (without holes),

$$H_0(X; \mathbb{Z}_2) = H^0(X; \mathbb{Z}_2) = H_2(X; \mathbb{Z}_2) = H^2(X; \mathbb{Z}_2) = \mathbb{Z}_2,$$
$$H_1(X; \mathbb{Z}_2) = H^1(X; \mathbb{Z}_2) = \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{r},$$

where

$$r = \begin{cases} 2g, & \text{if } X \text{is a sphere with } g \text{ handles}, \\ 2g + 1, & \text{if } X \text{ is a projective plane with } g \text{ handles}, \\ 2g + 2, & \text{if } X \text{ is a Klein bottle with } g \text{ handles}. \end{cases}$$

EXERCISE 1. Find the homology and cohomology of real projective spaces and real Grassmann manifolds with coefficients in $\mathbb{Z}_m$ where m is odd.

To finish the section, let us notice that if $f: X \to Y$ is a weak homotopy equivalence, then

$$f_*: H_n(X; G) \to H_n(Y; G) \text{ and } f^*: H^n(Y; G) \to H^n(X; G)$$

are isomorphisms for all G and n.

15.3 Coefficient Sequences

We begin studying relations between homologies and cohomologies with different coefficients. There is an obvious fact that any homomorphism $\varphi\colon G_1 \to G_2$ between Abelian groups induces, for every X and n, homomorphisms

$$\varphi_*\colon H_n(X;G_1) \to H_n(X;G_2) \text{ and } \varphi_*\colon H_n(X;G_1) \to H_n(X;G_2)$$

(in the same direction). However, as many examples (including some known to us) show, the homomorphism φ being a monomorphism, or an epimorphism, or just nontrivial, does not imply similar properties for any of the φ_*s. For a deeper understanding of the subject, let us consider the following situation. Let G be an Abelian group, H be a subgroup of G, and F be the quotient group G/H. Usually, all of this is presented as a short exact sequence,

$$0 \to H \to G \to F \to 0.$$

Besides the homomorphisms $H_n(X;H) \to H_n(X;G) \to H_n(X;F)$ and $H^n(X;H) \to H^n(X;G) \to H^n(X;F)$, there arise "connecting homomorphisms"

$$\delta_*\colon H_n(X;F) \to H_{n-1}(X;H) \text{ and } \delta^*\colon H^n(X;F) \to H^{n+1}(X;H).$$

Here is the construction of the first of them. For an $\alpha \in H_n(X;F)$, choose a representative $a \in C_n(X;F)$. Since $G \to F$ is an epimorphism, a possesses an inverse image $\widetilde{a} \in C_n(X;G)$. The projection $C_{n-1}(X;G) \to C_{n-1}(X;F)$ takes $\partial\widetilde{a}$ into $\partial a = 0$; hence, $\partial\widetilde{a}$ actually belongs to $C_{n-1}(X;H)$. This is a cycle, and its homology class in $H_{n-1}(X;H)$ is taken for $\partial_*(\alpha)$. The construction of the homomorphism δ^* is similar $[(\gamma \in H^n(X;F)) \mapsto (c \in C^n(X;F)) \mapsto (\widetilde{c} \in C^n(X;G)) \mapsto (\delta\widetilde{c} \in C^{n+1}(X;H)) \mapsto (\delta^*(\gamma) \in H^{n+1}(X;H))]$.

EXERCISE 2. Check that the preceding constructions provide well-defined homomorphisms ∂_* and δ^*.

EXERCISE 3. Prove that the *coefficient sequences*

$$\cdots \to H_n(X;H) \to H_n(X;G) \to H_n(X;F) \to H_{n-1}(X;H) \to \dots,$$
$$\cdots \to H^n(X;H) \to H^n(X;G) \to H^n(X;F) \to H^{n+1}(X;H) \to \dots$$

are exact.

HISTORICAL AND TERMINOLOGICAL REFERENCE. The homomorphisms ∂_* and δ^* were discovered, in a particular case, by M. Bockstein long before exact sequences became commonplace in algebraic topology. Here is how the Bockstein homomorphism was first described. Let $\alpha \in H_n(X;\mathbb{Z}_m)$. Take a representative a of α. All the coefficients involved in a are residues modulo m; we can regard them as integers $0, 1, \dots, m-1$. Then the cycle a becomes an integral chain $\widetilde{a}$. The boundary $\partial\widetilde{a}$ is divisible by m; let us divide. The result, $\dfrac{1}{m}\partial\widetilde{a}$, is an integral cycle. It represents some class $B_m(\alpha) \in H_{n-1}(X;\mathbb{Z})$ [by the way, $mB_m(\alpha) = 0$]; after reducing modulo m, it becomes a class $b_m(\alpha) \in H_{n-1}(X;\mathbb{Z}_m)$. We have constructed "Bockstein homomorphisms"

$$B_m\colon H_n(X;\mathbb{Z}_m) \to H_{n-1}(X;\mathbb{Z}) \text{ and } b_m\colon H_n(X;\mathbb{Z}_m) \to H_{n-1}(X;\mathbb{Z}_m).$$

In a very similar way, cohomological Bockstein homomorphisms

$$B^m\colon H^n(X;\mathbb{Z}_m) \to H^{n+1}(X;\mathbb{Z}) \text{ and } b^m\colon H^n(X;\mathbb{Z}_m) \to H^{n+1}(X;\mathbb{Z}_m)$$

are defined.

Actually, all of these Bockstein homomorphisms are connecting homomorphisms ∂_* and δ^* of coefficient sequences induced by the short exact sequences

$$0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z} \to \mathbb{Z}_m \to 0 \text{ and } 0 \to \mathbb{Z}_m \to \mathbb{Z}_{m^2} \to \mathbb{Z}_m \to 0.$$

From the exactness of the coefficient sequences, it follows then that (1) an element of $H_n(X;\mathbb{Z}_m)$ belongs to the kernel of B_m if and only if it is "integral," that is, belongs to the image of the reducing homomorphism $H_n(X;\mathbb{Z}) \to H_n(X;\mathbb{Z}_m)$; an element of $H_n(X;\mathbb{Z})$ belongs to the image of B_m if and only if it is annihilated by the multiplication by m; similarly for the cohomological Bockstein homomorphisms.

15.4 Algebraic Preparation to Universal Coefficients Formulas

Let A and B be Abelian groups. Then let $B = F_1/F_2$, where F_1 is a free Abelian group and F_2 is a subgroup of F_1 which must also be free (such a presentation exists for any Abelian group). What are the interrelations between $A \otimes F_1$, $A \otimes F_2$, and $A \otimes B$? To answer this question, we need a lemma which can be regarded as the most fundamental property of tensor products.

Lemma 1. *The tensor product operation is right exact. This means that if the sequence*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is exact, then the sequence

$$G \otimes A \xrightarrow{G\otimes\alpha} G \otimes B \xrightarrow{G\otimes\beta} G \otimes C \longrightarrow 0$$

is exact.

Proof. Recall that, by definition, the tensor product $K \otimes L$ is $F(K \times L)/R(K, L)$, where $F(K \times L)$ is the free Abelian group generated by the set $K \times L$ and $R(K, L)$ is the subgroup of $F(K \times L)$ generated by elements of the form $(k, \ell)+(k', \ell)-(k+k', \ell)$ and $(k, \ell) + (k, \ell') - (k, \ell + \ell')$. The image of (k, ℓ) in $K \otimes L$ is denoted as $k \otimes \ell$.

It is obvious that $G \otimes \beta$ is onto: $\sum_i (g_i \otimes c_i) = [G \otimes \beta] \left(\sum_i (g_i \otimes b_i)\right)$, where the b_i are chosen to satisfy the condition $\beta(b_i) = c_i$. It is also obvious that $(G \otimes \beta) \circ (G \otimes \alpha) = 0$. It remains to prove that $\operatorname{Ker}(G \otimes \beta) \subset \operatorname{Im}(G \otimes \alpha)$.

Let $[G \otimes \beta] \left(\sum_i (g_i \otimes b_i)\right) = 0$. This means that $\sum_i (g_i, \beta(b_i)) \in R(G, C)$; that is, $\sum_i (g_i, \beta(b_i))$ is a linear combination of elements of $F(G \times C)$ of the form $(g, c) + (g', c) - (g + g', c)$ and $(g, c) + (g, c') - (g, c + c')$. For all c, c' involved, find $b, b' \in B$ whose β-images are c, c', and the subtract from $\sum_i (g_i, \beta(b_i))$ the same linear combination with c, c' replaced by the corresponding b, b'. We get an element of $F(G \times B)$ which also represents $\sum_i (g_i \otimes b_i)$ but also belongs to the kernel of the map $F(G \times \beta)\colon F(G \times B) \to F(G \times C)$. This kernel is generated by differences $(g, b') - (g, b'')$ with $\beta(b' - b'') = 0$, that is, $b' - b'' \in\in \alpha$. Thus, $\sum_i (g_i \otimes b_i) = \sum_j (g')j \otimes (b'_j - b''_j))$ and hence $\sum_i (g_i \otimes b_i) = [G \otimes \alpha] \left(\sum_j (g'_j \otimes a_j)\right)$, where $\alpha(a_j) = b'_j - b''_j$.

Lemma 1 shows that the sequence

$$A \otimes F_2 \to A \otimes F_1 \to A \otimes B \to 0$$

is exact; that is, $A \otimes B$ is a quotient of $A \otimes F_1$ over the image of the natural map $A \otimes F_2 \to A \otimes F_1$, but this map is not necessarily a monomorphism.

Lemma 2. *The kernel* $\operatorname{Ker}(A \otimes F_2 \to A \otimes F_1)$ *does not depend on the choice of presentation* $B = F_2/F_1$.

Proof The proof consists in constructing a canonical isomorphism

$$\operatorname{Ker}(A \otimes F'_2 \to A \otimes F'_1) \cong \operatorname{Ker}(A \otimes F_2 \to A \otimes F_1)$$

for an arbitrary other presentation $B = F'_1/F'_2$. First, we construct homomorphisms $\alpha_1\colon F'_1 \to F_1$, $\alpha_2\colon F'_2 \to F_2$, making the diagram

$$\begin{tikzcd}
F'_2 \arrow[r, "i'"] \arrow[d, "\alpha_2"'] & F'_1 \arrow[dr, "p'"] \arrow[d, "\alpha_1"] \arrow[dl, "\beta"'] & \\
 & & B \\
F_2 \arrow[r, "i"'] & F_1 \arrow[ur, "p"'] &
\end{tikzcd}$$

(where the i, i' are inclusion maps and the p, p' are projections) commutative. Here α_1 takes a generator x of F'_1 into $y \in F_1$ such that $p(y) = p'(x)$ (which exists, since p is an epimorphism). This α_1 takes $\operatorname{Ker} p' = F'_2$ into $\operatorname{Ker} p = F_2$, thus giving rise to an $\alpha_2\colon F'_2 \to F_2$. Since y in the previous construction is determined (by x) up to an element of $\operatorname{Ker} p = F_2$, any other choice of α_1 has the form $\alpha_1 + i' \circ \beta$, where

β is a homomorphism $F_1' \to F_2$, and then the new α_2 is $\alpha_2 + \beta \circ i$. Take the tensor product of (the square part of) this diagram with A:

$$\begin{tikzcd} A \otimes F_2' \arrow[r, "A \otimes i'"] \arrow[d, "A \otimes \alpha_2"'] & A \otimes F_1' \arrow[d, "A \otimes \alpha_1"] \arrow[dl, "A \otimes \beta"'] \\ A \otimes F_2 \arrow[r, "A \otimes i"'] & A \otimes F_1 \end{tikzcd}$$

The map $A \otimes \alpha_2$ takes $\mathrm{Ker}(A \otimes i'$ into $\mathrm{Ker}(A \otimes i)$. This map does not depend on the choice of α_1 and α_2, since $A \otimes (\beta \circ i') = (A \otimes \beta) \circ (A \otimes i')$ is zero on $\mathrm{Ker}(A \otimes i')$. The map $\mathrm{Ker}(A \otimes i') \to \mathrm{Ker}(A \otimes i)$ is constructed in the same way, and the composition of these maps in any order is the identity, because of the same uniqueness (this time, applied to $F_1' = F_1$, $F_2' = F_2$).

Definition. The kernel $\mathrm{Ker}(A \otimes F_2 \to A \otimes F_1)$ is called the *periodic product* of A and B and is denoted as $\mathrm{Tor}(A, B)$.

EXERCISE 4. Show that the operation Tor is natural with respect to both arguments; that is, homomorphisms $A \to A', B \to B'$ induce a homomorphism $\mathrm{Tor}(A, B) \to \mathrm{Tor}(A', B')$ with all expectable properties (for A it is obvious, while for B this requires a construction like the one in the beginning of the proof of the lemma).

EXERCISE 5. Prove a natural isomorphism $\mathrm{Tor}(A, B) \to \mathrm{Tor}(B, A)$. (This might be harder than one can expect. The most common idea of proving that is the following. Consider two presentations $A = F_1/F_2$, $B = G_1/G_2$ with free Abelian F_1, F_2, G_1, G_2, form the complex

$$0 \to F_2 \otimes G_2 \to [(F_1 \otimes G_2) \oplus (F_2 \otimes G_1)] \to F_1 \otimes G_1 \to 0,$$

and prove that the homology groups H_2, H_1, and H_0 of this complex are 0, $\mathrm{Tor}(A, B)$, and $\mathrm{Hom}(A, B)$. This provides a definition of Tor symmetric in A, B.)

EXERCISE 6. Prove that if A (or B) is a free Abelian group, then $\mathrm{Tor}(A, B) = 0$.

EXERCISE 7. Prove that $\mathrm{Tor}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_m \otimes \mathbb{Z}_n$ [$= \mathbb{Z}_{\gcd(m,n)}$] [this isomorphism is *not* canonical; it depends on the choice of generators in $\mathbb{Z}_m$ and $\mathbb{Z}_n$]. Thus, for finitely generated Abelian groups A, B,

$$\mathrm{Tor}(A, B) \cong \mathrm{Tors}\, A \otimes \mathrm{Tors}\, B$$

($\mathrm{Tors}\, A$ = torsion of A, the group of elements of finite order).

EXERCISE 8. For infinitely generated A, B, the last isomorphism, in general, does not hold: Construct an example.

EXERCISE 9. Prove that is $A = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$, then $\mathrm{Tor}(A, B) = 0$ for any B.

The "dual" operation Ext is defined in a similar way. First, we dualize Lemma 1:

Lemma 3. *If the sequence*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is exact, then the sequence

$$\mathrm{Hom}(A,G) \xleftarrow{\mathrm{Hom}(\alpha,G)} \mathrm{Hom}(B,G) \xleftarrow{\mathrm{Hom}(\beta,G)} \mathrm{Hom}(C,G) \longleftarrow 0$$

is exact.

Proof The proof is left to the reader; it is easier than the proof of the Lemma 1.

EXERCISE 10. Prove that the operation $\mathrm{Hom}(G,-)$ is *left exact*. This means that if the sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is exact, then the sequence

$$0 \longrightarrow \mathrm{Hom}(G,A) \xrightarrow{\mathrm{Hom}(G,\alpha)} \mathrm{Hom}(G,B) \xrightarrow{\mathrm{Hom}(G,\beta)} \mathrm{Hom}(G,C)$$

is exact.

Let A, B be Abelian groups, and let $A = F_1/F_2$, where F_1 and F_2 are free Abelian groups. Lemma 3 says that the kernel of the map $\mathrm{Hom}(F_1,B) \to \mathrm{Hom}(F_2,B)$, $f \mapsto f\,|_{F_2}$ is $\mathrm{Hom}(A,B)$, but this map is not onto. The cokernel of this map, which is the quotient of $\mathrm{Hom}(F_2,B)$ over the image of this map, is taken for $\mathrm{Ext}(A,B)$.

EXERCISE 11. Prove that Ext is well defined (this is a dualization of Lemma 2).

EXERCISE 12. Show that the operation Ext is natural with respect to both arguments; that is, homomorphisms $A \to A', B \to B'$ induce a homomorphism $\mathrm{Ext}(A',B) \to \mathrm{Ext}(A,B')$ with all expectable properties. (Notice the reversion of the arrow $A \to A'$.)

EXERCISE 13. Prove that $\mathrm{Ext}(\mathbb{Z},B) = 0$ for any B; prove also that $\mathrm{Ext}(\mathbb{Z}_m,\mathbb{Z}_n) \cong \mathrm{Hom}(\mathbb{Z}_m,\mathbb{Z}_n) \cong \mathbb{Z}_m \otimes \mathbb{Z}_n \cong \mathbb{Z}_{(m,n)}$ (not canonically!), and $\mathrm{Ext}(\mathbb{Z}_m,\mathbb{Z}) \cong \mathbb{Z}_m$ (unlike $\mathrm{Tor}(\mathbb{Z}_m,\mathbb{Z}) = 0$).

EXERCISE 14. The set $\mathrm{Ext}(A,B)$ has another definition (due to Yoneda) as the set of equivalence classes of "extensions" of A by B, that is, short exact sequences

$$0 \to B \to C \to A \to 0$$

where C is an Abelian group. Prove the equivalence of the two definitions of Ext and make up a direct definition of a group structure in the set $\mathrm{Ext}(A,B)$ described as the set of extensions.

EXERCISE 15. Prove that if one of the groups A, B is $\mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$, then $\operatorname{Ext}(A, B) = 0$.

15.5 The Universal Coefficients Formula

Now we will show that the usual (integral) homology of X (actually, of any complex consisting of free Abelian groups) determine homology and cohomology of X with arbitrary coefficients.

Theorem. *For any X, n, and G,*

$$H_n(X;G) \cong (H_n(X) \otimes G) \oplus \operatorname{Tor}(H_{n-1}(X), G)$$

$$H^n(X;G) \cong (H^n(X) \otimes G) \oplus \operatorname{Tor}(H^{n+1}(X;\mathbb{Z}), G)$$

$$H^n(X;G) \cong \operatorname{Hom}(H_n(X), G) \oplus \operatorname{Ext}(H_{n-1}(X), G).$$

IMPORTANT ADDITION. *The isomorphisms of the theorem are not canonical. What is canonical are the following three exact sequences*:

$$0 \to H_n(X) \otimes G \to H_n(X;G) \to \operatorname{Tor}(H_{n-1}(X), G) \to 0,$$

$$0 \to H^n(X;\mathbb{Z}) \otimes G \to H^n(X;G) \to \operatorname{Tor}(H^{n+1}(X;\mathbb{Z}), G) \to 0,$$

$$0 \leftarrow \operatorname{Hom}(H_n(X), G) \leftarrow H^n(X;G) \leftarrow \operatorname{Ext}(H_{n-1}(X), G) \leftarrow 0.$$

Proof. The first two exact sequences are easily obtained from coefficient sequences. The first sequence is obtained in the following way. Let $G = F_1/F_2$, where F_1 and F_2 are free Abelian groups. Then $F_1 = \mathbb{Z} \oplus \mathbb{Z} \oplus \dots,$, and hence

$$H_n(X;F_1) = H_n(X;\mathbb{Z} \oplus \mathbb{Z} \oplus \dots) = H_n(X) \oplus H_n(X) \oplus \dots = H_n(X) \otimes F_1,$$

and, similarly, $H_n(X;F_2) = H_n(X) \otimes F_2$. Hence, the fragment

$$H_n(X;F_2) \to H_n(X;F_1) \to H_n(X;G) \to H_{n-1}(X;F_2) \to H_{n-1}(X;F_2)$$

of the coefficient sequence takes the form

$$\begin{aligned} H_n(X) \otimes F_2 \to H_n(X) \otimes F_1 &\to H_n(X;G) \\ &\to H_{n-1}(X) \otimes F_2 \to H_{n-1}(X) \otimes F_2. \end{aligned}$$

A five-term exact sequence $A \xrightarrow{\varphi} B \to C \to D \xrightarrow{\psi} E$ can be transformed into a short exact sequence $0 \to \operatorname{Coker} \varphi \to C \to \operatorname{Ker} \psi \to 0$ (where Coker is the quotient over the image, $\operatorname{Coker} \varphi = B/\operatorname{Im} \varphi$). This transformation converts the last sequence into the first of the three exact sequences in the theorem. The second sequence is obtained in the way from the cohomological coefficient sequence (and the isomorphisms $H^n(X; F_i) = H^n(X; \mathbb{Z}) \otimes F_i$). The last sequence can hardly be obtained in a similar way, because it contains both homology and cohomology. But there exists a different approach which yields isomorphisms from the theorem rather than the exact sequences.

Since for every n, $B_n(X) = \mathrm{Im}[\partial_{n+1}: C_{n+1}(X) \to C_n(X)]$ is a free Abelian group, there exists a (nonunique) homomorphism $s_n: B_n(X) \to C_{n+1}(X)$ such that $\partial_{n+1} \circ s_n = \mathrm{id}$. Thus,

$$C_{n+1}(X) = \mathrm{Ker}\, \partial_{n+1} \oplus \mathrm{Im}\, s_n = Z_{n+1}(X) \oplus B_n(X).$$

The boundary operator looks like this:

$$\begin{array}{ccc} C_{n+1}(X) = Z_{n+1}(X) & \oplus & B_n(X) \\ \Big\downarrow \partial_{n+1} & & \Big\downarrow \text{inclusion} \\ C_n(X) \quad = & & Z_n(X) \oplus B_{n-1}(X)\,. \end{array}$$

This shows that the whole singular complex $\mathcal{C} = \{C_n(X), \partial_n\}$ is isomorphic (not canonically) to the direct sum of very short complexes $\mathcal{C}(n)$,

$$\begin{array}{ccccc} \ldots 0 \to 0 \to & B_n(X) & \xrightarrow{\text{incl.}} & Z_n(X) & \to 0 \to 0 \ldots \\ & (n+1) & & (n) & \end{array}$$

[for this complex, the n-dimensional homology is $H_n(X)$; all the other homology groups are zero]. Since the tensor product has the distributivity property, the complex $\mathcal{C} \otimes G = \{C_n(X; G) = C_n(X) \otimes G, \partial_n \otimes G\}$ is the sum of complexes $\mathcal{C}(n) \otimes G$,

$$\ldots 0 \to 0 \to B_{n-1} \otimes G \to Z_{n-1} \otimes G \to 0 \to 0 \ldots.$$

Since $B_n(X)$ and $Z_n(X)$ are free Abelian groups and $Z_n(X)/B_n(X) = H_n(X)$, the homology groups of the complex $\mathcal{C} \otimes G$ are

$$\begin{gathered} \text{dimension } n+1\text{: } \mathrm{Tor}(H_n(X), G); \\ \text{dimension } n\text{: } H_n(X) \otimes G. \end{gathered}$$

The summation over n gives the first formula of the theorem: $H_n(X; G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}(H_{n-1}(X), G)$. The second formula is obtained in the same way; we leave this job to the reader.

To prove the last part of the theorem, consider again the decomposition of the singular complex $\mathcal{C}$ of X into the sum of "very short complexes" $\mathcal{C}(n)$:

$$\begin{tikzcd}
\cdots\ C_{n+2}(X) \arrow[r, "\partial"] & C_{n+1}(X) \arrow[r, "\partial"] \arrow[d] & C_n(X) \arrow[r, "\partial"] & C_{n-1}(X)\ \cdots & (\mathcal{C}) \\
\cdots\ 0 \arrow[r] & B_n(X) \arrow[r, "\text{incl.}"] & Z_n(X) \arrow[r] \arrow[u] & 0\ \cdots & (\mathcal{C}(n))
\end{tikzcd}$$

We see that although the decomposition $\mathcal{C} = \bigoplus \mathcal{C}(n)$ is not canonical, and hence there is neither a canonical projection $\mathcal{C} \to \mathcal{C}(n)$ or a canonical embedding $\mathcal{C}(n) \to \mathcal{C}$, there are still the canonical projection $C_{n+1}(X) \to B_n(X)$ and the canonical embedding $Z_n(X) \to C_n(X)$, as shown in the diagram. Now apply to this diagram the operation $\mathrm{Hom}(-, G)$. We obtain the diagram

$$\begin{tikzcd}
\mathrm{Hom}(C_{n+2}(X), G) \arrow[r] & 0 \\
\mathrm{Hom}(C_{n+1}(X), G) \arrow[r] \arrow[u] & \mathrm{Hom}(B_n(X), G) \arrow[u] \\
\mathrm{Hom}(C_n(X), G) \arrow[r] \arrow[u] & \mathrm{Hom}(Z_n(X), G) \arrow[u] \\
\mathrm{Hom}(C_{n-1}(X), G) \arrow[r] \arrow[u] & 0 \arrow[u]
\end{tikzcd}$$

For the (co)homology $H^m(\mathcal{C}; G)$ of the complex $\mathrm{Hom}(\mathcal{C}, G)$, we have

$$\begin{aligned} H^n(\mathcal{C}(n); G) &= \mathrm{Ker}[\mathrm{Hom}(Z_n(X), G) \to \mathrm{Hom}(B_n(X), G)] \\ &= \mathrm{Hom}(H_n(X), G), \end{aligned}$$

$$\begin{aligned} H^{n+1}(\mathcal{C}(n); G) &= \mathrm{Coker}[\mathrm{Hom}(Z_n(X), G) \to \mathrm{Hom}(B_n(X), G)] \\ &= \mathrm{Ext}(H_n(X), G) \end{aligned}$$

and $H^m(\mathcal{C}(n)) = 0$ for $m \neq n, n+1$. From this,

$$H^n(X; G) \cong \bigoplus_k H^n(\mathcal{C}; G) = \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G),$$

as stated. Moreover, as we have seen, there are canonical homomorphisms

$$H^n(X; G) \to \mathrm{Hom}(H_n(X), G),\ \mathrm{Ext}(H_{n-1}(X), G) \to H^n(X; G),$$

which form the exact sequence

$$0 \leftarrow \mathrm{Hom}(H_n(X), G) \leftarrow H^n(X; G) \leftarrow \mathrm{Ext}(H_{n-1}(X), G) \leftarrow 0.$$

This completes the proof of the theorem.

We can add that the map

$$C^n(X;G) = \mathrm{Hom}(C_n(X), G) \to \mathrm{Hom}(Z_n(X), G)$$

considered above is simply the restriction to $Z_n(X)$; moreover, if $c \in C^n(X;G)$ is a cocycle, then the restriction of c to $B_n(X)$ is zero, which provides an element of $\mathrm{Hom}(H_n(X), G)$ depending only on the cohomology class of c; this is how our homomorphism $H^n(X;G) \to \mathrm{Hom}(H_n(X), G)$ acts. In other words, this homomorphism sends a cohomology class $\gamma \in H^n(X;G)$ to a homomorphism $\alpha \mapsto \langle\gamma, \alpha\rangle$ of $H_n(X)$ into G. The fact that this homomorphism is onto yields the following important proposition.

Corollary 1. *For every homomorphism $f: H_n(X) \to G$, there exists a cohomology class $\gamma \in H^n(X;G)$ such that $f(\alpha) = \langle\gamma, \alpha\rangle$ for every $\alpha \in H_n(G)$.*

Remark also that this γ is defined up to an element of $\mathrm{Ext}(H_n(X), G)$; in particular, if $H_n(X)$ and G are finitely generated, then this Ext group is finite, so γ is defined by f up to adding an element of finite order.

Before the final exercises of this section, we will mention one more interesting corollary.

Corollary 2. *If the groups $H_n(X)$ are finitely generated, then*

$$H^n(X;\mathbb{Z}) \cong \text{Free part of } H_n(X) \oplus \text{Torsion part of } H_{n-1}(X).$$

In particular, $H^1(X;\mathbb{Z})$ is a free Abelian group.

EXERCISE 16. If $\mathbb{K} = \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$, then

$$H_n(X;\mathbb{K}) = H_n(X) \otimes \mathbb{K} \text{ and } H^n(X;\mathbb{K}) = \mathrm{Hom}(H_n(X), \mathbb{K}).$$

Thus, the transition from the integral coefficients to the rational, real, or complex coefficients kills the torsion. On the other hand, the Betti numbers of X become the dimension of homology or cohomology with coefficients in $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. (Actually, the same is true for any field of characteristic zero.)

EXERCISE 17. If $\mathbb{K}$ is a field, then homology and cohomology with coefficients in $\mathbb{K}$ possess a natural structure of vector spaces over $\mathbb{K}$. Prove that

$$H^n(X;\mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(H_n(X;\mathbb{K}), \mathbb{K}).$$

[It is better not to deduce this formula from the universal coefficients formula, but rather to prove it directly using the equality $C^n(X;\mathbb{K}) = \mathrm{Hom}_{\mathbb{K}}(C_n(X;\mathbb{K}), \mathbb{K})$.]

EXERCISE 18. Prove that if X is a finite CW complex and $\mathbb{K}$ is a field, then

$$\sum (-1)^m \dim_{\mathbb{K}} H_m(X;\mathbb{K})$$

does not depend on $\mathbb{K}$ and is equal to the Euler characteristic of X (see Sect. 13.7).

15.6 Künneth's Formula

By its contents, Künneth's formula is closer to the next lecture than to the current one. But by sight, this formula has so strong resemblance to the universal coefficients formulas (actually, these formulas can be deduced from the same general algebraic result; thus, they have a common ancestor) that it would be unfair to try to separate them.

Theorem 1. *Let X_1, X_2 be topological spaces. Then for any n,*

(1) *There is a (noncanonical) isomorphism*

$$H_n(X_1 \times X_2) \cong \bigoplus_{i+j=n} (H_i(X_1) \otimes H_j(X_2)) \quad \bigoplus \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(X_1), H_j(X_2)).$$

(2) *There is a canonically defined exact sequence*

$$0 \to \bigoplus\nolimits_{i+j=n} (H_i(X_1) \otimes H_j(X_2)) \to H_n(X_1 \times X_2) \to \bigoplus\nolimits_{i+j=n-1} \mathrm{Tor}(H_i(X_1), H_j(X_2)) \to 0.$$

We will deduce Theorem 1 from an algebraic result related to the tensor product of complexes.

Definition. Let

$$\begin{array}{ll} (\mathcal{C}) & \dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots, \\ (\mathcal{C}') & \dots \xrightarrow{\partial'_{n+1}} C'_n \xrightarrow{\partial'_n} C'_{n-1} \xrightarrow{\partial'_{n-1}} \dots \end{array}$$

be two *positive* complexes. Let

$$T_n = \bigoplus\nolimits_{i+j=n} (C_i \otimes C'_j)$$

and let $\tau_n : T_n \to T_{n-1}$ take $c \otimes c' \in C_i \otimes C_j \subset T_n$ into

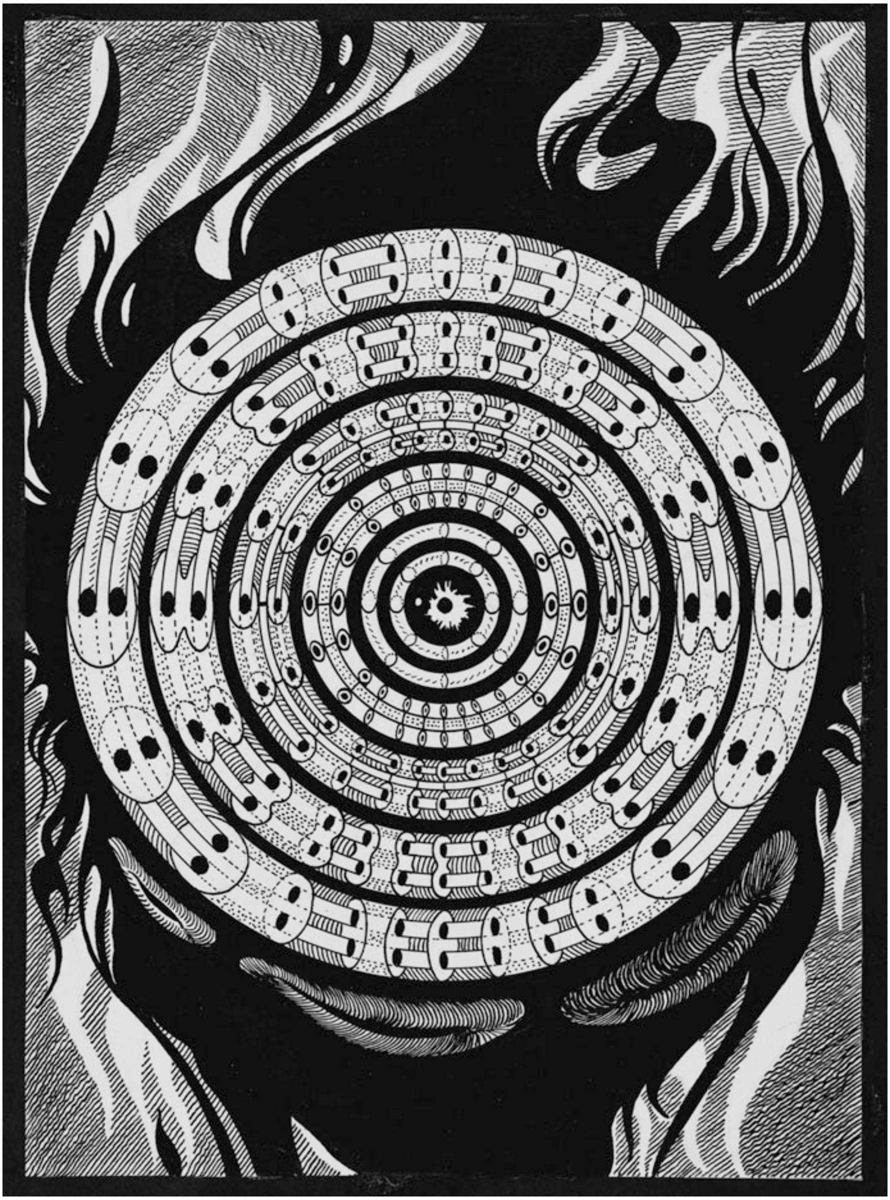
$$\tau_n(c \otimes c') = (\partial_i c \otimes c') + (-1)^i (c \otimes \partial'_j c') \in (C_{i-1} \otimes C_j) \oplus (C_i \otimes C_{j-1}) \subset T_{n-1}.$$

A direct verification (see below) shows that $\tau_{n-1} \circ \tau_n = 0$. The complex arising,

$$\dots \xrightarrow{\tau_{n+1}} T_n \xrightarrow{\tau_n} T_{n-1} \xrightarrow{\tau_{n-1}} \dots,$$

is called the *tensor product* of the complexes $\mathcal{C}$ and $\mathcal{C}'$ and is denoted as $\mathcal{C} \otimes \mathcal{C}'$.

VERIFICATION OF $\tau_{n-1} \circ \tau_n = 0$. Let $c \in C_i$, $c' \in C'_j$. Then



$$\begin{aligned}\tau_{n-1} \circ \tau_n(c \otimes c') &= \tau_{n-1}(\partial_i c \otimes c') + (-1)^i \tau_{n-1}(c \otimes \partial'_j c') \\ &= (\partial_{i-1} \circ \partial_i(c) \otimes c') + (-1)^{i-1}(\partial_i c \otimes \partial'_j c') \\ &\quad +(-1)^i((\partial_i c \otimes \partial'_j c') + (-1)^{j-1}(c \otimes \partial'_{j-1} \circ \partial'_j(c'))) \\ &= (-1)^{i-1}(\partial_i c \otimes \partial'_j c') + (-1)^i(\partial_i c \otimes \partial'_j c') = 0.\end{aligned}$$

Our next goal is to express the homology of the tensor product of two complexes in terms of homologies of these complexes.

Theorem 2. *If the complexes $\mathcal{C}$, and $\mathcal{C}'$ are free (that is, all C_n, C'_n are free Abelian groups), then, for every n,*

(1) *There is a (noncanonical) isomorphism*

$$\begin{aligned}&H_n(\mathcal{C} \otimes \mathcal{C}') \cong \\ &\bigoplus_{i+j=n} (H_i(\mathcal{C}) \otimes H_j(\mathcal{C}') \quad \oplus \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(\mathcal{C}), H_j(\mathcal{C}')).\end{aligned}$$

(2) *There is a canonically defined exact sequence*

$$\begin{aligned}0 \to \textstyle\bigoplus_{i+j=n}(H_i(\mathcal{C}) \otimes H_j(\mathcal{C}')) \to H_n(\mathcal{C} \otimes \mathcal{C}') \\ \to \textstyle\bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(\mathcal{C}), H_j(\mathcal{C}')) \to 0.\end{aligned}$$

Proof. Begin with part (2). Let $Z_n = \operatorname{Ker} \partial_n, B_{n-1} = \operatorname{Im} \partial_n$. Consider the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \xrightarrow{0} & Z_{n+1} & \xrightarrow{0} & Z_n & \xrightarrow{0} & Z_{n-1} & \xrightarrow{0} & \dots \\
 & & \Big\downarrow{\scriptstyle\subset} & & \Big\downarrow{\scriptstyle\subset} & & \Big\downarrow{\scriptstyle\subset} & & \\
\dots & \xrightarrow{\partial} & C_{n+1} & \xrightarrow{\partial} & C_n & \xrightarrow{\partial} & C_{n-1} & \xrightarrow{\partial} & \dots \\
 & & \Big\downarrow{\scriptstyle\partial} & & \Big\downarrow{\scriptstyle\partial} & & \Big\downarrow{\scriptstyle\partial} & & \\
\dots & \xrightarrow{0} & B_n & \xrightarrow{0} & B_{n-1} & \xrightarrow{0} & B_{n-2} & \xrightarrow{0} & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & . &
\end{array}$$

The rows of this diagram are complexes, the columns are exact sequences, and the diagram is commutative. Thus, this diagram can be regarded as a short exact sequence of complexes:

$$0 \to \mathcal{Z} \to \mathcal{C} \to \mathcal{B} \to 0,$$

where $\mathcal{Z}$ and $\mathcal{B}$ are complexes with trivial differential composed of groups Z_n and B_n [but the nth group of the complex $\mathcal{B}$ is B_{n-1}]. Since the complex $\mathcal{C}'$ is free, the sequence remains exact after tensoring with $\mathcal{C}'$:

$$0 \to \mathcal{Z} \otimes \mathcal{C}' \to \mathcal{C} \otimes \mathcal{C}' \to \mathcal{B} \otimes \mathcal{C}' \to 0.$$

Since $\mathcal{Z}$ and $\mathcal{B}$ have trivial differentials and consist of free Abelian groups,

$$H_n(\mathcal{Z} \otimes \mathcal{C}') = \bigoplus_{i+j=n} (Z_i \otimes H_j(\mathcal{C}')),\ H_n(\mathcal{B} \otimes \mathcal{C}') = \bigoplus_{i+j=n-1} (B_i \otimes H_j(\mathcal{C}')).$$

Thus, the homology sequence corresponding to the last short exact sequence of complexes takes the form

$$\begin{aligned} &\bigoplus_{i+j=n} (B_i \otimes H_j(\mathcal{C}')) \xrightarrow{\varphi} \bigoplus_{i+j=n} (Z_i \otimes H_j(\mathcal{C}')) \to H_n(\mathcal{C} \otimes \mathcal{C}') \\ &\qquad \to \bigoplus_{i+j=n-1} (B_i \otimes H_j(\mathcal{C}')) \xrightarrow{\psi} \bigoplus_{i+j=n-1} (Z_i \otimes H_j(\mathcal{C}')). \end{aligned}$$

It is easy to see also that the connecting homomorphisms φ and ψ are induced by the inclusion maps $B_i \to Z_i$ [before tensoring with $\mathcal{C}'$, they consist first in applying ∂^{-1} and then ∂; tensoring with $\mathcal{C}'$ does not change anything]. Since the Abelian groups B_i and Z_i are free and $H_i(\mathcal{C}) = Z_i/B_i$, the exact sequence $0 \to \operatorname{Coker}\varphi \to H_n(\mathcal{C} \otimes \mathcal{C}') \to \operatorname{Ker}\psi \to 0$ is precisely the exact sequence from part (2) of Theorem 2.

To prove part (1), first notice that if $H_n(\mathcal{C}) = 0$ for $n \neq i$ and $H_n(\mathcal{C}') = 0$ for $n \neq j$, then part (2) shows that the homology of $\mathcal{C} \otimes \mathcal{C}'$ is zero, except

$$\begin{aligned} H_{i+j}(\mathcal{C} \otimes \mathcal{C}') &= H_i(\mathcal{C}) \otimes H_j(\mathcal{C}'), \\ H_{i+j-1}(\mathcal{C} \otimes \mathcal{C}') &= \operatorname{Tor}(H_i(\mathcal{C}), H_j(\mathcal{C}')), \end{aligned}$$

so the isomorphism of part (1) holds. In general,

$$\mathcal{C} \cong \bigoplus \mathcal{C}(i), \text{ where } \mathcal{C}(i) \text{ is } \ldots 0 \to 0 \to \underset{(i+1)}{B_i} \xrightarrow{\text{incl.}} \underset{(i)}{Z_i} \to 0 \to 0 \ldots,$$

$$\mathcal{C}' \cong \bigoplus \mathcal{C}'(j), \text{ where } \mathcal{C}'(j) \text{ is } \ldots 0 \to 0 \to \underset{(j+1)}{B'_j} \xrightarrow{\text{incl.}} \underset{(j)}{Z'_j} \to 0 \to 0 \ldots$$

(noncanonical isomorphisms; compare with Sect. 15.5), and all the homology groups of $\mathcal{C}(i)$ and $\mathcal{C}'(j)$ are zero besides $H_i(\mathcal{C}(i)) = H_i(\mathcal{C})$ and $H_j(\mathcal{C}'(j)) = H_j(\mathcal{C}')$. This implies part (1) in full generality.

Proof of Theorem 1. In the case when X_1 and X_2 are CW complexes, it is sufficient to remark that the cellular chain complex of $X_1 \times X_2$ is the tensor product of the cellular complexes of X_1 and X_2 ($e \times e' \leftrightarrow e \otimes e'$). To extend the result to arbitrary topological spaces, we use two previous results: (1) Every topological space is weakly homotopy equivalent to a CW complex (Sect. 11.6); and (2) homology is weakly homotopy invariant (Sect. 14.1).

Remarks. (1) It is not true, in general, that the singular complex of the product $X_1 \times X_2$ of two topological spaces is isomorphic the tensor product of the singular complexes of X_1 and X_2. But these complexes are homotopy equivalent (there exists a homotopy equivalence canonically defined up to a homotopy between them). This fact, known as the Eilenberg–Zilber theorem, is proved in many textbooks in topology.

(2) A comparison of the universal coefficients formula with Künneth's formula gives the following result (which may be useful in Chap. 3):

$$H_n(X_1 \times X_2) = \bigoplus_{i+j=n} H_i(X_1; H_j(X_2)).$$

EXERCISE 19. The last equality can be modified to the case of homology and cohomology with coefficients:

$$\begin{aligned} H_n(X_1 \times X_2; G) &= \textstyle\bigoplus_{i+j=n} H_i(X_1; H_j(X_2; G)) \\ H^n(X_1 \times X_2; G) &= \textstyle\bigoplus_{i+j=n} H^i(X_1; H^j(X_2; G)). \end{aligned}$$

(These equalities, as well as the equality in the preceding remark, can be proven without any references to the universal coefficients and Künneth's formulas: They hold, actually, at the level of cellular chains. This provides a direct way to deduce the noncanonical part of Künneth's formula from the similar part of the universal coefficients formulas.)

Here is a small but significant application of Künneth's formula.

EXERCISE 20. Find the homology of $\mathbb{R}P^2 \times \mathbb{R}P^2$. (If the result seems unexpected to you, check it using a direct cellular computation.)

Like the universal coefficients formula, Künneth's formula simplifies a lot in the case of coefficients in a field.

EXERCISE 21. Prove that if $\mathbb{K}$ is a field, then

$$\begin{aligned} H_n(X_1 \times X_2; \mathbb{K}) &= \textstyle\bigoplus_{i+j=n} H_i(X_1; \mathbb{K}) \otimes_{\mathbb{K}} H_j(X_2; \mathbb{K}), \\ H^n(X_1 \times X_2; \mathbb{K}) &= \textstyle\bigoplus_{i+j=n} H^i(X_1; \mathbb{K}) \otimes_{\mathbb{K}} H^j(X_2; \mathbb{K}). \end{aligned}$$

In conclusion, here are two more formulas.

EXERCISE 22. $B_n(X_1 \times B_2) = \sum_{i+j=n} B_i(X_1)B_j(X_2)$.

EXERCISE 23. $\chi(X_1 \times X_2) = \chi(X_1)\chi(X_2)$. (In both exercises, we assume that the right-hand sides of the formulas are defined.)

Lecture 16 Multiplications

16.1 Introduction

Although homology is geometrically much more transparent than cohomology, cohomology is immensely more useful because it possesses many naturally defined additional structures. The first of these structures is a multiplication: If G is a ring, then for $\alpha \in H^{n_1}(X;G)$ and $\beta \in H^{n_2}(X;G)$ there exists a naturally defined "product" $\alpha\beta \in H^{n_1+n_2}(X;G)$ which has good algebraic properties. Nothing like this is possible for homology (see Exercise 14 ahead). We will discuss these products (and some other products) in this lecture and will describe many other structures in later chapters (starting with Chap. 4).

The simplest way to introduce the cohomological multiplication is as follows. Let G be a commutative ring, and let X_1, X_2 be two CW complexes. For cellular cochains $c_1 \in \mathcal{C}^{n_1}(X_1;G), c_2 \in \mathcal{C}^{n_2}(X_2;G)$, we define a cellular cochain $c_1 \times c_2 \in \mathcal{C}^{n_1+n_2}(X_1 \times X_2;G)$ in the most natural way: For the oriented cells $e_1 \subset X_1, e_2 \subset X_2$ of dimensions n_1, n_2, the value of $c_1 \times c_2$ on $e_1 \times e_2$ is $c_1(e_1)c_2(e_2)$ (product in G). It is easy to check that $\delta(c_1 \times c_2) = (\delta c_1) \times c_2 + (-1)^{n_1} c_1 \times \delta c_2$; thus, if c_1, c_2 are cocycles, then $c_1 \times c_2$ is also a cocycle. The same formula shows that the cohomology class of the cocycle $c_1 \times c_2$ depends only on the cohomology classes of cocycles c_1, c_2, so we get a valid (bilinear, associative) multiplication

$$[\gamma_1 \in H^{n_1}(X_1;G), \gamma_2 \in H^{n_2}(X_2;G)] \mapsto \gamma_1 \times \gamma_2 \in H^{n_1+n_2}(X_1 \times X_2;G).$$

A similar construction exists for homology. Namely, if $a_1 = \sum_i g_i e_{1i} \in \mathcal{C}_{n_1}(X_1;G), a_2 = \sum_j g_j e_{2j} \in \mathcal{C}_{n_2}(X_2;G)$, then we put

$$a_1 \times a_2 = \sum_{i,j} (g_i g_j)(e_{1i} \times e_{2j}) \in \mathcal{C}_{n_1+n_2}(X_1 \times X_2;G).$$

A check shows that $\partial(a_1 \times a_2) = (\partial a_1) \times a_2 + (-1)^{n_1} a_1 \times \partial a_2$, which gives rise to a homological multiplication

$$[\alpha_1 \in H_{n_1}(X_1;G), \alpha_2 \in H_{n_2}(X_2;G)] \mapsto \alpha_1 \times \alpha_2 \in H_{n_1+n_2}(X_1 \times X_2;G).$$

The two $\times$-products (usually called *cross-products*) are connected by the formula

$$\langle \gamma_1 \times \gamma_2, \alpha_1 \times \alpha_2 \rangle = (-1)^{n_1 n_2} \langle \gamma_1, \alpha_1 \rangle \langle \gamma_2, \alpha_2 \rangle.$$

EXERCISE 1. Another definition of the homological $\times$-product can be obtained from Künneth's formula: This formula yields a canonical map $H_{n_1}(X_1) \otimes H_{n_2}(X_2) \to H_{n_1+n_2}(X_1 \times X_2)$, and the image of $\alpha_1 \otimes \alpha_2$ with respect to this map is taken for $\alpha_1 \times \alpha_2$. Prove the equivalence of the two definitions.

At this moment, however, the difference between homology and cohomology becomes important. For any topological space X, there exists the *diagonal map* $\Delta: X \to X \times X,\ \Delta(x) = (x, x)$. This maps induces homomorphisms

$$\Delta_*: H_n(X; G) \to H_n(X \times X; G),$$
$$\Delta^*: H^n(X \times X; G) \to H^n(X; G);$$

of these homomorphisms; the first one is useless for us now, but the second one provides cohomological multiplication: For $\gamma_1 \in H^{n_1}(X; G), \gamma_2 \in H^{n_2}(X; G)$, we put

$$\gamma_1 \smile \gamma_2 = \Delta^*(\gamma_1 \otimes \gamma_2) \in H^{n_1+n_2}(X; G).$$

(The classical notation $\smile$, "cup," is not very convenient, so often instead of $\gamma_1 \smile \gamma_2$ we will simply write $\gamma_1\gamma_2$.)

However, this way of defining the cohomological product has two important disadvantages. First, we must still prove the independence of the CW structure. Second, the diagonal map is not cellular, and to apply it to a cellular cochain we need to choose a cellular approximation, which cannot be done in a canonical way, at least, in the context of arbitrary CW complexes. To avoid these difficulties we will use the opposite order of the definition. First, we will define a $\smile$-product (usually called the *cup-product*) by a singular, topologically invariant, construction, and then we will use it to define the cross-product.

Terminological Remark. The cup-product was initially called the *Kolmogorov–Alexander product*, after the two remarkable mathematicians who (independently of each other) conceived of this operation in the mid-1930s. Unfortunately, the next generation of topologists found this term too long.

16.2 The Cup-Product: A Direct Construction

In the standard simplex Δ^n, $n = n_1 + n_2$ with the vertices $v_0, \dots, v_n$, consider two faces of dimensions n_1 and n_2: $\Gamma_-^{n_1}\Delta^n$ with the vertices $v_0, \dots, v_{n_1}$ and $\Gamma_+^{n_2}\Delta^n$ with vertices $v_{n_1}, \dots, v_n$. These faces have dimensions n_1 and n_2 and have one common vertex, v_{n_1}. Accordingly, for an n-dimensional singular simplex $f: \Delta^n \to X$, we will consider faces $\Gamma_-^{n_1}f = f\ |_{\Gamma_-^{n_1}\Delta^n}$ and $\Gamma_+^{n_2}f = f\ |_{\Gamma_+^{n_2}\Delta^n}$, which are singular simplices of dimensions n_1 and n_2.

Let X be an arbitrary topological space and let G be a commutative ring. Then let $c_1 \in C^{n_1}(X; G)$ and $c_2 \in C^{n_2}(X; G)$. We define a cochain $c_1 \smile c_2 \in C^{n_1+n_2}(X; G)$ by the formula

$$[c_1 \smile c_2](f) = c_1(\Gamma_-^{n_1}f)c_2(\Gamma_+^{n_2}f),$$

where f is $(n_1 + n_2)$-dimensional singular simplex of X.

Proposition (Properties of the Cochain Cup-Product). *Let* $c_1 \in C^{n_1}(X;G)$, $c_2 \in C^{n_2}(X;G)$. *Then*

(0) $\delta(c_1 \smile c_2) = (\delta c_1) \smile c_2 + (-1)^{n_1} c_1 \smile \delta c_2$.
(1) $c_1 \smile (c_2 \smile c_3) = (c_1 \smile c_2) \smile c_3$ $[c_3 \in C^{n_3}(X;G)]$.
(2) *Let* ω *be the backward transformator (Example 2 in Sect. 12.5). Then for any* $(n_1 + n_2)$*-dimensional singular chain* a*,*

$$[c_1 \smile c_2](a) = (-1)^{n_1 n_2}[c_2 \smile c_1](\omega^X_{n_1+n_2} a).$$

(3) *For a continuous map* $g: X \to Y$*,*

$$g^\#(c_1 \smile c_2) = (g^\# c_1) \smile (g^\# c_2).$$

(4) *For a ring homomorphism* $h: G \to H$*,*

$$h_*(c_1 \smile c_2) = (h_* c_1) \smile (h_* c_2).$$

Proof The proof is obvious [only property (0) requires a simple calculation] and is left to the reader.

Remark. The noncommutativity (even the non-plus-minus-commutativity) of the chain cup-product is an unavoidable property which has important consequences (which will show themselves in Chap. 4).

Property (0) shows that the cup-product of two cocycles is a cocycle whose cohomology class depends only on the cohomology classes of the factors. This gives rise to the cohomological cup-product

$$[\gamma_1 \in H^{n_1}(X_1;G), \gamma_2 \in H^{n_2}(X_2;G)] \mapsto \gamma_1 \times \gamma_2 \in H^{n_1+n_2}(X_1 \times X_2;G).$$

Theorem (Properties of the Cohomology Cup-Product). *Let* $\gamma_1 \in H^{n_1}(X;G)$, $\gamma_2 \in H^{n_2}(X;G)$. *Then*

(1) $\gamma_1 \smile (\gamma_2 \smile \gamma_3) = (\gamma_1 \smile \gamma_2) \smile \gamma_3$ $[\gamma_3 \in H^{n_3}(X;G)]$.
(2) $\gamma_1 \smile \gamma_2 = (-1)^{n_1 n_2} \gamma_2 \smile \gamma_1$.
(3) *For a continuous map* $g: X \to Y$*,*

$$g^*(\gamma_1 \smile \gamma_2) = (g^* \gamma_1) \smile (g^* \gamma_2).$$

(4) *For a ring homomorphism* $h: G \to H$*,*

$$h_*(\gamma_1 \smile \gamma_2) = (h_* \gamma_1) \smile (h_* \gamma_2).$$

This follows from the proposition [the proof of property (2) uses the transformator lemma; see Sect. 12.5].

Notice that there is an obvious generalization of the previous construction: If $\gamma_1 \in H^{n_1}(X; G_1)$, $\gamma_2 \in H^{n_2}(X; G_2)$ and there is a pairing $\mu: G_1 \times G_2 \in G$, then there arises a cup-product $\gamma_1 \smile_\mu \gamma_2 = \gamma_1 \smile \gamma_2 \in H^{n_1+n_2}(X; G)$. For example, if $\gamma_1 \in H^{n_1}(X; G)$ (where G is just an Abelian group) and $\gamma_2 \in H^{n_2}(X; \mathbb{Z})$, then there is a cup-product $\gamma_1 \smile \gamma_2 \in H^{n_1+n_2}(X; G)$.

EXERCISE 2. Prove that if X is connected and $\gamma \in H^0(X; G) = G$, then $\gamma \smile \gamma_1 = \gamma\gamma_1$ for any $\gamma_1 \in H^n(X; G)$. In particular, if $1 \in G$ is the unity of the ring G, then $1 \in G = H^0(X; G)$ is the unity of the cohomological multiplication.

EXERCISE 3. Construct a relative version of cup-product: If $\gamma_1 \in H^{n_1}(X, A; G)$ and $\gamma \in H^{n_2}(X, B; G)$, then $\gamma_1 \smile \gamma_2 \in H^{n_1+n_2}(X, A \cup B; G)$. [To prove this, it is convenient to regard $H_n(X, A \cup B)$ not as the homology of the complex consisting of the groups $C_n(X)/C_n(A \cup B)$, but rather as the complex of groups $C_n(X)/(C_n(A) \oplus C_n(B))$; the homology remains the same (for sufficiently good A and B) by the refinement lemma.]

16.3 The Cross-Product: A Construction via the Cup-Product

As before, let X_1, X_2 be topological spaces, let G be a commutative ring, and let $\gamma_1 \in H^{n_1}(X_1; G)$, $\gamma_2 \in H^{n_2}(X_2; G)$ be cohomology classes. Put

$$\gamma_1 \times \gamma_2 = (p_1^*\gamma_1) \smile (p_2^*\gamma_2) \in H^{n_1+n_2}(X_1 \times X_2; G),$$

where p_1 and p_2 are projections of $X_1 \times X_2$ onto X_1 and X_2.

EXERCISE 4. Make up a definition of the relative cross-product,

$$\begin{aligned}&[\gamma_1 \in H^{n_1}(X_1, A_1; G), \gamma_2 \in H^{n_2}(X_2, A_2; G)]\\ &\qquad\qquad \mapsto \gamma_1 \times \gamma_2 \in H^{n_1+n_2}(X_1 \times X_2, (A_1 \times X_2) \cup (X_1 \times A_2); G).\end{aligned}$$

EXERCISE 5. Check all kinds of naturalness for the cross-product.

Theorem. *This definition of the cross-product is equivalent to that in Sect. 16.1.*

Proof. It turns out to be sufficient to compute explicitly the cross-product in one particular case. Since standard simplices and their products are homeomorphic to balls,

$$\begin{gathered}H^{n_1}(\Delta^{n_1}, \partial\Delta^{n_1}; \mathbb{Z}) = \mathbb{Z},\ H^{n_2}(\Delta^{n_2}, \partial\Delta^{n_2}; \mathbb{Z}) = \mathbb{Z};\\ H^{n_1+n_2}(\Delta^{n_1} \times \Delta^{n_2}, \partial(\Delta^{n_1} \times \Delta^{n_2}); \mathbb{Z}) = \mathbb{Z}.\end{gathered}$$

Similar formulas hold for homology.

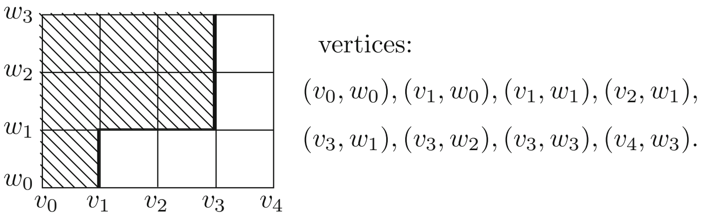


Fig. 64 Triangulation of a product of simplices

What we want to check is that the cross-product of the generators of the groups $H^{n_1}(\Delta^{n_1}, \partial\Delta^{n_1}; \mathbb{Z}) = \mathbb{Z}$, $H^{n_2}(\Delta^{n_2}, \partial\Delta^{n_2}; \mathbb{Z}) = \mathbb{Z}$ is, up to a sign, the generator of $H^{n_1+n_2}(\Delta^{n_1} \times \Delta^{n_2}, \partial(\Delta^{n_1} \times \Delta^{n_2}); \mathbb{Z}) = \mathbb{Z}$.

Obviously, the singular simplex id: $\Delta^{n_1} \to \Delta^{n_1}$ is a relative cycle representing the generator of $\mathbb{Z} = H_{n_1}(\Delta^{n_1}, \partial\Delta^{n_1})$, and similarly for Δ^{n_2}. As to $\mathbb{Z} = H^{n_1+n_2}(\Delta^{n_1} \times \Delta^{n_2}, \partial(\Delta^{n_1} \times \Delta^{n_2}); \mathbb{Z})$, to describe the generator, we will construct a triangulation (actually, quite standard) of the product $\Delta^{n_1} \times \Delta^{n_2}$, generalizing the triangulation of the product $\Delta^n \times I$ constructed in Sect. 12.2; see Fig. 59.

Let $v_0, v_1, \dots, v_{n_1}$ be the vertices of Δ^{n_1}, and let $w_0, w_1, \dots, w_{n_2}$ be the vertices of Δ^{n_2}. In $\Delta^{n_1} \times \Delta^{n_2}$, take $(n_1 + n_2)$-dimensional affine simplices whose vertices make a sequence of the form

$$(v_{i_0}, w_{j_0}), (v_{i_1}, w_{j_1}), (v_{i_2}, w_{j_2}), \dots, (v_{i_{n_1+n_2}}, w_{j_{n_1+n_2}}),$$

where

$$\begin{gathered} 0 = i_0 \le i_1 \le i_2 \le \dots \le i_{n_1+n_2} = n_1; \\ 0 = j_0 \le j_1 \le j_2 \le \dots \le j_{n_1+n_2} = n_2; \\ i_s + j_s = s. \end{gathered}$$

In other words, in an $(n_1 + 1) \times (n_2 + 1)$ grid with horizontal bars labeled by $w_0, \dots, w_{n_2}$ and vertical bars labeled by $v_0, \dots, v_{n_1}$, we choose a path from (v_0, w_0) to (v_{n_1}, w_{n_2}) and take the sequence of crossings of the bars on this path (see an example in Fig. 64).

There are $\binom{n_1 + n_2 + 2}{n_1 + 1}$ such paths, and accordingly $\Delta^{n_1} \times \Delta^{n_2}$ falls into the union of this amount of $(n_1 + n_2)$-dimensional simplices. These simplices can be described in terms of barycentric coordinates: to which of them the point $((t_0, \dots, t_{n_1}), (u_0, \dots, u_{n_2}) \in \Delta^{n_1} \times \Delta^{n_2}$ belongs depends on the ordering of numbers

$$t_0, t_0 + t_1, \dots, t_0 + t_1 + \dots + t_{n_1-1}; u_0, u_0 + u_1, \dots, u_0 + u_1 + \dots + u_{n_2-1}.$$

For example, the seven-dimensional simplex corresponding to the path in Fig. 64 is described in $\Delta^3 \times \Delta^4$ by the inequalities

$$0 \le t_0 \le u_0 \le t_0 + t_1 \le t_0 + t_1 + t_2 \le u_0 + u_1$$
$$\le u_0 + u_1 + u_2 \le t_0 + t_1 + t_2 + t_3 \le 1$$

(the rule is as follows: We move along the path and after a horizontal edge we place the sum of ts, and after a vertical edge we place the sum of us). Since the vertices of each simplex of the subdivision are ordered, there arise canonical maps of the standard simplex onto the simplices of the subdivision, that is, singular simplices of $\Delta^{n_1} \times \Delta^{n_2}$. Let $c(n_1, n_2) \in C_{n_1+n_2}(\Delta^{n_1} \times \Delta^{n_2})$ be the sum of these singular simplices with the coefficients ± 1 where the sign is determined by the parity of the number of squares of grid below the chosen path (left unshadowed in Fig. 64; for the path shown there this number is 5 and the sign is minus). It is obvious that $c(n_1, n_2)$ is a relative cycle modulo $\partial(\Delta^{n_1} \times \Delta^{n_2})$: Two of our simplices have a common $(n_1 + n_2 - 1)$-dimensional face in the interior of $\Delta^{n_1} \times \Delta^{n_2}$ if and only if the two paths have precisely one square between them; then they appear in $c(n_1, n_2)$ with opposite signs, and the faces have the same number in them; so the faces cancel. To prove that $\alpha_1 \times \alpha_2$ is plus–minus the standard generator of $H^{n_1+n_2}(\Delta^{n_1} \times \Delta^{n_2}, \partial(\Delta^{n_1} \times \Delta^{n_2}); \mathbb{Z}) = \mathbb{Z}$, it is sufficient to check that $\langle \alpha_1 \times \alpha_2, c(n_1, n_2) \rangle = \pm 1$. For an $(n_1 + n_2 - 1)$-dimensional singular simplex f of $\Delta^{n_1} \times \Delta^{n_2}$, the value of $\alpha_1 \times \alpha_2$ of f (here by α_1, α_2 we mean rather cochains than cohomology classes) is $\alpha_1(p_1 \circ \Gamma_-^{n_1} f)\alpha_2(p_2 \circ \Gamma_+^{n_2} f)$. But for a simplex f with vertices

$$(v_{i_0}, w_{j_0}), (v_{i_1}, w_{j_1}), (v_{i_2}, w_{j_2}), \dots, (v_{i_{n_1+n_2}}, w_{j_{n_1+n_2}}),$$

the simplex $p_1(\Gamma_-^{n_1} f)$ has the vertices $v_{i_0}, \dots, v_{i_{n_1}}$ and the simplex $p_2(\Gamma_+^{n_2} f)$ has the vertices $w_{j_{n_1}}, \dots, w_{j_{n_1+n_2}}$. The only case when these two simplices are not contained in $\partial\Delta^{n_1}$ and $\partial\Delta^{n_1}$ is when

$$i_0 = 0, \dots, i_{n_1-1} = n_1 - 1, i_{n_1} = i_{n_1+1} = \dots = i_{n_1+n_2} = n_1;$$
$$j_0 = j_1 = \dots = j_{n_1} = 0, j_{n_1+1} = 1, \dots, j_{n_1+n_2} = n_2.$$

Thus, only one summand in $c(n_1, n_2)$ makes a contribution into $\langle \alpha_1 \times \alpha_2, c(n_1, n_2) \rangle$, and this contribution is ± 1.

The rest of the proof uses only the naturalness of the cross-product. It consists of six steps.

Step 1. The cross-product

$$H^{n_1}(S^{n_1}, \mathrm{pt}; \mathbb{Z}) \times H^{n_2}(S^{n_2}, \mathrm{pt}; \mathbb{Z}) \to H^{n_1+n_2}(S^{n_1} \times S^{n_2}, S^{n_1} \vee S^{n_2}; \mathbb{Z})$$

is, up to a sign, the standard multiplication $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. Indeed, the projections $(\Delta^{n_1}, \partial\Delta^{n_1}) \to (S^{n_1}, \mathrm{pt}), (\Delta^{n_2}, \partial\Delta^{n_2}) \to (S^{n_2}, \mathrm{pt}), (\Delta^{n_1} \times \Delta^{n_2}, \partial(\Delta^{n_1} \times \Delta^{n_2})) \to (S^{n_1} \times S^{n_2}, S^{n_1} \vee S^{n_2})$ induce isomorphisms in the cohomology of dimensions $n_1, n_2, n_1 + n_2$.

Step 2. The cross-product

$$H^{n_1}(S^{n_1}; \mathbb{Z}) \times H^{n_2}(S^{n_2}; \mathbb{Z}) \to H^{n_1+n_2}(S^{n_1} \times S^{n_2}; \mathbb{Z})$$

is, up to a sign, the standard multiplication $\mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}$. Indeed, the maps $(S^{n_1}, \mathrm{pt}) \to (S^{n_1}, \emptyset), \ldots$ induce isomorphisms in the cohomology of appropriate dimensions.

Step 3. Similar statements for the bouquets of spheres (we leave precise statements to the reader).

Step 4. The ring $\mathbb{Z}$ can be replaced by an arbitrary ring G. This follows from the naturalness of the cross-product with respect to ring homomorphisms $\mathbb{Z} \to G$.

Step 5. X_1, X_2 are CW complexes of the respective dimensions n_1, n_2, and cohomology classes $\gamma_1 \in H^{n_1}(X_1; G), \gamma_2 \in H^{n_2}(X_2; G)$ are represented by cellular cocycles c_1, c_2; then $\gamma_1 \times \gamma_2 \in H^{n_1+n_2}(X_1 \times X_2; G)$ is represented by the cellular cocycle

$$[c_1 \times c_2](e_1 \times e_2) = \pm c_1(e_1)c_2(e_2).$$

For the proof we can consider the projections $X_1 \to X_1/\operatorname{sk}_{n_1-1} X_1, X_2 \to X_2/\operatorname{sk}_{n_2-1} X_2$; the induced cohomology homomorphisms are epimorphisms.

Step 6. The general case. For the transition to this case we consider the inclusion maps $\operatorname{sk}_{n_1} X_1 \to X_1, \operatorname{sk}_{n_2} X_2 \to X_2, \operatorname{sk}_{n_1} X_1 \times \operatorname{sk}_{n_2} X_2 \to X_1 \times X_2$; the induced cohomology homomorphisms in the appropriate dimensions are monomorphisms.

This completes the proof.

16.4 Cup-Product and Diagonal Map

Now let us briefly investigate the connection between the definition of the cup-product in Sect. 16.2 and the preliminary definition from the introduction (Sect. 16.1). The first statement is almost obvious.

Theorem. *For any X, G, and $\gamma_1 \in H^{n_1}(X; G), \gamma_2 \in H^{n_2}(X; G)$,*

$$\gamma_1 \smile \gamma_2 = \Delta^*(\gamma_1 \times \gamma_2),$$

where $\Delta\colon X \to X \times X$ is the diagonal map.

Proof. Obviously, $p_1 \circ \Delta = p_2 \circ \Delta = \mathrm{id}$. Hence,

$$\Delta^*(\gamma_1 \times \gamma_2) = \Delta^*(p_1^*\gamma_1 \smile p_2^*\gamma_2) = (p_1 \circ \Delta)^*\gamma_1 \smile (p_2 \circ \Delta)^*\gamma_2 = \gamma_1 \smile \gamma_2.$$

In addition to that, we remark that actually the definition of cup-product in Sect. 16.2 can be regarded as a combination of the definition in Sect. 16.1 and a particular choice of a cellular approximation of the diagonal map. Let us describe the latter, first in the case when X is a triangulated space. First, in the product $\Delta^n \times \Delta^n$, let us consider the CW subcomplex $\bigcup_{p+q=n}(\Gamma_-^p \Delta^n \times \Gamma_+^q \Delta^n)$; for $n_1 = n_2 = 2$, it is shown in Fig. 65 (surely, a picture of a four-dimensional figure on a two-dimensional paper sheet cannot be awfully clear). The dashed triangle is the diagonal image of Δ^2; it is not a cellular subspace of $\Delta^2 \times \Delta^2$. The cellular

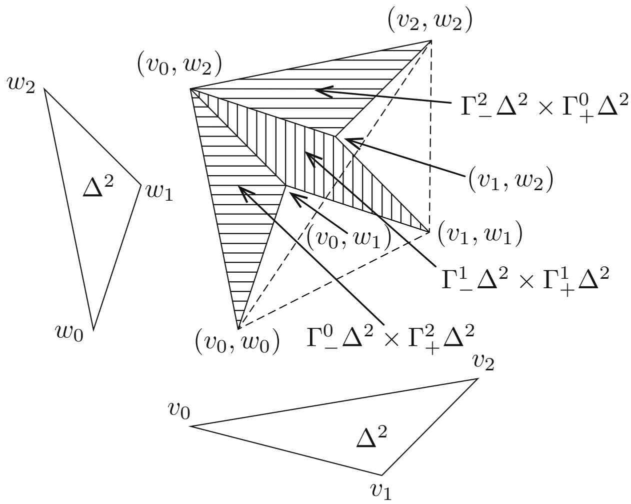


Fig. 65 A cellular approximation of the diagonal in $\Delta^2 \times \Delta^2$

approximations of the diagonal edges $[(v_0, w_0), (v_1, w_1)], [(v_1, w_1), (v_2, w_2)]$, and $[(v_2, w_2), (v_0, w_0)]$ are broken lines $[(v_0, w_0), (v_0, w_1), (v_1, w_1)]$, $[(v_1, w_1), (v_1, w_2), (v_2, w_2)]$, and $[(v_2, w_2), (v_2, w_0), (v_0, w_0)]$; the diagonal triangle is approximated by the union of three pieces: two triangles and one parallelogram, as shown in Fig. 65.

In general, the approximation $\Delta_0 \colon \Delta^n \xrightarrow{\cong} \bigcup_{p+q=n}(\Gamma^p_- \Delta^n \times \Gamma^q_+ \Delta^n) \subset \Delta^n \times \Delta^n$ is defined by the formula

$$\begin{aligned}(t_0, \dots, t_n) \mapsto ((2t_0, \dots, 2t_{p-1}, 2(t_p + \dots + t_n) - 1, 0 \dots, 0),\\ (0, \dots, 0, 2(t_0 + \dots + t_p) - 1, 2t_{p+1}, \dots, 2t_n))),\\ \text{if } t_0 + \dots + t_p \geq \tfrac{1}{2},\ t_p + \dots + t_n \geq \tfrac{1}{2}.\end{aligned}$$

It is clear that the restriction of Δ_0 to any face of Δ^n (of any dimension) is a similar map for this face.

If X is an *ordered* triangulated space (see Sect. 13.10), then this construction can be applied to each simplex of the triangulation, and we obtain a canonical cellular approximation $\Delta_0 \colon X \to X \times X$ of the diagonal map (here we mean the CW structure of $X \times X$ which is obtained as the product of two copies of the triangulation of X regarded as a CW structure; thus, the cells of $X \times X$ are products of simplices). Now it is clear that for the two cochains $c_1 \in \mathcal{C}^{n_1}(X; G), c_2 \in \mathcal{C}^{n_2}(X; G)$, the cochain $c_1 \smile c_2 \in \mathcal{C}^{n_1+n_2}(X \times X; G)$ is nothing but $(\Delta_0)_\#(c_1 \times c_2)$; this sheds light on the connection between the definitions of cup-product given in Sects. 16.1 and 16.2. We can add that the construction above can be applied not only to triangulated spaces; for example, it works perfectly well for the cellular realization $\mathcal{S}ing(X)$ of the singular complex of an arbitrary topological space, and hence gives an explanation for the construction of the $\smile$-product of singular cochains.

16.5 First Application: The Hopf Invariant

To demonstrate at once the power of the cohomological multiplication, we will immediately, before any serious computations of this multiplication, prove a highly nontrivial statement concerning the homotopy groups of spheres.

Theorem. *The group $\pi_{4n-1}(S^{2n})$ is infinite for any $n \geq 1$. Moreover, the Whitehead square $[\iota_{2n}, \iota_{2n}]$ of the generator of $\pi_{2n}(S^{2n})$ has an infinite order in $\pi_{4n-1}(S^{2n})$.* (Compare this theorem with the results of Sects. 9.9 and 10.5.)

The proof of this theorem is based on the *Hopf invariant*, which is an integer assigned to every element of $\varphi \in \pi_{4n-1}(S^{2n})$. Its definition is as follows. Consider a spheroid $f\colon S^{4n-1} \to S^{2n}$ and form the space $X_\varphi = S^{2n} \cup_f D^{4n}$ (aka the cone of f). The space X_φ depends, up to a homotopy equivalence, only on φ (which justifies the notation). It has a natural CW structure with three cells of dimensions $0, 2n$, and $4n$. Thus,

$$H^q(X_\varphi; \mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ for } q = 0, 2n, 4n, \\ 0 \;\text{ for } q \neq 0, 2n, 4n. \end{cases}$$

The groups $H^{2n}(X_\varphi; \mathbb{Z}), H^{4n}(X_\varphi; \mathbb{Z})$ (isomorphic to $\mathbb{Z}$) have natural generators (determined by the canonical orientations of S^{2n} and D^{4n}), and we denote these generators by a and b. Since the cup-square $a^2 = a \smile a$ has dimension $4n$, we have $a^2 = hb$, where $h \in \mathbb{Z}$. The number $h = h(\varphi)$ is, by definition, the Hopf invariant of φ. [1] Our theorem is covered by the following two lemmas.

Lemma 1. *The Hopf invariant is additive: $h(\varphi + \psi) = h(\varphi) + h(\psi)$.*

Lemma 2. *The Hopf invariant is nontrivial; in particular,*

$$h([\iota_{2n}, \iota_{2n}] = 2.$$

Proof of Lemma 1. In addition to the spaces $X_\varphi, X_\psi, X_{\varphi+\psi}$ (constructed using the spheroids $f, g, f + g\colon S^{4n-1} \to S^{2n}$), we will consider the space

$$Y_{\varphi,\psi} = (S^{2n} \cup_f D^{4n}) \cup_g D^{4n} = S^{2n} \cup_{f \vee g} (D^{4n} \vee D^{4n}).$$

This space has a CW structure with four cells of dimensions $0, 2n, 4n, 4n$ and has the following cohomology:

$$H^q(Y_{\varphi,\psi}; \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } q = 4n, \\ \mathbb{Z} & \text{for } q = 0, 2n, \\ 0 & \text{for } q \neq 0, 2n, 4n. \end{cases}$$

[1] In the homotopy theory, there are interesting generalizations of the Hopf invariant; see Whitehead [88] and Hilton [44].

Denote the canonical generators of the cohomology groups $H^{2n}(Y_{\varphi,\psi};\mathbb{Z})$ and $H^{4n}(Y_{\varphi,\psi};\mathbb{Z})$ by a' and b'_1, b'_2. There are natural CW embeddings $X_\varphi \to Y_{\varphi,\psi}$ and $X_\psi \to Y_{\varphi,\psi}$. There is also a natural map $X_{\varphi+\psi} \to Y_{\varphi,\psi}$; it consists of the identity map $S^{2n} \to S^{2n}$ and the map $D^{4n} \to D^{4n} \vee D^{4n}$ which collapses the equatorial plane to a point (these maps compose a continuous map $X_{\varphi+\psi} \to Y_{\varphi,\psi}$ because the diagram

$$\begin{tikzcd}
S^{4n-1} \arrow[r, "\text{proj.}"] \arrow[dr, "f+g"'] & S^{4n-1}/\text{equator} = S^{4n-1} \vee S^{4n-1} \arrow[d, "f \vee g"] \\
& S^{2n}
\end{tikzcd}$$

is commutative by the definition of the sum of spheroids). The induced cellular chain maps for all three maps described above are obvious; the cohomology maps act like this:

$$\begin{aligned}
X_\varphi \to Y_{\varphi,\psi}: &\quad a' \mapsto a,\ b'_1 \mapsto b,\ b'_2 \mapsto 0\\
X_\psi \to Y_{\varphi,\psi}: &\quad a' \mapsto a,\ b'_1 \mapsto 0,\ b'_2 \mapsto b\\
X_{\varphi+\psi} \to Y_{\varphi,\psi}: &\quad a' \mapsto a,\ b'_1 \mapsto b,\ b'_2 \mapsto b.
\end{aligned}$$

We must have $(a_1)^2 = h_1 b'_1 + h_2 b'_2$, where $h_1, h_2 \in \mathbb{Z}$. By the naturalness of the cup-product,

$$a^2 = h_1 b \text{ in } X_\varphi,\ a^2 = h_2 b \text{ in } X_\psi,\ a^2 = (h_1 + h_2) b \text{ in } X_{\varphi+\psi}.$$

On the other hand,

$$a^2 = h(\varphi) b \text{ in } X_\varphi,\ a^2 = h(\psi) b \text{ in } X_\psi,\ a^2 = h(\varphi+\psi) b \text{ in } X_{\varphi+\psi}.$$

Hence, $h_1 = h(\varphi), h_2 = h(\psi), h_1 + h_2 = h(\varphi+\psi)$, from which $h(\varphi+\psi) = h(\varphi) + h(\psi)$.

Proof of Lemma 2. Consider the product $S^{2n} \times S^{2n}$. Its cohomology is $H^{2n}(S^{2n} \times S^{2n};\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ (the generators c_1, c_2) and $H^{4n}(S^{2n} \times S^{2n};\mathbb{Z}) = \mathbb{Z}$ (the generator d). The multiplication: $c_1^2 = c_2^2 = 0$ (proof: Consider the projections $S^{2n} \times S^{2n} \to S^{2n}$) and $c_1 c_2 = d$ (follows from step 2 of the proof in Sect. 16.3 plus the definition of the cup-product in Sect. 16.2).

Make a factorization of $S^{2n} \times S^{2n}$ using the relation $(x_0, x) \sim (x, x_0)$ for all $x \in S^{2n}$, where x_0 is the zero-dimensional cell of S^{2n}. That is, we glue to each other the two two-dimensional cells of $S^{2n} \times S^{2n}$. The resulting space X has three cells, of dimensions $0, 2n$, and $4n$; that is, it has the form $S^{2n} \cup_f D^{4n}$, where f is a certain map $S^{4n-1} \to S^{2n}$. Moreover, if we compare this construction with the definition of the Whitehead product in Sect. 10.5, we notice that this f is nothing but the canonical spheroid representing the Whitehead product $[\iota_{2n}, \iota_{2n}]$. Thus, $X = X_{[\iota_{2n},\iota_{2n}]}$. The cohomology of X is $H^{2n}(X;\mathbb{Z}) = H^{4n}(X;\mathbb{Z}) = \mathbb{Z}$, and if a, b are canonical generators of these cohomology groups, then $a^2 = h([\iota_{2n}, \iota_{2n}])b$. But the cohomology homomorphism induced by the projection $S^{2n} \times S^{2n} \to X$ takes a and b into $c_1 + c_2$ and d. Thus, in the cohomology of $S^{2n} \times S^{2n}$, $(c_1 + c_2)^2 = h([\iota_{2n}, \iota_{2n}])d$, and, since $(c_1 + c_2)^2 = c_1^2 + 2c_1c_2 + c_2^2 = 2d$, we have $h([\iota_{2n}, \iota_{2n}]) = 2$.

Remark 4. As we will see in Chap. 3, $\pi_{4n-1}(S^{2n}) = \mathbb{Z} \oplus$ a finite group. In particular, $\pi_3(S^2) = \mathbb{Z}$ (we already know this), $\pi_7(S^4) = \mathbb{Z} \oplus \mathbb{Z}_{12}, \pi_{11}(S^6) = \mathbb{Z}, \pi_{15}(S^8) = \mathbb{Z} \oplus \mathbb{Z}_{120}$. It is also true that all the homotopy groups of spheres are finite besides $\pi_n(S^n) = \mathbb{Z}$ and $\pi_{4n-1}(S^{2n})$.

Remark 5. Lemma 2 shows that the image of the Hopf homomorphism $h: \pi_{4n-1}(S^{2n}) \to \mathbb{Z}$ is either the whole group $\mathbb{Z}$ or the group of even integers. The choice between these two options is reduced to the question: Does $\pi_{4n-1}(S^{2n})$ contain an element with the Hopf invariant one? This question has several remarkable equivalent statements. For example, it is possible to show that S^m possesses an H-space structure if and only if m is odd, that is, $m = 2n - 1$, and $\pi_{4n-1}(S^{2n})$ contains an element with the Hopf invariant one. The same condition is necessary and sufficient for the existence in $\mathbb{R}^{m+1}$ of a bilinear multiplication with a unique division. The combination of Lemma 2 and Exercise 7 in Lecture 10 shows that the Hopf invariant of the Hopf class $\eta_2 \in \pi_3(S^2)$ equals 1 (this corresponds to the complex number multiplication in $\mathbb{R}^2$ or to the natural group structure in S^1). In 1960, J. Adams showed that elements with the Hopf invariant one are contained only in $\pi_3(S^2), \pi_7(S^4)$, and $\pi_{15}(S^8)$ (we mentioned his results in Sect. 1.4; we will discuss two proofs of it: in Chaps. 5 and 6).

16.6 An Addendum: Other Multiplications

A: Homological ×-Product

We already mentioned this in the introduction. Its definition corresponds to the general spirit of this lecture: Singular simplices $f_1: \Delta^{n_1} \to X_1, f_2: \Delta^{n_2} \to X_2$ give rise to a map $f_1 \times f_2: \Delta^{n_1} \times \Delta^{n_2} \to X_1 \times X_2$; then we triangulate the product $\Delta^{n_1} \times \Delta^{n_2}$ as in the proof of the theorem in Sect. 16.3. Then we define the product of the singular simplices f_1 and f_2 the singular chain of $X_1 \times X_2$, which is the sum with the coefficients ± 1 (the same as in Sect. 16.3) of the singular simplices which are restrictions of the map $f_1 \times f_2$ to the $(n_1 + n_2)$-dimensional simplices of the triangulation. This chain is also denoted as $f_1 \times f_2$. By bilinearity, this $\times$-product is extended to singular chains: $\left(\sum_i g_{1i}f_{1i}\right) \times \left(\sum_j g_{2j}f_{2j}\right) = \sum_{i,j} g_{1i}g_{2j}\left(f_{1i} \times f_{2j}\right)$ (where g_{1i}, g_{2j} are elements of the coefficient ring G). A verification shows that $\partial(c_1 \times c_2) = (\partial c_1) \times c_2 + (-1)^{n_1} c_1 \times \partial c_2$ (where $n_1 = \dim c_1$). Thus, there arises a homology multiplication: For $\alpha_1 \in H_{n_1}(X_1; G), \alpha_2 \in H_{n_2}(X_2; G)$, there is the product $\alpha_1 \times \alpha_2 \in H_{n_1+n_2}(X_1 \times X_2; G)$. The proof of coincidence of this product with the homological cross-product described in Sect. 16.1 is a replica of the proof of the similar cohomological result in Sect. 16.3.

EXERCISE 6. Prove that for $\alpha_1 \in H_{n_1}(X_1; G), \alpha_2 \in H_{n_2}(X_2; G), \gamma_1 \in H^{n_1}(X_1; G)$, $\gamma_2 \in H^{n_2}(X_2; G)$,

$$\langle \gamma_1 \times \gamma_2, \alpha_1 \times \alpha_2 \rangle = (-1)^{n_1 n_2} \langle \gamma_1, \alpha_1 \rangle \langle \gamma_2, \alpha_2 \rangle.$$

B: Cap-Product

This is a mixed operation involving both homology and cohomology. Let $a = \sum_i g_i f_i \in C_{n_1}(X; G), c \in C^{n_2}(X; G)$, where $n_1 \geq n_2$. Put

$$a \frown c = \sum_i g_i c\left(\Gamma_-^{n_2}\right) \Gamma_+^{n_1 - n_2} \in C_{n_1 - n_2}(X; G)$$

(we use the notation introduced in Sect. 16.2).

EXERCISE 7. Prove the formula

$$(\partial a) \frown c = a \frown \delta c + (-1)^{n_2} \partial(a \frown c).$$

EXERCISE 8. Deduce from this that if a is a cycle representing a homology class $\alpha \in H_{n_1}(X; G)$ and c is a cocycle representing a cohomology class $\gamma \in H^{n_2}(X; G)$, then $a \frown c$ is a cycle whose homology class is fully determined by α and γ.

In the notation of Exercise 9, the homology class of $a \frown c$ is denoted as $\alpha \frown \gamma$. Thus, we get the *cap-product*

$$[\alpha \in H_{n_1}(X; G), \gamma \in H^{n_2}(X; G)] \mapsto \alpha \frown \gamma \in H_{n_1 - n_2}(X; G).$$

EXERCISE 9. Prove that if $n_1 = n_2$ and X is connected, then $\alpha \frown \gamma = \langle \gamma, \alpha \rangle \in G = H_0(X; G)$.

EXERCISE 10. Prove the "mixed associativity": $\alpha \frown (\gamma_1 \smile \gamma_2) = (\alpha \frown \gamma_1) \frown \gamma_2$.

EXERCISE 11. Prove the naturalness of the cap-product: If $\alpha \in H_{n_1}(X; G), \gamma \in H^{n_2}(Y; G)$, and $f: X \to Y$ is a continuous map, then $(f_* \alpha) \frown \gamma = f_*(\alpha \frown f^* \gamma)$.

C: Pontryagin–Samelson Multiplication

EXERCISE 12. Prove that if n_1, n_2 are positive integers, then there is no way to introduce for all X a nonzero bilinear multiplication

$$H_{n_1}(X; G) \times H_{n_2}(X; G) \to H_{n_1 + n_2}(X; G)$$

natural with respect to continuous maps.

However, it is possible to define a multiplication in homology groups of X if X itself possesses a multiplication making it a topological group or, at least, an H-space. The definition is obvious: If $\mu: X \times X \to X$ is the multiplication in X and $\alpha_1 \in H_{n_1}(X; G), \alpha_2 \in H_{n_2}(X; G)$ where G is a ring, then $\alpha_1\alpha_2 = \mu_*(\alpha_1 \times \alpha_2)$. This product is called the Pontryagin–Samelson product. We have no opportunity to discuss this product in detail, but we recommend to the reader, after reading Chap. 3, to return to this product and to calculate it for the homology groups of major topological groups and H-spaces.

Final Remark. All multiplications considered in this lecture can be generalized, in an obvious way, from the case of ring coefficients to the case when there is a pairing $G_1 \times G_2 \to G$, the factors lie in the homology/cohomology with coefficients in G_1 and G_2, and the product belongs to the homology/cohomology with coefficients in G.

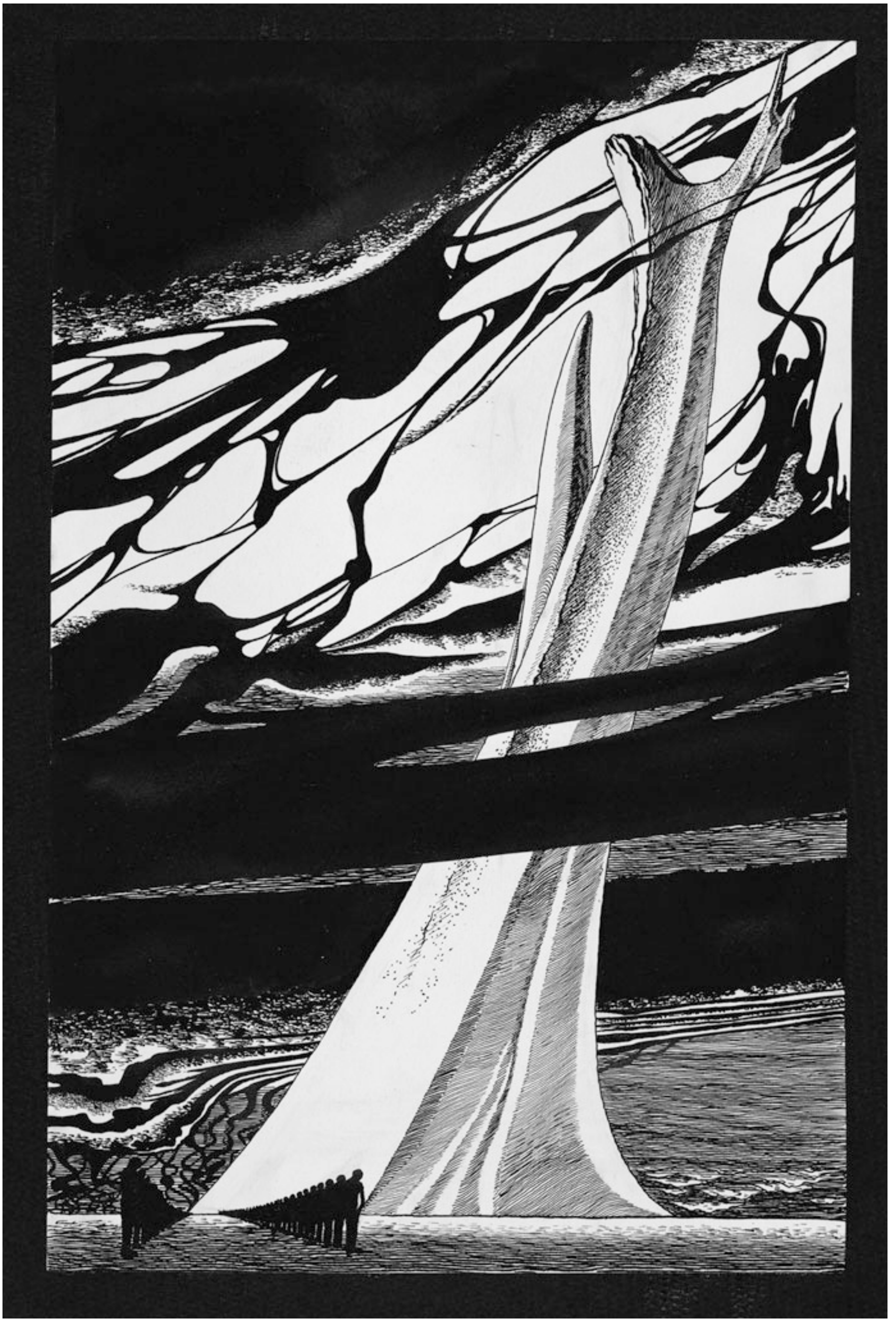
Lecture 17 Homology and Manifolds

Among the natural computational tools used by homology theory, the most efficient ones are delivered by the topology of smooth manifolds, and we cannot help considering this subject. However, the foundations of the theory of manifolds, rooted in geometry and analysis, require a thick volume by themselves. The most common way to overcome this difficulty is to replace the notion of a smooth manifold by various combinatorial substitutes like homology manifolds or pseudomanifolds (see Sects. 17.2 and 17.3 ahead). By doing this, we can achieve a rigor of the proofs at the expense of geometric visuality. To compensate for the latter, we will sometimes provide geometric explanations based on statements which are easy to believe, but not always easy to prove.

We begin with a short sightseeing tour in the theory of smooth manifolds.

17.1 Smooth Manifolds

A Hausdorff topological space with a countable base of open sets (these topological assumptions are not in the spirit of this book, but we have to impose them, since without them many statements that follow would be plainly wrong) is called an n-dimensional (topological) *manifold* if every point of it possesses a neighborhood homeomorphic to the space $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_- = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \leq 0\}$. A point of an n-dimensional manifold X which has no neighborhood homeomorphic to $\mathbb{R}^n$ is called a *boundary point*. Boundary points of X form an $(n-1)$-dimensional manifold ∂X called the *boundary* of X. Obviously, ∂X is a manifold without boundary: $\partial\partial X = \emptyset$.



Examples of manifolds: Euclidean spaces, spheres, balls, classical surfaces, projective spaces, Grassmann manifolds, flag manifolds, Lie groups, Stiefel manifolds, products of the spaces listed above, open sets in these spaces, closed domains with smooth boundaries in these spaces, and so on.

A homeomorphism between $\mathbb{R}^n$ or $\mathbb{R}^n_-$ (or an open set in one of these spaces) and an open set U in a manifold X determines coordinates in U which are called *local coordinates*. If the domains U, V of local coordinate systems $f\colon U \to \mathbb{R}^n_{(-)}$, $g\colon V \to \mathbb{R}^n_{(-)}$ (also called *charts*) overlap, then there arises a *transition map*

$$\begin{array}{ccccc} f(U \cap V) & \xrightarrow{f^{-1}} & U \cap V & \xrightarrow{g} & g(U \cap V) \\ \cap & & & & \cap \\ \mathbb{R}^n & & & & \mathbb{R}^n, \end{array}$$

which is described by usual functions of n variables. These functions can be smooth (as usual in topology, we understand the word *smooth* as belonging to the class $\mathcal{C}^\infty$), analytic, algebraic, etc. A set of charts which cover the manifold is called an *atlas*. An atlas is called smooth (analytic) if such functions are all transition functions between charts of this atlas. Two smooth (analytic) atlases are called smoothly (analytically) equivalent if their union is smooth (analytic) atlas. A class of equivalent smooth (analytic) atlases is called a *smooth (analytic) structure* on a manifold. A manifold with a smooth (analytic) structure is called a smooth (analytic) manifold. The boundary of a smooth (analytic) manifold is, in a natural way, a smooth (analytic) manifold. In the following, we will not consider analytic manifolds any seriously.

All manifolds listed above possess a natural smooth structure. Add one more example: Smooth surfaces in a Euclidean space, that is, closed subsets of $\mathbb{R}^m$ locally determined by systems of equations

$$f_i(x_1, \dots, x_m) = 0,\ i = 1, \dots, k$$

and, possibly, one inequality

$$f_{k+1}(x_1, \dots, x_m) \geq 0,$$

where $f_1, \dots, f_k$ $(, f_{k+1})$ are smooth functions whose gradients in their common domain are linearly independent.

There are two fundamental theorems in the theory of smooth manifolds (also called *differential topology*).

Theorem 1. *Every smooth manifold is diffeomorphic (that is, homeomorphic with preserving the smooth structure) to a smooth surface in an Euclidean space.*

Theorem 2. *Every compact smooth manifold is homeomorphic to a triangulated subset of an Euclidean space, and the homeomorphism can be made smooth on every simplex of the triangulation.*

Remarks. (1) In both theorems, the dimension of the Euclidean space can be as small as twice the dimension of the manifold.

(2) Theorem 2 also holds for noncompact manifolds, but the triangulation in this case has to be infinite.

We do not prove these theorems. Theorem 1 is proved in many textbooks in differential topology. Its proof is not hard. The situation with Theorem 2 is worse. Since the 1920s, the topologist regarded this fact as obvious. There are many geometric approaches to this result which look promising. For example, take a compact smooth surface in an Euclidean space and decompose this space into a union of small cubes. If the decomposition satisfies some general position condition with respect to the surface, we can expect that the intersections of the surface with the cubes will be close to convex polyhedra and we can easily triangulate these polyhedra. Or, choose a random finite subset of the smooth surface which is sufficiently dense, and take the Dirichlet domain; again we should get a subdivision of the surface into smooth polyhedra. However, numerous attempts to make this proof rigorous turned out to be unsuccessful. The first flawless proof of this theorem (actually, of a stronger relative result) was given in the 1930s by H. Whitney. This proof was based on entirely different ideas and did not look easy. We know two textbook presentations of this proof, in the books Whitney [89] and Munkres [64].

EXERCISE 1. Construct a realization as smooth surfaces in Euclidean spaces of projective spaces, Grassmann manifolds, flag manifolds, and Stiefel manifolds.

EXERCISE 2. Prove that all classical surfaces can be presented as smooth surfaces in $\mathbb{R}^n$ with $n \leq 4$.

EXERCISE 3. Construct smooth triangulations of classical surfaces; try to minimize the number of simplices needed.

EXERCISE 4. Prove that the number of n-dimensional simplices adjacent to an $(n-1)$-dimensional simplex of a smooth triangulation of an n-dimensional smooth manifold is 2 if this $(n-1)$-dimensional simplex is not contained in the boundary, and is 1 otherwise.

EXERCISE 5 (a generalization of Exercise 4). Let s be a k-dimensional simplex of a smooth triangulation of an n-dimensional smooth manifold. Consider the simplices of the triangulation which contain s, and in each of these simplices take the face opposite s (that is, spanned by the vertices not belonging to s). Prove that the union of these faces (which is called the *link* of the simplex s) is homeomorphic to S^{n-k-1} if s is not contained in the boundary and is homeomorphic to D^{n-k-1} otherwise. (For a warmup, begin with the case when $n = 3$ and $k = 1$.)

Remark. The notion of a link will be used later, so the reader who is not interested in this exercise still has to understand the definition of a link.

An atlas of a smooth manifold is called *oriented* if for every two overlapping charts the transition map has a positive determinant at every point. Two oriented atlases determine (belong to) the same *orientation* if their union is an oriented atlas.

A manifold is called *orientable* (*oriented*) if it possesses (is furnished by) an oriented atlas, that is, an orientation.

EXERCISE 6. Which projective spaces and Grassmann manifolds are orientable? (*Answer*: Only real projective spaces and Grassmann manifolds can be nonorientable. Namely, $\mathbb{R}P^n$ is orientable if and only if n is odd, and $G(n,k)$ is orientable if and only if n is even.)

EXERCISE 7. Prove that spheres with handles are orientable and projective planes and Klein bottles are nonorientable; drilling holes does not affect the orientability.

EXERCISE 8. Prove that a connected orientable manifold of positive dimension has precisely two orientations.

EXERCISE 9. Prove that every connected chart of an orientable manifold can be included in an oriented atlas; thus, if an orientable manifold is connected, then every connected chart determines an orientation.

EXERCISE 10. Prove that a manifold is orientable if and only if a neighborhood of every closed curve on this manifold is orientable.

EXERCISE 11. Prove that every simply connected manifold is orientable.

EXERCISE 12. Prove that every connected nonorientable manifold possesses an orientable twofold covering.

EXERCISE 13. Prove that the boundary of an orientable manifold is orientable.

It is also possible to define orientations using the language of triangulations. An orientation of an n-dimensional simplex is the order of its vertices given up to an even permutation. An orientation of an n-dimensional simplex induces orientations of its $(n-1)$-dimensional faces (using an even permutation of the order of vertices, we make the number of the vertex complementary to the face to be n, after which we orient the face by the order of remaining vertices). (Some modification is needed in the cases of $n = 0, 1$: An orientation of a zero-dimensional simplex is just $+$ or $-$, the orientation of faces v_0 and v_1 of a one-dimensional simplex $[v_0, v_1]$ are $-$ and $+$.) If two n-dimensional simplices share an $(n-1)$-dimensional face, then their orientations are *coherent* if they induce opposite orientations on this face. A triangulated n-dimensional manifold is orientable if all its n-dimensional simplices can be coherently oriented.

EXERCISE 14. An orientation of a connected orientable n-dimensional manifold is determined by an orientation of any of its n-dimensional simplices. [It may be reasonable to do this exercise after reading (the beginning of) the next section.]

17.2 Pseudomanifolds and Fundamental Classes

Definition. A triangulated space X is called an *n-dimensional pseudomanifold* if it satisfies the following three axioms.

1 (Dimensional homogeneity). X is the union of its n-dimensional simplices.
2 (Strong connectedness). For any two n-dimensional simplices s, s' of X, there exists a finite chain of n-dimensional simplices, $s_0, s_1, \dots, s_k$, such that $s_0 = s, s_k = s'$, and for every $i = 1, \dots, k$, the simplices s_{i-1}, s_i share an $(n-1)$-dimensional face.
3 (Nonbranching property). Every $(n-1)$-dimensional simplex of X is a face of precisely two n-dimensional simplices of X.

If X is a connected smooth n-dimensional manifold without boundary furnished with a smooth triangulation, then the triangulation obviously satisfies Axiom 1, satisfies Axiom 3 as stated in Exercise 4, and satisfies Axiom 2 as stated in Exercise below.

EXERCISE 15. Prove that a smoothly triangulated smooth connected manifold without boundary is strongly connected (see Axiom 2). [All we need to establish is that two interior points of n-dimensional simplices can be joined by a path avoiding an $(n-2)$-dimensional skeleton.]

Thus, a *smoothly triangulated connected smooth manifold without boundary is a pseudomanifold.* The converse is wrong: A pseudomanifold is not always a manifold. See the simplest example in Fig. 66.

There are fewer artificial examples of pseudomanifolds topologically different from manifolds: complex algebraic varieties, and Thom spaces of vector bundles (these will be extensively studied later, in Lecture 31 and further lectures).

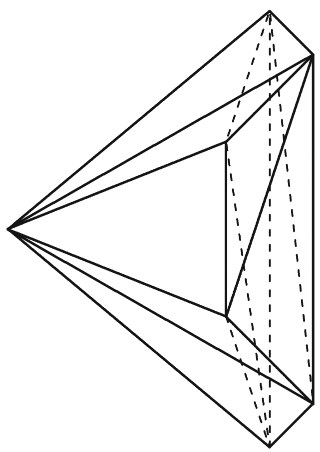


Fig. 66 A pseudomanifold which is not a manifold (a pinched torus)

An orientation of a pseudomanifold is defined as in the end of the previous section (Exercise 14 is also applied to this case). If a pseudomanifold is a manifold, then an orientation of this pseudomanifold is the same as an orientation of the manifold (in the sense of Sect. 17.1).

Theorem. *Let X be an n-dimensional pseudomanifold. Then*

$$H_n(X) = \begin{cases} \mathbb{Z}, & \text{if } X \text{ is compact and orientable,} \\ 0 & \text{otherwise;} \end{cases}$$
$$H_n(X;\mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & \text{if } X \text{ is compact,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We consider the classical complex $\{\mathcal{C}_n(X), \partial_n\}$, corresponding to an arbitrary ordering of vertices (see Sect. 13.10). Since $\mathcal{C}_{n+1}(X) = 0$, $H_n(X) = Z_n(X)$, the group of n-dimensional cycles of the classical complex. Let $c = \sum_i k_i s_i$ be such a cycle (k_i are integers, s_i are n-dimensional simplices). If the simplices s_i and s_j share an $(n-1)$-dimensional face, then this face does not belong to any other simplex, and $\partial c = 0$ implies $k_i = \pm k_j$ (the sign depends on the orientations). Since X is strongly connected, this shows that c involves all n-dimensional simplices of X, with all the coefficients of the form $\pm k$, where k is a nonnegative integer, the same for all the simplices. From this we immediately see that if the number of simplices is infinite, then there are no nonzero cycles, and $H_n(X) = 0$. If the number of simplices is finite, then let us reverse the orientations of simplices with a negative value of the coefficient. Since c is a cycle, these new orientations induce opposite orientations on every $(n-1)$-dimensional face; that is, they are coherent. We see that a nonzero cycle exists if and only if X is orientable. This proves our result for $H_n(X)$. The case of $\mathbb{Z}_2$-coefficients is similar, but it does not involve signs, and hence does not involve orientations.

This proof provides a canonical generator for the group $H_n(X)$ for a compact oriented pseudomanifold X: This is the homology class of the cycle, which is the sum of all n-dimensional simplices of X with orientations compatible with the orientation of X and with the coefficients all equal to 1. This homology class is called the *fundamental class* of X (and the cycle is called the *fundamental cycle*). In the orientation-free case, we have fundamental classes and fundamental cycles with coefficients in $\mathbb{Z}_2$ (certainly, only for compact pseudomanifolds). Notation: $[X] \in H_n(X)$ or $H_n(X;\mathbb{Z}_2)$.

Since connected smooth manifolds without boundary are pseudomanifolds, the preceding theorem holds for them. In particular, for compact connected smooth manifolds without boundary there are fundamental classes. (It is time to mention a broadly used term: A compact manifold without boundary is called *closed*.) This has an obvious generalization to the disconnected case: For a closed oriented n-dimensional manifold X, $H_n(X) = \bigoplus_\alpha H_n(X_\alpha)$, where X_α are components of X, and $[X]$ is simply $\{[X_\alpha]\}$.

EXERCISE 16. Prove that if X is a connected n-dimensional manifold with nonempty boundary, then $H_n(X) = H_n(X; \mathbb{Z}_2) = 0$.

EXERCISE 17. Prove that if X has a boundary, then the same construction as above gives a class $[X, \partial X] \in H_n(X, \partial X)$ or $H_n(X, \partial X; \mathbb{Z}_2)$ and $\partial_*[X, \partial X] = [\partial X]$.

EXERCISE 18. Prove the relation $[X_1 \times X_2] = [X_1] \times [X_2]$ in all possible versions (including the boundary one).

EXERCISE 19. Prove that for any homology class $\alpha \in H_n(Y)$ of an arbitrary topological space Y there exists a compact oriented (not necessarily connected) pseudomanifold X and a continuous map $f: X \to Y$ such that $f_*[X] = \alpha$. Prove a similar statement for an $\alpha \in H_n(Y; \mathbb{Z}_2)$ and nonoriented pseudomanifolds. (Actually, the $\mathbb{Z}_2$-case is easier, and so it may be advisable to begin with it; a construction in Sect. 13.11 may serve as a pattern for both the oriented and nonoriented cases.)

There arises a natural question regarding the possibility to present a homology class of a topological space as an image of the fundamental class of a manifold. The answer is negative, for homology classes with coefficients in $\mathbb{Z}$ as well as for those with coefficients in $\mathbb{Z}_2$. We will return to the discussion of this in the last lecture of this book.

A more popular question arises in the topology of manifolds: If Y is a manifold and $\alpha \in H_n(Y)$, then when is it possible to find a closed oriented n-dimensional submanifold X of Y (we assume that the reader understands what it is) such that the homomorphism induced by the inclusion map sends $[X]$ into α (as people say, X *realizes* α)? Again, a similar question exists for the $\mathbb{Z}_2$ homology classes and nonoriented submanifolds. There are many remarkable results regarding submanifold realizations; for example, for any homology class α of a manifold, there exists a number N such that $N\alpha$ can be realized by a submanifold. (For this result and other results, see the classical paper by Thom [84].)

EXERCISE 20. Prove that the generators of groups

$$H_m(\mathbb{R}P^n; \mathbb{Z}_2),\ H_m(\mathbb{R}P^n),\ H_{2m}(\mathbb{C}P^n),\ H_{4m}(\mathbb{H}P^n)$$

are realized by projective subspaces of $\mathbb{R}P^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$. (Compare also to Exercise 11 in Lecture 14.)

Mention in conclusion that if X, Y are oriented pseudomanifolds of the same dimension, and $f: X \to Y$ is a continuous map, then $f_*[X] = k \cdot [Y]$, where k is an integer. This k is called the degree of f and is denoted as $\deg f$; it is a homotopy invariant. In the nonoriented case, the degree $\deg f$ may be defined as an element of $\mathbb{Z}_2$. We have already had this notion in the particular case $X = Y = S^n$ (see Sects. 10.3 and 13.3). In the manifold case, there exists a description of the degree similar to the description given in Sect. 10.3 for spheres; we formulate the result in the form of an exercise.

EXERCISE 21. Let $f: X \to Y$ be a (piecewise) smooth map between two closed oriented n-dimensional manifolds, and let $y: Y$ be a regular value of this map. Then there is a neighborhood U of y such that $f^{-1}(U)$ is a disjoint union of a finite collection of sets U_i with all restrictions $f|_{U_i}$ being homeomorphisms $U_i \to U$. Prove that $\deg f$ is the number of i for which this homeomorphism preserves the orientation minus the number of i for which it reverses the orientation.

17.3 Homology Manifolds

The most general definition of a homology manifold is formulated in terms of *local homology*: For a topological space X, its mth local homology at the point $x_0 \in X$ is defined as $H^{\text{loc}}_{m,x_0}(X) = H_m(X, X - x_0)$.

Definition. A space X is called an n-dimensional homology manifold if, for any m, $H^{\text{loc}}_{m,x_0}(X) = \widetilde{H}_m(S^n)$, that is,

$$H^{\text{loc}}_{m,x_0}(X) = \begin{cases} \mathbb{Z}, & \text{if } m = n, \\ 0, & \text{if } m \neq n. \end{cases}$$

For us, the most important will be the case when X is triangulated. Recall that the star $\mathrm{St}(s)$ of a simplex s of triangulation is the union of simplices that contain s. The link $\mathrm{Lk}(s)$ is the union of faces of simplices that contain s opposite to s. Figure 67 shows examples of stars and links of a vertex and a one-dimensional simplex of the standard triangulation of the plane.

Proposition 1. (1) *A triangulated space X is an n-dimensional homology manifold if and only if for every vertex v of X, the link* $\mathrm{Lk}(v)$ *is a homological* $(n-1)$-*dimensional sphere* (that is, has the same homology groups as S^{n-1}).

(2) *A triangulated space X is an n-dimensional homology manifold if and only if for every simplex s of X, the link* $\mathrm{Lk}(s)$ *is a homological* $(n-k-1)$-*dimensional sphere where* $k = \dim s$.

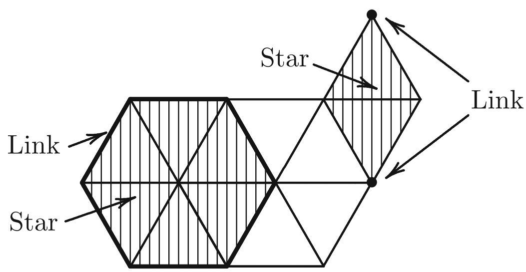


Fig. 67 Stars and links

Proof. Open stars of vertices, $\mathrm{st}(v) = \mathrm{St}(v) - \mathrm{Lk}(v)$, for an open cover of X. Also, $\mathrm{St}(v)$ is a cone over $\mathrm{Lk}(v)$ with the vertex v. Thus, if $x_0 \in \mathrm{st}(v)$, then

$$\begin{aligned} H^{\mathrm{loc}}_{m,x_0}(X) = H_m(X, X - x_0) &= H_m(X, X - \mathrm{st}(v)) \\ &= H_m(\mathrm{St}(v), \mathrm{Lk}(v)) = \widetilde{H}_{m-1}(\mathrm{Lk}(v)) \end{aligned}$$

[the four equalities follow from the definition of local homology, homotopy invariance of homology, excision theorem, and reduced homology sequence of the pair $(\mathrm{St}(v), \mathrm{Lk}(v))$]. This proves (1).

To prove (2), notice that for a simplex s, $\mathrm{St}(s) = s * \mathrm{Lk}(s)$. Hence, for every interior point x_0 of s,

$$\begin{aligned} H^{\mathrm{loc}}_{m,x_0}(X) &= H_m(X, X - x_0) = H_m(s * \mathrm{Lk}(s), (\partial s) * \mathrm{Lk}(s)) \\ &= \widetilde{H}_{m-1}((\partial s) * \mathrm{Lk}(s)) = \widetilde{H}_{m-1}(\Sigma^k \mathrm{Lk}(s)) = \widetilde{H}_{m-k-1}(\mathrm{Lk}(s)), \end{aligned}$$

where $k = \dim s$. This proves (2).

Proposition 2. *Every connected n-dimensional homology manifold is an n-dimensional pseudomanifold.*

Proof. Let X be an n-dimensional homology manifold. Since the link of every vertex of X is an $(n-1)$-dimensional homological sphere, this link contains simplices of dimension $\geq n-1$; hence, every vertex is a vertex of an n-dimensional simplex. There cannot be simplices of dimension $> n$, because the link of every n-dimensional simplex must be empty (homological S^{-1}). Every simplex of dimension $< n$ must have a nonempty link, so it must be a face of a simplex of a bigger dimension. Hence, X must be the union of n-dimensional simplices (dimensional homogeneity axiom holds). The link of an $(n-1)$-dimensional simplex s consists of isolated points, one for every n-dimensional simplex containing s; since the link is a homological S^0, this number is 2 (unbranching axiom holds). A path connecting two points of X can be made straight within every simplex; since the links of simplices of dimension $\leq n-2$ are connected, the path can be pushed from every point of a simplex of dimension $\leq n-2$ to simplices of bigger dimensions. Hence, there is a path disjoint from the $(n-2)$nd skeleton of X (the strong connectedness holds).

Remark 1. Proposition 2 shows that everything said in Sect. 17.2 about pseudomanifolds can be applied to homological manifolds. In particular, homological manifolds can be orientable or nonorientable, there are fundamental cycles and classes, and the theorem of Sect. 17.2 holds for a connected homology manifold.

Remark 2. This argumentation shows a difference between pseudomanifolds and homology manifolds. While in homology manifolds all links are homological spheres of appropriate dimensions, in n-dimensional pseudomanifolds this holds for links of simplices of dimensions n and $n-1$. Add to that that a pseudomanifold in Fig. 66 is not a homology manifold.

Remark 3. A smooth manifold without boundary is a homology manifold (and in the smooth case, links are homeomorphic to spheres, not just are homological spheres).

Remark 4. A homology manifold is not always a topological manifold. For example, there are manifolds with the same homology as a sphere, but not simply connected (the best known example is the *Poincaré sphere* defined in $S^5 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 + |z_3|^2 = 1\}$ by the equation $z_1^5 + z_2^3 + z_3^2 = 0$). The suspension over such a manifold is a homology manifold, but no neighborhoods of vertices are homeomorphic to a Euclidean space.

17.4 Poincaré Isomorphism

The main result of the homological theory of manifolds is the following:

Theorem. *Let X be a compact n-dimensional homology manifold, and let $0 \le m \le n$. If X is orientable, then for any G,*

$$H_m(X; G) \cong H^{n-m}(X; G).$$

In the general case,

$$H_m(X; \mathbb{Z}_2) \cong H^{n-m}(X; \mathbf{Z}_2).$$

In both cases, there are canonical isomorphisms

$$D: H^{n-m}(X; G) \to H_m(X; G)$$

which act by the formula $D(\alpha) = [X] \frown \alpha$, where $[X]$ is the fundamental class (see Sect. 17.2) and $\frown$ denotes the cap-product (see Sect. 16.6).

Remarks. (1) The isomorphism D is usually referred to as the *Poincaré isomorphism.*

(2) By Remark (3) in Sect. 17.3, the theorem holds for closed (compact and boundary-less) smooth manifold.

The proof of the theorem will consist of two parts: First we will give (the most classical) construction of Poincaré isomorphism, and then we will prove the formula involving the cap-product. This formula will show, in particular, that the isomorphism provided by the classical construction does not depend on the triangulation.

For a simplex s of the triangulation of X, denote as $\mathrm{Bast}(s)$ the union of all simplices of the barycentric triangulation whose intersection with s is the center of s. Using the fact that the simplices of the barycentric triangulation correspond to the increasing chains $s_0 \subset \cdots \subset s_j$ of the initial triangulation, we can describe

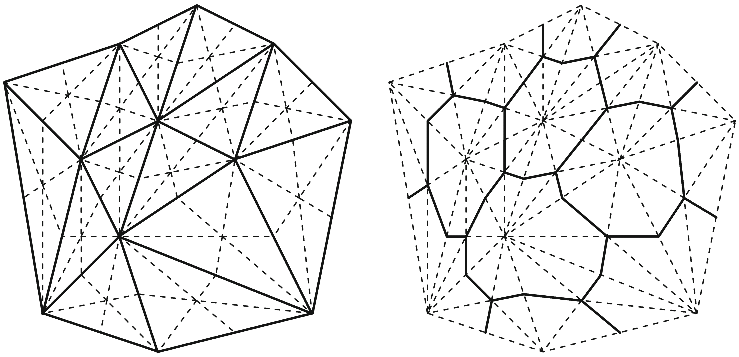


Fig. 68 Barycentric stars

Bast(s) as the union of simplices of barycentric triangulation corresponding to chains as above with $s_0 = s$. Obviously, Bast(s) is the union of its simplices of the maximal dimension, $n - k$ (where $k = \dim s$), that is, simplices corresponding to chains $s = s_0 \subset s_1 \cdots \subset s_{n-k}$ with $\dim s_i = k + i$. This important that $\dim \mathrm{Bast}(s) = n - \dim(s)$.

The reader may see in Fig. 68 (where $n = 2$) what barycentric stars look like. Barycentric stars of vertices are polyhedra of dimension 2 ("centered" at these vertices), barycentric stars of one-dimensional simplices have dimension 1, and barycentric stars of two-dimensional simplices are centers of these simplices (this is true for any dimension n: The barycentric star of an n-dimensional simplex is its center).

Besides barycentric stars, there are barycentric links: For a simplex s, Balk(s) is the union of faces of barycentric simplices in Bast(s) opposite the center of s. Obviously, Bast(s) is the cone over Balk(s) and Balk(s) is homeomorphic to Lk(s) (the reader who has any doubt can observe all this in Fig. 68). Also, there are *open barycentric stars*, bast(s) = Bast(s) − Balk(s). Obviously, X is a disjoint union of open barycentric stars of all its simplices.

If X is a homology manifold, then

$$\begin{aligned} H_m(\mathrm{Bast}(s), \mathrm{Balk}(s)) &= H_m(C(\mathrm{Balk}(s)), \mathrm{Balk}(s)) \\ &= \widetilde{H}_{m-1}(\mathrm{Balk}(s)) = \widetilde{H}_{m-1}(\mathrm{Lk}(s)) \\ &= \begin{cases} \mathbb{Z}, \text{ if } m = n - \dim(s), \\ 0 \quad \text{otherwise} \end{cases} \end{aligned}$$

In other words, although the decomposition of X into open barycentric cells is not necessarily a CW structure, still it can be used for computing homology in the same way. We can define "skeletons" $\mathrm{sk}^m_{\mathrm{bast}}(X)$ as unions of barycentric stars of dimensions $\leq m$ (that is, barycentric stars of simplices of dimensions $\geq n - m$), and the complex $\{C_m^{\mathrm{bast}}(X), \partial_m\}$, where

$$C_m^{\mathrm{bast}}(X) = H_m(\mathrm{sk}_{\mathrm{bast}}^m(X), \mathrm{sk}_{\mathrm{bast}}^{m-1}(X)),$$

$$\partial_m = \partial_*\colon H_m(\mathrm{sk}_{\mathrm{bast}}^m(X), \mathrm{sk}_{\mathrm{bast}}^{m-1}(X)) \to H_{m-1}(\mathrm{sk}_{\mathrm{bast}}^{m-1}(X), \mathrm{sk}_{\mathrm{bast}}^{m-2}(X)),$$

has homology equal to that of X.

Our next remark is that if the homology manifold X is oriented, then there exists a natural way to establish a correspondence between orientations of a simplex s and of the barycentric star $\mathrm{Bast}(s)$. Namely, let the orientation of s be determined by an order of its vertices, $v_0, v_1, \dots, v_k$. Consider an $(n-k)$-dimensional (barycentric) simplex u belonging to $\mathrm{Bast}(s)$; it corresponds to a sequence $s = s_0 \subset \cdots \subset s_{n-k}$ with $\dim s_i = k + i$. For $i = 1, \dots, n-k$, let v_{k+i} be the vertex of s_i not belonging to s_{i-1}. Then $v_0, \dots, v_k, v_{k+1}, \dots, v_n$ is the full set of vertices of the n-dimensional simplex s_{n-k}, and we assign to u the orientation determined by the order $v_k, \dots, v_n$ of its vertices if the order $v_0, \dots, v_n$ of vertices of the simplex v_{n-k} determines the orientation of v_{n-k} compatible with the orientation of X, and we assign the opposite orientation otherwise. If the simplex u shares an $(n-k-1)$-dimensional face with another simplex $u' \subset \mathrm{Bast}(s)$, then u' corresponds to a sequence $s = s_0 \subset \dots s_{j-1} \subset s'_j \subset s_{j+1} \cdots \subset s_{n-k}$ with $s'_j \neq s_j$. If $j < n-k$, then the simplex s_{n-k} stays the same, but the vertices v_j, v_{j+1} are swapped; thus, the orientation of u' is determined by the order of vertices $v_0, \dots, v_{j+1}, v_j, \dots, v_{n-k}$ only if the orientation of u is *not* determined by the order of vertices $v_0, \dots, v_j, v_{j+1}, \dots, v_{n-k}$; their common $(n-k-1)$-dimensional face has the vertices $v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_{n-k}$, and it obtains opposite orientations from u and u'. The case $j = n-k$ is similar: In this case $s'_{n-k} \neq s_{n-k}$, the simplices s'_{n-k} and s_{n-k} have a common $(n-1)$-dimensional face, let it be t, and t obtains opposite orientations from s_{n-k} and s'_{n-k}. The orientations of the common face of u and u' are determined by the orientations of s and t (precisely as the orientation of u is determined by the orientations of s and s_{n-k}) and thus they are also opposite each other.

$C_{n-k}^{\mathrm{bast}}(X; G)$ is the group of linear combinations $\sum_i g_i \,\mathrm{Bast}(s_i)$ where the summation is taken over oriented k-dimensional simplices s_i and $g_i \in G$. If X is a compact oriented n-dimensional homology manifold, consider an isomorphism

$$D\colon C^k_{\mathrm{class}}(X; G) \to C_{n-k}^{\mathrm{bast}}(X; G), \quad D(s^*) = \mathrm{Bast}(s),$$

where s^* is a k-dimensional cochain of the classical complex of X which takes value 1 on s and value 0 on every other k-dimensional simplex, and the orientations of s and $\mathrm{Bast}(s)$ are compatible as above. Fact: For a cochain $c \in C^k_{\mathrm{class}}(X; G)$,

$$D(\delta c) = (-1)^k \partial D(c) \tag{$*$}$$

(see ahead). This shows that D established a dimension-reversing isomorphism between cohomology and homology of X; this is Poincaré isomorphism (also denoted by D).

It remains to establish two facts: the relation ($*$) and the relation $D(\alpha) = [X] \frown \alpha$. Begin with the first. The boundary of $\mathrm{Bast}(s) \in C^{\mathrm{bast}}_{n-k}(X)$ consists of barycentric simplices lying in $\mathrm{Balk}(s)$ [faces inside $\mathrm{Bast}(s)$ are cancelled as follows from the preceding argumentations regarding the orientations]. The face of the barycentric simplex corresponding to the sequence $s = s_0, s_1, \dots, s_{n-k}$ lying in $\mathrm{Balk}(s)$ corresponds to the sequence $s_1, \dots, s_{n-k}$ and thus is contained in $\mathrm{Bast}(s_1)$. In this way, we see that $\mathrm{Bast}(s_1)$ is contained in the boundary of $\mathrm{Bast}(s)$ if and only if s is a face of s_1. The coefficient is $(-1)^k$ (this requires comparing the orientations, which we leave to the reader). Now, go to the second relation. Let bX be the barycentric subdivision of X with the ordering of vertices described in Sect. 13.10, and let $c \in C^k_{\mathrm{class}}(bX;G)$ and $[X]$ be the fundamental cycle of bX. The cellular map $\mathrm{id}\colon X \to bX$ induces a map

$$\mathrm{id}^{\#}\colon C^k_{\mathrm{class}}(bX;G) \to C^k_{\mathrm{class}}(X;G),$$

and the cochain $\mathrm{id}^{\#} c$ takes on a k-dimensional simplex s of X on s, the value equal to the sum of the values, with appropriate signs, of c of k-dimensional simplices of bX contained in s. On the other hand, the chain $[bX] \frown c$ is the sum of faces of n-dimensional simplices of bX spanned by the last vertices (see the definition of $\frown$ in Sect. 16.6). These are simplices in barycentric stars of k-dimensional simplices of X; each barycentric star of s appears in $[X] \frown c$ with the coefficient equal to the sum of values of c on the barycentric parts of s, that is, to $\mathrm{id}^{\#} c(s)$. Thus, $\mathrm{id}_{\#}(D(\mathrm{id}^{\#} c)) = [bX] \frown c$, where the last $\mathrm{id}_{\#}$ is

$$\mathrm{id}_{\#}\colon C^{\mathrm{bast}}_{n-k}(bX;G) \to C^{\mathrm{class}}_{n-k}(X;G).$$

This finishes the proof in the oriented case. In the nonoriented case everything is the same with the usual simplification—we do not need to care about orientations and signs (since the coefficient group is $\mathbb{Z}_2$).

Corollary. *The Euler characteristic of a closed homology manifold of odd dimension equals 0.*

For the proof, it is more convenient to use Poincaré isomorphism with coefficients in $\mathbb{Z}_2$, since it also holds in the nonorientable case. If $n = \dim X$, then

$$\begin{aligned}\chi(X) &= \textstyle\sum_m (-1)^m \dim_{\mathbb{Z}_2} H_m(X;\mathbb{Z}_2) = \sum_m (-1)^m \dim_{\mathbb{Z}_2} H^{n-m}(X;\mathbb{Z}_2)\\ &= \textstyle\sum_m (-1)^m \dim_{\mathbb{Z}_2} H_{n-m}(X;\mathbb{Z}_2) = \sum_m (-1)^{n-m} \dim_{\mathbb{Z}_2} H_m(X;\mathbb{Z}_2)\\ &= \textstyle(-1)^n \sum_m (-1)^m \dim_{\mathbb{Z}_2} H_m(X;\mathbb{Z}_2) = -\chi(X).\end{aligned}$$

17.5 Intersection Numbers and Poincaré Duality

The results of Sect. 15.5 give the possibility to restate Poincaré isomorphisms between homology and cohomology as (noncanonical) isomorphisms between homology and homology. Namely,

$$H_m(X;\mathbb{Z}_2) \cong H_{n-m}(X;\mathbb{Z}_2)$$

for an arbitrary n-dimensional homology manifold X and

$$\text{Free Part of } H_m(X) \cong \text{Free Part of } H_{n-m}(X)$$
$$\text{Torsion Part of } H_m(X) \cong \text{Torsion Part of } H_{n-m-1}(X)$$

in the oriented case. It turns out that these noncanonical isomorphisms reflect a very canonical duality called *Poincaré duality* which is much more classical than Poincaré isomorphisms. We will postpone (until Sect. 17.7) a discussion of torsion parts and concentrate our attention on the free parts of homology groups.

Poincaré duality is based on the notion of the *intersection number*. Let $c_1 = \sum_i k_i \operatorname{Bast}(s_i)$ be some m-dimensional chain of the barycentric star complex of some compact triangulated oriented n-dimensional homology manifold X, and let $c_2 = \sum_j \ell_j s_j$ be some $(n-m)$-dimensional chain of the classical complex of X. Thus, both summations are taken over the set of $(n-m)$-dimensional simplices of X. The integer

$$\phi(c_1, c_2) = \sum_i k_i \ell_i = \langle D^{-1} c_1, c_2 \rangle$$

is called the intersection number of c_1 and c_2. It follows from the last formula and the properties of Poincaré isomorphism that the intersection number of two cycles depends only on the homology classes of these cycles, and we can speak of intersection numbers of homology classes: If $\alpha_1 \in H_m(X)$ and $\alpha_2 \in H_{n-m}(X)$, then $\phi(\alpha_1, \alpha_2) = \langle D^{-1}\alpha_1, \alpha_2 \rangle$, or $\phi(\alpha_1, \alpha_2) = \alpha_2 \frown D^{-1}\alpha_1 \in H_0(X) = \mathbb{Z}$ (see Exercise 10 in Sect. 16.6). Differently, the homology invariance of intersection numbers can be deduced from the formula $\phi(\partial c_1, c_2) = \phi(c_1, \partial c_2)$, which follows, in turn, from relation $(*)$ in Sect. 17.4:

$$\phi(\partial c_1, c_2) = \langle D^{-1}\partial c_1, c_2 \rangle = \langle \delta D^{-1} c_1, c_2 \rangle = \langle D^{-1} c_1, \partial c_2 \rangle = \phi(c_1, \partial c_2).$$

Another interesting relation arises from the “mixed associativity” of cup- and cap-products (see Exercise 11 in Sect. 16.6):

$$\begin{aligned}\phi(\alpha_1, \alpha_2) &= \alpha_2 \frown D^{-1}\alpha_1 = ([X] \frown D^{-1}\alpha_2) \frown D^{-1}\alpha_1 \\ &= [X] \frown (D^{-1}\alpha_2 \smile D^{-1}\alpha_1) = D(D^{-1}\alpha_2 \smile D^{-1}\alpha_1).\end{aligned}$$

This provides a more symmetric definition of the intersection number, which implies, in particular [in view of commutativity relation for the cup-product; see the theorem in Sect. 16.2, part (2)], the commutativity relation

$$\phi(\alpha_1, \alpha_2) = (-1)^{m(n-m)}\phi(\alpha_2, \alpha_1) \quad (\alpha_1 \in H_m(X), \alpha_2 \in H_{m-k}(X)).$$

In the nonoriented case, the intersection number can be defined for cycles and homology classes modulo 2; they take values in $\mathbb{Z}_2$. It is also possible to define "intersection numbers" corresponding to an arbitrary pairing $G_1 \times G_2 \to G$.

A remarkable property of the intersection numbers is their geometric visualizability. A simplex and its barycentric star transversely intersect each other at one point, so the intersection number of two cycles may be regarded as the number of their intersection points taken with the signs determined by their orientations. This statement has a convenient differential statement.

Theorem 1. *Let X be a smooth closed oriented n-dimensional manifold, and let $\alpha_1 \in H_m(X), \alpha_2 \in H_{n-m}(X)$. Let Y_1 and Y_2 be closed oriented submanifolds of X of dimensions m and $n - m$ which realize α_1 and α_2 in the sense that $\alpha_1 = (i_1)_*[Y_1]$ and $\alpha_2 = (i_2)_*[Y_2]$ where i_1, i_2 are inclusion maps. We assume also that Y_1, Y_2 are in general position (which means that they intersect in finitely many points and transverse to each other at each of these points). We assign a sign to every intersection point: plus if the orientations of Y_1 and Y_2 (in this order) compose the orientation of X at this point, and minus otherwise. Then the intersection number $\phi(\alpha_1, \alpha_2)$ equals to the number of the intersection points of Y_1 and Y_2 counted with the signs described above.*

Similar statements hold for homology classes modulo 2 (in which case no orientation is needed) and for manifolds with pseudomanifold-like singularities (away from the intersection points).

As usual (see the warning in the beginning of this lecture), we do not give a rigorous proof of these statements; but from the point of view of common sense they are obvious. We can make the simplices of a triangulation of X much smaller than the distances between the intersection points of Y_1 and Y_2 and then approximate Y_1 and Y_2 by cycles of, respectively, classical and barycentric star complexes. Then the statements become obvious.

Notice that the general position condition is not really harmful: We can make the position of Y_1 and Y_2 general by a small perturbation of one of those.

Example. Natural generators y_r, y_{n-r} of the groups $H_{2r}(\mathbb{C}P^n)$, $H_{2(n-r)}(\mathbb{C}P^n)$ have the intersection number 1. Indeed, they are realized by projective subspaces $\mathbb{C}P^r, \mathbb{C}P^{n-r}$ of $\mathbb{C}P^n$ which (in the general position) intersect in one point. Regarding the sign, we will make an *important remark*. If X is a *complex manifold*, that is, its charts are maps into $\mathbb{C}^n$ and the transition maps are holomorphic, then X possesses a natural, "complex," orientation. The matter is that the Jacobian of a holomorphic map $\mathbb{C}^n \to \mathbb{C}^n$ regarded as a smooth map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is equal to the square of the absolute value of the complex Jacobian and, hence,

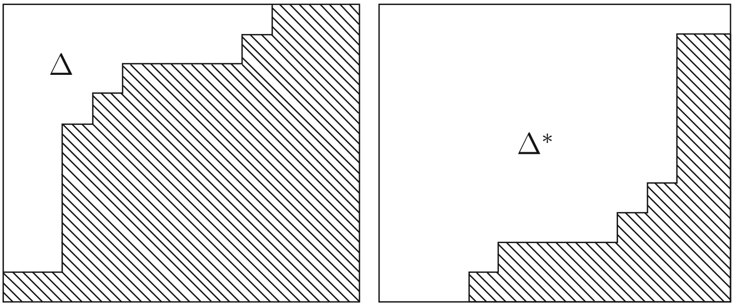


Fig. 69 Dual Young diagrams

is always positive. Moreover, if Y_1, Y_2 are complex (that is, locally determined by holomorphic equations) submanifolds of X of complementary dimensions in a general position, then every point in Y_1, Y_2 contributes $+1$ into the intersection number of the homology classes. Thus, $\phi(y_r, y_{n-r}) = 1$, not -1.

EXERCISE 22. Let Δ be a Young diagram inscribed into a rectangle $k \times (n-k)$, and let Δ^* be the "dual" Young diagram obtained from the complement of Δ in the rectangle by the reflection in the center of the rectangle (see Fig. 69). Then the intersection number of the homology classes of $\mathbb{C}G(n,k)$ corresponding to the Young diagrams Δ, Δ' (see Sects. 5.4.C and 13.8.C) is 1 if $\Delta' = \Delta^*$ and is 0 otherwise. (The same is true for modulo 2 intersection numbers for real Grassmann manifolds; the proof is the same).

The fact that the intersection number of two cycles depends only on the homology classes of these cycles is often used in solving geometric problems. Of a huge set of problems of this kind we give two.

EXERCISE 23. Prove that on any smooth closed orientable surface in $\mathbb{R}^4 = \mathbb{C}^2$, there exist at least two different points for which the tangent planes are complex lines. (*Hint*: The orientation takes care of the existence of more than one such point.)

EXERCISE 24. Prove that if X_1, X_2 are two closed orientable surfaces in $\mathbb{R}^4$, then there are at least four pairs of points $(x_1 \in X_1,\ x_2 \in X_2)$ such that the tangent planes to $X_1 m, x_2$ at x_1, x_2 are parallel.

Return to our definition of the intersection number. Together with Corollary 1 in Sect. 15.5, it implies the following statement.

Theorem 2. *Let X be compact oriented homology manifold.* (1) *For every homomorphism $f: H_m(X) \to \mathbb{Z}$, there exists a homology class $\alpha \in H_{n-m}(X)$ such that $f(\alpha) = \phi(\alpha, \beta)$ for every $\beta \in H_m(X)$.* (2) *The class β is determined by f uniquely, up to adding an element of finite order.*

A similar result holds in the nonoriented case for homology and intersection numbers modulo 2; moreover, in this case β, for a given f, is genuinely unique.

Thus, the intersection numbers determine a nondegenerate duality between the free parts of the groups $H_m(X)$ and $H_{n-m}(X)$ in the oriented case and between the vector spaces $H_m(X;\mathbb{Z}_2)$ and $H_{n-m}(X;\mathbb{Z}_2)$ in general. This duality is called *Poincaré duality*. (One can notice that in the topological literature confusion exists between the terms "Poincaré isomorphism" and "Poincaré duality." It is especially surprising, since in other cases mathematicians have a tendency to be supersensitive to the difference between a vector space and a dual vector space.)

Notice that in the middle-dimensional homology of an even-dimensional manifold, Theorem 2 has the following, more algebraic restatement.

Theorem 3. *Let X be a connected closed orientable manifold of even dimension $2k$, and let $H_k^0(X)$ be the free part of $H_k(X)$. Then the integral bilinear form ϕ (the intersection index) on $H_k^0(X)$ is unimodular [that is, the matrix $\|\phi(\alpha_i,\alpha_j)\|$ where $\alpha_1,\alpha_2,\dots$ is a system of generators in $H_k^0(X)$ has determinant ± 1].*

This matrix is symmetric if k is even and is skew-symmetric if k is odd. Since any skew-symmetric matrix of odd order is degenerate, we have the following:

Corollary. *The middle Betti number of any closed orientable manifold of dimension* $\equiv 2 \bmod 4$ *is even; hence, the Euler characteristic of such a manifold is even.*

For nonorientable manifolds neither is true; examples: the first Betti number of the Klein bottle is 1, and the Euler characteristic of the real projective plane is 1.

Proof of Theorem 3. Consider the homomorphism $\omega_i\colon H_k^0(X)\to\mathbb{Z}$, $\omega_i(\alpha_j)=\delta_{ij}$. By part (2) of Theorem 2, there exists a $\beta_i\in H^k(X;\mathbb{Z})$ such that $\langle\beta_i,\alpha\rangle=\omega_i(\alpha)$, in particular, $\langle\beta_i,\alpha_j\rangle=\phi(D\beta_i,\alpha_j)=\delta_{ij}$. Let $D\beta_i=\sum_k b_{ik}\alpha_k+$ a finite order element (where b_{ki} are integers). Then

$$\phi(D\beta_i,\alpha_j)=\sum_k b_{ik}\phi(\alpha_k,\alpha_j)=\delta_{ij}.$$

That is, the product of integer matrices $\|b_{ij}\|$ and $\|\phi(\alpha_i,\alpha_j)\|$ is the identity matrix; hence, each of them has the determinant ± 1.

Theorem 3 demonstrates the importance of the theory of integral unimodular ($\det=\pm 1$) forms in topology of manifolds, especially of dimensions divisible by 4: For an oriented closed manifold of such dimension, there arises a unimodular integral quadratic form as the intersection form in the middle dimension. For example, the famous Pontryagin theorem states that a homotopy type of a simply connected closed four-dimensional manifold is fully determined by this form. A lot is known about the classification of such forms (the best source is Milnor and Husemoller [58]), but the question of which forms can be intersection forms for smooth closed four-dimensional simply connected manifolds is very far from being resolved.

In conclusion, let us prove a useful statement on Poincaré duality in products of manifolds.

Theorem 4. *Let X_1, X_2 be a compact oriented homology manifold of dimensions n_1, n_2, and let $\gamma_1 \in H^{q_1}(X_1; G)$, $\gamma_2 \in H^{q_2}(X_2; G)$. Then*

$$D_{X_1 \times X_2}(\gamma_1 \times \gamma_2) = (-1)^{(n_1 - q_1)q_2} D_{X_1}\gamma_1 \times D_{X_2}\gamma_2.$$

(Here D_X denotes Poincaré isomorphism in X.)

Proof. We use the obvious relation $(\alpha_1 \times \alpha_2) \frown p_1^*\gamma = (a \frown \gamma) \times \beta)$, where $\alpha_1 \in H_{q_1}(X_1), \alpha_2 \in H_{q_2}(X_2), \gamma \in H^r(X_1; G), p_i : X_1 \times X_2 \to X_i$ is the projection (this relation holds at the chain–cochain level), and the relation $(\alpha_1 \times \alpha_2) \frown p_2^*\gamma = (-1)^{q_1 r}\alpha_1 \times (\alpha_2 \frown \gamma)$, which is obtained from the previous relation by applying the swapping homeomorphism $X_1 \times X_2 \leftrightarrow X_2 \times X_1$.

Back to the theorem:

$$\begin{aligned}
[X_1 \times X_2] \frown (\gamma_1 \times \gamma_2) &= [X_1 \times X_2] \frown (p_1^*\gamma_1 \smile p_2^*\gamma_2) \\
&= ([X_1 \times X_2] \frown p_1^*\gamma_1) \frown p_2^*\gamma_2 \\
&= (([X_1] \times [X_2]) \frown p_1^*\gamma_1) \frown p_2^*\gamma_2 \\
&= (([X_1] \frown \gamma_1)) \times [X_2]) \frown p_2^*\gamma_2 \\
&= (-1)^{(n_1 - q_1)q_2}([X_1] \frown \gamma_1) \times ([X_2] \frown \gamma_2).
\end{aligned}$$

17.6 Application: The Lefschetz Formula

Let X be a compact topological space with finitely generated homology $\bigoplus_n H_n(X)$, and let $f : X \to X$ be a continuous map. The number

$$\mathcal{L}(f) = \sum_n (-1)^n \operatorname{Tr} f_{*n}$$

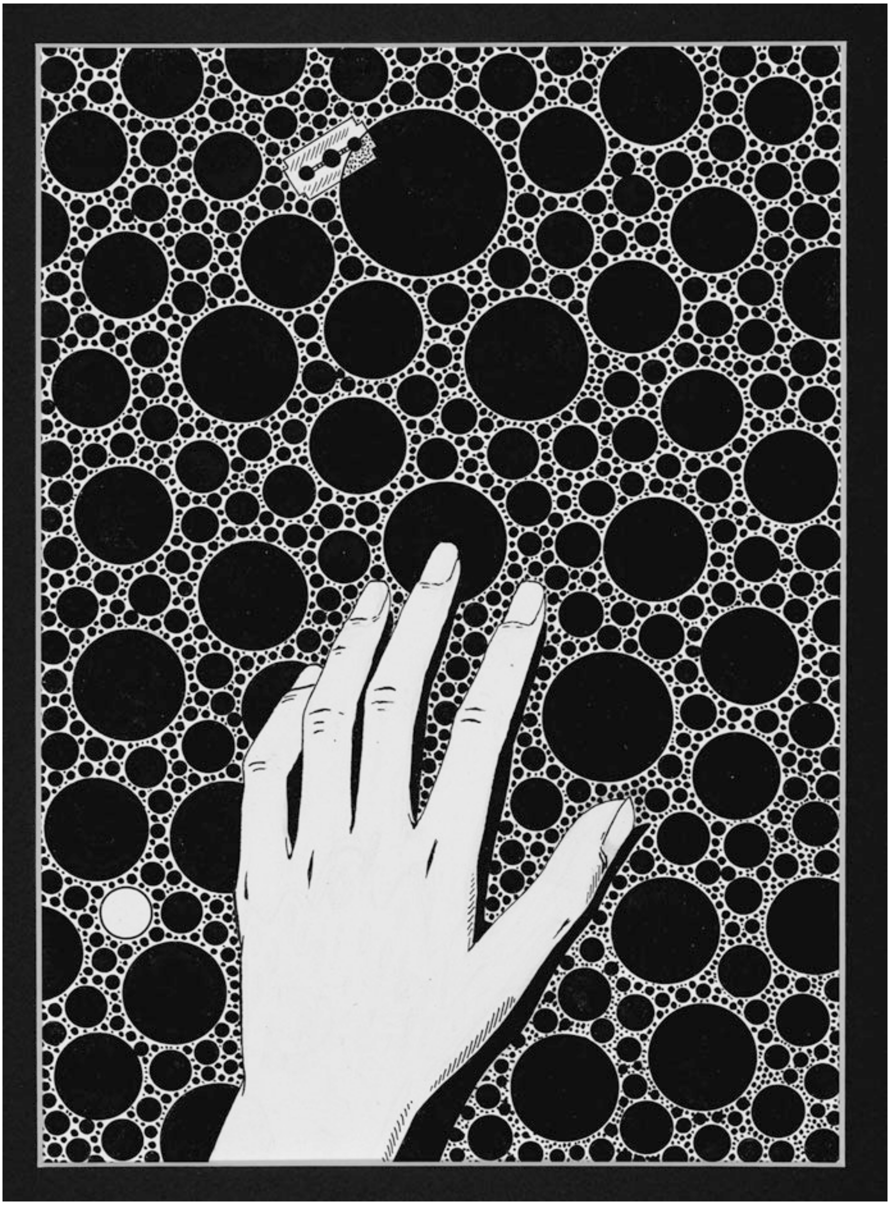
is called the *Lefschetz number* of f [here $\operatorname{Tr} f_{*n}$ denotes the trace of the lattice homomorphism

$$f_{*n} : H_n(X)/\operatorname{Tors} H_n(X) \to H_n(X)/\operatorname{Tors} H_n(X)].$$

Obviously, $\mathcal{L}(f)$ is a homotopy invariant of f. The goal of this section is to establish a relation between the Lefschetz number of f and the behavior of fixed points of f.

Algebraic Lemma. *Let*

$$(\mathcal{C}) \qquad \ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots$$



be a complex with finitely generated $\bigoplus_n C_n$*, and let* $f = \{f_n: C_n \to C_n\}$ *be an endomorphism of* $\mathcal{C}$*. Let* $f_{*n}: H_n(\mathcal{C}) \to H_n(\mathcal{C})$ *be the induced homology endomorphism. Then*

$$\sum_n (-1)^n \operatorname{Tr} f_n = \sum_n (-1)^n \operatorname{Tr} f_{*n}.$$

EXERCISE 25. Prove the algebraic lemma.

For example, if X is a finite CW complex, then the Lefschetz number of a continuous map $f: X \to X$ can be calculated as the alternated sum of traces of homomorphisms $g_\#: \mathcal{C}_n(X) \to \mathcal{C}_n(X)$ induced by a cellular approximation g of f. This observation alone yields the first, and maybe the most important, application of Lefschetz numbers (not related to manifolds, the more so to Poincaré duality).

Theorem 1. *Let* X *be a finitely triangulated space, and let* $f: X \to X$ *be a continuous map. If* f *has no fixed points, then* $\mathcal{L}(f) = 0$.

Proof. We assume that X is furnished with a metric in which every simplex is isometric to the standard simplex. Then there is a positive δ such that $\operatorname{dist}(x, f(x)) > \delta$ for every $x \in X$. By applying to X the barycentric subdivision sufficiently many times, we can make the diameters of the simplices much less than δ. After this, a simplicial approximation g of f will be such that $g(s) \cap s = \emptyset$ for every simplex s of X. In this case, the simplicial chain $g_\#(s)$ will not involve s, so all the diagonal entries of the matrix of $g_{\#n}$ will be zero. Hence, all the traces are zero, and the Lefschetz number is 0.

Let us return to manifolds (but, for now, not to Poincaré duality).

Theorem 2. *Let* X *be a compact smooth manifold (not necessarily orientable, and maybe with a nonempty boundary), and let* ξ *be a vector field on* X*. Suppose that* ξ *has no zeroes and that on the boundary* ∂X *it is directed inside* X*. Then* $\chi(X) = 0$.

This result implied the immensely popular "hairy ball theorem": There is no nowhere vanishing vector field on S^2 (one cannot comb a hairy ball).

Proof of Theorem 2. A vector field ξ on X (with or without zeroes) determines a "flow" $f_t: X \to X$, and for a sufficiently small positive ε the fixed points of f_ε are zeroes of ξ. Since f_ε is homotopic to the identity, $\mathcal{L}(f_\varepsilon) = \mathcal{L}(\mathrm{id}) = \chi(X)$, and if ξ has no zeroes, then $\chi(X) = 0$.

(We will see in Lecture 18 that the converse is also true: If a closed manifold, orientable or not, has zero Euler characteristic, then it possesses a nowhere vanishing vector field.)

So far, regarding Lefschetz numbers, we were interested only in their being zero or not zero. But in reality, in the case of manifolds, the Lefschetz number gives some count of fixed points. This can be expressed by the following proposition.

Theorem 3. *Let* X *be a triangulated compact orientable n-dimensional homology manifold* (we will discuss later how much the orientability is really needed) *and let*

$f: X \to X$ be a continuous map. Let $F: X \to X \times X$, $F(x) = (x, f(x))$ be the graph of f, and let $\Delta: X \to X \times X$ be the diagonal map, $\Delta(x) = (x, x)$. Then

$$\phi(F_*[X], \Delta_*[X]) = \mathcal{L}(f).$$

Before proving this theorem, let us briefly discuss its meaning. The intersection points of $F(X)$ and $\Delta(X)$ correspond precisely to fixed points of f. In the smooth case, the intersection number is described in Theorem 1 of Sect. 17.5. First, we need to assume that all the intersections of the graph and the diagonal are transverse. This condition may be formulated in the language of calculus. If x_0 is a fixed point of a smooth map $f: X \to X$, then there arises the differential, $d_{x_0}f: T_{x_0}X \to T_{x_0}X$. The graph and the diagonal are transverse at x_0 if the matrix of $d_{x_0}f - \mathrm{id}$ is nondegenerate, that is, if $f_{x_0}f$ has no eigenvalues equal to 1. If this condition holds, then every intersection point acquires some sign, and the intersection number, equal to the Lefschetz number by Theorem 3, is the "algebraic number of fixed points." The sign can be described as the parity of the number of real eigenvalues of $d_{x_0}f$ less than 1.

A very similar thing can be said about the vector fields. A nondegenerate zero of a vector field can be assign a sign, and then the algebraic number of zeroes of a vector field must be equal to the Euler characteristic of the manifold.

Now, let us turn to proving Theorem 3. We will need a couple of lemmas.

Lemma 1. $\phi(f_*\alpha_1, \alpha_2) = (-1)^{\dim \alpha_1} \phi(F_*[X], \alpha_1 \times \alpha_2)$.

(On the left-hand side the intersection number is taken in X, while on the right-hand side it is taken in $X \times X$.)

Proof of Lemma 1. Let $\alpha_1 = D\gamma_1, \alpha_2 = D\gamma_2$. Then

$$\begin{aligned}
\phi(F_*[X], \alpha_1 \times \alpha_2) &= \phi((\mathrm{id} \times f)_* \circ \Delta_*[X], \alpha_1 \times \alpha_2) \\
&= \langle D^{-1}(\alpha_1 \times \alpha_2), (\mathrm{id} \times f)_* \circ \Delta_*[X] \rangle \\
&= \pm \langle \gamma_1 \times \gamma_2, (\mathrm{id} \times f)_* \circ \Delta_*[X] \rangle = \pm \langle \Delta^*(\gamma_1 \times f^*\gamma_2), [X] \rangle \\
&= \pm \langle \gamma_1 \smile f^*\gamma_2, [X] \rangle = \pm [X] \frown (\gamma_1 \smile f^*\gamma_2) \\
&= \pm ([X] \frown \gamma_1) \frown f^*\gamma_2 = \pm \alpha_1 \frown f^*\gamma_2 = \pm \langle f^*\gamma_2, \alpha_1 \rangle \\
&\qquad\qquad = \pm \langle \gamma_2, f_*\alpha_1 \rangle = \pm \phi(f_*\alpha_1, \alpha_2)
\end{aligned}$$

(the signs are determined in Theorem 3 of Sect. 17.5).

Lemma 2. *Let $\alpha_1, \dots, \alpha_N$ be a basis in the free part of the full homology group of a compact oriented homology manifold X [first, the basis in $H_0(X)$, then $H_1(X)$, and so on], and let $\alpha_1^*, \dots, \alpha_N^*$ be the dual basis [that is, $\phi(\alpha_i^*, \alpha_j) = \delta_{ij}$]. Then, up to a summand of finite order, $\Delta_*[X] = \sum_i (\alpha_i^* \times \alpha_i)$.*

Proof. By part (2) of Theorem 2 in Sect. 17.5, it is sufficient to prove that

$$\phi(\Delta_*[X], \alpha_p \times \alpha_q) = \phi\left(\sum\nolimits_i (\alpha_i^* \times \alpha_i), \alpha_p \times \alpha_q\right)$$

for every p, q. But

$$\phi(\Delta_*[X], \alpha_p \times \alpha_q) = (-1)^{\dim \alpha_p} \phi(\alpha_p, \alpha_q)$$

by the lemma, and

$$\begin{aligned}\phi\left(\textstyle\sum_i(\alpha_i^* \times \alpha_i), \alpha_p \times \alpha_q\right) &= \textstyle\sum_i \phi((\alpha_i^* \times \alpha_i), (\alpha_p \times \alpha_q)) \\ &= \textstyle\sum_i (-1)^{\dim \alpha_i \dim \alpha_p} \phi((\alpha_i^*, \alpha_p)\phi(\alpha_i, \alpha_q)) \\ &= (-1)^{(\dim \alpha_p)^2} \phi(\alpha_p, \alpha_q)\end{aligned}$$

by Exercise 7 in Sect. 16.6. This proves Lemma 2.

Proof of Theorem 3. Since the intersection numbers are not sensitive to terms of finite order, we can replace in Theorem 3 $\Delta_*[X]$ by $\sum_i \alpha_i^* \times \alpha_i$ and $F_*[X] = (\mathrm{id} \times f)* \circ \Delta_*[X]$ by $\sum_j \alpha_j^* \times f_*\alpha_j$. Also, since the diagonal Δ is invariant with respect to the coordinate swapping map $X \times X \to X \times X$, we have $\sum_i \alpha_i^* \times \alpha_i = \sum_i (-1)^{d_i(n-d_i)} \alpha_i \times \alpha_i^*$ where $d_i = \dim \alpha_i$. Put $f_*\alpha_j = \sum_k a_{jk}\alpha_k$ and perform the calculations:

$$\begin{aligned}\phi(F_*[X],\ \Delta_*[X]) &= \phi\left(\textstyle\sum_{j,k} \alpha_j^* \times a_{jk}\alpha_k, \displaystyle\sum_i (-1)^{d_i(n-d_i)} \alpha_i \times \alpha_i^*\right) \\ &= \sum_{i,j,k} (-1)^{d_i(n-d_i)} (-1)^{d_i d_k} a_{jk} \phi(a_j^*, a_i)(-1)^{(n-d_i)d_k} \phi(\alpha_i^*, \alpha_k) \\ &= \sum_{i,j,k} (-1)^{d_i(n-d_i)+d_i d_k+(n-d_i)d_k} a_{jk}\delta_{ji}\delta_{ik} = \sum_i (-1)^{d_i^2} a_{ii} = \mathcal{L}(f).\end{aligned}$$

Let us now briefly discuss the applicability of the Lefschetz theory to the nonorientable and boundary cases. We begin with vector fields. For a nonoriented (even nonorientable) closed manifold the equality between the algebraic number of zeroes of a vector field and the Euler characteristic obviously holds modulo 2. But in reality, mod 2 reduction is not needed. First, the definition of signs attributed to zeroes of vector fields does not require orientation. Second, a connected nonorientable manifold X has an orientable twofold covering, $\widehat{X}$, and a vector field ξ on X can be lifted to a vector field $\widehat{\xi}$ on $\widehat{X}$. It is clear also that $\chi(\widehat{X}) = 2\chi(X)$ (follows from Corollary in Sect. 13.7) and the (algebraic) number of zeroes of $\widehat{\xi}$ is twice the same number for ξ. This implies the statement.

EXERCISE 26. Let X be a connected closed nonorientable manifold, and let $f: X \to X$ be a smooth map which takes orientation preserving loops into orientation preserving loops and orientation reversing loops into orientation reversing loops. Prove that if all fixed points of f are nondegenerate, then the algebraic number of these points is $\mathcal{L}(f)$.

Another extension of the Lefschetz theory may be obtained by admitting, for a manifold considered, a nonempty boundary. Namely, if X is a compact manifold

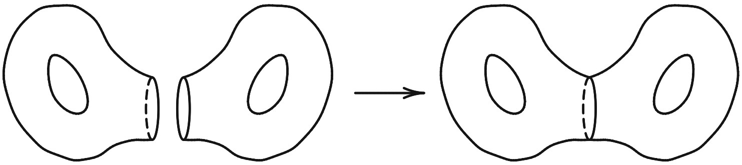


Fig. 70 Doubling a manifold with boundary

with the boundary ∂X, then we can *double* X by attaching to it a second copy of X to the common boundary of the two copies (see Fig. 70).

Let $f: X \to X$ be a continuous map without fixed points on ∂X, and let XX be the double of X. We can extend f to a map $ff: XX \to X \subset XX$ defining this map on the second half to be the same as on the first half [thus $ff(XX)$ is contained in the first half of XX]. It is obvious that ff has the same fixed points as f and $\mathcal{L}(f) = \mathcal{L}(ff)$; hence, the statement of the relation of Lefschetz numbers with fixed points holds for compact manifolds with boundary (orientable or not). Also, we can state that the algebraic number of zeroes of a vector field ξ on a manifold X with boundary such that ξ has no zeroes and directed inside X on ∂X is equal to $\chi(X)$.

EXERCISE 27. There exists a different approach to the Lefschetz theory. First we prove Theorem 1: The Lefschetz number of a fixed-point–free map is zero. Then we consider a map $f: X \to X$ with a nondegenerate fixed point, and, at a neighborhood of this point, we modify both X and f in such a way that the fixed point disappears and the Lefschetz number is changed in a controllable way. Try to recover the details.

In conclusion, let us give one of countless applications of the Lefschetz theory.

EXERCISE 28. The n-dimensional torus T^n can be regarded as $\mathbb{R}^n/\mathbb{Z}^n$. Hence, a linear map $\mathbb{R}^n \to \mathbb{R}^n$ determined by an integral matrix A can be factorized to some continuous map $T^n \to T^n$; denote it as f_A. (Certainly, every continuous map $T^n \to T^n$ is homotopic to a unique map of the form f_A; you may try to prove this.) Calculate the Lefschetz number for f_A (the best possible answer expresses this Lefschetz number in terms of the eigenvalues of A).

EXERCISE 29. Denote the Lefschetz number from Exercise 28 as $\mathcal{L}_A$. Prove that a map homotopic to f_A has at least $|\mathcal{L}_A|$ different fixed points.

EXERCISE 30. Prove that a map $f: T^n \to T^n$ homotopic to f_A with $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ has infinitely many periodic points. [A point $y \in Y$ is called a periodic point of a map $g: Y \to Y$ if $g^n(y) = y$ for some n.]

[The last two statements are taken from the note by Ginzburg [43] (Russian).]

17.7 Secondary Intersection Numbers and Secondary Poincaré Duality

Let us return to Poincaré duality. The duality between

$$\operatorname{Tors} H_m(X) \text{ and } \operatorname{Tors} H_{n-m-1}(X)$$

is based on *secondary intersection numbers*, which are defined ahead. (We need to warn the reader that the main results of this section will be given in the form of exercises.)

Let X be a compact oriented n-dimensional homology manifold, and let $\alpha \in H_m(X)$ and $\beta \in H_{n-m-1}(X)$ be homology classes of finite order. Let a and b be cycles representing α and β in the barycentric star and classical complexes of X, and assume that $Na = \partial c$. We define $\omega(\alpha, \beta)$ to be the rational number $\frac{1}{N}\phi(c, b)$ reduced modulo 1 [thus $\omega(\alpha, \beta) \in \mathbb{Q}/\mathbb{Z}$].

EXERCISE 31. Check that $\omega(\alpha, \beta)$ is well defined. (It is this statement that requires the assumption that β has a finite order.)

EXERCISE 32. Prove that if $N\alpha = 0$ and $M\beta = 0$, then $K\omega(\alpha, \beta) = 0$, where $K = \gcd(M, N)$.

EXERCISE 33. Prove that $\omega(\beta, \alpha) = \pm\omega(\alpha, \beta)$ (what is the sign?).

The main property of secondary intersection numbers is the following *secondary Poincaré duality*.

Theorem. *The correspondence* $\alpha \mapsto \{\beta \mapsto \omega(\alpha, \beta)\}$ *yields an isomorphism*

$$\operatorname{Tors} H_m(X) \xrightarrow{\cong} \operatorname{Hom}(\operatorname{Tors} H_{n-m-1}(X), \mathbb{Q}/\mathbb{Z}).$$

EXERCISE 34. Prove this theorem.

17.8 Inverse Homomorphisms

Let X and Y be compact oriented homology manifolds of, possibly, different dimensions m and n, and let $f: X \to Y$ be a continuous map. Poincaré isomorphism allows us to construct "wrong direction" homology and cohomology homomorphisms

$$f^!: H_q(Y; G) \xrightarrow{D^{-1}} H^{n-q}(Y; G) \xrightarrow{f^*} H^{n-q}(X; G) \xrightarrow{D} H_{m-n+q}(X; G),$$
$$f_!: H^q(X; G) \xrightarrow{D} H_{m-q}(X; G) \xrightarrow{f_*} H_{m-q}(Y; G) \xrightarrow{D^{-1}} H^{n-m+q}(Y; G).$$

Both homomorphisms change dimensions by $m - n$: The homomorphism $f^!$ "increases" the dimension by $m-n$ (we use quotation marks because $m-n$ may be negative or zero), and the homomorphism $f_!$ "decreases" the dimension by $m-n$. We will not say much about the cohomology homomorphism $f_!$. It can be regarded as the simplest case of a general construction called "direct image." Its analytic sense (and it belongs rather to analysis than to topology), at least in the case when f is the projection of a smooth fibration, can be best described by the words "fiberwise integration" (people familiar with the de Rham theory can easily understand them). As to the homology homomorphism $f^!$ (called the *inverse Hopf homomorphism*), it has a transparent geometric sense which is described, in the smooth case, by the following proposition.

Theorem. *Let a homology class $\alpha \in H_q(Y)$ be represented by a q-dimensional submanifold Z of Y* (that is, $\alpha = i_*[Z]$, where $i\colon Z \to Y$ is the inclusion map), *and let f be transversely regular with respect to Z* (that is, the composition

$$T_yY \xrightarrow{d_yf} T_{f(y)}X \xrightarrow{\text{proj}} T_{f(y)}X/T_{f(y)}Z$$

is onto for every point $y \in f^{-1}(Z)$). *Then $f^{-1}(Z)$ is a $(q+m-n)$-dimensional submanifold of X which represents the homology class $f^!(\alpha) \in H_{q+m-n}(X)$.*

We will not prove this theorem but will restate it in a form in which it can be easily translated into an easy-to-prove statement concerning homology manifolds. Let W be an oriented $(q+m-n)$-dimensional submanifold of X transverse to $f^{-1}(Z)$ which may have pseudomanifold-like singularities not in a neighborhood of $f^{-1}(Z)$. Then, at least in a neighborhood of Z, $f(W)$ is an $(n-q)$-dimensional manifold of Y, and f establishes a (sign-preserving) bijection between $W \cap f^{-1}Z$ and $f(W) \cap Z$. Now let us turn to the homology manifold case.

Proposition 1. *Let X, Y, and f be as above, and let $\alpha \in H_q(Y)$, $\beta \in H_{m-q}(X)$. Then*

$$\phi_X(f^!\alpha, \beta) = \phi_Y(\alpha, f_*\beta)$$

(ϕ_X and ϕ_Y denote the intersection number in X and Y).

Proof.

$$\begin{aligned}\phi_X(f^!\alpha, \beta) = \phi_X(Df^*D^{-1}\alpha, \beta) &= \langle f^*D^{-1}\alpha, \beta\rangle \\ &= \langle D^{-1}\alpha, f_*\beta\rangle = \phi_Y(\alpha, f_*\beta).\end{aligned}$$

By part (2) of Theorem 2 in Sect. 17.5, this relation determines $f^!\alpha$ up to a summand of finite order.

Here is one more illustration of the fact that geometrically $f^!$ may be regarded as a preimage.

Proposition 2. *Let X, Y be compact oriented homological manifolds, and let $p: X \times Y \to Y$ be the projection. Then, for any $\alpha \in H_m(Y)$,*

$$p^! \alpha = [X] \times \alpha.$$

Proof. Let $\alpha = D\gamma, \gamma \in H^{n-m}(Y; \mathbb{Z})$. Then

$$p^! \alpha = D_{X \times Y} p^* \gamma = D_{X \times Y}(1 \times \gamma) = D_X 1 \times D_Y \gamma = [X] \times \alpha.$$

EXERCISE 34. Prove the formula $\langle \alpha, f^! \beta \rangle = \langle f_! \alpha, \beta \rangle$.

Let us now turn to the case when $\dim X = \dim Y$.

Proposition 3. *Let X, Y be connected compact oriented manifolds of the same dimension n, and let $f: X \to Y$ be a continuous map of degree d. Then the compositions*

$$H_m(Y) \xrightarrow{f^!} H_m(X) \xrightarrow{f_*} H_m(Y),$$
$$H^m(Y; \mathbb{Z}) \xrightarrow{f^*} H^m(X; \mathbb{Z}) \xrightarrow{f_!} H^m(Y; \mathbb{Z})$$

are both multiplication by d.

Here is a proof of the first statement. Let $\alpha \in H_m(Y)$, $\alpha = D_Y \gamma, \gamma \in H^{n-m}(Y; \mathbb{Z})$. Then $f_* f^! \alpha = f_* D_X f^* \gamma = f_*([X] \frown f^* \gamma) = f_*[X] \frown \gamma = d[Y] \frown \gamma = d D_Y \gamma = d\alpha$ (we used Exercise 12 of Sect. 16.6).

EXERCISE 35. Prove the second statement of Proposition 3.

Corollary. *If $d = \pm 1$, then f_* is an epimorphism, and f^* is a monomorphism.*

GENERALIZATION. *If $d \neq 0$, then every homology class of Y multiplied by d belongs to the image of f_*, and every cohomology class of Y belonging to* $\operatorname{Ker} f^*$ *is annihilated by the multiplication by d.*

For example, there is no map $S^2 \to S^1 \times S^1$ of a nonzero degree, but there is a map $S^1 \times S^1 \to S^2$ of degree 1: factorization over $S^1 \vee S^1$.

Everything said in this section has an obvious nonorientable $\mathbb{Z}_2$-analog.

17.9 Poincaré Duality and the Cup-Product

Again, we begin with a statement for the smooth case.

Theorem 1. *Let Y_1, Y_2 be closed oriented submanifolds of a smooth closed oriented manifold X transverse to each other; the latter means that the inclusion map i_1 of Y_1 in X is transversely regular to Y_2. Then the intersection $Z = Y_1 \cap Y_2 = i_1^{-1}(Y_2)$ is a submanifold of X whose dimension k is related to the dimensions n, m_1, m_2 of X, Y_1, Y_2 by the formula $k = m_1 + m_2 - n$. Let $\alpha_1 \in H^{n-m_1}(X; \mathbb{Z}), \alpha_2 \in H^{n-m_2}(X; \mathbb{Z})$,* *and* $\beta \in H^{2n-m_1-m_2}(X;\mathbb{Z})$ *be cohomology classes such that homology classes* $D\alpha_1, D\alpha_2$*, and* $D\beta$ *are represented by* Y_1, Y_2*, and* Z*. Then*

$$\alpha_1 \smile \alpha_2 = \beta.$$

There is a similar $\mathbb{Z}_2$-statement for the nonorientable case.

Proof of Theorem 1.

$$\begin{aligned} D(\alpha_1 \smile \alpha_2) &= [X] \frown (\alpha_1 \smile \alpha_2) = ([X] \frown \alpha_1) \frown \alpha_2 = (D\alpha_1) \frown \alpha_2 \\ &= i_{1*}[Y_1] \frown \alpha_2 = i_{1*}([Y_1] \frown i_1^*\alpha_2) = i_{1*}(Di_1^*\alpha_2) \\ &= i_{1*}(Di_1^*D^{-1}(i_{2*}[Y_2]) = i_{1*}(i_1^!(i_{2*}[Y_2])) \\ &= i_{1*}[i_1^{-1}(Y_2)] = i_*[Z] = D\beta. \end{aligned}$$

(Here i_2 and i are inclusion maps of Y_2 and Z in X; we used in this the proof of Theorem 1 from Sect. 17.9, which was not proven there; if we use instead Theorem 2, then the equality $\alpha_1 \smile \alpha_2 = \beta$ will be proven in a broader context of homology manifolds, but only modulo summand of a finite order.)

This theorem provides a very powerful tool for determining multiplicative structure in cohomology, mainly for manifolds, but actually for all spaces, because of the naturality of the multiplicative structure.

Example. If $q + r \le n$, then the product of canonical generators of the groups $H^{2q}(\mathbb{C}P^n;\mathbb{Z})$ and $H^{2r}(\mathbb{C}P^n;\mathbb{Z})$ is the canonical generator of $H^{2(q+r)}(\mathbb{C}P^n;\mathbb{Z})$; indeed, Poincaré isomorphism takes the three generators into the homology classes of projective subspaces of dimensions $n-q, n-r$, and $n-q-r$, and, in general position, the intersection of the first two is the third. Thus, the ring $H^*(\mathbb{C}P^n;\mathbb{Z}) = \bigoplus_i H^i(\mathbb{C}P^n;\mathbb{Z})$ has the following structure: There is $1 \in H^0(\mathbb{C}P^n;\mathbb{Z})$ and the generator $x \in H^2(\mathbb{C}P^n;\mathbb{Z})$; the group $H^{2q}(\mathbb{C}P^n;\mathbb{Z})$ with $1 \le q \le n$ is generated by x^q. If n is finite, then $x^{n+1} = 0$. In more algebraic terms, $H^*(\mathbb{C}P^n;\mathbb{Z})$ is the ring of polynomials of one variable x factorized by the ideal generated by x^{n+1},

$$H^*(\mathbb{C}P^n;\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}),\ \dim x = 2;$$

similarly,

$$\begin{aligned} H^*(\mathbb{H}P^n;\mathbb{Z}) &= \mathbb{Z}[x]/(x^{n+1}),\ \dim x = 4; \\ H^*(\mathbb{R}P^n;\mathbb{Z}_2) &= \mathbb{Z}_2[x]/(x^{n+1}),\ \dim x = 1; \\ H^*(\mathbb{C}\mathrm{a}P^2;\mathbb{Z}) &= \mathbb{Z}[x]/(x^3),\ \dim x = 8. \end{aligned}$$

In all cases, excluding $\mathbb{R}P^n$, the ring $\mathbb{Z}$ may be replaced by any commutative ring.

EXERCISE 36. Prove that the integral cohomology ring of the sphere S_g^2 with g handles is as follows: there are generators $a_1, \dots, a_g, b_1, \dots, b_g$ of $H^1(S_g^2;\mathbb{Z})$ such that $a_1b_1 = a_2b_2 = \dots = a_gb_g$ is the generator of $H^2(S_g^2;\mathbb{Z})$ and all other products of generators of $H^1(S_g^2;\mathbb{Z})$ are zeroes. Describe the multiplicative structure in $\mathbb{Z}_2$-cohomology of the projective plane with handles and the Klein bottle with handles.

EXERCISE 37. Prove that any continuous map $\mathbb{C}P^n \to \mathbb{C}P^m$ with $n > m$ induces a trivial map in cohomology of any positive dimension (with any coefficients). Prove a similar statement for real projective spaces.

EXERCISE 38. Prove that if $g < h$, then there are no continuous maps $S_g^2 \to S_h^2$ of a nonzero degree.

Theorem 1 shows that the multiplicative structure in cohomology of a closed orientable manifold is rich (many nonzero products). Actually, we already have a strong statement of this kind: Theorems 2 and 3 of Sect. 17.5 show that if X is a compact oriented n-dimensional homology manifold, then for every infinite order class $\alpha \in H^m(X;\mathbb{Z})$ there exists a $\beta \in H^{n-m}(X;\mathbb{Z})$ such that $\langle \alpha \smile \beta, [X]\rangle = 1$. If $\dim X = 2k$ and $\alpha_1, \alpha_2, \dots$ is a basis in the free part of $H^k(X;\mathbb{Z})$, then the matrix $\|\langle \alpha_i \smile \alpha_j, [X]\rangle\|$ is unimodular (that is, its determinant is ± 1).

The remaining part of this lecture is devoted to several modifications (generalizations) of Poincaré duality.

17.10 The Noncompact, Relative, and Boundary Cases of Poincaré Isomorphism

Suppose that a connected triangulated space X is an oriented n-dimensional homology manifold which, however, is not assumed to be compact; that is, the triangulation may be not finite. In this case we still have a correspondence between (oriented) simplices and barycentric stars of complementary dimensions, but no isomorphism between chains and cochains, since chains are supposed to be finite linear combinations of simplices (or barycentric stars), and cochains are allowed to take nonzero values on infinitely many simplices. To construct Poincaré isomorphism, we need to modify the definition either of chains or of cochains. Both modifications are well known in topology; moreover, they exist on the singular level. Here, we restrict ourselves to a brief description of these modifications.

Let X be a locally compact topological space. An n-dimensional *open singular chain* of X is a possibly infinite, linear combination of n-dimensional singular simplices of X with integer coefficients, $\sum_i k_i f_i$, $f_i\colon \Delta^n \to X$, such that for any compact subset $K \subset X$ the coefficients k_i may be nonzero only for finitely many singular simplices f_i such that $f_i(\Delta^n) \cap K \neq \emptyset$. Open chains form a group $C_n^{\text{open}}(X)$, and the usual definition of the boundary operator gives homomorphisms $\partial\colon C_n^{\text{open}}(X) \to C_{n-1}^{\text{open}}(X)$ with $\partial\partial = 0$ and, finally, *open homology groups* $H_n^{\text{open}}(X)$. *Proper* (preimages of compact sets are compact) continuous maps $f\colon X \to Y$ induce chain and homology homomorphisms $f_\#\colon C_n^{\text{open}}(X) \to C_n^{\text{open}}(Y)$ and $f_*\colon H_n^{\text{open}}(X) \to H_n^{\text{open}}(Y)$ with all usual properties (including *proper* homotopy invariance for open homology). In particular, if X is a locally finite CW complex, then $H_n^{\text{open}}(X)$ can be calculated by means of cellular chains which are not assumed to be finite.

There is also a similar (dual) definition of *compact* or *compactly supported* cohomology of a locally compact topological space X. Namely, a cochain $c \in C^n(X;G)$ is called compactly supported if there exists a compact set $K \subset X$ such that $c(f) = 0$ for any singular simplex $f\colon \Delta^n \to X$ such that $f(\Delta^n) \cap K = \emptyset$. There arise groups of compactly supported cochains, $C^n_{\text{comp}}(X;G)$, coboundary operators, $\delta\colon C^n_{\text{comp}}(X;G) \to C^{n+1}_{\text{comp}}(X;G)$, and compact(ly supported) cohomology $H^n_{\text{comp}}(X;G)$. For compactly supported cochains and cohomologies, homomorphisms $f^\#$ and f^* are induced by proper continuous maps. For locally finite CW complexes, compact cohomology can be calculated by means of complexes of finite cochains. Remark also that the usual definition of multiplications gives (in the presence of a pairing $G_1 \times G_2 \to G$) the following binary operations:

$$\big[\gamma_1 \in H^{q_1}_{\text{comp}}(X;G_1), \gamma_2 \in H^{q_2}(X;G_2)\big] \mapsto \gamma_1 \smile \gamma_2 \in H^{q_1+q_2}_{\text{comp}}(X;G);$$
$$\big[\alpha \in H^{\text{open}}_{q_1}(X;G_1), \gamma \in H^{q_2}(X;G_2)\big] \mapsto \alpha \frown \gamma \in H^{\text{open}}_{q_1-q_2}(X;G);$$
$$\big[\alpha \in H^{\text{open}}_{q_1}(X;G_1), \gamma \in H^{q_2}_{\text{comp}}(X;G_2)\big] \mapsto \alpha \frown \gamma \in H_{q_1-q_2}(X;G).$$

All these operations are defined in the usual way on the chain/cochain level.

Consider again a connected triangulated oriented n-dimensional homology manifold X. The barycentric star construction of Sect. 17.4 provides Poincaré isomorphisms

$$D\colon H^m(X;G) \to H^{\text{open}}_{n-m}(X;G) \text{ and } D\colon H^m_{\text{comp}}(X;G) \to H_{n-m}(X;G);$$

both can be expressed by the formula $D\gamma = [X] \frown \gamma$, where the fundamental class $[X]$ is an element of $H_n^{\text{open}}(X)$. These isomorphisms may not look appealing because they involve exotic homology and cohomology groups. However, in many important cases this may be avoided. This possibility is provided by the following general proposition.

Proposition 1. *Let X be a compact topological space and let $A \subset X$ be a closed subset. Then there are natural* (make the statement precise: in what sense natural?) *isomorphisms*

$$H_n^{\text{open}}(X-A;G) \cong H_n(X,A;G) \text{ and } H^n_{\text{comp}}(X-A;G) \cong H^n(X,A;G).$$

In particular, if X is locally compact and $X^\bullet$ is the one-point compactification of X, then

$$H_n^{\text{open}}(X;G) \cong \widetilde{H}_n(X^\bullet;G) \text{ and } H^n_{\text{comp}}(X;G) \cong \widetilde{H}^n(X^\bullet;G).$$

Proposition 1 shows that the preceding Poincaré isomorphisms, in the case when the given homology manifold is a complement to a CW subcomplex A of a compact CW complex X, take the form

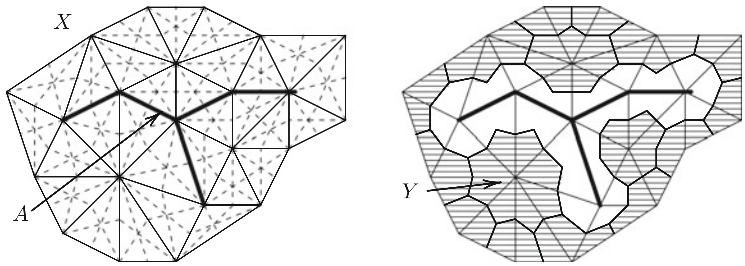


Fig. 71 A barycentric star complex approximation of $X - A$

$$D\colon H^m(X - A; G) \to H_{n-m}(X, A; G)$$
$$\text{and } H^m(X, A; G) \to H_{n-m}(X - A; G).$$

(Moreover, both isomorphisms can be described as cap-products with the "fundamental class" $[X, A] \in H_n(X, A)$.) We do not prove this proposition, and we do not even offer it as an exercise. Instead, we will give a direct construction of the last isomorphisms, at least in the triangulated case.

Let X be a compact triangulated space, and let A be a triangulated subspace of X such that $X - A$ is a homology manifold. We assume that A satisfies the "regularity condition": If all vertices of some simplex s of X belong to A, then s is contained in A. Let Y be the union of barycentric stars of simplices of X not contained in A (see Fig. 71). Then Y is a closed subset of X, even a triangulated subspace of the barycentric subdivision of X; moreover, Y is homotopy equivalent to $X - A$ (we do not give a formal proof of this homotopy equivalence, but we hope that Fig. 71 may serve as a convincing confirmation of that). The correspondence between simplices and their barycentric stars provides isomorphisms between free Abelian groups generated by simplices in X not contained in A and barycentric stars in Y. These isomorphisms may be considered as either $\mathcal{C}^m_{\text{bast}}(Y; \mathbb{Z}) \cong \mathcal{C}^{\text{class}}_{n-m}(X, A)$ or $\mathcal{C}^m_{\text{class}}(X, A; \mathbb{Z}) \cong \mathcal{C}^{\text{bast}}_{n-m}(Y)$; in both cases, the commutativity with ∂ and δ [similar to $(*)$ in Sect. 17.4] holds, so there arise homology/cohomology isomorphisms

$$D\colon H^m(X - A; \mathbb{Z}) \to H_{n-m}(X, A) \text{ and } D\colon H^m(X, A; \mathbb{Z}) \to H_{n-m}(X - A)$$

as stated above (it is easy to extend them to an arbitrary coefficient group G).

EXERCISE 39. Prove that both isomorphisms can be expressed as $[X, A] \frown$. (For one of them, we will have to reverse the ordering of vertices in the barycentric subdivision.)

EXERCISE 40. For homology classes $\alpha \in H_m(X - A)$, $\beta \in H_{n-m}(X, A)$, define the intersection number $\phi(\alpha, \beta)$ which has the usual geometric sense. (This must be a replica of Sect. 17.5.) Prove the relative Poincaré duality: The homomorphism

$$\operatorname{Free} H_m(X-A) \to \operatorname{Hom}(\operatorname{Free} H_{n-m}(X,A), \mathbb{Z}),\ \alpha \mapsto \{\beta \mapsto \phi(\alpha,\beta)\}$$

is an isomorphism. Do similar work with the torsion subgroup and the secondary intersection numbers.

There are two especially important cases of the relative Poincaré duality: the case when X is a sphere and the case when X is a manifold with boundary and $A = \partial X$. We postpone the first case to the next section and will consider the second case now.

Although there exists a theory of homology manifolds with boundary (see, for example, Mitchel [62]), we will not discuss it here; instead of this, we will restrict ourselves to the smooth case. Let X be a connected oriented compact $(n+1)$-dimensional smooth manifold with a boundary ∂X; we suppose that X possesses a smooth triangulation such that simplices contained in ∂X form a smooth triangulation of ∂X. Since, obviously, $X - \partial X$ is a homology manifold, the previous construction yields (for an arbitrary coefficient group G) Poincaré isomorphisms

$$D\colon H^m(X;G) \to H_{n+1-m}(X,\partial X;G),$$
$$D\colon H^m(X,\partial X;G) \to H_{n+1-m}(X;G)$$

(we use the obvious fact that $X - \partial X$ is homotopy equivalent to X). Both isomorphisms have the form $\gamma \mapsto [X,\partial X] \frown \gamma$, where $[X,\partial X] \in H_{n+1}(X,\partial X)$ is the fundamental class of X [represented in the classical complex by the sum of all $(n+1)$-dimensional simplices of X oriented in accordance to the orientation of X].

Proposition 2. *Poincaré isomorphisms described above, together with Poincaré's isomorphisms for the manifold ∂X, form an isomorphism between homology and cohomology sequences of the pair $(X,\partial X)$; more precisely, there arises a plus–minus commutative diagram*

$$\begin{tikzcd}
\cdots\ H_m(\partial X;G) \arrow[r] & H_m(X;G) \arrow[r] & H_m(X,\partial X;G) \arrow[r] & H_{m-1}(\partial X;G)\ \cdots \\
\cdots\ H^{n-m}(\partial X;G) \arrow[r] \arrow[u, "D"] & H^{n+1-m}(X,\partial X;G) \arrow[r] \arrow[u, "D"] & H^{n+1-m}(X;G) \arrow[r] \arrow[u, "D"] & H^{n+1-m}(\partial X;G)\ \cdots \arrow[u, "D"]
\end{tikzcd}$$

Proof. We will prove the plus–minus commutativity of the first square; for the third square the proof is more or less the same, while the commutativity of the second square is obvious.

Take a $c \in C^{n-m}(\partial X;G)$ and extend it to $\widetilde{c} \in C^{n-m}(X;G)$. Here we use the notations $[X,\partial X]$ and $[\partial X]$ for chains; thus, $[X,\partial X] \in C_{n+1}(X)$ and $\partial[X,\partial X] = [\partial X] \in C_n(\partial X) \subset C_n(X)$. As we know from Sect. 16.6 (Exercise 8),

$$\partial([X,\partial X] \frown \widetilde{c}) = \pm(\partial[X,\partial X] \frown \widetilde{c}) \pm ([X,\partial X] \frown \delta\widetilde{c}). \qquad (*)$$

Since $\partial[X,\partial X] = [\partial X] \in C_n(\partial X) \subset C_n(X)$, the cap-product $\partial[X,\partial X] \frown \widetilde{c} \in C_m(X;G)$ belongs to $C_m(\partial X;G)$ and, in this capacity, is $[\partial X] \frown \left(\widetilde{c}|_{\partial X}\right) = [\partial X] \frown c$. If c is a cocycle representing a class $\gamma \in H^{n-m}(\partial X;G)$, then $\partial[X,\partial X] \frown \widetilde{c}$ and $[X,\partial X] \frown \delta\widetilde{c}$ are cycles (in $C_m(X;G)$) representing $i_*([\partial X] \frown \gamma) = i_* \circ D\gamma$ and

$[X, \partial X] \frown \delta^*\gamma = D \circ \delta^*\gamma$. Since the sum or difference of these cycles is a boundary [formula ($*$)], this proves the plus–minus commutativity of the first square.

We will reformulate the last proposition by passing from Poincaré isomorphisms to Poincaré duality. To avoid separately considering free parts and torsion, we will assume that the coefficient domain is $\mathbb{Q}$, and, for brevity's sake, we will omit the indication of the coefficient domain. We will replace the bottom line of the diagram in Proposition 2 by the dual (with respect to $\langle\ ,\ \rangle$) homology sequence. We get the following "duality diagram."

$$\begin{array}{ccccccccc}
\cdots & H_m(\partial X) & \xrightarrow{i_*} & H_m(X) & \xrightarrow{j_*} & H_m(X, \partial X) & \xrightarrow{\partial_*} & H_{m-1}(\partial X) & \cdots \\
 & \textit{dual} & & \textit{dual} & & \textit{dual} & & \textit{dual} & \\
\cdots & H_{n-m}(\partial X) & \xleftarrow{\partial_*} & H_{n+1-m}(X, \partial X) & \xleftarrow{j_*} & H_{n+1-m}(X) & \xleftarrow{i_*} & H_{n+1-m}(\partial X) & \cdots
\end{array}$$

The spaces of each vertical are dual to each other with respect to the intersection number, while the arrows of each vertical are plus–minus dual to each other. The last fact (equivalent to Proposition 2) means the following:

$$\begin{aligned}
&\phi(i_*\alpha, \beta) = \pm\phi(\alpha, \partial_*\beta) \text{ for every } \alpha \in H_m(\partial X),\ \beta \in H_{n-m+1}(X, \partial X),\\
&\phi(j_*\alpha, \beta) = \pm\phi(\alpha, j_*\beta) \text{ for every } \alpha \in H_m(X),\ \beta \in H_{n-m+1}(X),\\
&\phi(\partial_*\alpha, \beta) = \pm\phi(\alpha, i_*\beta) \text{ for every } \alpha \in H_m(X, \partial X),\ \beta \in H_{n-m+1}(\partial X).
\end{aligned}$$

These results appear the most interesting when n is even: $n = 2k$. Consider the fragment

$$H_{k+1}(X, \partial X) \xrightarrow{\partial_*} H_k(\partial X) \xrightarrow{i_*} H_k(X)$$

of the homology sequence of the pair $(X, \partial X)$ (with the coefficient in $\mathbb{Q}$). The middle space is self-dual, the left and right groups are dual to each other, as well as the homomorphisms i_* and ∂_* (all the dualities are with respect to the intersection number ϕ). The exactness of the sequence implies the equality $\dim H_k(\partial X) = \operatorname{rank} \partial_* + \operatorname{rank} i_*$, and the duality shows that $\operatorname{rank} \partial_* = \operatorname{rank} i_*$. Together, these equalities show that $B_k(\partial X) = \dim H_k(\partial X) = 2 \operatorname{rank} \partial_*$. In other words, the space $H_k(\partial X)$ is even-dimensional (we already know this in the case when k is odd; see Theorem 3 of Sect. 17.5), and the dimension of $\operatorname{Ker} i_* = \operatorname{Im} \partial_* \subset H_k(\partial X)$ is half of $\dim H_k(\partial X)$. For example, the torus T can be presented as a boundary of an orientable compact three-dimensional manifold in many different ways (for example, the torus is the boundary of the solid torus). But if $T = \partial X$ (where X is a compact orientable three-dimensional manifold), then the inclusion homomorphism $i_*\colon H_1(T) \to H_1(X)$ must have a one-dimensional kernel, not less and not more (if X is a solid torus, then i_* annihilates the homology class of the meridian, but not the homology class of the parallel).

Furthermore, if $\alpha, \beta \in H_{k+1}(X, \partial X)$, then, since ∂_* and i_* are ϕ-dual to each other,

$$\phi(\partial_*\alpha, \partial_*\beta) = \phi(\alpha, i_*\partial_*\beta) = \phi(\alpha, 0) = 0,$$

which shows that the restriction of the form ϕ to this subspace is zero. In the case when k is odd, the form ϕ determines a symplectic structure in $H_k(\partial X)$, and the last statement means that $\operatorname{Ker} i_* = \operatorname{Im} \partial_*$ is a *Lagrangian subspace* of $H_k(\partial X)$. This, however, does not impose any condition on the manifold ∂X. The case when k is even, however, is very much different. A real vector space V with a nondegenerate symmetric bilinear form ω can have a subspace W of dimension one half of $\dim V$ with a zero restriction $\omega|_W$ if and only if the *signature* of ω (the difference between the positive and negative inertia indices) is zero. For a compact oriented 4ℓ-dimensional manifold Y, the signature of the form ϕ in $H_{2\ell}(Y)$ is called *the signature of* Y and is denoted as $\tau(Y)$.

EXERCISE 41. Prove that τ is multiplicative: If Y_1 and Y_2 are two closed oriented manifolds of dimensions divisible by 4, then $\tau(Y_1 \times Y_2) = \tau(Y_1)\tau(Y_2)$.

EXERCISE 42. Prove that if Y_1 and Y_2 are two closed orientable manifolds whose dimensions are not divisible by 4, but sum up to a number divisible by 4, then $\tau(Y_1 \times Y_2) = 0$.

EXERCISE 43. Prove that the reversion of the orientation leads to the negation of the signature.

EXERCISE 44. Let Y_1 and Y_2 be two connected orientable closed manifolds of the dimension 4ℓ, and let $Y = Y_1 \# Y_2$ be the connected sum of Y_1, Y_2 (that is, Y is obtained from Y_1, Y_2 by drilling holes in both of them and then attaching to the boundaries of the holes the tube $S^{4\ell-1} \times I$). Prove that $\tau(Y) = \tau(Y_1) + \tau(Y_2)$.

Theorem. *If a closed oriented 4ℓ-dimensional manifold Y is a boundary of a compact oriented manifold X, then $\tau(Y) = 0$ [in particular, $B_{2\ell}(Y)$ is even].*

Proof. We showed that $B_{2\ell}(\partial X)$ must be even and that $H_{2\ell}(\partial X)$ contains a subspace of dimension $\frac{1}{2}B_{2\ell}(\partial X)$ with zero restriction of ϕ. Hence, $\tau(\partial X) = 0$.

Example. The manifold $\mathbb{C}P^{2\ell}$ cannot be a boundary of a compact orientable $(4\ell + 1)$-dimensional manifold, because $B_{2\ell}(\mathbb{C}P^{2\ell}) = 1$ is odd. But the connected sum $\mathbb{C}P^{2\ell} \# \mathbb{C}P^{2\ell}$ (see Exercise 44), which has even middle Betti number, is also not a boundary since its signature is not zero (it is 2). The same is true for a connected sum of a number of copies of $\mathbb{C}P^{2\ell}$. But the connected sum $\mathbb{C}P^{2\ell} \# (-\mathbb{C}P^{2\ell})$ (where the minus sign stands for the orientation reversion) has zero signature and may be a boundary. Actually, it is a boundary (see Exercise 45 ahead).

EXERCISE 45. Let Y be a connected closed oriented manifold. Prove that the manifold $Y \# (-Y)$ is a boundary of some compact manifold. (*Hint*: Drill a hole in Y and then multiply by I.)

17.11 Alexander Duality

Let $A \subset S^n$ be a simplicial subset of S^n, that is, a union of some simplices of some triangulation of S^n. The goal of this section is to construct *Alexander isomorphisms*,

$$L\colon \widetilde{H}^m(A;G) \xrightarrow{\cong} \widetilde{H}_{n-1-m}(S^n - A;G)$$
$$\text{and } L\colon \widetilde{H}^m(S^n - A;G) \xrightarrow{\cong} \widetilde{H}_{n-1-m}(A;G),$$

and then to reformulate them as a duality between homology groups of A and $S^n - A$. We begin with an obvious remark: If A is empty of is equal to S^n, then the existence of the isomorphisms follows from the definition of groups $\widetilde{H}_{-1}$ and $\widetilde{H}^{-1}$ (which demonstrates one more time that these definitions are right). From now on, we assume that neither A, nor $S^n - A$, is empty. For brevity's sake, we will always omit the indication to the coefficient group (which may be arbitrary).

Remember that, according to Sect. 17.10, the cap-product $[S^n, A]\frown$ yields isomorphisms

$$D\colon H^m(S^n - A) \to H_{n-m}(S^n, A)$$
$$\text{and } D\colon H^{m+1}(S^n, A) \to H_{n-1-m}(S^n - A).$$

Consider the reduced homology sequence of the pair (S^n, A):

$$\ldots \widetilde{H}_{n-m}(S^n) \to H_{n-m}(S^n, A) \to \widetilde{H}_{n-1-m}(A) \to \widetilde{H}_{n-1-m}(S^n) \ldots . \qquad (*)$$

If $m \neq 0, 1$, then the first and last groups in this exact sequence are zeroes, and we obtain an isomorphism $\partial_*\colon H_{n-m}(S^n, A) \to \widetilde{H}_{n-1-m}(A)$ and the composition

$$L = \partial_* \circ D\colon H^m(S^n - A) \xrightarrow{\cong} \widetilde{H}_{n-1-m}(A)$$

as was promised [for these m, $H^m(S^n - A) = \widetilde{H}^m(S^n - A)$]. It remains to settle the cases $m = 0, 1$.

Lemma. *If $A \neq S^n$, then the inclusion homomorphism $H_n(A) \to H_n(S^n)$ is zero.*

Proof. If $x_0 \notin A$, then this homomorphism factorizes as $H_n(A) \to H_n(S^n - x_0) \to H_n(S^n)$, and $H_n(S^n - x_0) = 0$, since $S^n - x_0$ is homeomorphic to $\mathbb{R}^n$.

[Actually, $H_n(A) = 0$, since $H_{n+1}(S^n, A) = 0$; but we do not need this.]

If $m = 1$, then the last homomorphism of the sequence $(*)$ is zero, and ∂_* remains an isomorphism. If $m = 0$, we get the exact sequence

$$\xrightarrow{0} \widetilde{H}_n(S^n)\ (= \mathbb{Z}) \to H_n(S^n, A) \to \widetilde{H}_{n-1}(A) \to 0,$$

which provides an isomorphism $H_n(S^n, A)/\mathbb{Z} \to \widetilde{H}_{n-1}(A)$ which gives, in combination with D, the promised isomorphism

$$L\colon \widetilde{H}^0(S^n - A) = H^0(S^n - A)/\mathbb{Z} \xrightarrow{D} H_n(S^n, A)/\mathbb{Z} \to \widetilde{H}_{n-1}(A)$$

(the reader is granted the right to replace $\mathbb{Z}$ everywhere with G).

The isomorphism $L\colon \widetilde{H}^m(A) \to \widetilde{H}_{n-1-m}(S^n - A; G)$ is obtained from the isomorphism $D\colon H^{m+1}(S^n, A) \to H_{n-1-m}(S^n - A)$ precisely in the same way, with use of the reduced *co*homology sequence of the pair (S^n, A).

Like Poincaré isomorphism, Alexander isomorphism may be turned into a homology–homology duality, with the role of intersection numbers played by so-called *linking numbers*. From the point of view of Alexander isomorphism, the definition of linking numbers is immediately clear. Let $A \subset S^n$ be as above, and let $\alpha \in H_p(S^n - A)$, $\beta \in H_q(A)$ be two homology classes with $p + q = n - 1$. Then

$$\lambda(\alpha, \beta) = \langle L^{-1}\alpha, \beta\rangle$$

is called the linking number of α and β, and the isomorphism L (rather L^{-1}) becomes a duality

$$\operatorname{Free} H_q(A) \xrightarrow{\cong} \operatorname{Hom}(\operatorname{Free} H_p(S^n - A), \mathbb{Z}),\ \beta \mapsto \{\alpha \mapsto \lambda(\alpha, \beta)\}.$$

But, like intersection numbers, linking numbers have a clear geometric sense, which we will describe now.

Let a, b be two cycles of a compact oriented n-dimensional homology manifold X whose dimensions p, q sum up to $n-1$. [It is convenient to assume that $a \in \mathcal{C}_p^{\text{class}}(X)$ and $b \in \mathcal{C}_q^{\text{bast}}(X)$.] Suppose also that both a, b are homological to zero. Choose a c with $\partial c = b$ and put

$$\lambda(a, b) = \phi(a, c)$$

(see Fig. 72).

EXERCISE 46. Prove that $\lambda(a, b)$ does not depend on the choice of c.

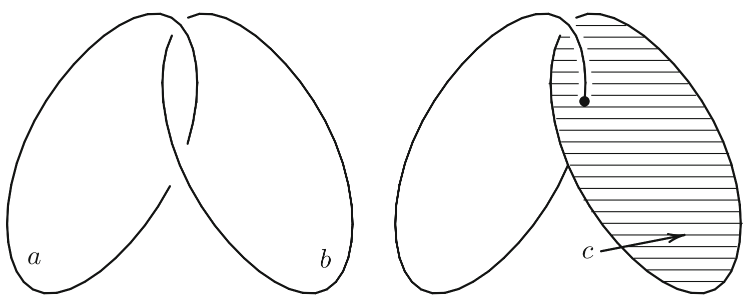


Fig. 72 Definition of the linking number $\lambda(a, b)$

EXERCISE 47. Prove that $\lambda(a,b) = (-1)^{pq+1}\lambda(b,a)$. (For example, the linking number of two disjoint oriented closed curves in $\mathbb{R}^3$ is symmetric with respect to these curves.)

Let us now transfer the definition of a linking number into a context closer to the Alexander duality. Let A, B be disjoint closed subsets of a compact oriented n-dimensional homology manifold X (we can conveniently assume that both are union of simplices of X), and let $\alpha \in H_p(A)$, $\beta \in H_q(B)$ be homology classes which are annihilated by homology homomorphisms induced by the inclusions $A \to X$, $B \to X$. Then $\beta = \partial_*\gamma$ for some $\gamma \in H_{q+1}(X,B)$, and we put $\lambda(\alpha,\beta) = \phi(\alpha,\gamma)$ (in the last formula, we can think of α on the right-hand side as of the image of α in the homology of $X - B$).

EXERCISE 46′. Prove that $\lambda(\alpha,\beta)$ does not depend on the choice of γ.

EXERCISE 47′. Prove that $\lambda(\alpha,\beta) = (-1)^{pq+1}\lambda(\beta,\alpha)$.

In particular, we can take S^n for X, and the complement to a thin neighborhood of A (which is as above) for B (that is, B may look like Y in Fig. 71). Then linking numbers are defined for any $\alpha \in \widetilde{H}_p(A)$, $\beta \in \widetilde{H}_q(B)$ with $p + q = n - 1$.

Theorem. *The equality*

$$\lambda(\alpha,\beta) = \langle L^{-1}\alpha, \beta\rangle$$

holds.

This follows from the definition of L: $L = \partial_* \circ D$.

Thus, linking numbers provide Alexander duality similar to the Poincaré duality.

EXERCISE 48. Make up the definition of "secondary linking numbers" $\mu(\alpha,\beta) \in \mathbb{Q}/\mathbb{Z}$ for $\alpha \in \operatorname{Tors} H_p(A)$, $\beta \in \operatorname{Tors} H_q(S^n - A)$ with $p + q = n - 2$ and prove that

$$\operatorname{Tors} H_p(A) \to \operatorname{Hom}(\operatorname{Tors} H_q(S^n - A), \mathbb{Q}/\mathbb{Z}),\ \alpha \mapsto \{\beta \mapsto \mu(\alpha,\beta)\}$$

(where $p + q = n - 2$) is an isomorphism.

In conclusion, several exercises.

EXERCISE 49. (The Alexander isomorphism in $\mathbb{R}^n$) Let A be a compact polyhedron in $\mathbb{R}^n$. Prove that $H_p(A) \cong \widetilde{H}_q(\mathbb{R}^n - A)$ for $p + q = n - 1$.

EXERCISE 50. Let A be a k-component link (= the union of k disjoint non-self intersecting closed curves in S^3). Find the homology of $S^3 - A$.

EXERCISE 51. (A continuation of Exercise 50) Assume that the linking numbers of the components of A are known. Find the multiplicative structure in the integral cohomology of $S^3 - A$.

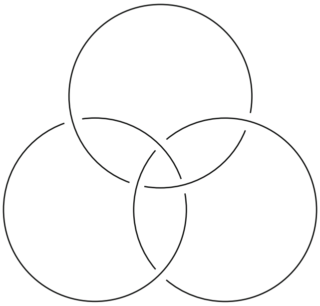


Fig. 73 Borromeo rings

EXERCISE 52. The following is a description of a "secondary multiplicative structure in cohomology" provided by "Massey products." Let $\alpha \in H^p(X;G)$, $\beta \in H^q(X;G)$, $\gamma \in H^r(X;G)$ be cohomology classes of some topological space with coefficients in a ring. Assume that $\alpha \smile \beta = 0$ and $\beta \smile \gamma = 0$. Let $a \in C^p(X;G)$, $b \in C^q \in C^q(X;G)$, $c \in C^r(X;G)$ be (singular) cocycles representing α, β, γ, and let $a \smile b = \delta u$, $b \smile c = \delta v$. Then $h = u \smile c - (-1)^p a \smile v \in C^{p+q+r-1}(X;G)$ is a cocycle, and its cohomology class is determined by α, β, and γ up to a summand of the form $\alpha \smile \sigma + \tau \smile \gamma$ with $\sigma \in H^{q+r-1}(X;G)$, $\tau \in H^{p+q-1}(X;G)$. This (not always and not uniquely) defined cohomology class is called the (triple) Massey product of α, β, γ and is denoted as $\langle \alpha, \beta, \gamma \rangle$. Check all this and compute the cohomology, with cup-products and Massey products, of the complement of the "Borromeo rings" (see Fig. 73).

There exists an extensive theory of "triple linking numbers" and their relations to Massey products (with further generalizations); see Milnor [54] and Turaev [87].

17.12 Integral Poincaré Isomorphism for Nonorientable Manifolds

These isomorphisms have the form

$$H^m(X;\mathbb{Z}) \cong H^{n-m}(X;\mathbb{Z}_T),\ H^m(X;\mathbb{Z}_T) \cong H_{n-m}(X;\mathbb{Z}).$$

Here X is a connected compact n-dimensional nonorientable homology manifold, and homology and cohomology with coefficients in $\mathbb{Z}_T$ ("twisted" integers) are defined in the following way. Let $\widetilde{X}$ be the oriented twofold covering of X. Then there is a canonical orientation reversing involution $t : \widetilde{X} \to \widetilde{X}$. There arise a transformation $t_\#: C_q(\widetilde{X}) \to C_q(\widetilde{X})$ with the square 1, and a decomposition

$$C_q(\widetilde{X}) = C_q^+(\widetilde{X}) \oplus C_q^-(\widetilde{X}),$$

where $C_q^{\pm}(\widetilde{X}) = \{c \in C_q(\widetilde{X}) \mid t_\#(c) = \pm c\}$. Obviously, $C_q^{+}(\widetilde{X})$ is the same as $C_q(X)$; we take the other summand, $C_q^{-}(\widetilde{X})$, for $C_q(X; \mathbb{Z}_T)$. The groups $C_q(X; \mathbb{Z}_T)$ form, in the obvious way, a complex. The homology of this complex is denoted as $H_q(X; \mathbb{Z}_T)$, and the corresponding cohomology is taken for $H^q(X, \mathbb{Z}_T)$. We will not discuss in any detail these homology and cohomology with "twisted coefficients"; moreover, we will have to do it in a much bigger generality in Chap. 3. Now we restrict ourselves to a recommendation to the reader to reconstruct Poincaré isomorphism given above [they are cap-products with a "fundamental class" $[X] \in H_n(X; \mathbb{Z}_T)$], and Poincaré duality with appropriately defined intersection numbers and secondary intersection numbers.

Lecture 18 The Obstruction Theory

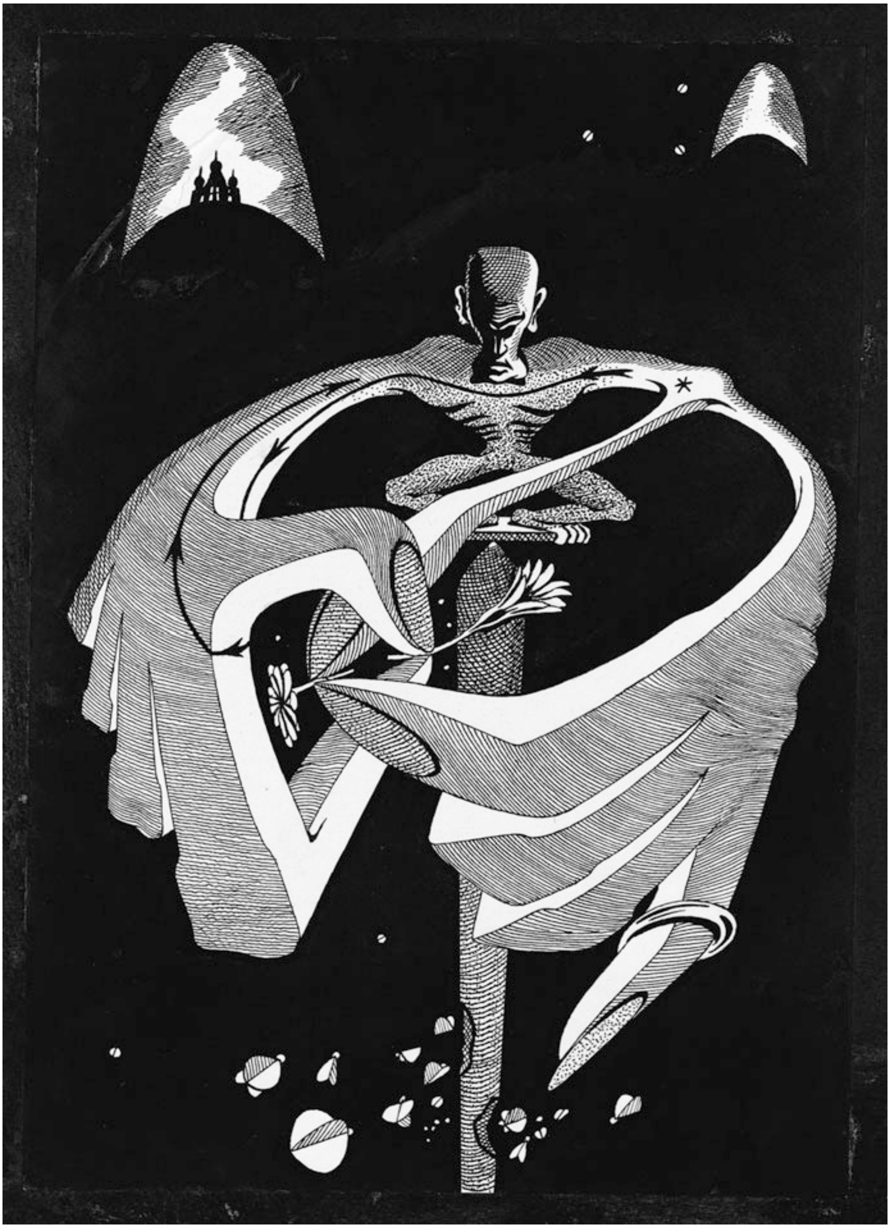
18.1 Obstructions to Extending a Continuous Map

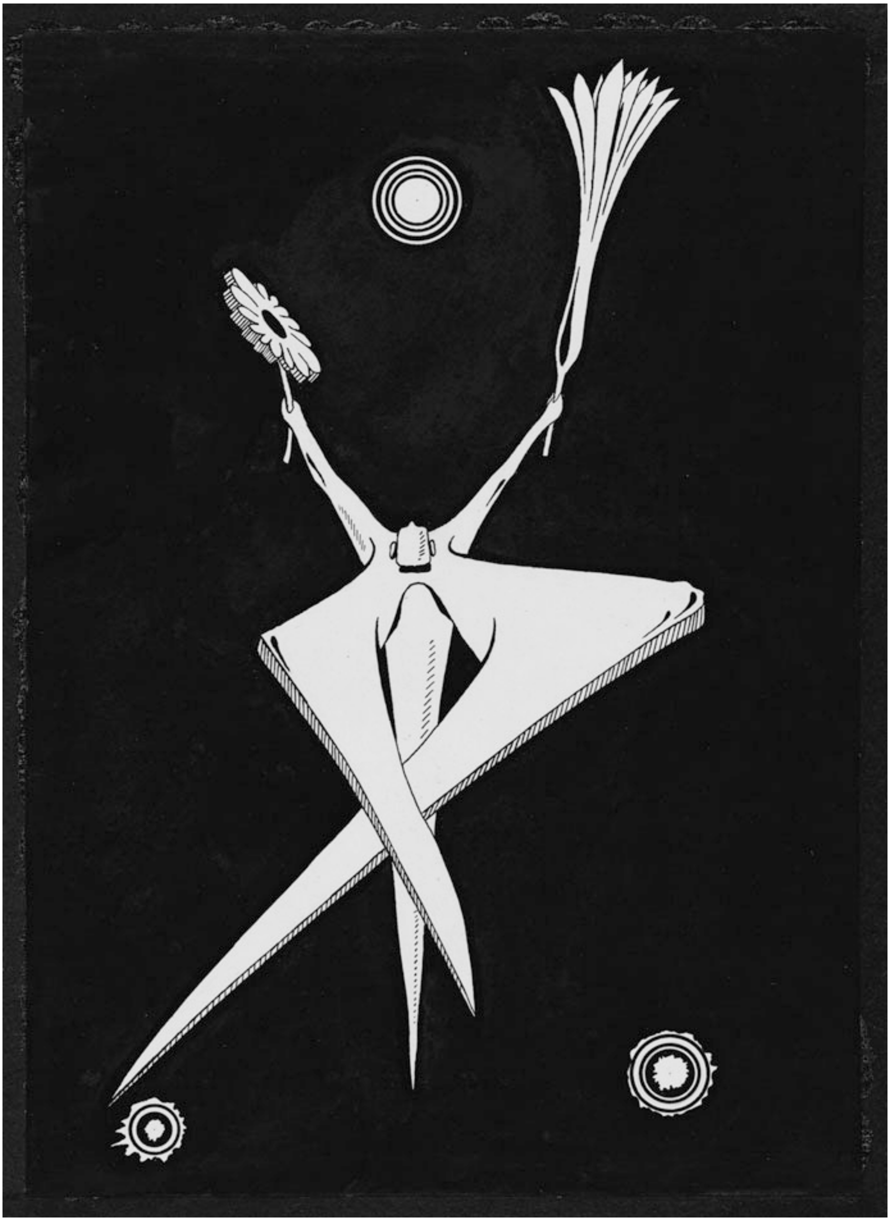
Most problems in homotopy topology consist in a homotopy classification of continuous maps between two topological spaces. A natural intermediate problem is the question of whether a given continuous map $A \to Y$ can be extended to a continuous map $X \to Y$ for some $X \supset A$ (with a subsequent classification of such extensions). This is what the obstruction theory was designed for. We will begin with a technically important particular case.

Let X be a CW complex, and let Y be a connected topological space which is assumed homotopically simple (that is, the action of the fundamental group in all homotopy groups is trivial; later, we will discuss several possibilities of removing or, at least, weakening this condition). Consider the problem of extending a continuous map $f: X^n \to Y$ to a continuous map $X^{n+1} \to Y$ (where X^n, X^{n+1} are skeletons). Let $e \subset X$ be a cell of dimension $n+1$, and let $h: D^{n+1} \to X$ be a corresponding characteristic map. There arises a continuous map $f_e = f \circ h|_{S^n}: S^n \to Y$. It is obvious that f can be continuously extended to $X^n \cup e$ if and only if f_e is homotopic to a constant, that is, if f_e represents the class $0 \in \pi_n(Y)$ (since Y is homotopically simple, we do not need to fix a base point in Y).

Furthermore, the possibility of extension of f to X^{n+1} is the same as the possibility of its extension to every $(n+1)$-dimensional cell of X. If we construct, as above, a map $f_e: S^n \to Y$ for every e and denote by φ_e the class of f_e in $\pi_n(Y)$, we arrive at the following, essentially tautological, statement: A continuous map $f: X^n \to Y$ can be extended to a continuous map $X^{n+1} \to Y$ if and only if every φ_e is equal to 0.

The function $e \mapsto \varphi_e$ can be regarded as an $(n+1)$-dimensional cellular cochain c_f of X with coefficients in $\pi_n(Y)$. (This cochain does not depend on the choice of characteristic maps. Indeed, from the homotopy point of view there are only two characteristic maps corresponding to the two orientations of e; the replacement of h





by a characteristic map of the opposite orientation changes the sign at φ_e, but also reverses the orientation of e, so the cochain c_f stays unchanged.) Thus,

$$c_f \in \mathcal{C}^{n+1}(X; \pi_n(Y)),$$

and f can be extended to X^{n+1} if and only if $c_f = 0$. The cochain c_f is called the *obstruction cochain* to the extension of f to X^{n+1}.

Notice that the obstruction cochains have a naturality property: If $\varphi: X' \to X$ is a cellular map and $\psi: Y \to Y'$ is a continuous map, then $c_{\psi \circ f \circ \varphi} = \varphi^{\#}\psi_{\#}c_f$.

Up to now, everything said was a sheer triviality. Here is the first nontrivial statement.

Theorem 1. *The obstruction cochain is a cocycle:* $\delta c_f = 0$.

Proof. The statement may be regarded as a variation on the theme of $\partial\partial = 0$ [we need to prove that $c_f(\partial a) = 0$, but the cochain c_f itself is defined by means of boundaries], but the accurate proof requires some work. For example, it can be deduced from the relative Hurewicz theorem (Sect. 14.4). According to this theorem, if X satisfies some conditions (we will discuss them later), then the Hurewicz homomorphism $h: \pi_q(X^q, X^{q-1}) \to H_q(X^q, X^{q-1})$ is an isomorphism. Consider the diagram

$$\begin{tikzcd}
\mathcal{C}_{n+2}(X) = H_{n+2}(X^{n+2}, X^{n+1}) \arrow[r, "h^{-1}"] \arrow[dd, "\partial"] & \pi_{n+2}(X^{n+2}, X^{n+1}) \arrow[d, "\partial"] \\
 & \pi_{n+1}(X^{n+1}) \arrow[d] \\
\mathcal{C}_{n+1}(X) = H_{n+1}(X^{n+1}, X^n) \arrow[r, "h^{-1}"] \arrow[ddr, "c_f"'] & \pi_{n+1}(X^{n+1}, X^n) \arrow[d, "\partial"] \\
 & \pi_n(X^n) \arrow[d, "f_*"] \\
 & \pi_n(Y)
\end{tikzcd}$$

0

This diagram is commutative by the definition of the cochain c_f and the homomorphism $\partial: \mathcal{C}_{n+2}(X) \to \mathcal{C}_{n+1}(X)$. Also, the part of the vertical column marked by a brace is a fragment of the homotopy sequence of the pair (X^{n+1}, X^n), and hence the composition os homomorphism within this part is 0. Thus, $c_f \circ \partial = \delta c_f = 0$.

However, the reference to the relative Hurewicz theorem forces us to respect its assumptions, that is, to assume that X is simply connected and that $n + 1 > 1$.

We will ignore the second assumption (it is easy to see that our arguments are valid when $n = 0$), and we can get rid of the simply connectedness assumption in the following way. Let $p: \widetilde{X} \to X$ be the universal covering of X. The CW decomposition of X induces a CW decomposition of $\widetilde{X}$, and the map $p^{\#}: \mathcal{C}^q(X) \to \mathcal{C}^q(\widetilde{X})$ is a monomorphism. For a map $f: X^n \to Y$, the obstruction cochain $c_{f \circ p} \in \mathcal{C}^{n+1}(\widetilde{X}; \pi_n(Y))$ is $p^{\#}c_f$, $p^{\#}\delta c_f = \delta p^{\#} c_f = \delta c_{f\circ p} = 0$, and hence $\delta c_f = 0$.

The cohomology class $C_f \in H^{n+1}(X; \pi_n(Y))$ of the cocycle c_f is called the *cohomology obstruction*, or simply the *obstruction* to extension of f to X^{n+1}.

Theorem 2. *The condition $C_f = 0$ is necessary and sufficient to the existence of extending $f\,|_{X^{n-1}}$ to X^{n+1}. In other words, $C_f=0$ if and only if it is possible to extend f to X^{n+1} after, possibly, a changing f on $X^n - X^{n-1}$.*

[One can apply this theorem to successive extensions of f from a skeleton to a skeleton. Say, let us have a continuous map $f: X^n \to Y$. There arises an obstruction $C_f \in H^{n+1}(X; \pi_n(Y)$. If it is 0, we can extend f to X^{n+1} at the price of some modification of f on X^n not touching f on X^{n-1}. In this case (that is, if $C_f = 0$), we get a new obstruction in $H^{n+2}(X; \pi_{n+1})$. If it is zero, we extend f to X^{n+2} (maybe, after changing the previous extension), and get the next obstruction in $H^{n+3}(X; \pi_{n+2}(Y))$, and so on. One should remember, however, that every new obstruction depends from the previous extension, and hence these obstructions are defined with a growing indeterminacy.]

Before proving Theorem 2, we will give a new definition which will be useful in the proof but will also have a considerable independent value. Let $f, g: X^n \to Y$ be two continuous maps which agree on X^{n-1}. Consider an arbitrary n-dimensional cell e with a characteristic map $h: D^n \to X$. The maps $f \circ h, g \circ h: D^n \to Y$ agree on S^{n-1} [since $h(S^{n-1}) \subset X^{n-1}$, and f and g agree on X^{n-1}] and together compose a map $k_e: S^n \to Y$ (which is $f \circ h$ on the lower hemisphere and $g \circ h$ on the upper hemisphere). We define the *difference cochain*

$$d_{f,g} \in \mathcal{C}^n(X; \pi_n(Y)),$$

whose value on e is the class of k_e in $\pi_n(Y)$. It is clear that the condition $d_{f,g} = 0$ is necessary and sufficient for the existence of a homotopy between f and g which is fixed on X^{n-1} (in the terminology of Chap. 1, an X^{n-1}-homotopy; see Sect. 5.7). In the important case when f and g are defined on the whole X and agree on X^{n-1}, the condition $d_{f,g} = 0$ is necessary and sufficient for the existence of an X^{n-1}-homotopy of f making f agree with g on X^n (for this statement, we need to use Borsuk's theorem, Sect. 5.5). Notice also that the difference cochains have a naturality property similar to that of the obstruction cochains: $d_{\psi\circ f\circ\varphi, \psi\circ f\circ\varphi} = \varphi^{\#}\psi_{\#} d_{f,g}$.

Lemma 1. *For any continuous map $f: X^n \to Y$ and any cochain $d \in \mathcal{C}^n(X; \pi_n(Y))$, there exists a continuous map $g: X^n \to Y$ which agrees with f on X^{n-1} and is such that $d_{f,g} = d$.*

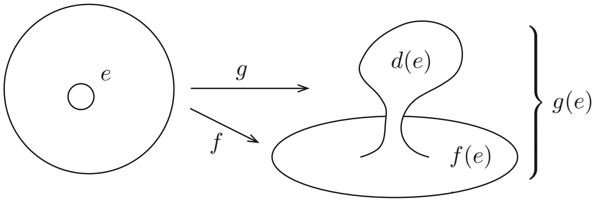


Fig. 74 Proof of Lemma 1

Proof. Consider an n-dimensional cell e of X and distinguish a small ball in e. Than change the map f on this ball in such a way that the two maps of the ball, the old one and the new one, compose a spheroid of the class $d(e)$ (see Fig. 74). Having such a change made on each n-dimensional cell, we get the map g with the required properties.

Lemma 2. $\delta d_{f,g} = c_g - c_f$.

Proof. Consider, for simplicity's sake, the case when f and g are different on only one n-dimensional cell $e \subset X$ (the general case, essentially, is not different from this case). Let σ be an $(n+1)$-dimensional cell of X; we want to show that

$$c_g(\sigma) - c_f(\sigma) = [\sigma : e] d_{f,g}(e).$$

Let $h: D^{n+1} \to X$ be a characteristic map for σ. We can assume that $h^{-1}(e)$ consists of several open balls, of which every one is mapped by h homeomorphically onto e, with preserving or reversing the orientation, and $[\sigma : e]$ is the difference of the number of balls where the orientation is preserved and the number of balls where it is reversed (compare the description of the incidence numbers in Sect. 13.6). This makes the desired equality obvious: A spheroid representing $c_g(\sigma)$ is obtained from a spheroid representing $c_f(\sigma)$ by adding spheroids of the class $\pm d_{f,g}(e)$, and the algebraic number of these spheroids is $[\sigma : e]$.

Proof of Theorem 2. If $C_f = 0$, then $c_f = \delta d$ and, by Lemma 1, there exists a map $g: X^n \to Y$ such that $g \mid_{X^{n-1}} = f \mid_{X^{n-1}}$ and $d_{f,g} = -d$. But then, by Lemma 2, $c_g = c_f + \delta d_{f,g} = \delta d - \delta d = 0$; thus, g can be extended to X^{n+1}. Conversely, if there exists a map $g: X^n \to Y$ which agrees with f on X^{n-1} and can be extended to X^{n+1}, then $c_g = 0$ and $c_f = c_f - c_g = \delta d_{f,g}$, and hence $C_f = 0$.

Remark. The two lemmas of this proof are not less important than the theorem; we will use them later.

EXERCISE 1. Prove that $d_{g,f} = -d_{f,g}$ and $d_{f,h} = d_{f,g} + d_{g,h}$.

18.2 The Relative Case

Let A be a CW subcomplex of a CW complex X, and let the continuous map f be defined on $A \cup X^n$. The obstruction cochain c_f to an extension of this map to $A \cup X^{n+1}$ is contained in $\mathcal{C}^{n+1}(X, A; \pi_n(Y))$, it is a cocycle, and its cohomology class $C_f \in H^{n+1}(X, A; \pi_n(Y))$ is called an obstruction. The theory of these relative obstructions is absolutely parallel to its absolute prototype; in particular, it contains the notion of difference cochains, and there are precise analogies (for both the statements and the proofs) of all theorems and lemmas of the previous section. We will point out the following important consequence of the relative theory in the absolute theory.

Let $f, g: X \to Y$ (or $X^{n+1} \to Y$) be two maps with $f \mid_{X^{n-1}} = g \mid_{X^{n-1}}$ [or with a fixed homotopy connecting $f \mid_{X^{n-1}}$ and $g \mid_{X^{n-1}}$]. We consider the problem of constructing a homotopy between f and g fixed (or coinciding with the given homotopy) on X^{n-1}. This problem is equivalent to extending to $X \times I$ (or to $X^{n+1} \times I$) the map which is given on $(X \times 0) \cup (X^{n-1} \times I) \cup (X \times 1)$ by the formula

$$(x, t) \mapsto \begin{cases} f(x), & \text{if } t = 0 \text{ or } x \in X^{n-1}, \\ g(x), & \text{if } t = 1 (\text{or } x \in X^{n-1}) \end{cases}$$

(this formula is for the case when f and g agree on X^{n-1}; if a homotopy between $f \mid_{X^{n-1}}$ and $g \mid_{X^{n-1}}$ is given, the formula will be slightly different; we leave the details to the reader). The obstruction to an extension of this map to $(X \times 0) \cup (X^n \times I) \cup (X \times 1)$ lies in $\mathcal{C}^{n+1}(X \times I, (X \times 0) \cup (X \times 1); \pi_n(Y)) = \mathcal{C}^n(X; \pi_n(Y))$, and it is easy to see that it is nothing but $d_{f,g}$. By the way, $\delta d_{f,g} = c_g - c_f = 0$, since f and g are both defined on the whole X (or, at least, on X^{n+1}). If we apply to this situation the relative version of Theorem 2 of Sect. 18.1, we will get the following result.

Theorem. *If $f, g: X \to Y$ are two continuous maps which agree on X^{n-1}, then the difference cochain $d_{f,g}$ is a cocycle whose cohomology class $D_{f,g} \in H^n(X; \pi_n(Y))$ is equal to 0 if and only if $f \mid_{X^n}$ and $g \mid_{X^n}$ are X^{n-2}-homotopic.*

18.3 The First Application: Cohomology and Maps into $K(\pi, n)$s

The main result of this section was promised in Lecture 4. Let π be an Abelian group.

Recall that the construction of a $K(\pi, n)$ space begins with taking a bouquet of n-dimensional spheres set into a correspondence with some system of generators of π (see Sect. 11.7); then we attach to this bouquet cells of dimensions $> n$. If we assign to every n-dimensional cell of $K(\pi, n)$ the corresponding element of π, we get a cochain $c \in \mathcal{C}^n(K(\pi, n); \pi)$ [we admit here a certain abuse of notation, using the symbol $K(\pi, n)$ for a CW complex obtained by some concrete construction].

Lemma. *c is a cocycle.*

First Proof (Direct). The cells of dimension $n+1$ correspond to the defining relations between the chosen generators. If the cell σ corresponds to the relation $\sum k_i g_i = 0$ between the generators g_i, then for the n-dimensional cell e_i corresponding to the generator g_i, the incidence number $[\sigma : e_i]$ is k_i. Then

$$\delta c(\sigma) = \sum_i [\sigma : e_i] c(e_i) = \sum_i k_i g_i = 0.$$

Second Proof (Indirect). Actually, $c = d_{\text{const,id}}$; thus, $\delta c = 0$ by Lemma 2 of Sect. 18.1.

The cohomology class $F_\pi \in H^n(K(\pi, n); \pi)$ of the cocycle c is called the *fundamental cohomology class* of $K(\pi, n)$. Another description of this class: According to the universal coefficients formula,

$$H^n(K(\pi, n); \pi) = \text{Hom}(H_n(K(\pi, n)), \pi),$$

and, by Hurewicz's theorem, $H_n(K(\pi, n)) = \pi_n(K(\pi, n)) = \pi$. The class F_π corresponds to the identity homomorphism

$$\text{id}_\pi \in \text{Hom}(H_n(K(\pi, n)), \pi).$$

EXERCISE 2. Prove the equivalence of the two definitions of the fundamental homology class.

Notice that the second definition of the fundamental class can be applied to an arbitrary $(n-1)$-connected space X. In this case, it yields a cohomology class $F_X \in H^n(X; \pi_n(X))$. We will return to this class later.

Now we turn to the main result of this section.

Theorem. *Let X be a CW complex. For any Abelian group π and for any $n > 0$, the map*

$$\pi(X, K(\pi, n)) \to H^n(X; \pi), \ [f] \to f^*(F_\pi), \tag{$*$}$$

is a bijection.

Proof. First, let $\gamma \in H^n(X; \pi)$, and let $c \in \mathcal{C}^n(X; \pi)$ be a cocycle of the class γ. We want to construct a continuous map $f: X \to K(\pi, n)$ which takes the cocycle of class F_π (constructed above) into c. By Lemma 1 of Sect. 18.1, there exists a map $f: X^n \to K(\pi, n)$ such that $f(X^{n-1})$ is the (only) vertex of $K(\pi, n)$ and $d_{\text{const},f} = c$. Then, obviously, $f^\#: \mathcal{C}^n(K(\pi, n); \pi) \to \mathcal{C}^n(X; \pi)$ takes $d_{\text{const,id}}$ into $d_{\text{const},f} = c$ (by the naturality property of the difference cochains; see Sect. 18.1). Then we extend this map f to $X^{n+1}, X^{n+2}, \ldots$, and it is possible, since $\pi_{n+1}(K(\pi, n)), \pi_{n+2}(K(\pi, n)), \ldots$ are all zeroes. We obtain a map $f: X \to K(\pi, n)$. By construction, f^* takes F_π into γ. Thus, the map ($*$) is onto.

Now let $f, g: X \to K(\pi, n)$ be two continuous maps with $f^* F_\pi = g^* F_\pi$. We want to prove that $f \sim g$; we can assume that f and g are cellular maps (in particular, they are constant on X^{n-1}). Then $f^* F_\pi$ and $g^* F_\pi$ are represented by $f^\# d_{\text{const,id}} = d_{\text{const},f}$ and $g^\# d_{\text{const,id}} = d_{\text{const},g}$. Hence, the cocycles $d_{\text{const},f}$ and $d_{\text{const},g}$ are cohomological, so the difference $d_{\text{const},g} - d_{\text{const},f} = d_{f,g}$ is cohomological to 0, or $D_{f,g} = 0$. According to the theorem in Sect. 18.2, this shows that f and g are X^{n-2}-homotopic (the homotopy being fixed on X^{n-2} is not important to us) on X^n. They are also homotopic on further skeletons, since the further difference cochains belong to the cochain groups with trivial coefficients. Thus, the map $(*)$ is one-to-one.

Corollary 1. *A CW complex of the type $K(\pi, n)$ is homotopically unique. Hence, a topological space of the type $K(\pi, n)$ is weakly homotopically unique.*

Proof. Let X, X' be CW complexes of the type $K(\pi, n)$, and let $F_\pi \in H^n(X; \pi), F'_\pi \in H^n(X'; \pi)$ be the fundamental classes. According to the theorem, there exist continuous maps $f: X \to X'$, $g: X' \to X$ such that $f^*(F'_\pi) = F_\pi$ and $g^*(F_\pi) = F'_\pi$. Since $(g \circ f)^*(F_\pi) = f^* \circ g^*(F_\pi) = F_\pi = (\text{id}_X)^* (F_\pi)$, we have $g \circ f \sim \text{id}_X$ and, similarly, $f \circ g \sim \text{id}_{X'}$.

EXERCISE 3. Since $K(\pi, n) \sim \Omega K(\pi, n+1)$ is an H-space, the set $\pi(X, K(\pi, n))$ is a group (see Lecture 4), and the bijection $H^n(X; \pi) \leftrightarrow \pi(X, K(\pi, n))$ is a bijection between two groups. Prove that it is a group isomorphism.

Actually, for every Abelian group π and every n, there exists an Abelian topological group of the type $K(\pi, n)$. The reader may try to prove it by an appropriate enhancing of the construction of the (second) loop space.

Corollary 2. *For a CW complex X, there is a group isomorphism $H^1(X; \mathbb{Z}) \cong \pi(X, S^1)$ (where S^1 is regarded as an Abelian topological group).*

EXERCISE 4. Prove that every continuous map $\underbrace{S^1 \times \cdots \times S^1}_{n} \to \underbrace{S^1 \times \cdots \times S^1}_{m}$ is homotopic to a linear map (that is, to a map obtained by a factorization from a linear map $\mathbb{R}^n \to \mathbb{R}^m$ determined by an integral matrix).

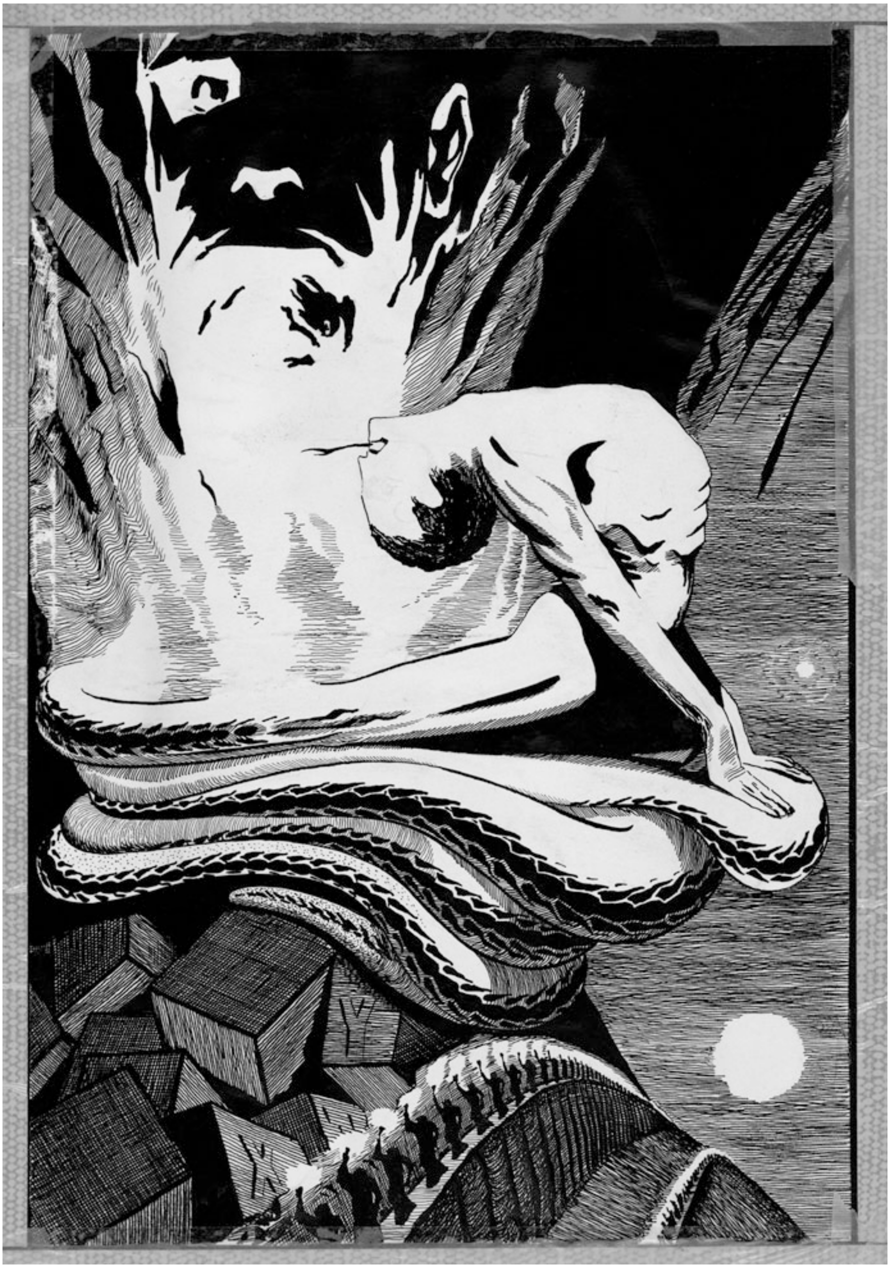
18.4 The Second Application: Hopf's Theorems

Theorem 1 (Hopf). *For every n-dimensional CW complex X, there is a bijection*

$$H^n(X; \mathbb{Z}) \leftrightarrow \pi(X, S^n), \ [f] \mapsto f^*(s),$$

where $s = 1 \in \mathbb{Z} = H^n(S^n; \mathbb{Z})$.

Proof. This classical theorem (proved, actually, before the appearance of not only the obstruction theory, but also cohomology) is, from a modern point of view, a corollary of the theorem in Sect. 18.3. Indeed, the construction of the space



$K(\pi, n)$, as given in Sect. 11.7, begins with a bouquet of n-dimensional spheres corresponding to generators of π; if $\pi = \mathbb{Z}$, we can take one sphere. On the next step, we attach $(n+1)$-dimensional cells corresponding to relations between the chosen generators; but in the case $\pi = \mathbb{Z}$ there are no relations, and no $(n+1)$-dimensional cells are needed. Then we attach cells of dimensions $\geq n+2$. We see that the $(n+1)$st skeleton of (such constructed) $K(\mathbb{Z}, n)$ is S^n. Hence, by the cellular approximation theorem, if X is n-dimensional, every map $X \to K(\mathbb{Z}, n)$ is homotopic to a map $X \to S^n \subset K(\mathbb{Z}, n)$ and every two maps $X \to S^n \subset K(\mathbb{Z}, n)$ homotopic in $K(\mathbb{Z}, n)$ are homotopic in S^n.

[There is a more direct proof which is a replica of the proof of the theorem in Sect. 18.3. The main difference is that the higher obstruction and difference cochains are equal to zero not because the higher homotopy groups of S^n are zeroes (which is not true), but because X has no cells of higher dimensions.]

Theorem 2 (Hopf). *Let an n-dimensional CW complex X contain as a CW subcomplex a sphere S^{n-1}. This sphere is a retract of X if and only if the inclusion homomorphism $H^{n-1}(X;\mathbb{Z}) \to H^{n-1}(S^{n-1};\mathbb{Z})$ is an epimorphism.*

Proof. The only if part is obvious: If $r: X \to S^{n-1}$ is a retraction, then the composition

$$H^{n-1}(S^{n-1};\mathbb{Z}) \xrightarrow{r^*} H^{n-1}(X;\mathbb{Z}) \xrightarrow{j^*} H^{n-1}(S^{n-1};\mathbb{Z}),$$

where j is the inclusion map, is the identity, and hence j^* is an epimorphism. Assume now that j^* is an epimorphism and fix a class $\alpha \in H^{n-1}(X;\mathbb{Z})$ such that $j^*(\alpha) = 1 \in \mathbb{Z} = H^{n-1}(S^{n-1};\mathbb{Z})$. Let $a \in \mathcal{C}^{n-1}(X;\mathbb{Z})$ be a cocycle of the class α. Construct a map $q: X \to S^{n-1}$ in the following way. All the cells of dimensions $\leq n-2$ we map into a point. On every $(n-1)$-dimensional cell e define the map as the spheroid of the class $a(e)$. This requirement means precisely that the map $q^\#$ takes $1 \in \mathbb{Z} = \mathcal{C}^{n-1}(S^{n-1};\mathbb{Z})$ into a. On the other side, it means that the cochain a is the difference cochain between the already constructed part of the map q and the map const: $X^{n-1} \to S^{n-1}$. Hence,

$$0 = \delta a = \delta d_{q,\text{const}} = c_q - c_{\text{const}} = c_q,$$

so the map q can be extended to $X^n = X$. The composition

$$S^{n-1} \xrightarrow{j} X \xrightarrow{q} S^{n-1}$$

induces the identity map in cohomology: $(q \circ j)^*(1) = j^*(q^*(1)) = j^*(\alpha) = 1$, and hence homotopic to id. We can extend the homotopy between this map and id to the homotopy of the map q. As a result, we will get a map $r: X \to S^{n-1}$ which is the identity on S^{n-1}, that is, a retraction.

18.5 Obstructions to Extensions of Sections

Let $\xi = (E, B, F, p)$ be a locally trivial fibration. We assume that the fiber F is homotopically simple (for example, simply connected), and the base B is simply connected. (The last assumption can be weakened to the assumption of the *homotopical simplicity of the fibration*. The latter means that for every continuous map $S^1 \to B$, the induced fibration over S^1 is trivial. In the next lecture, we will encounter important examples of this situation.)

Assume that the base B is a CW complex and that there given a section $s: B^n \to E$ [which means that $p \circ s = \mathrm{id}$] over the nth skeleton of the base. We are going to describe an obstruction to extending this section to B^{n+1}. Let e be an $(n+1)$-dimensional cell over B. The fibration $h^*\xi$ over D^{n+1}, induced by means of a characteristic map $h: D^{n+1} \to B$ for the cell e, is trivial. The section s induces a section $S^n \to D^{n+1} \times F$ of the restriction of the last fibration to $S^n \subset D^{n+1}$, and hence an element of $\pi_n(D^{n+1} \times F) = \pi_n(F)$ (rather of the fiber $p^{-1}(x)$ over some point $x \in e$, but the simply connectedness of the base, or the homotopical simplicity of the fibration ξ, provides a canonical homomorphism between homotopy groups of all fibers—the reader will reconstruct a detailed explanation of this). We get a cochain $c_s \in C^{n+1}(B; \pi_n(F))$. This is the *obstruction cochain to extending s to B^{n+1}*. The properties of this obstruction cochain are the same as those of the obstruction cochains considered in Sect. 18.1. Namely:

(1) The section s can be extended to a section over the $(n+1)$st skeleton of B if and only if $c_s = 0$.
(2) $\delta c_s = 0$.
(3) The cohomology class $C_s \in H^{n+1}(B; \pi_n(F))$ of c_s (which is called the *obstruction*) is equal to 0 if and only if the section s can be extended to a section over B^{n+1}.

There are also *difference cochains* $d_{s,s'}$ whose definition and properties are the same as before.

Obstructions to extending maps may be regarded as particular cases of obstructions to extending sections. Namely, a continuous map $f: X \to Y$ can be represented by the graph $F: X \to X \times Y$, $F(x) = (x, f(x))$, which, in turn, is a section of the trivial fibration $(X \times Y, X, Y, p)$, where $p: X \times Y \to X$ is the projection of the product onto a factor. Obstructions to extending a map are the same as obstructions to extending its graph. On the other hand, the theory of obstructions to sections cannot be reduced to the theory of obstructions to maps. In particular, the latter does not have any analogy of the next construction.

Suppose that $\pi_0(F) = \pi_1(F) = \cdots = \pi_{n-1}(F) = 0$, and $\pi_n(F) \neq 0$. Then there are no obstructions to extending a section from B^0 (where it obviously exists) to $B^1, \ldots, B^{n-1}$ and the first obstruction emerges in $H^{n+1}(B; \pi_n(F))$: It is the obstruction to extending the section from B^{n-1} to B^n. This obstruction could depend, however, on the sections on the previous skeletons; however, the next proposition states that it is not the case.

Proposition 1. *Let $\pi_0(F) = \pi_1(F) = \cdots = \pi_{n-1}(F) = 0$, and let $s, s' \colon B^n \to E$ be two sections. Then $C_s = C_{s'} \in H^{n+1}(B; \pi_n(F))$.*

To prove this, we need a slightly modified version of the homotopy extension property (Borsuk's theorem; see Sect. 5.5).

Lemma (Borsuk's Theorem for Sections). *Let $\xi = (E, B, F, p)$ be a locally trivial fibration with a CW base, let $S\colon B \to E$ be a section of ξ, let A be a CW subspace of B, and let $s_t\colon A \to E$ be a homotopy consisting of sections of $\xi\ |_A$ such that $s_0 = S\ |_A$. Then there exists a homotopy $S_t\colon B \to E$ consisting of sections of ξ and such that $S_0 = S$, $S_t\ |_A = s_t$.*

Proof of Lemma. This lemma is not different from Borsuk's theorem in the case when the fibration is (standard) trivial: $E = B \times F$, p is the projection of the product onto a factor. Indeed, in this case, a section is the same as a continuous map $B \to F$. Passing to the general case, we can restrict ourselves to the situation when A and B differ by one cell: $B = A \cup e$, where e is a cell of B. Take a characteristic map $h\colon D^n \to B$ (where $n = \dim e$). Then the sections S, s_t of ξ and $\xi\ |_A$ give rise to sections S', s'_t of the fibrations $h^*\xi, h^*\xi\ |_{S^{n-1}}$ [such that $s'_0 = S'\ |_{S^{n-1}}$]. Since the fibration $h^*\xi$ is trivial (Feldbau's theorem, Sect. 9.2), the lemma has already been proved for this fibration, which provides a homotopy S'_t consisting of sections of this fibration such that $S'_0 = S'$ and $S'_t\ |_{S^{n-1}} = s'_t$. The homotopies s_t and S'_t together form a homotopy $S_t\colon B \to E$ with the required properties.

Proof of Proposition 1. It is clear that a homotopy of a section $s\colon B^k \to E$ will not affect either c_s or C_s. Suppose that the given sections s, s' are homotopic over B^k for some k, $0 \le k < n-1$ (since the fiber F is connected, this is obviously true for $k = 0$). A homotopy of s' to s on B^k can be extended, by the lemma, to a homotopy of s' on B^n, without any changes for $c_{s'}$ and $C_{s'}$ so we can assume that $s' = s$ on B^k. The difference cochain $d_{s,s'} \in \mathcal{C}^{k+1}(B; \pi_{k+1}(F))$ is zero, because $\pi_{k+1}(F) = 0$; thus, $s' \sim s$ on B^{k+1}. In this way, we can reduce the general case of the proposition to the case when $s' = s$ on B^{n-1}. Then we have a difference cochain $d_{s,s'} \in \mathcal{C}^n(B, \pi_n(X))$, and $\delta d_{s,s'} = c_{s'} - c_s$. Thus, the cocycles c_s and $c_{s'}$ are cohomological and hence $C_s = C_{s'}$.

Proposition 1 shows that the *first obstruction* to extending a section to the nth skeleton of the base is determined by the fibration, so we obtain a well-defined class $C(\xi) \in H^{n+1}(B; \pi_n(F))$ (recall that n is the number of the first nontrivial homotopy group of F); this class is called the *characteristic class* of ξ; we will also use the term *primary characteristic class* to distinguish it from numerous characteristic classes of vector bundles, which will be studied in Lecture 19.

One can say that a fibration as above has a section over the nth skeleton of the base if and only if its characteristic class is zero.

EXERCISE 5 (The main property of characteristic classes). Let ξ be a fibration as above, and let $f\colon B' \to B$ be a continuous map of some CW complex into B. Then

$$C(f^*\xi) = f^*(C(\xi)).$$

EXERCISE 6. Prove that a characteristic class is homotopy invariant (we leave to the reader not only the proof, but also a precise statement of this fact). In particular, the characteristic class does not depend on the CW structure of the base.

EXERCISE 7. Using previous exercises, make up a definition of a characteristic class in the case when the base is not a CW complex.

Example. (Since this example concerns smooth manifolds, the definitions and statements will not be genuinely rigorous.) Let X be a connected closed oriented n-dimensional manifold and let T be the manifold of all nonzero tangent vectors of M. The projection $p: T \to X$ (which assigns to a tangent vector the tangency point) gives rise to a locally trivial fibration $\tau_X = (T, X, \mathbb{R}^n - 0, p)$. Since the fiber is homotopy equivalent to S^{n-1}, there arises a characteristic class $C(\tau_X) \in H^n(X; \mathbb{Z})$. (It is easy to understand that the fibration τ_X is simple if and only if the manifold X is orientable.)

Proposition 2. $\langle C(\tau_X), [X]\rangle = \chi(X)$.

Proof. A section of the fibration τ_X is the same as a nowhere vanishing vector field on X. It is easy to understand that a generic vector field on X has only isolated zeroes. Take a local coordinate system with the origin at the isolated zero x_0 of a vector field ξ, take a small sphere $S \approx S^{n-1}$ centered at x_0, and consider the map $S = S^{n-1} \to S^{n-1}$ which takes $x \in S$ into $\xi(x)/\|\xi(x)\|$. Denote by $d_\xi(x_0)$ the degree of this map. We can assume (although it is actually not necessary) that all the zeroes of ξ are *nondegenerate*, that is, $d(x_0) = \pm 1$. Now consider a smooth triangulation of X such that all zeroes of ξ lie inside n-dimensional simplices, at most one in every simplex. Then ξ is a section of the fibration τ_X over the $(n-1)$st skeleton of X, and the obstruction c_ξ to extending this section to an n-dimensional simplex s is zero if s does not contain zeroes of ξ and is $d(x_0)$ if s contains a zero x_0 of ξ. Since the fundamental cycle of $[X]$ is the sum of all (oriented) n-dimensional simplices of the triangulation, $\langle c_\xi, [X]\rangle = \sum_{x_0 \in \{\text{zeroes of } \xi\}} d(x_0)$. The left-hand side of this equality is $\langle C(\tau_X), [X]\rangle$, the right-hand side, as explained in Sect. 17.6 (see Theorem 3 and the discussion after it), is $\chi(X)$. This completes the proof of Proposition 2.

Corollary. *A connected closed orientable manifold possesses a nowhere vanishing vector field if and only if* $\chi(X) = 0$.

The only if part of this statement has been proved before: See Theorem 2 in Sect. 17.6. The if part was promised there. The orientability condition is not needed; it also was explained in Sect. 17.6.

In conclusion, a couple of additional exercises.

EXERCISE 8. Make up a theory of obstructions to extending sections in the context of Serre fibrations (see Sect. 9.4).

EXERCISE 9. Let X be a CW complex with $\pi_0(X) = \pi_1(X) = \cdots = \pi_{n-1}(X) = 0$, $\pi_n(X) \neq 0$. Prove that the characteristic class of the Serre fibration $EX \to X$ with the fiber ΩX which belongs to $H^n(X; \pi_{n+1}(\Omega X)) = H^n(X; \pi_n(X))$ is just the fundamental class of X.

Lecture 19 Vector Bundles and Their Characteristic Classes

19.1 Vector Bundles and Operations over Them

A: Definitions

We consider three types of vector bundles: *real, oriented,* and *complex*. A real n-dimensional vector bundle with the base B is a locally trivial fibration with the base B and the fiber homeomorphic to $\mathbb{R}^n$ with an additional structure: Each fiber is furnished by a structure of an n-dimensional vector space, in such a way that the vector space operations $(\lambda, x) \mapsto \lambda x$ and $(x, y) \mapsto x + y$ depend continuously on the fiber, in the sense that the arising maps $\mathbb{R} \times E \to E$ and $\{(x, y) \in E \times E \mid p(x) = p(y)\} \to E$ (where E is the total space and p is the projection of the fibration) are continuous. Complex vector bundles are defined precisely in the same way, only the field $\mathbb{R}$ is replaced by the field $\mathbb{C}$; oriented vector bundles are real vector bundles whose fibers are furnished with orientation depending continuously on the fiber. The last property can be formalized in the following way. For simplicity's sake, assume that B is connected. Let $\widetilde{E}$ be the set of all bases in all fibers of the fibration; there is a natural topology in $\widetilde{E}$. The fibration is orientable if and only if $\widetilde{E}$ has two (not one) components; a choice of one of these components is an *orientation* of the fibration.

For vector bundles of all three kinds there are natural definitions of *equivalences, restrictions* (over subspaces of the base) and *induced bundles* (by a continuous map of some space into the base). A *trivial bundle* is a bundle equivalent (in its class) to the projection bundle $B \times \mathbb{R}^n \to B$ or $B \times \mathbb{C}^n \to B$ furnished by the obvious structure.

Important Example. The *Hopf* or *tautological* vector bundle over $\mathbb{R}P^n$ is the one-dimensional vector bundle whose total space is the set of pairs (ℓ, x), where $\ell \in \mathbb{R}P^n$ is a line in $\mathbb{R}^{n+1}$ and $x \in \ell$ is a point on this line [topology in this set is defined by the inclusion into $\mathbb{R}P^n \times \mathbb{R}^{n+1}$]. Precisely in the same way, the Hopf, or tautological, one-dimensional complex vector bundle over $\mathbb{C}P^n$ is defined. An obvious generalization of this construction provides tautological vector bundles over the Grassmannians $G(m, n), G_+(m, n)$, and $\mathbb{C}G(m, n)$, which are n-dimensional, respectively, real, oriented, and complex vector bundles.

B: Realification and Complexification

One can make a complex vector bundle real by removing a part of its structure, namely the multiplication by nonreal scalars. If ξ is an n-dimensional complex vector bundle, then the realification provides a $2n$-dimensional real vector bundle which is denoted as $\mathbb{R}\xi$. The bundle $\mathbb{R}\xi$ possesses a canonical orientation: If $x_1, \dots, x_n$ is a complex basis in a fiber of ξ, then $x_1, ix_1, \dots, x_n, ix_n$ is a real basis in the same space, and the orientation of this basis does not depend on the choice of the complex basis $x_1, \dots, x_n$ [this follows from the fact that the image of the natural embedding $c\colon GL(n, \mathbb{C}) \to GL(2n, \mathbb{R})$ consists of matrices with positive determinant; the last statement follows from the fact that $GL(n, \mathbb{C})$ is connected, or, more convincingly, from the formula $\det(cA) = |\det A|^2$; compare with the "important remark" in the example after Theorem 1 in Sect. 17.5]. The definition of the complexification $\mathbb{C}\xi$ of a real vector bundle $(E, B, \mathbb{R}^n, p)$ is a bit more complicated. In the product $\mathbb{C} \times E$, make an identification $(rx, \lambda) = (x, r\lambda)$ for every $x \in E, r \in \mathbb{R}, \lambda \in \mathbb{C}$. The resulting space $\mathbb{C}E$ is the space of our fibration; the projection $\mathbb{C}E \to B$ is defined by the formula $(x, \lambda) = p(x)$, and the vector operations act as $(x, \lambda) + (x, \lambda') = (x, \lambda + \lambda')$ and $\mu(x, \lambda) = (x, \mu\lambda)$ (it is obvious that these formulas are compatible with the preceding factorization). It is clear also that $\mathbb{C}\xi$ is an n-dimensional complex vector bundle.

There is one more operation related to the two previous ones. Let ξ be a complex vector bundle. Denote by $\overline{\xi}$ a complex vector bundle (of the same dimension as ξ) which differs from ξ only by the operation of multiplication by scalars: λx with respect to the structure of $\overline{\xi}$ is the same as $\overline{\lambda} x$ in ξ.

EXERCISE 1. Let ξ be a complex vector bundle. Prove that the following two statements are equivalent:

(i) The vector bundles ξ and $\overline{\xi}$ are equivalent to each other.
(ii) There exists a real vector bundle η such that ξ is equivalent to $\mathbb{C}\eta$.

C: Direct Sums and Tensor Products

If ξ_1, ξ_2 are two vector bundles of the same type (real, complex, oriented) and with the same base, then the (direct or Whitney) sum $\xi_1 \oplus \xi_2$ and the tensor product $\xi_1 \otimes \xi_2$ are defined as vector bundles with the same base whose fibers are, respectively, direct sums or tensor products of the fibers of the bundles ξ_1 and ξ_2. Here is a more formal definition of the sum (here and below, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$). Let $\xi_1 = (E_1, B_1, \mathbb{K}^{n_1}, p_1), \xi_2 = (E_2, B_2, \mathbb{K}^{n_2}, p_2)$ be two vector bundles (the bases may not be the same). Put $\xi_1 \times \xi_2 = (E_1 \times E_2, B_1 \times B_2, \mathbb{K}^{n_1+n_2}, p_1 \times p_2)$; this is a vector bundle over $B_1 \times B_2$ of dimension $n_1 + n_2$. If $B_1 = B_2 = B$, then we define $\xi_1 \oplus \xi_2$ as the restriction of $\xi_1 \times \xi_2$ to the diagonal $B \subset B \times B$. Another formal definition: Let $B_1 = B_2 = B$ and let $p_2^* \xi_1 = (\widetilde{E}, E_2, \mathbb{K}^{n_1}, \widetilde{p})$ be the bundle over E_2 induced by ξ_1. Then $\xi_1 \oplus \xi_2 = (\widetilde{E}, B, \mathbb{K}^{n_1+n_2}, p_2 \circ \widetilde{p})$.

There exists a different approach to the definition of $\oplus$ and $\otimes$ (see Sect. 19.4). At the moment, we speak of tensor products of vector bundles not specifying any formal definition; we hope that the reader will be able to create this definition without our help (Exercise 5).

EXERCISE 2. Prove the equivalence of the two definitions of $\xi_1 \oplus \xi_2$. (This will show, in particular, that the second definition is actually symmetric with respect to ξ_1 and ξ_2.)

EXERCISE 3. Introduce an orientation into the sum of two oriented bundles.

EXERCISE 4. Make up a formal definition of a tensor product of two (real or complex) vector bundles.

EXERCISE 5. For real or complex vector bundles ξ_1, ξ_2 with the same base, make up a definition of a vector bundle $\mathrm{Hom}(\xi_1, \xi_2)$.

Two vector bundles of the same type, but, possibly, of different dimensions, are called *stably equivalent* if they become equivalent after adding trivial bundles. To make up a more formal definition, notice that a standard trivial n-dimensional bundle $B \times \mathbb{K}^n \to B$ is usually denoted simply as n. With this notation,

$$\xi \sim_{\text{stab}} \eta \Leftrightarrow \exists m, n\colon \xi \oplus n \sim \eta \oplus m.$$

In conclusion, let us point out a connection of the sum construction with previous constructions.

EXERCISE 6. Make up a canonical real vector bundle equivalence $\mathbb{R}\mathbb{C}\xi \sim \xi \oplus \xi$ (where ξ is a real vector bundle).

EXERCISE 7. Make up a canonical complex vector bundle equivalence $\mathbb{C}\mathbb{R}\xi \sim \xi \oplus \overline{\xi}$ (where ξ is a complex vector bundle).

D: Linear Maps Between Vector Bundles, Subbundles, and Quotient Bundles

A linear map of a vector bundle $\xi_1 = (E_1, B_1, \mathbb{K}^{n_1}, p_1)$ into a vector bundle $\xi_2 = (E_2, B_2, \mathbb{K}^{n_2}, p_2)$ (as before, $\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$) is a pair of continuous maps $F\colon E_1 \to E_2$, $f\colon B_1 \to B_2$ such that $f \circ p_1 = p_2 \circ F$ and for every $x \in B$, the appropriate restriction of F is a linear map $p_1^{-1}(x) \to p_2^{-1}(f(x))$. The subbundle of a vector bundle $\xi = (E, B, \mathbb{K}^n, p)$ is a vector bundle $\xi' = \left(E', B, \mathbb{K}^{n'}, p|_{E'}\right)$ with $E' \subset E$ whose fibers are subspaces of the fibers of ξ. The inclusion map $E' \to E$ and the identity map $B \to B$ compose a linear map (inclusion) $\xi' \to \xi$. If ξ' is a subbundle of ξ, then a fiberwise factorization creates a quotient bundle ξ/ξ'. More formally, the total space of ξ/ξ' is obtained from E by a factorization over the equivalence relation: $x_1 \sim x_2$ if $p(x_1) = p(x_2)$ and $x_2 - x_1 \in E'$. There is an obvious linear map (projection) $\xi \to \xi/\xi'$.

Let us mention two important subbundles: $S^k\xi \subset \underbrace{\xi \otimes \cdots \otimes \xi}_{k}$ and $\Lambda^k\xi \subset \underbrace{\xi \otimes \cdots \otimes \xi}_{k}$.

E: Coordinate Presentation of a Vector Bundle

Let ξ be an n-dimensional vector bundle (of one of our three types). Fix an open covering $\{U_i\}$ of the base B such that the restrictions $\xi|_{U_i}$ are all trivial vector bundles; let $\varphi_i: p^{-1}(U_i) \to \mathbb{K}^n$ be a trivialization, that is, a map which is a vector space isomorphism on every $p^{-1}(x), x \in U_i$. For every $y \in U_i \cap U_j$, there arises a composition

$$\mathbb{K}^n \xrightarrow{\varphi_j^{-1}} p^{-1}(y) \xrightarrow{\varphi_i} \mathbb{K}^n;$$

the function which assigns this composition to y is a continuous map $\varphi_{ij}: U_i \cap U_j \to G$ where $G = GL(n, \mathbb{K})$ [$GL_+(n, \mathbb{R})$ in the case of an oriented bundle]. Moreover, (i) $\varphi_{ii}(y) = I$ for $y \in U_i$, (ii) $\varphi_{ji}(y) = \left(\varphi_{ij}(y)\right)^{-1}$ for $y \in U_i \cap U_j$, and (iii) $\varphi_{ik}(y)\varphi_{kj}(y) = \varphi_{ij}(y)$ for $y = U_i \cap U_j \cap U_k$. It is easy to understand that a set of maps $\varphi_{ij}: U_i \cap U_j \to G$ with properties (i)–(iii) gives rise to a vector bundle. This presentation of a vector bundle is called the *coordinate presentation.*

An obvious generalization of the so presented vector bundles consists in specifying a topological group G and a G-space F. Suppose that there are an open covering $\{U_i\}$ of a space B and a set of continuous functions $\varphi_{ij}: U_i \cap U_j \to G$ with properties (i)–(iii) just listed. In the disjoint union $\coprod_i (U_i \times F)$, make, for every $i, j, y \in U_i \cap U_j$, an identification $\left[(y, f) \in U_j \times F\right] \sim \left[(y, \varphi_{ij}(y)f) \in U_i \times F\right]$; the space arising we take for E. The projections $U_i \times F \to U_i \subset B$ form a projection $p: E \to F$, and there arises a locally trivial fibration (E, B, F, p) with a certain additional structure similar to a structure of a vector bundle. Such fibrations are called fiber bundles (or Steenrod fibrations); according to this terminology, G is the *structure group*, and F is the *standard fiber*. The reader can find details in the classical book by Steenrod [80], or in a variety of more modern books, for example, Husemoller [49]; here we only mention some examples.

There are many obvious examples. Take a coordinate presentation of a real, complex, or oriented vector bundle and assume that the functions φ_{ij} take values not in the group $GL(n, \mathbb{R}), GL(n, \mathbb{C})$ or $GL_+(n; \mathbb{R})$, but in some subgroup of one of these groups, say, in $O(n), SO(n)$, or $U(n)$. It is clear that the fiber bundles arising have an adequate description as real, complex, or oriented vector bundles with an additional structure, for the examples above, with an Euclidean or Hermitian structure, in every fiber. If the subgroup is the group of block diagonal matrices, $GL(p, \mathbb{K}) \times GL(q, \mathbb{K}) \subset GL(n; \mathbb{K}), n = p + q$, then the fiber bundle arising is the usual n-dimensional vector bundle presented as the sum of two vector bundles, of dimensions p and q. In a similar way, we can present vector bundles with a fixed nonvanishing section, or with a fixed subbundle, and so on. An example of a different nature: Take an arbitrary G and put $F = G$ with the left translation action; the fibrations arising are called *principal*. Some other examples will appear in the next sections.

19.2 Tangent and Normal Bundles

The notion of a *tangent vector* to a smooth manifold is very important, and for this reason it has many equivalent definitions. The most natural definition is based on local coordinates. Let x be a point of an n-dimensional manifold X, and let $\varphi\colon U \to \mathbb{R}^n$ be a chart such that $x \in U$; then a tangent vector to X at x is defined as a vector v of the space $\mathbb{R}^n$ at the point $\varphi(x)$. If there is another chart, $\psi\colon V \to \mathbb{R}^n$, also covering x, then the tangent vector corresponding to the chart φ and the vector v is identified with the tangent vector corresponding to the chart ψ and the vector $w = d_{\varphi,\psi}(v)$, where $d_{\varphi,\psi}$ is the differential of the map $\varphi(U \cap V) \to \psi(U \cap V)$, $y \mapsto \psi(\varphi^{-1}(y))$. Another possibility, which does not require a fixation of a chart, is to define a tangent vector at x as a class of parametrized smooth curves $\gamma\colon(-\varepsilon, \varepsilon) \to X$ such that $\gamma(0) = x$, where the curves γ, γ' are equivalent if $\mathrm{dist}(\gamma(t), \gamma'(t)) = o(t)$ (the distance is calculated with respect to any local coordinate system). An algebraically more convenient approach consists in defining a tangent vector of X at x as a linear map $v\colon \mathcal{C}^\infty(X) \to \mathbb{R}$ ($\mathcal{C}^\infty(X)$ is the space of real $\mathcal{C}^\infty$-functions) such that $v(fg) = v(f)g(x) + f(x)v(g)$ (in other words, tangent vectors are identified with directional derivatives). Finally, if X is presented as a smooth surface in an Euclidean space, then a tangent vector to X is simply a tangent vector to this surface. To make this definition compatible with previous definitions, we can say that a tangent vector at some point to the Euclidean space regarded as a smooth manifold is simply a vector of this space at this point, and tangent vectors to a submanifold are tangent vectors to the manifold tangent to the submanifold.

The set of tangent vectors to an n-dimensional manifold X at a point x is an n-dimensional vector space which is denoted as T_xX. The union of all spaces T_xX possesses a natural topology and, moreover, a structure of a $2n$-dimensional smooth manifold; this manifold is denoted as TX. The natural projection $TX \to X$ makes TX a total space of a vector bundle over X; this vector bundle is called the *tangent bundle* of X and is denoted as $\tau(X)$. A section of a tangent bundle is a vector field on the manifold. A manifold whose tangent bundle is trivial is called *parallelizable*; a manifold is parallelizable if it is possible to choose bases in all tangent spaces depending continuously of a point or, equivalently, if there exist $n = \dim X$ vector fields on X which are linearly independent at every point. For example, the circle is parallelizable, the torus is parallelizable, while the two-dimensional sphere is not parallelizable. The three-dimensional sphere is parallelizable: If it is presented as the space of unit quaternions, then the basis at the space T_xS^3 is formed by quaternions ix, jx, kx where i, j, k are quaternion units. If you replace quaternions by octonions, you will prove that the sphere S^7 is parallelizable. There is a remarkable fact that no spheres besides S^1, S^3, S^7 are parallelizable: This is one of the versions of the Frobenius conjecture proven by Adams (two different proofs, both belonging to Adams, will be presented in Chaps. V and VI later). Notice that the problem of parallelization of spheres is equivalent to the problem of existence of spheroids with the invariant Hopf equal to one (see Remark 5 in Sect. 16.5).

EXERCISE 8. Prove that the orientability of a manifold X (in the sense of Sect. 17.1) is equivalent to the orientability of the tangent bundle $\tau(X)$.

If Y is a submanifold of a manifold X, then there arise two vector bundles with the base Y: $\tau(Y)$ and $\tau(X)|_Y$, and $\tau(Y) \subset \tau(X)|_Y$ (a tangent vector to a submanifold is also a tangent vector to the manifold). The quotient bundle $\tau(X)|_Y/\tau(Y)$ is called the *normal bundle* of Y in X and is denoted as $\nu_X(Y)$ or $\nu(Y)$. The word "normal" is an indication of the fact that if X is a submanifold of an Euclidean space, then the total space of $\nu(Y)$ may be regarded as consisting of vectors at points of Y which are tangent to X and normal to Y.

Mark an isomorphism $\tau(Y) \oplus \nu(Y) = \tau(X)|_Y$. In particular, if $X = \mathbb{R}^n$, then $\tau(Y) \oplus \nu(Y) = n$.

Notice that the construction of normal bundles with all properties listed can be applied not only to submanifolds, that is, to embeddings of a manifold Y to a manifold X, but also to *immersions* $\iota\colon Y \to X$; the only significant change is that the restriction bundle $\tau(X)|_Y$ should be replaced by the induced bundle $\iota^*\tau(X)$.

EXERCISE 9. Deduce from the last equality that normal bundles of a manifold corresponding to different embeddings or immersions of this manifold to Euclidean spaces (possibly, of different dimensions) are stably equivalent.

EXERCISE 10. Prove that the normal bundle to an n-dimensional oriented surface embedded (or immersed) into the $(n + 1)$-dimensional Euclidean space is trivial. Deduce from this that the tangent bundle to such a surface (for example, to an arbitrary sphere with handles) is stably trivial (that is, stably equivalent to a trivial bundle). A manifold whose tangent bundle is stably trivial is called *stably parallelizable*. Obviously, a manifold is stably parallelizable if and only if its normal bundle is stably trivial.

FYI (this is not an exercise). A closed connected manifold is stably parallelizable if and only if it is parallelizable in the complement to a point. A noncompact connected manifold if stably parallelizable if and only if it is parallelizable. A manifold is stably parallelizable if and only if it is orientable and admits an immersion in the Euclidean space of the dimension bigger by 1.

EXERCISE 11. Let ζ be the Hopf bundle over $\mathbb{R}P^n$. Prove that

$$\tau(\mathbb{R}P^n) \oplus 1 \sim \underbrace{\zeta \oplus \cdots \oplus \zeta}_{n+1} = (n + 1)\zeta.$$

Prove a similar statement for $\mathbb{C}P^n$ [notice that the bundle $\tau(\mathbb{C}P^n)$ possesses a natural structure of a complex vector bundle].

19.3 Associated Fibrations and Characteristic Classes

A: An Introduction

Choose one of the three types of vector bundles, and choose integers n and q and an Abelian group G. A characteristic class c of n-dimensional vector bundles on the chosen type with values in q-dimensional cohomology with the coefficients in G is a function which assigns to every n-dimensional vector bundle ξ of the chosen type with a CW base B a cohomology class $c(\xi) \in H^q(B; G)$ such that if $f: B' \to B$ is a continuous map of another CW complex into B, then $c(f^*\xi) = f^*c(\xi)$. Here f^* on the left-hand side of the formula means the inducing operation for vector bundles, and on the right-hand side it means the induced cohomology homomorphism.

The term "characteristic class" is not new for us: In Sect. 18.5, we called the first obstruction to extending a section of a locally trivial fibration a characteristic class (or a primary characteristic class) of this fibration, and the equality $c(f^*\xi) = f^*c(\xi)$ held for that characteristic classes. However, that construction cannot be applied to vector bundles directly, because their fiber is contractible. (Recall that the coefficient domain for the characteristic classes of Sect. 18.5 is the first nontrivial homotopy group of the fiber.) What we still can do is to apply the construction to some fibration which can be constructed from the given vector bundle. An ample variety of such fibrations is delivered by the construction of an *associated fibration.*

B: A Construction of Associated Fibrations

This construction was actually described in Sect. 19.1.E. We take a coordinate presentation $\{\{U_i\}, \{\varphi_{ij}: U_i \cap U_j \to G\}\}$ of a vector bundle with the base B [where $G = GL(n, \mathbb{R}), GL_+(n; \mathbb{R})$ or $GL(n, \mathbb{C})$] and choose an arbitrary space F with an action of the group G. After this, we construct the total space E of a new fibration as

$$\textstyle\coprod_i (U_i \times F) \mid [(y,f) \in U_j \times F] \sim [(y, \varphi_{ij}(y)f) \in U_i \times F] \\ \text{for all } y \in U_i \cap U_j, f \in F.$$

The fibration (E, B, F, p) [where $p: E \to B$ is the projection $(y,f) \mapsto y$] is the associated (by the given vector bundle) fibration with the standard fiber F. However, usually we will not need this general construction: Almost always, we will restrict ourselves to one particular case of it, which is described ahead. Let $\xi = (E, B, \mathbb{R}^n, p)$, or $(E, B, \mathbb{C}^n, p)$, be a given vector bundle, and let $1 \le k \le n$. Put

$$E_k = \{(x_1, \ldots, x_k) \in E \times \cdots \times E \mid p(x_1) = \cdots = p(x_k); \\ x_1, \ldots, x_k \text{ are linearly independent}\}.$$

There is an obvious projection $p_k\colon E_k \to B$, and there arises a locally trivial fibration $\xi_k = (E_k, B, R_k, p_k)$ where R_k is the space of all linearly independent k-frames in $\mathbb{R}^n$ or $\mathbb{C}^n$. (This is the fibration associated with ξ with the standard fiber R_k.) The case $k = 1$ is especially simple: E_1 is $E - B$, where B is embedded into E as the zero section, and R_1 is $\mathbb{R}^n - 0$ or $\mathbb{C}^n - 0$.

Point out a small defect of this construction (rather more aesthetic than mathematical). The fibers are noncompact spaces which would have better been replaced by homotopy equivalent classical manifolds: Stiefel manifolds and spheres. This can be done with the help of the following simple lemma.

Lemma. *If a vector bundle has a CW base, then it is possible to introduce in all fibers an Euclidean or Hermitian structure which depends continuously on the point of the base; moreover, this can be done in a homotopically unique way.*

Proof. The set of all Euclidean (Hermitian) structures in fibers of a vector bundle is a total space of a fibration whose fiber is the space of all Euclidean (Hermitian) structures in a given vector space (this is also a fibration associated with the vector bundle). Obviously, the fiber of this fibration is contractible (it is a convex subset of the space of all symmetric bilinear (Hermitian) forms in this vector space. This fibration has a section (all the obstructions are zeroes) and this section is homotopically unique (all difference cochains are zeroes). This is precisely the statement of the lemma.

Using these Euclidean or Hermitian structures in the fibers, we can replace the fibration ξ_k into the fibration ξ_k^0 whose total space is the space of all orthonormal (unitary) frames in the fibers of ξ. The fiber of ξ_k^0 is the Stiefel manifold $V(n,k)$ or $\mathbb{C}V(n,k)$; in particular, ξ_1^0 is the fibration whose fiber is the sphere S^{n-1} (S^{2n-1} in the complex case); this fibration is called *spherical.*

C: Classical Characteristic Classes of Vector Bundles

Let ξ be an n-dimensional oriented (real) vector bundle with the CW base B. Consider the corresponding spherical fibration ξ_1^0. It is easy to see that the orientability of the bundle ξ implies the orientability of the fibration ξ_1^0; that is, the fibration ξ_1^0 is homologically simple. (The reader may prove that a Steenrod bundle whose structure group is connected is always simple.) Thus, there arises the first obstruction to extending a section of ξ_1^0, and this first obstruction is an element of $H^n(B;\mathbb{Z})$. Regarded as a characteristic class of the bundle ξ, this element is called the *Euler class* of ξ; the notation: $e(\xi)$.

Pass to the fibrations ξ_k^0.

Lemma. *Let* $1 \le k < n$. *Then*

(i) $\pi_i(V(n,k)) = 0$ *for* $i < n - k$.

(ii) $\pi_{n-k}(V(n,k)) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 1 \text{ or } n-k \text{ is even;} \\ \mathbb{Z}_2 & \text{in all other cases.} \end{cases}$

Proof. The case $k = 1$ is trivial: $\pi_i(V(n,1)) = \pi_i(S^{n-1})$ is zero for $i < n-1$ and $\mathbb{Z}$ for $i = n-1$. Let $k \geq 2$, and consider the fibration

$$V(n,k) \xrightarrow{V(n-1,k-1)} S^{n-1}$$

[the projection assigns to $\{v_1, \dots, v_k\} \in V(n,k)$ the last vector v_k]. Consider the fragment

$$\pi_{i+1}(S^{n-1}) \to \pi_i(V(n-1,k-1)) \to \pi_i(V(n,k)) \to \pi_i(S^{n-1})$$

of the homotopy sequence of this fibration. If $i < n-2$, then the first and last terms are zeroes, and we get an isomorphism $\pi_i(V(n-1,k-1)) \cong \pi_i(V(n,k))$. Thus, if $i < n-k$ and $k > 1$, then

$$\pi_i(V(n,k)) \cong \pi_i(V(n-1,k-1)) \cong \cdots \cong \pi_i(V(n-k+1,1)) = \pi_i(S^{n-k}) = 0.$$

For $i = n-k$, this chain of isomorphisms becomes shorter:

$$\pi_{n-k}(V(n,k)) \cong \pi_{n-k}(V(n-1,k-1)) \cong \cdots \cong \pi_{n-k}(V(n-k+2,2)),$$

and the general case of the lemma is reduced to the case of $V(n,2)$. We need to prove that $\pi_{n-2}(V(n,2)) = \mathbb{Z}$ for n even and $\mathbb{Z}_2$ for n odd. For $k = 2$ and $i = n-2$, our homotopy sequence becomes

$$\begin{array}{ccccccc} \pi_{n-1}(S^{n-1}) & \to & \pi_{n-2}(S^{n-2}) & \to & \pi_{n-2}(V(n,2)) & \xrightarrow{\partial_*} & \pi_{n-2}(S^{n-1}) \\ \| & & \| & & & & \| \\ \mathbb{Z} & & \mathbb{Z} & & & & 0. \end{array}$$

Thus, $\pi_{n-2}(V(n,2)) = \operatorname{Coker}[\partial_*\colon \pi_{n-1}(S^{n-1}) \to \pi_{n-2}(S^{n-2})]$. The space $V(n,2)$ is the space T_1S^{n-1} of unit tangent vectors to the sphere S^{n-1}, the fibration $V(n,2) \xrightarrow{S^{n-2}} S^{n-1}$ is the natural fibration of the space of unit tangent vectors. The construction of the homomorphism ∂_* is the following. We take a homotopy of an $(n-2)$-dimensional spheroid of S^{n-1} sweeping an $(n-1)$-dimensional spheroid, lift this homotopy to T_1S^{n-1}, and obtain a spheroid of the fiber. If we apply this construction to the identity spheroid $S^{n-1} \to S^{n-1}$, the lifting provides a vector field on S^{n-1}, and the resulting element of $\pi_{n-2}(S^{n-2})$ is the value of the obstruction to extending a vector field on S^{n-1}. As proved in Sect. 18.5 (see Proposition 2), this value is the Euler characteristic of S^{n-1}, that is, 2 for n odd and 0 for n even. Thus, the homomorphism $\partial_*\colon \pi_{n-1}(S^{n-1}) \to \pi_{n-2}(S^{n-2})$ is trivial if n is even and is a multiplication by 2 if n is odd. This completes the proof of the lemma.

The lemma shows that the first obstruction to extending a section of fibration ξ_k^0 (or ξ_k) takes value in $H^{n-k+1}(B; \mathbb{Z}$ or $\mathbb{Z}_2)$. Reduced modulo 2, this obstruction is a characteristic class of ξ with the values in $H^j(B; \mathbb{Z}_2)$, $j = n - k + 1$. This class is called the jth *Stiefel–Whitney class* of ξ and is denoted as $w_j(\xi)$. We also put $w_i(\xi) = 0$ for $i > \dim \xi$ and $w_0(\xi) = 1 \in H^0(B; \mathbb{Z}_2)$.

Notice that the orientability of the vector bundle ξ which was needed for the simplicity of the fibration ξ_k becomes unnecessary after reducing modulo 2; thus, the Stiefel–Whitney classes are defined for arbitrary real vector bundles.

For an n-dimensional oriented vector bundle ξ, $w_n(\xi) = \rho_2 e(\xi)$, where ρ_2 is the reduction modulo 2.

The complex version of the previous construction is a simplified version of it.

Lemma. *Let* $1 \leq k < n$. *Then*

$$\pi_i(\mathbb{C}V(n,k)) \cong \begin{cases} 0 & \text{for } i < 2(n-k)+1, \\ \mathbb{Z} & \text{for } i = 2(n-k)+1. \end{cases}$$

Proof This repeats the first, easier, part of the proof of the previous lemma and is based on the equality $\mathbb{C}V(n,1) = S^{2n-1}$ and the homotopy sequence

$$\pi_{i+1}(S^{2n-1}) \to \pi_i(\mathbb{C}V(n,k)) \to \pi_i(\mathbb{C}V(n-1,k-1) \to \pi_i(S^{2n-1})$$

of the fibration $\mathbb{C}V(n,k) \xrightarrow{\mathbb{C}V(n-1,k-1)} S^{2n-1}$.

Let ξ be an n-dimensional complex vector bundle with a CW base B. The lemma shows that the first obstruction to extending a section in the fibration ξ_k^0 (or ξ_k) is a class $c_j(\xi) \in H^{2j}(B; \mathbb{Z})$ where $j = n-k+1$. We get a characteristic class of complex vector bundles which is called the jth *Chern class*. Precisely as in the real case, we put $c_i(\xi) = 0$ for $i > \dim_{\mathbb{C}} \xi$ and $c_0(\xi) = 1$.

Finally, if ξ is again an n-dimensional vector bundle, then we put $p_j(\xi) = (-1)^j c_{2j}(\mathbb{C}\xi) \in H^{4j}(B; \mathbb{Z})$ and call the classes $p_j(\xi)$ *Pontryagin classes* of the bundle ξ. [The sign $(-1)^j$ has a historic origin. The reason why we restrict ourselves to even-numbered Chern classes is that the odd-numbered Chern classes of a complexification of a real vector bundle have order at most 2; see Exercise 15 in Sect. 19.5 later.] It is possible to define Pontryagin classes directly: We can associate with an n-dimensional vector bundle a fibration whose standard bundle is the space of all systems of $n-2j+2$ vectors of rank $> n-2j$; the first obstruction to extending sections in this fibration is $p_j(\xi)$ (the reader can try to prove this although it is not awfully interesting).

EXERCISE 12. Prove that $w_1(\xi) = 0$ if and only if the bundle ξ is orientable.

EXERCISE 13. Prove that if ξ is an n-dimensional complex vector bundle, then

$$e(\mathbb{R}\xi) = c_n(\xi),\ w_{2j}(\mathbb{R}\xi) = \rho_2 c_j(\xi),\ w_{2j+1}(\mathbb{R}\xi) = 0.$$

D: Geometric Construction of Euler, Stiefel–Whitney, and Chern Classes

Let ξ be an n-dimensional oriented vector bundle with a CW base B. Then there exists a nowhere vanishing section of ξ over the $(n-1)$st skeleton B^{n-1} of B. We can extend this section to B^n, but it may have zeroes over n-dimensional cells. If we assume these zeroes to be transverse intersections with the zero section, then we can count the "algebraic number" of these zeroes (that is, we assign a $+$ or $-$ sign to every zero), and a function which assigns this number to every cell is an n-dimensional integral cellular cocycle. Its cohomology class is the Euler class $e(\xi)$ (this is the construction of the first obstruction).

If ξ is not assumed oriented, then the previous construction gives a cohomology class modulo 2, and this is $w_n(\xi)$. We can construct in this way the other Stiefel–Whitney classes. Namely, let us assume that ξ has an Euclidean structure (in the fibers), and consider again a nowhere vanishing section of ξ over B^{n-1}. Let us try to construct a second nowhere vanishing section of ξ orthogonal to the first section. This can be done over B^{n-2}, but if we want to extend the second section to B^{n-1}, we have to admit that it will have zeroes over $(n-1)$-dimensional cells. Assuming these zeroes transverse, we can count their number modulo 2 in every $(n-1)$-dimensional cell, and in this way we get an $(n-1)$-dimensional cellular cocycle with coefficients in $\mathbb{Z}_2$, and the cohomology class of this cocycle is $w_{n-1}(\xi)$. Then we construct a third section orthogonal to the first two, it can be made nowhere vanishing over B^{n-3}, but to extend this third section to B^{n-2}, we have to admit transverse zeroes over $(n-2)$-dimensional cells, and in this way we obtain a cocycle representing $w_{n-2}(\xi)$. And so on.

The Chern classes of complex vector bundles may be constructed in a similar way; we leave the details to the reader.

19.4 Characteristic Classes and Classifying Spaces

A: The Classification Theorem

In Sect. 19.1.A, we mentioned tautological bundles over Grassmannians. They will be of primary importance now.

The theory here has three absolutely parallel versions for the three types of vector bundles. We will consider in detail the real case; the transition to the two other cases does not require any efforts: One should just replace the Grassmannians $G(\infty, n)$ by $G_+(\infty, n)$ and $\mathbb{C}G(\infty, n)$.

Recall that the total space of the tautological bundle (which we denote as η or η_n) over the Grassmannian $G(\infty, n)$ is the space of pairs (π, x) where $\pi \in G(\infty, n)$ is an n-dimensional subspace of $\mathbb{R}^N$ and $x \in \pi \subset \mathbb{R}^\infty$; the projection acts as $(\pi, x) \mapsto \pi$.

Theorem. *Let X be a finite CW complex. Then*

(i) *For every n-dimensional vector bundle ξ over X, there exists a continuous map $f: X \to G(\infty, n)$ such that $f^*\eta = \xi$.*
(ii) *This map f is unique up to a homotopy; that is, if $f_1^*\eta \sim f_2^*\eta$, then $f_1 \sim f_2$ (the second $\sim$ means a homotopy).*
(iii) *Conversely, if $f_1 \sim f_2$, then $f_1^*\eta \sim f_2^*\eta$.*

Corollary. *The correspondence $f \mapsto f^*\eta$ establishes a bijection between the set $\pi(X, G(\infty, n))$ of homotopy classes of continuous maps $X \to G(\infty, n)$ and equivalence classes of n-dimensional vector bundles with the base X.*

Proof of Theorem. First, notice that since X is compact and $G(\infty, n) = \overrightarrow{\lim}\, G(N, n)$, a continuous map $X \to G(\infty, n)$ is the same as a continuous map $X \to G(N, n)$ (with sufficiently large N) composed with the inclusion map $G(N, n) \to G(\infty, n)$. Same for homotopies: Maps $X \to G(N_1, n) \to G(\infty, n)$ and $X \to G(N_2, n) \to G(\infty, n)$ are homotopic if and only if maps $X \to G(N_1, n) \to G(M, n)$ and $X \to G(N_2, n) \to G(M, n)$ are homotopic for sufficiently large M.

Second, notice that statements (i) and (ii) are covered by the following relative version of statement (i):

(i$'$) *Let X be a finite CW complex, and ξ be an n-dimensional vector bundle over X. Then let A be a CW subcomplex of X, and let $g: A \to G(\infty, n)$ be a continuous map such that $g^*\eta \sim \xi|_A$. Then there exists a continuous map $f: X \to G(\infty, n)$ such that $f^*\eta \sim \xi$ and $f|_A = g$.*

We begin with proving statement (i), that is, (i$'$) with $A = \emptyset$, and then we will explain what we need to add to handle the case $A \neq \emptyset$. A *linear functional* on the total space E of ξ is a continuous function $E \to \mathbb{R}$ which is linear on every fiber of the bundle ξ. To construct a linear functional on E, it is sufficient to take some linear function $\varphi: p^{-1}(x) \to \mathbb{R}$ (where $x \in X$), then extend it to a linear functional $p^{-1}(U) \to \mathbb{R}$ where U is a neighborhood of x such that the restriction $\xi|_U$ is trivial [there is a retraction $\rho: p^{-1}(U) \approx U \times p^{-1}(x) \to \pi^{-1}(x)$ which is linear on every fiber, and the composition $\varphi \circ \rho$ is a required functional], and then we multiply the last functional by a continuous function $X \to \mathbb{R}$, which is 1 in a neighborhood V of x such that $\overline{V} \subset U$ and is 0 in the complement of U. We apply this construction to some linearly independent functionals $\varphi_i: p^{-1}(x) \to \mathbb{R}$, $i = 1, \ldots, n$, and we get linear functionals $\varphi_{x,i}: E \to \mathbb{R}$ whose restrictions to $p^{-1}(x)$ are linearly independent; hence, for some neighborhood U_x of x the restrictions of these functionals to $p^{-1}(y)$ are linearly independent for all $y \in U_x$. Since X is compact, there exist some $x_1, \ldots, x_m$ such that the sets U_{x_j}, $j = 1, \ldots, m$ cover X. Then the functionals $\varphi_{x_j,i}$ have the following property: *For every $z \in X$ there are n of these functionals which are linearly independent on $p^{-1}(z)$.*

Together, the $N = mn$ functionals $\varphi_{x_j,i}: E \to \mathbb{R}$ form a map $\Phi: E \to \mathbb{R}^N$, and for every $z \in X$, the restriction $\Phi|_{p^{-1}(z)}$ is a linear monomorphism. The image $\Phi(p^{-1}(z))$ is an n-dimensional subspace of $\mathbb{R}^N$, and we define the map $f: X \to G(N, n)$ by the formula $f(z) = \Phi(p^{-1}(z))$. Since Φ maps isomorphically the fiber of ξ over $z \in X$ onto the fiber of η over $f(z)$, we have $f^*\eta = \xi$, as required.

Now, let us adjust this proof to the relative version. We assume that there are some $A \subset X$, $g: A \to G(M,n) \subset G(\infty,n)$, and an equivalence between $\xi|_A$ and $g^*\eta$. The last equivalence is the same as a continuous map $\psi: p^{-1}(A) \to \mathbb{R}^M$ which maps isomorphically $p^{-1}(z)$ ($z \in A$) onto the subspace $g(z)$ of $\mathbb{R}^M$. First, we extend this map to a continuous map $\Psi: E \to \mathbb{R}^M$ which is linear on each fiber of ξ [to do this, we need to extend each of the M coordinate functions of ψ to a function $E \to \mathbb{R}$ linear on fibers; this is the same as extending from A to X a section of a certain vector bundle (composed of dual spaces $(p^{-1}(x))^*$) which does not meet any obstruction, since the fibers of a vector bundle are contractible]. The linear maps $\Psi|_{p^{-1}(y)}$ are isomorphisms for $y \in A$, and hence they are isomorphisms for $y \in W$, where W is some open neighborhood of A. To finish the construction, we take a $\Phi: X \to \mathbb{R}^N$ as constructed above and multiply this Φ by a continuous function $h: X \to \mathbb{R}$ which is 0 on A and 1 in the complement of W, $\Phi' = h\Phi$. The functions Ψ and Φ' together form a map $\Omega: E \to \mathbb{R}^{M+N}$, which is a linear monomorphism on every fiber of ξ such that $\Omega|_{p^{-1}(A)}$ is the composition $p^{-1}(A) \xrightarrow{\psi} \mathbb{R}^M \xrightarrow{\subset} \mathbb{R}^{M+N}$. This Ω gives rise to a continuous map $f: X \to G(M+N,n)$ such that $f^*\eta = \xi$ and $f|_A$ is the composition $A \xrightarrow{g} G(M,n) \xrightarrow{\subset} G(M+N,n)$. This completes a proof of (i′).

It remains to prove (iii). Our proof is based on the following simple observation. We say that n-dimensional subspaces π_1, π_2 of $\mathbb{R}^n$ are *close* to each other if no nonzero vector of π_1 is orthogonal to π_2 (this condition is symmetric in π_1, π_2); equivalently: π_1 is close to π_2 if the orthogonal projection $\pi_1 \to \pi_2$ is an isomorphism. Obviously, every $\pi \in G(N,n)$ has a neighborhood U in $G(N,n)$ such that every $\pi \in U$ is close to π.

Lemma. *Let $f_1, f_2: X \to G(N,n)$ (no restrictions on X) be two continuous maps such that, for every $x \in X$, the subspaces $f_1(x), f_2(x)$ of $\mathbb{R}^n$ are close to each other. Then $f_1^*\eta \sim f_2^*\eta$.*

Proof of Lemma. Let $p_1: E_1 \to X, p_2: E_2 \to X$ be the bundles $f_1^*\eta, f_2^*\eta$. For every $x \in X$, the definition of the inducing operation provides isomorphisms $\eta_1: p_1^{-1}(x) \to f_1(x)$, $\eta_2: p_2^{-1}(x) \to f_2(x)$; also, there is the orthogonal projection $\pi: f_1(x) \to f_2(x)$. The composition $\eta_2^{-1} \circ \pi \circ \eta_1: p_1^{-1}(x) \to p_2^{-1}(x)$ is an isomorphism depending continuously on x; and these isomorphisms form an equivalence $f_1^*\eta \sim f_2^*\eta$.

Proof of (iii). If X is compact (otherwise, arbitrary), and $\{f_t: X \to G(N,n)\}$ is a homotopy, then there exists an m such that, for every $i, 0 \le i < m$, the maps $f_{i/m}, f_{(i+1)/m}$ satisfy the condition of the lemma. Hence, for every $i, f_{i/m}^*\eta \sim f_{(i+1)/m}^*\eta$. Hence, $f_0^*\eta \sim f_1^*\eta$, which is the statement of (iii).

B: More General Constructions

The space $G(\infty,n)$ is called a *classifying space* for real n-dimensional vector bundles, and η is called a *universal bundle*; a similar terminology is applied to $G_+(\infty,n)$ and $\mathbb{C}G(\infty,n)$. There exists a far-reaching generalization of the preceding construction. For a topological group G, there exists a principal fibration (see Sect. 19.1.E) (EG, BG, G, p_G) with a cellular base and contractible space EG; for a given G, a principal fibration with these properties is unique up to a homotopy equivalence. The space BG is called the *classifying space* for G; in particular, $BGL(n,\mathbb{R}) = BO(n) = G(\infty,n)$, $BGL_+(n,\mathbb{R}) = BSO(n) = G_+(\infty,n)$, $BGL(n,\mathbb{C}) = BU(n) = \mathbb{C}G(\infty,n)$. If F is a space with a faithful action of G, then, for a finite CW complex X, there is a bijection between the set of equivalence classes of Steenrod bundles over X with the structure group G and the standard fiber F and the set $\pi(X, BG)$ of homotopy classes of continuous maps $X \to BG$. This construction belongs to J. Milnor [55]. It has further generalizations to the cases when G is not a topological group, but an H-space or a topological groupoid.

C: Immediate Applications of the Classification Theorem

Some definitions and theorems of the previous sections can be clarified with the help of the classification theorem of Sect. 19.4.A. For example, the lemma of Sect. 19.3.B, which states that every vector bundle whose base is a finite CW complex can be furnished by an Euclidean or Hermitian structure in the fibers, follows immediately from the theorem of Sect. 19.4.A. Namely, if we fix an Euclidean structure in $\mathbb{R}^N$ [or a Hermitian structure in $\mathbb{C}^N$], then all n-dimensional subspaces inherit this structure. This provides an Euclidean or Hermitian structure in the fibers of η, and hence in the fibers of all vector bundles induced by η, that is, of all vector bundles whose bases are finite CW complexes.

The definition of the sum of vector bundles can be done in the following way: If $f: X \to G(N,n)$ and $g: X \to G(M,m)$ are two continuous maps, then there arises a map $f \oplus g: X \to G(M+N, m+n)$, $(f \oplus g)(x) = f(x) \oplus g(x) \subset \mathbb{R}^N \oplus \mathbb{R}^M$, and $f^*\eta \oplus g^*\eta = (f \oplus g)^*\eta$, which gives an alternative construction of the sum of vector bundles. The same for tensor products: We consider a map $f \otimes g: X \to \mathbb{R}^{NM}$, $f \otimes g(x) = f(x) \otimes g(x) \subset \mathbb{R}^N \otimes \mathbb{R}^M = \mathbb{R}^{NM}$, and $f^*\eta \otimes g^*\eta = (f \otimes g)^*\eta$, which can be regarded as a definition of a tensor product of vector bundles (same with complex vector bundles). In a similar way, for a vector bundle ξ, we can define $S^r\xi$, $\Lambda^r\xi$, ξ^*, etc.

D: Characteristic Classes and Cohomology of Classifying Spaces

Theorem. *The group of q-dimensional characteristic classes of n-dimensional real [resp. n-dimensional oriented, resp. n-dimensional complex] vector bundles with coefficients in G is isomorphic to the group $H^q(G(\infty,n); G)$ [resp. $H^q(G_+(\infty,n); G)$, resp. $H^q(\mathbb{C}G(\infty,n); G)$].*

Proof. We restrict ourselves to the real case; the proof in the other two cases is the same. If c is a characteristic class of the type considered, we can compute it for the bundle η over $G(\infty, n)$ [or over $G(N, n)$ with $N \gg n, q$]. We get a $c(\eta) \in H^q(G(\infty, n); G)$. We need to check two things: (i) If $c(\eta) = 0$, then $c = 0$; (ii) for every $\gamma \in H^q(G(\infty, n); G)$, there exists a characteristic class c such that $c(\eta) = \gamma$.

(i) Let $c(\eta) = 0$. If ξ is an n-dimensional vector bundle whose base X is a finite CW complex, then $\xi = f^*\eta$ for some $f: X \to G(\infty, n)$, and therefore $c(\xi) = c(f^*\eta) = f^*c(\eta) = f^*(0) = 0$. If the base X of ξ is an arbitrary CW complex, and $0 \neq c(\xi) = \alpha \in H^q(X; G)$, then there exists a finite CW subcomplex Y of X such that $\alpha|_Y \neq 0$; then $0 = c(\xi|_Y) = c(\xi)|_Y = \alpha|_Y \neq 0$, which is a contradiction.

(ii) Let $\gamma \in H^q(G(\infty, n); G)$; we want to define a characteristic class c with $c(\eta) = \gamma$. Let ξ be an n-dimensional real vector bundle with a CW base X. Then, for every finite CW subcomplex Y of X, we can define $c(\xi|_Y) \in H^q(Y; G)$ as $f^*\gamma$, where $f: Y \to G(\infty, n)$ is a continuous map with $f^*\eta = \xi|_Y$. Then, obviously, there exists a unique $\alpha \in H^q(X; G)$ such that $\alpha|_Y = c(\xi|_Y)$ for every finite $Y \subset X$. We set $c(\xi) = \alpha$.

(Both parts of this proof implicitly use the following property of cohomology. Let X be a CW complex, and let $\mathcal{F}$ be the category of finite CW subcomplexes of X and inclusions. Then $H^q(X; G) = \overleftarrow{\lim}_{\mathcal{F}} H^q(Y; G)$. This follows, for example, from a similar property for homology and the universal coefficients formula for cohomology. We leave the details to the reader.)

GENERALIZATION. *Characteristic classes of Steenrod fibrations with the structure group G taking values in the q-dimensional cohomology of the base with coefficients in A correspond bijectively to elements of $H^q(BG; A)$.*

E: Completeness of Systems of Euler, Stiefel–Whitney, Chern, and Pontryagin Characteristic Classes

Theorem. (i) *Every characteristic class of n-dimensional real vector bundles with coefficients in $\mathbb{Z}_2$ is a polynomial of the Stiefel–Whitney classes $w_1, \dots, w_n$, and different polynomials are different characteristic classes.*

(ii) *Every characteristic class of n-dimensional complex vector bundles with coefficients in $\mathbb{Z}$ is a polynomial of the Chern classes $c_1, \dots, c_n$, and different polynomials are different characteristic classes.*

(iii) *Every characteristic class of n-dimensional real vector bundles with coefficients in $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$ is a polynomial of the (images with respect to the inclusion of $\mathbb{Z}$ into the coefficient domain) of the Pontryagin classes $p_1, \dots, p_{[n/2]}$, and different polynomials are different characteristic classes.*

(iv) *Every characteristic class of n-dimensional orientable vector bundles with coefficients in $\mathbb{Q}$, or $\mathbb{R}$, or $\mathbb{C}$ is a polynomial of the (images with respect* *to the inclusion of $\mathbb{Z}$ into the coefficient domain) of the Pontryagin classes $p_1, \dots, p_{[n/2]}$ and, if n is even, the Euler class e, and different polynomials are different characteristic classes.*

We postpone the details of the proof to the next section. Here we only notice that the proof of every part consists of two parts. First, we need to show that the corresponding group

$$H^q(G(\infty, n), G_+(\infty, n), \text{ or } \mathbb{C}(\infty, n);\ \mathbb{Z}_2, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \text{ or } \mathbb{C})$$

has precisely the same size as the group of polynomials of the form indicated. This can be easily deduced from the computation of the cohomology of Grassmannians in Sect. 13.8.C. Next, we need to check that no one of these polynomials is zero as a characteristic class [in the nonfield case (ii) we will need slightly more]. For this purpose, we need a sufficient supply of explicit computations of characteristic classes. At the moment, we do not have such a supply, but it will appear in the next section.

19.5 The Most Important Properties of the Euler, Stiefel–Whitney, Chern, and Pontryagin Classes

A: The Properties of the Stiefel–Whitney Classes

Theorem. *The Stiefel–Whitney classes possess the following properties.*

(1) *For the Hopf (tautological) bundle ζ over $\mathbb{R}P^n$ $(n \geq 2)$, $0 \neq w_1(\zeta) \in H^1(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2$ and $w_i(\zeta) = 0$ for $i > 1$.*
(2) *For arbitrary real vector bundles ξ, η with (the same) CW base,*

$$w_i(\xi \oplus \eta) = \sum_{p+q=i} w_p(\xi) w_q(\eta).$$

Remark. Statements (1) and (2) are often considered as axioms for Stiefel–Whitney classes: Together with the property that Stiefel–Whitney classes are characteristic classes, these axioms uniquely determine them. We will not return to this axiomatic definition of Stiefel–Whitney classes, but the reader will be able to deduce all necessary statements from the results of the current section. In details, this axiomatic approach to all classical characteristic classes is developed in the book *Characteristic Classes* by Milnor, Stasheff [60].

Proof of Part (1) is immediate. The restriction of ζ to $\mathbb{R}P^1 = S^1$ is the Möbius bundle, and obviously it has no nowhere vanishing section. Thus, ξ has no section over the first skeleton, which means that the first obstruction $w_1(\zeta) \in H^1(\mathbb{R}P^n; \mathbb{Z}_2)$ is not zero.

Part (2) is equivalent to the statement (2′): *for arbitrary real vector bundles* ξ, η *with, possibly different, CW bases,*

$$w_i(\xi \times \eta) = \sum_{p+q=i} w_p(\xi) \times w_q(\eta).$$

The proof of (2′) consists of three steps.

Step 1. Stiefel–Whitney classes invariant with respect to stable equivalence, which is the same as the statement $w_i(\xi \oplus 1) = w_i(\xi)$. This follows from the inductive construction of Stiefel–Whitney classes outlined in Sect. 19.3.D. For the first section of $\xi \oplus 1$ we can take the natural nonzero section of the summand 1. Then the second section of $\xi \oplus 1$ is the first section of ξ, the third section of $\xi \oplus 1$ is the second section of ξ, and so on. We see that if $\dim \xi = n$, then, for every k, $w_{(n+1)-(k+1)+1}(\xi \oplus 1) = w_{n-k+1}(\xi)$, which is our statement.

Step 2. Let $\dim X = \dim \xi = p$, $\dim Y = \dim \eta = q$ (where X and Y are the bases of ξ and η); statement (2′) in this case means $w_{p+q}(\xi \times \eta) = w_p(\xi) \times w_q(\eta)$. Fix a section of ξ which has no zeroes on X^{p-1} and has transverse zeroes on p-dimensional cells; for a p-dimensional cell e of X, let $n_e \in \mathbb{Z}_2$ be the number of zeroes of the section on e reduced modulo 2. Similarly, fix a section of η, without zeroes on Y^{q-1} and with transverse zeroes on q-dimensional cells, and let $m_f \in \mathbb{Z}_2$ be the number of zeroes of η on a cell f reduced modulo 2. Then $w_p(\xi)$ is represented by the cocycle $e \mapsto n_e$, and $w_q(\eta)$ is represented by the cocycle $f \mapsto m_f$. The two sections together form a section of $\xi \times \eta$ with no zeroes on $(X \times Y)^{p+q-1}$ and with transverse zeroes on cells $e \times f$, the number of which modulo 2 is $n_e m_f$. Thus, $(e \times f \mapsto n_e m_f)$ is a cocycle of the class $w_{p+q}(\xi \times \eta)$ which shows that $w_{p+q}(\xi \times \eta) = w_p(\xi) \times w_q(\eta)$.

Step 3. The general case. Fix p, q with $p + q = i, p \leq \dim \xi, q \leq \dim \eta$, and consider the restrictions $\xi|_{X^p}, \eta|_{Y^q}$. We know that ξ has $\dim \xi - p$ linearly independent sections over X^p and η has $\dim \eta - q$ linearly independent sections over Y^q. From this, we conclude that

$$\xi|_{X^p} = \xi_p \oplus (\dim \xi - p), \ \ \eta|_{Y^q} = \eta_q \oplus (\dim \eta - q)$$

where ξ_p and η_q are bundles over X^p and Y^q of dimensions p and q. Certainly, it is also true that

$$(\xi \times \eta)|_{X^p \times Y^q} = (\xi_p \times \eta_q) + (\dim \xi + \dim \eta - p - q).$$

Let

$$u = w_i(\xi \times \eta) - \sum_{p'+q'=i} w_{p'}(\xi) \times w_{q'}(\eta) \in H^i(X \times Y; \mathbb{Z}_2).$$

Then

$$\begin{aligned} u|_{X^p\times Y^q} &= w_i((\xi\times\eta)|_{X^p\times Y^q}) - \textstyle\sum_{p'+q'=i} w_{p'}(\xi|_{X^p})\times w_{q'}(\eta|_{Y^q}) \\ &= w_i(\xi_p\times\eta_q) - \textstyle\sum_{p'+q'=i} w_{p'}(\xi_p)\times w_{q'}(\eta_q) \\ &= w_i(\xi_p\times\eta_q) - w_p(\xi_p)\times w_q(\eta_q) = 0 \end{aligned}$$

(here the first equality is obvious, the second equality follows from the result of step 1, the third equality follows from triviality of Stiefel–Whitney classes in dimensions exceeding the dimension of the bundle, and the last equality is the result of step 2). We see that $u|_{X^p\times Y^q} = 0$ for any p, q with $p+q=i$. Consider the homomorphism

$$\begin{aligned} H^i(X\times Y;\mathbb{Z}_2) &= \textstyle\bigoplus_{p+q=i} H^p(X;\mathbb{Z}_2)\otimes H^q(Y;\mathbb{Z}_2) \\ &\to \textstyle\bigoplus_{p+q=i} H^p(X^p;\mathbb{Z}_2)\otimes H^q(Y^q;\mathbb{Z}_2) \\ &= \textstyle\bigoplus_{p+q=i} H^i(X^p\times Y^q;\mathbb{Z}_2); \end{aligned}$$

here the two equalities follow from Künneth's formula, and the arrow denotes the sum of homomorphisms induced by the inclusion maps $X^p\to X$, $Y^q\to Y$. On the one hand, every homomorphism $H^p(X;\mathbb{Z}_2)\to H^p(X^p;\mathbb{Z}_2)$ is a monomorphism (since $H^p(X,X^p;\mathbb{Z}_2)=0$), and similarly for Y; thus, the preceding homomorphism is a monomorphism. On the other hand, this homomorphism acts as

$$\gamma\mapsto(\gamma|_{X^i\times Y^0}, \gamma|_{X^{i-1}\times Y^1}, \dots, \gamma|_{X^1\times Y^{i-1}}, \gamma|_{X^0\times Y^i}).$$

Hence, it takes u to 0, and hence $u=0$. This completes the proof.

It is convenient to write the formulas from (2) and (2′) as

$$w(\xi\oplus\eta)=w(\xi)w(\eta),\ \ w(\xi\times\eta)=w(\xi)\times w(\eta)$$

where w is the formal sum $1+w_1+w_2+\dots$.

B: The Splitting Principle for the Stiefel–Whitney Classes

We begin with a computation of the Stiefel–Whitney classes for a very important example.

Proposition. *Consider the vector bundle* $\underbrace{\zeta\times\dots\times\zeta}_{n}$ *over the space* $\underbrace{\mathbb{R}P^\infty\times\dots\times\mathbb{R}P^\infty}_{n}$. *Let* $x_1,\dots,x_n\in H^1(\mathbb{R}P^\infty\times\dots\times\mathbb{R}P^\infty;\mathbb{Z}_2)$ *be the generators of* $H^*(\mathbb{R}P^\infty\times\dots\times\mathbb{R}P^\infty;\mathbb{Z}_2)$. *Then*

$$w_i(\zeta\times\dots\times\zeta)=e_i(x_1,\dots,x_n),$$

where e_i is ith elementary symmetric polynomial.

Proof. Since $w(\zeta) = 1 + x$, the preceding formula (2′) shows that

$$\begin{aligned} w(\zeta \times \cdots \times \zeta) = (1+x) \times \cdots \times (1+x) &= (1+x_1)\dots(1+x_n) \\ &= 1 + \textstyle\sum_{i=1}^n e_i(x_1, \dots, x_n), \end{aligned}$$

as required.

Now we can prove a result announced in the previous section.

Proof of the Theorem in Sect. 19.4.E, Part (i). It is well known in algebra that every symmetric polynomial in n variables with coefficients in an arbitrary integral domain R has a unique presentation as a polynomial in the elementary symmetric polynomial; the uniqueness statement means that no nonzero polynomial in $e_1, \dots, e_n$ is equal to zero. If $W = P(w_1, \dots, w_n)$ is a nonzero polynomial of the Stiefel–Whitney classes, then $W(\zeta \times \cdots \times \zeta) = P(e_1(x_1, \dots, x_n), \dots, e_n(x_1, \dots, x_n)) \neq 0$, which shows that W is not zero as a characteristic class. Hence, the dimension (over $\mathbb{Z}_2$) of the full space of q-dimensional characteristic classes with coefficients in $\mathbb{Z}_2$ of n-dimensional real vector bundles is at least the number of partitions $q = 1 \cdot r_1 + 2 \cdot r_2 + \dots + n \cdot r_n$ with nonnegative r_is. But this number is precisely the number of q-dimensional cells in the standard (Schubert) CW decomposition of $G(\infty, n)$, which, in turn, does not exceed $\dim_{\mathbb{Z}_2} H^q(G(\infty, n); \mathbb{Z}_2)$, that is, the dimension of the space of characteristic classes. Thus, all these numbers and dimensions are the same. This proves that all the characteristic classes of n-dimensional vector bundles with coefficients in $\mathbb{Z}_2$ are polynomials in $w_1, \dots, w_n$, as stated in part (i) of the theorem in Sect. 19.4.E.

Remark 1. This proof shows that $\dim_{\mathbb{Z}_2} H^q(G(\infty, n); \mathbb{Z}_2)$ is actually equal to the number of q-dimensional Schubert cells, which means, in turn, that all the incidence numbers in the cellular complex corresponding to the Schubert cell decomposition of the Grassmannian are even. This fact is not new for us; it was offered as Exercise 11 in Sect. 13.8.C. Now we have a proof of this fact, thus a (rather indirect) solution of that exercise.

Remark 2. We see also that a nonzero characteristic class with coefficients in $\mathbb{Z}_2$ of n-dimensional vector bundles takes a nonzero value on the bundle $\zeta \oplus \cdots \oplus \zeta$. This provides a method of finding relations between characteristic classes: A relation holds if it holds for $\zeta \oplus \cdots \oplus \zeta$. Usually, this statement is formulated in a seemingly weaker, but actually equivalent form: *To establish a relation between characteristic classes it is sufficient to check it for splitting bundles, that is, for bundles isomorphic to sums of one-dimensional bundles.* This proposition is known under the name of the *splitting principle* (later, we will deal with different versions of this principle).

EXERCISE 14. Prove the following version of the splitting principle (and explain why it is equivalent to the splitting principle). The $\mathbb{Z}_2$-cohomology homomorphism induced by the map

$$\underbrace{\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty}_{n} \to G(\infty, n),$$

$$\big(\{\ell_1 \subset \mathbb{R}^\infty\}, \ldots, \{\ell_n \subset \mathbb{R}^\infty\}\big) \mapsto \ell_1 \times \cdots \times \ell_n \subset \mathbb{R}^\infty \times \cdots \times \mathbb{R}^\infty = \mathbb{R}^\infty$$

is a monomorphism; moreover, its image in $H^*(\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty; \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \ldots, x_n]$ is precisely the space of symmetric polynomials.

C: Stiefel–Whitney Classes and Operations over Vector Bundles

Formulas expressing the Stiefel–Whitney classes of the bundles $\xi \otimes \eta$, $\Lambda^k \xi$, $S^k \xi$, and so on via the Stiefel–Whitney classes of ξ and η (and the dimensions of ξ and η) exist, but more complicated and less convenient, than the formulas for the Stiefel–Whitney classes of the sum (or direct product). We will give a brief overview of this subject.

Lemma. *Let ξ, η be one-dimensional real vector bundle over the same CW base. Then*

$$w_1(\xi \otimes \eta) = w_1(\xi) + w_1(\eta).$$

Proof. Fix sections s, t of ξ and η over the 1-skeleton X^1 of the base X of ξ η. We may assume that these sections have no zeroes over X^0 and have transverse zeroes over one-dimensional cells, and the zeroes of s are different from the zeroes of t. Then $s \otimes t$ is a section of $\xi \otimes \eta$, and the set of zeroes of $s \otimes t$ is the union of the set of zeroes of s and the set of zeroes of t. Let m_e, n_e be residues modulo 2 of the numbers of zeroes of the sections s and t within a one-dimensional cell of X. Then the functions $e \mapsto m_e$, $e \mapsto n_e$, $e \mapsto m_e + n_e$ are cocycles representing $w_1(\xi), w_1(\eta), w_1(\xi \otimes \eta)$, whence our result.

For our next statement, we will need some notations from algebra of symmetric polynomials. Consider the ith symmetric polynomial of mn variables $y_j + z_k$, $1 \le j \le m$, $1 \le k \le n$, and express it as a polynomial in elementary symmetric polynomials separately in $y_1, \ldots y_m$ and $z_1, \ldots, z_n$ (we assume that $i \le m$ and $i \le n$):

$$e_i(y_j + z_k) = E_{m,n;i}(e_1(y), e_2(y), \ldots; e_1(z), e_2(z), \ldots);$$

for example,

$$\begin{aligned}
E_{m,n;1} &= \sum_{j=1}^{m}\sum_{k=1}^{n}(y_j+z_k) = n\sum_{j=1}^{m} y_j + m\sum_{k=1}^{n} z_k = ne_1(y)+me_1(z);\\
E_{m,n;2} &= \sum_{(j',k')\neq(j,k)} (y_j+z_k)(y_{j'}+z_{k'})\\
&= n(n-1)\sum_{j=1}^{m} y_j^2 + n^2\sum_{j'\neq j} y_jy_{j'} + 2(mn-1)\sum_{j=1}^{m}\sum_{k=1}^{n} y_jz_k\\
&\qquad + m(m-1)\sum_{k=1}^{n} z_k^2 + m^2\sum_{k'\neq k} z_kz_{k'}\\
&= n(n-1)(e_1(y)^2-2e_2(y)) + 2n^2e_2(y) + 2(mn-1)e_1(y)e_1(z)\\
&\quad + m(m-1)(e_1(z)^2-2e_2(z)) + 2m^2e_2(z),
\end{aligned}$$

that is,

$$\begin{aligned}
E_{m,n;2} = \frac{n(n-1)}{2}\,e_1(y)^2 + ne_2(y) + (mn-1)e_1(y)e_1(z)\\
+\frac{m(m-1)}{2}\,e_1(z)^2 + me_2(z).
\end{aligned}$$

These examples show that it is possible to find explicit expressions for the polynomials $E_{m,n;i}$, but the formula may be complicated.

In addition, consider the elementary symmetric polynomials of $\binom{n}{r}$ variables $x_{j_1}+\cdots+x_{j_r}$ $1\le j_1<\cdots<j_r\le n$. Obviously, they are symmetric polynomials in $x_1,\ldots,x_n$, and we can write

$$e_i(x_{j_1}+\cdots+x_{j_r}\ 1\le j_1<\cdots<j_r\le n) = F_{n;r;i}(e_1(x),e_2(x),\ldots),$$

where $F_{n;r;i}$ is a polynomial. For example,

$$F_{n;r;1} = \binom{n-1}{r-1}e_1(x),\ F_{n;2;2} = \frac{(n-1)(n-2)}{2}\,e_1(x)^2 + (n-2)e_2(x).$$

The polynomials F are related to the polynomials E. Namely, if we put $x_1 = y_1,\ldots,x_m = y_m, x_{m+1} = z_1,\ldots,x_{m+n} = z_n$, then, obviously, $\{x_j+x_k|1\le j<k\le m+n\} = \{y_j+y_k|1\le j<k\le m\}\bigcup\{y_j+z_k|1\le j\le m,\ 1\le k\le n\}\bigcup\{z_j+z_k|1\le j<k\le n\}$, which shows that

$$\begin{aligned}
F_{m+n;2;i}(e_1(x),e_2(x),\ldots) = \sum_{p+q+r=i} F_{m;2;p}(e_1(y),e_2(y),\ldots)\\
\cdot E_{m,n;q}(e_1(y),e_2(y),\ldots,e_1(z),e_2(z),\ldots)\cdot F_{n;2;r}(e_1(z),e_2(z),\ldots).
\end{aligned}\tag{$*$}$$

And one more family of polynomials:

$$G_{n;r;i}(e_1(x), e_2(x), \dots) = e_i(x_{j_1} + \dots + x_{j_r} | 1 \leq j_1 \leq \dots \leq j_r \leq n);$$

a computation shows that

$$\begin{aligned} G_{n;r;1} &= \binom{n+r-1}{r-1} e_1(x), \\ G_{n;2} &= \frac{(n-1)(n+2)}{2} e_1^2 + (n+2)e_2, \text{ if } n > 1. \end{aligned}$$

The formula (∗) with the polynomials F replaced by polynomials G is also true.

Now, let us formulate the main results of this section.

Theorem 1. *Let ξ and η be real vector bundles of dimensions m and n over the same CW base. Then*

$$w_i(\xi \otimes \eta) = E_{m,n;i}(w_1(\xi), w_2(\xi), \dots; w_1(\eta), w_2(\eta), \dots);$$

thus, in particular, $w_1(\xi \otimes \eta)$ and $w_2(\xi \otimes \eta)$ are, respectively,

$$nw_1(\xi) + mw_1(\eta)$$

and

$$\begin{aligned} \frac{n(n-1)}{2} w_1(\xi)^2 + nw_2(\xi) &+ (mn-1)w_1(\xi)w_1(\eta) \\ &+ \frac{m(m-1)}{2} w_1(\eta)^2 + mw_2(\eta). \end{aligned}$$

Theorem 2. *Let ξ be an n-dimensional real vector bundle with a CW base. Then*

$$w_i(\Lambda^r \xi) = F_{n;r;i}(w_1(\xi), w_2(\xi), \dots);$$

in particular,

$$\begin{aligned} w_1(\Lambda^r \xi) &= \binom{n-1}{r-1} w_1(\xi), \\ w_2(\Lambda^2 \xi) &= \frac{(n-1)(n-2)}{2} w_1(\xi) + (n-2)w_2(\xi). \end{aligned}$$

Theorem 3. *Let ξ be an n-dimensional real vector bundle with a CW base. Then*

$$w_i(S^r \xi) = G_{n;r;i}(w_1(\xi), w_2(\xi), \dots);$$

in particular,

$$\begin{aligned} w_1(S^r\xi) &= \binom{n+r-1}{r-1} w_1(\xi), \\ w_2(S^2\xi) &= \frac{(n-1)(n+2)}{2} w_1(\xi) + (n+2)w_2(\xi), \text{ if } n > 1. \end{aligned}$$

Proofs. We begin with Theorem 1 in the case when $\xi = \eta$. The class $w_i(\xi \otimes \xi)$ is a characteristic class of a real vector bundle. Hence, by part (i) of the theorem in Sect. 19.4.E, it must be a polynomial in Stiefel–Whitney classes. To identify this polynomial, we need to compute the class for the bundle $\xi = \underbrace{\zeta \times \cdots \times \zeta}_{n}$ over $\underbrace{\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty}_{n}$; this bundle is the same as $\zeta_1 \oplus \cdots \oplus \zeta_n$, where ζ_j is the bundle induced by ζ with respect to the projection of $\mathbb{R}P^\infty \times \cdots \times \mathbb{R}P^\infty$ onto the jth factor. Then $\xi \otimes \xi = \bigoplus_{j,k}(\zeta_j \otimes \zeta_k)$ and

$$\begin{aligned} w(\xi \otimes \xi) = \prod_{j,k} w(\zeta_j \otimes \zeta_k) &= \text{ (by Lemma) } \prod_{j,k}(1 + x_j + x_k) \\ &= 1 + \sum_{i \geq 1} e_i(x_j + x_k | 1 \leq j \leq n, 1 \leq k \leq n) \\ &= 1 + \sum_{i \geq 1} E_{n,n;i}(e_1(x), e_2(x), \dots, e_1(x), e_2(x), \dots) \\ &= 1 + \sum_{i \geq 1} E_{n,n;i}(w_1(\xi), w_2(\xi), \dots, w_1(\xi), w_2(\xi), \dots), \end{aligned}$$

which is the statement of the theorem (for $\xi = \eta$).

Next, we prove Theorem 2. The proof is the same as the previous proof, and it is based on the relation, for $\xi = \zeta_1 \oplus \cdots \oplus \zeta_n$,

$$\Lambda^r \xi = \bigoplus_{1 \leq j_1 < \cdots < j_r \leq n} (\zeta_{j_1} \otimes \cdots \otimes \zeta_{j_r}),$$

which gives, by the lemma,

$$\begin{aligned} w(\Lambda^r \xi) &= \prod_{1 \leq j_1 < \cdots < j_r \leq n} (1 + (j_1 + \cdots + j_r)) \\ &= 1 + \sum_{1 \leq j_1 < \cdots < j_r \leq n} e_1(x_{j_1} + \cdots + x_{j_r} | 1 \leq j_1 < \cdots < j_r \leq n). \end{aligned}$$

The rest of the proof repeats the previous proof.

The proof of Theorem 3 is so close to the proof of Theorem 2 that we do not feel any necessity in detailing it [just mention that it is based on the relation $S^r\xi = \bigoplus_{1\le j_1\le\cdots\le j_r\le n} (\zeta_{j_1}\otimes\cdots\otimes\zeta_{j_r})$].

Finally, let us prove Theorem 1 in the general case. Notice that for any vector bundles ξ and η,

$$\Lambda^2(\xi\oplus\eta) = \Lambda^2\xi\oplus(\xi\otimes\eta)\oplus\Lambda^2\eta,$$

and hence

$$w(\Lambda^2(\xi\oplus\eta)) = w(\Lambda^2\xi)w(\xi\otimes\eta)w(\Lambda^2\eta).$$

Since $w = 1 + w_1 + w_2 + \ldots$ is an invertible element of the ring $H^*(B;\mathbb{Z}_2)$, this formula determines $w(\xi\otimes\eta)$ if $w(\Lambda^2\xi)$, $w(\Lambda^2\eta)$ and $\Lambda^2(\xi\oplus\eta)$ are known. The formula from Theorem 1 follows from the formula of Theorem 2 and the relation ($*$).

D: Properties of the Euler, Chern, and Pontryagin Classes

For the Euler classes, a multiplication formula $e(\xi\otimes\eta) = e(\xi)e(\eta)$ holds.

All the major properties of the Stiefel–Whitney classes can be repeated with appropriate changes for the Chern classes. In particular, the class c_1 of the Hopf bundle $\zeta_{\mathbb{C}}$ is the standard generator of the group $H^2(\mathbb{C}P^\infty;\mathbb{Z})$. There are the multiplication formula

$$c_i(\xi\oplus\eta) = \sum_{p+q=i} c_p(\xi)c_q(\eta)$$

and the splitting principle. Like Stiefel–Whitney classes, the Chern classes are invariant with respect to stable equivalence. The computation of the Chern classes of tensor product, exterior powers, and symmetric powers of complex vector bundles repeats the computations in Sect. 19.5.C.

EXERCISE 15. Prove that $c_i(\overline{\xi}) = (-1)^i c_i(\xi)$. Deduce from this that for every *real* vector bundle ξ and every *odd* i the equality $2c_i(\mathbb{C}\xi) = 0$ (compare the comment to the definition of the Pontryagin classes in Sect. 19.3.C).

EXERCISE 16. Define a polynomial Q_r of r variables by the formula

$$N_r = Q_r(e_1,\ldots,e_r),$$

where the e_i are elementary symmetric polynomials and N_i are sums of ith powers of variables (so $Q_1 = e_1$, $Q_2 = e_1^2 - 2e_2$, $Q_3 = e_1^3 - 3e_1e_2 + 3e_3, \dots$). For a complex vector bundle ξ with the base X, put

$$\mathrm{ch}_r(\xi) = \frac{1}{r!}Q_r(c_1(\xi), \dots, c_r(\xi)) \in H^{2r}(X; \mathbb{Q}).$$

The (nonhomogeneous) characteristic class ch with coefficients in $\mathbb{Q}$ defined by the formula

$$\mathrm{ch} = \mathrm{ch}_0 + \mathrm{ch}_1 + \mathrm{ch}_2 + \cdots \in H^{\mathrm{even}}(X; \mathbb{Q})$$

is called the *Chern character*. Notice that $\mathrm{ch}_0(\xi) \in H^0(X; \mathbb{Q})$ is just $\dim \xi$.

Prove that

$$\mathrm{ch}(\xi \oplus \eta) = \mathrm{ch}(\xi) + \mathrm{ch}(\eta) \text{ and } \mathrm{ch}(\xi \otimes \eta) = \mathrm{ch}(\xi)\,\mathrm{ch}(\eta).$$

For the Pontryagin classes, the multiplication formulas and all the other formulas are deduced from the corresponding formulas for the Chern classes and hold "modulo 2-torsion"; for example,

$$2\left(p_i(\xi \oplus \eta) - \sum\nolimits_{p+q=i} p_p(\xi)p_q(\eta)\right) = 0.$$

EXERCISE 17. Prove that stably equivalent bundles have equal Pontryagin classes.

E: More Relations Between Stiefel–Whitney, Chern, and Euler Classes

In conclusion, we give two more formulas expressing the Stiefel–Whitney and Chern classes via the Euler class. Let ξ be an n-dimensional real vector bundle with a CW base X and ζ be the Hopf bundle over $\mathbb{R}P^\infty$. Consider the bundle $\xi \otimes \zeta$ over $X \times \mathbb{R}P^\infty$ (more precisely, it is the tensor product of bundles induced by ξ and ζ with respect to the projections of the product $X \times \mathbb{R}P^\infty$ onto the factors). Then

$$\rho_2 e(\xi \otimes \zeta) = w_n(\xi \otimes \zeta) = \sum_{i=0}^{n} (w_i(\xi) \times x^{n-i}) \in H^n(X \times \mathbb{R}P^\infty; \mathbb{Z}_2),$$

where $x \in H^1(\mathbb{R}P^\infty; \mathbb{Z}_2)$ is the generator. Similarly, if ξ is an n-dimensional complex vector bundle with a CW base X and $\zeta_{\mathbb{C}}$ is the (complex) Hopf bundle over $\mathbb{C}P^\infty$, then

$$e(\xi \otimes \zeta_{\mathbb{C}}) = c_n(\xi \otimes \zeta) = \sum_{i=0}^{n} (c_i(\xi) \times x^{n-i}) \in H^{2n}(X \times \mathbb{C}P^\infty; \mathbb{Z}),$$

where $x \in H^2(\mathbb{C}P^\infty; \mathbb{Z})$ is the generator.

These formulas may be regarded as definitions of the Stiefel–Whitney and Chern classes.

EXERCISE 18. Prove these formulas.

19.6 Characteristic Classes in Differential Topology

We can only touch on this vast subject.

A: Geometric Interpretation of the First Obstruction

Let (E, B, F, p) be a homotopically simple locally trivial fibration where E and B are smooth manifolds and the manifold B is closed, n-dimensional and oriented, and p is a submersion, that is, a smooth map whose differential at every point has rank equal to n. Assume also that $\pi_0(F) = \cdots = \pi_{k-2}(F) = 0$ and $\pi_{k-1}(F) = \pi$. Then the first obstruction to extending a section of our fibration lies in $H^k(B; \pi)$. Suppose also that we were able to construct a section over $B - X$ where X is a submanifold of B (possibly, with singularities of codimension ≥ 2) of dimension $n - k$ or a union of a finite number of such submanifolds which are connected and transversally intersect each other, $X = \bigcup X_i$ (simple general position argumentations show that it is always possible to do this). For every i, choose a nonsingular point x_i of X_i and construct a small $(k-1)$-dimensional sphere s_i centered at x_i in a k-dimensional surface transversally intersecting X_i at x_i. Since there is a section over s_i, and the fibration is trivial in a proximity of x_i, we obtain a continuous map $S^{k-1} \to p^{-1}(x_i)$ which determines, since the fibration and the fiber are homotopically simple, an element $\alpha_i \in \pi_{k-1}(F) = \pi$.

Claim: The homology class $\sum_i \alpha_i [X_i] \in H_{n-k}(B; \pi)$ is the Poincaré dual of the first obstruction to extending a section in our fibration. The proof is left to the reader. (*Hint*: Triangulate the manifold B in such a way that X is disjoint from the simplices of dimension less than k and intersects each k-dimensional simplex transversally in at most one point.)

B: Differential Topology Interpretations of the Euler Class

For a closed oriented manifold X, the value of the Euler class of the tangent bundle $e(X) = e(\tau_X)$ on the fundamental class $[X]$ is equal to the Euler characteristic $\chi(X)$ of X (this is Proposition 2 of Sect. 18.5). This implies that a closed manifold possesses a nonvanishing vector field if and only if its Euler characteristic is zero (corollary in Sect. 18.5).

Some other properties of the Euler class are given here as exercises.

EXERCISE 19. Prove that a closed manifold X (orientable or not) possesses a continuous family of tangent lines (equivalently: The tangent bundle $\tau(X)$ possesses a one-dimensional subbundle) if and only if $\chi(X) = 0$.

EXERCISE 20. Let $(E, B, \mathbb{R}^n, p)$ be a smooth vector bundle (that is, a vector bundle such that E and B are smooth manifolds, p is a submersion, and the vector space operations in E are smooth). Suppose that B is closed and oriented. Let $s: B \to E$ be a section of ξ in a general position with the zero section. Show that the intersection $B \cap s(B)$ (we assume that B is embedded into E as the zero section) represents the homology class of B which is the Poincaré dual of the Euler class $e(\xi)$ of ξ.

EXERCISE 21. Let Y be a closed oriented submanifold of a closed oriented manifold X, and let $\nu_X(Y) = (\tau(X))|_Y/\tau(Y)$ be the corresponding normal bundle. Prove the formula

$$D(e(\nu_X(Y))] = i^![Y],$$

where D is Poincaré isomorphism (in Y), $i: Y \to X$ is the inclusion map, and $[Y]$ is a homology class of X represented by Y. Corollary: If $[Y] = 0$, then $e(\nu_X(Y)) = 0$; in particular, the Euler class of the normal bundle of a manifold embedded into an Euclidean space or a sphere is zero.

EXERCISE 22. The last statement does not hold for immersions. Show, in particular, that if f is an immersion of a closed oriented manifold of even dimension n into $\mathbb{R}^{2n}$ with transverse self-intersections, then the algebraic number of the self-intersection points (the reader will have to make up the definition of a sign corresponding to a transverse self-intersection) is equal to one half of the "normal Euler number," that is, of the value of the Euler class of the normal bundle on the fundamental class of the manifold. Example: Construct an immersion of S^2 into $\mathbb{R}^4$ with one transverse self-intersection (such a two-dimensional figure-eight) and find the Euler class of the corresponding normal vector bundle.

C: Differential Topology Interpretations of the Stiefel–Whitney Classes

In this section, we deal only with cohomology and homology with coefficients in $\mathbb{Z}_2$ and understand accordingly Poincaré isomorphism D.

The Stiefel–Whitney classes of the tangent bundle of a smooth manifold X are called the Stiefel–Whitney classes of X and are denoted as $w_i(X)$. [In a similar way, people consider the Pontryagin classes $p_i(X)$ of a smooth manifold X and the Chern classes $c_i(X)$ of a complex manifold X.] Since the normal bundle of a smooth manifold embedded into a Euclidean space does not depend, up to a stable equivalence, on the embedding, we can speak of the "normal Stiefel–Whitney classes," $\overline{w}_i(X)$, of a smooth manifold X. It follows from the multiplication formula and the fact that the sum of the tangent and normal bundles is trivial that

$$\sum_{p+q=i} w_p(X)\overline{w}_q(Y) = 0 \text{ for } i > 0,$$

or $\overline{w} = w^{-1}$ (we already remarked in the end of Sect. 19.5.C that w is invertible in the cohomology ring). Thus, the normal Stiefel–Whitney classes are expressed via the usual (tangent) Stiefel–Whitney classes.

EXERCISE 23. Consider a generic smooth map (the reader is supposed to clarify the meaning of the word *generic*) of a closed n-dimensional manifold X into $\mathbb{R}^q$, $q \leq n$; let $Y \subset X$ be the set of points where this map is not a submersion (the rank of the differential is less than q). Prove that Y is a $(q-1)$-dimensional submanifold of X [maybe, with singularities, but the class $[Y] \in H_{q-1}(X; \mathbb{Z}_2)$ is defined] and that

$$D^{-1}[Y] = w_{n-q+1}(X).$$

EXERCISE 24. Consider a generic smooth map of a closed n-dimensional manifold X into $\mathbb{R}^q$, $q \geq n$; let $Y \subset X$ be the set of points where this map is not an immersion (the rank of the differential is less than n). Prove that Y is a $(2n-q-1)$-dimensional submanifold of X (maybe, with singularities) and that

$$D^{-1}[Y] = \overline{w}_{q+1-n}(X).$$

EXERCISE 25. If an n-dimensional manifold X possesses an immersion into $\mathbb{R}^{n+q}$, then $\overline{w}_i(X) = 0$ for $i > q$. (For closed manifolds, this follows from Exercise 24, but actually this fact is much easier than Exercise 24, and it is more natural to prove it directly.)

EXERCISE 26. If an n-dimensional manifold X possesses an embedding into $\mathbb{R}^{n+q}$, then $\overline{w}_i(X) = 0$ for $i \geq q$. (To prove this, one needs to use, in addition to Exercise 25, the corollary part of Exercise 21.)

EXERCISE 27. Prove that if $2^k \leq n < 2^{k+1}$, then $\mathbb{R}P^n$ has no immersion in $\mathbb{R}^{2^{k+1}-2}$ and no embedding in $\mathbb{R}^{2^{k+1}-1}$. (To prove this, one needs to use, besides Exercises 25 and 26, Exercise 12 (Sect. 19.2) and the theorem in Sect. 19.5.A.

Remark 1. Thus, if $n = 2^k$, the n-dimensional manifold $\mathbb{R}P^n$ cannot be embedded into $\mathbb{R}^{2n-1}$. This is a very rare phenomenon. The classical Whitney theorem asserts that an n-dimensional manifold (with a positive n) can always be embedded into $\mathbb{R}^{2n}$ (this result should not be confused with an earlier theorem of Whitney stating that any smooth map of an n-dimensional manifold into any manifold of dimension $\geq 2n+1$ can be smoothly approximated by smooth embeddings); embeddings into

$\mathbb{R}^{2n-1}$ are almost always possible: For a nonexistence of such an embedding, it is necessary and sufficient that n is a power of 2, and there exists a one-dimensional cohomology class with coefficients in $\mathbb{Z}_2$ whose nth power is not zero (these conditions imply the nonorientability).

Remark 2. Further information concerning embeddability of (real and complex) projective spaces into Euclidean spaces can be obtained with the help of K-theory (see Sect. 42.6 in Chap. 6).

EXERCISE 28. Let X be a triangulated smooth manifold. Denote by C_i the i-dimensional classical chain of the barycentric subdivision of the triangulation of X equal to the sum of all i-dimensional simplices of this subdivision. Prove that C_i is a cycle and that

$$D^{-1}[C_i] = w_i(X)$$

($[C_i]$ is the homology class of C_i).

The values of the cohomology classes of the form $w_{i_1}(X)\dots w_{i_r}(X)$ with $i_1 \dots i_r = n$ on the fundamental class of closed n-dimensional manifold (they are residues modulo 2) are called *Stiefel–Whitney numbers* of the manifold X; notation: $w_{i_1\dots i_r}[X]$. For example, two-dimensional manifolds have two Stiefel–Whitney numbers: $w_{11}[X]$ and $w_2[X]$.

EXERCISE 29. Find Stiefel–Whitney numbers of classical surfaces.

Remark. The reader will see that for any classical surface X, $w_{11}[X] = w_2[X]$. A classical theorem in the topology of a manifold asserts any connected closed two-dimensional manifold is a classical surface. Hence, the two Stiefel–Whitney numbers, $w_{11}[X]$ and $w_2[X]$, are always the same. Later in this section, we will see that there are more relations between Stiefel–Whitney numbers.

Theorem. *If a closed manifold is a boundary of a compact manifold, then all its Stiefel–Whitney numbers are zeroes.*

Proof. If $X = \partial Y$ and $i\colon X \to Y$ is the inclusion map, then $\tau(X) = \tau(Y)|_X \oplus 1$ (the normal bundle of the boundary is always trivial!). Hence, $w_j(X) = i^* w_j(Y)$ for every j, and

$$\begin{aligned}\langle w_{j_1}(X)\dots w_{j_r}(X), [X]\rangle &= \langle i^*(w_{j_1}(Y)\dots w_{j_r}(Y)), [X]\rangle \\ &= \langle w_{j_1}(Y)\dots w_{j_r}(Y), i_*[X]\rangle = 0\end{aligned}$$

since $i_*[X] = 0$ (the fundamental cycle of the boundary of a compact manifold is the boundary of the fundamental cycle of this manifold).

This theorem provides a powerful necessary condition for a closed manifold to be a boundary of a compact manifold.

EXERCISE 30. Prove that if $n+1$ is not a power of 2, then neither $\mathbb{R}P^n$ nor $\mathbb{C}P^n$ is a boundary of a compact manifold.

The most striking fact, however, is that this necessary condition is also sufficient for a closed manifold to be a boundary of a compact manifold (R. Thom, Fields Medal of 1952). We will discuss the proof of this result briefly in Chaps. 5 and 6. As we mentioned before, the Stiefel–Whitney numbers are not linearly independent ($w_1[X] = 0$ for any one-dimensional X, $w_{11}[X] = w_2[X]$ for any two-dimensional X). The fact is that a maximal linear independent system of Stiefel–Whitney numbers of a closed n-dimensional manifold is formed by the numbers $w_{j_1..j_r}[X]$ such that $j_1 + \cdots + j_r = n, j_1 \leq \cdots \leq j_r$ and no one of the numbers $j_s + 1$ is a power of 2. (Corollary: Every closed three-dimensional manifold is the boundary of some compact four-dimensional manifold; this is a classical theorem of Rokhlin.)

D: Differential Topology Interpretations of the Pontryagin Classes

The following statement is similar to Exercise 23. Let X be a closed oriented n-dimensional manifold and $f: X \to \mathbb{R}^{n-2q+2}$ be a generic smooth map. Let $Y \subset X$ be the set of points where the rank of the differential of f does not exceed $n-2q$ (that is, is at least 2 less than its maximal possible value). Then Y is an oriented $(n-4q)$-dimensional manifold (maybe, with singularities), and the class $[Y] \in H_{n-4q}(X)$ is the Poincaré dual to the Pontryagin class $p_q(X) \in H^{4q}(X;\mathbb{Z})$ of (the tangent bundle of) the manifold X. A similar statement holds for the normal Pontryagin classes (compare to Exercise 24.)

(The orientedness, and even orientability, of manifold X is actually not needed, but, in general, we will need the version of Poincaré isomorphism developed in Sect. 17.12.)

If X is a closed oriented manifold of dimension $4m$, then the value of the class $p_{j_1}(X)\dots p_{j_r}(X), j_1 + \cdots + j_r = m$ on the fundamental homology class of X is called a *Pontryagin number* and is denoted as $p_{j_1..j_r}[X]$. (It is convenient to assume that X is not necessarily connected; the fundamental class of a disconnected X is defined as the sum of the fundamental classes of the components.) If X is a boundary of a compact oriented manifold, then all the Pontryagin numbers of X are zeroes (this fact is proved precisely as the similar fact for the Stiefel–Whitney numbers). There also is a *Thom theorem* which asserts that if all the Pontryagin numbers of a closed orientable manifold are zeroes (for example, if its dimension is not divisible by 4), then a union of several copies of X (taken all with the same orientation) is a boundary of some compact manifold. Moreover, every set of integers $\{p_{j_1..j_r} \mid j_1 + \cdots + j_r = m\}$ becomes, after a multiplication of all the numbers in the set by the same positive integer, the set of Pontryagin numbers of some closed oriented manifold of dimension $4m$. (Actually, this theorem is way easier than the similar theorem for the Stiefel–Whitney numbers; we will see this in Chap. 6.)

A useful corollary of the Thom theorem (and the fact that if $Y = X_1 \bigsqcup X_2$ is the disjoint union of two closed oriented $4m$-dimensional manifolds, then

$$p_{j_1..j_r}[Y] = p_{j_1..j_r}[X_1] + p_{j_1..j_r}[X_2]$$

for every $j_1, \dots, j_r$ with $j_1 + \dots + j_r = m$) is the following statement.

EXERCISE 31. Suppose that for every closed oriented n-dimensional manifold there is assigned an integer $\sigma(X)$ with the following properties: (1) If X is a boundary of a compact oriented manifold, then $\sigma(X) = 0$; (2) $\sigma(X_1 \bigsqcup X_2) = \sigma(X_1) + \sigma(X_2)$. Prove that

$$\sigma(X) = \sum_{j_1+\dots+j_r=n/4} a_{j_1..j_r} p_{j_1..j_r}[X],$$

where $a_{j_1..j_r}$ are some *rational* numbers not depending on X. In particular, $\sigma(X) = 0$ if n is not divisible by 4.

This statement has only one broadly known application, but what an application it is! Denote by $\sigma(X)$ the signature of the intersection index form in the $2m$-dimensional homology of a $4m$-dimensional closed oriented manifold X. The theorem in Sect. 17.10 shows that σ satisfies condition (1); condition (2) for the signature is obvious. Hence, the signature is a rational linear combination of Pontryagin numbers. In particular, $\sigma(X) = ap_1[X]$ if $\dim X = 4$, $\sigma(X) = bp_2[X] + cp_{11}[X]$ if $\dim X = 8$, and so on. To find $a, b, c, \dots$, we need to have a sufficient supply of computations in concrete examples. For example, $H_{2m}(\mathbb{C}P^{2m}) = \mathbb{Z}$. The matrix of the intersection form is just (1); hence, $\sigma(\mathbb{C}P^{2m}) = 1$. Furthermore,

$$\tau(\mathbb{C}P^{2m}) \oplus 1 = (2m+1)\zeta_{\mathbb{C}}$$

(this is the complex version of Exercise 12), and hence

$$\mathbb{C}\tau(\mathbb{C}P^{2m}) \oplus 1_{\mathbb{C}} = (2m+1)(\zeta_{\mathbb{C}} \oplus \overline{\zeta}_{\mathbb{C}})$$

(see Exercise 8), and

$$\begin{aligned}(p_0 - p_1 + p_2 - \dots + (-1)^m p_m)(\mathbb{C}P^{2m}) &= [(1+x)(1-x)]^{2m+1} \\ &= (1-x^2)^{2m+1}\end{aligned}$$

where $x \in H^2(\mathbb{C}P^{2m}) = \mathbb{Z}$ is the canonical generator (see Sect. 19.5.D, including Exercise 16). Hence,

$$p_i(\mathbb{C}P^{2m}) = \begin{cases} \binom{2m+1}{i} x^{2i}, & \text{if } i \le m, \\ 0, & \text{if } i > m. \end{cases}$$

In particular, $p_1(\mathbb{C}P^2) = 3x^2$, $p_1[\mathbb{C}P^2] = 3$, and, since $\sigma(\mathbb{C}P^2) = 1$, then for every (closed, orientable) four-dimensional manifold X,

$$\sigma(X) = \frac{1}{3}p_1[X]. \tag{$*$}$$

(In particular, the Pontryagin number $p_1[X]$ of every closed orientable four-dimensional manifold X is divisible by 3.) Furthermore, $p_{11}[\mathbb{C}P^4] = 25, p_2[\mathbb{C}P^4] = 10, \sigma(\mathbb{C}P^4) = 1$. In addition,

$$(p_0 + p_1 + p_2)(\mathbb{C}P^2 \times \mathbb{C}P^2) = (p_0 + p_1)(\mathbb{C}P^2) \times (p_0 + p_1)(\mathbb{C}P^2)$$

(the multiplication formula for the Pontryagin classes holds only modulo 2-torsion, but there is no torsion in the cohomology of complex projective spaces), and hence

$$\begin{gathered} p_1(\mathbb{C}P^2 \times \mathbb{C}P^2) = (1 \times 3x^2) + (3x^2 \times 1), \\ p_1^2(\mathbb{C}P^2 \times \mathbb{C}P^2) = 18(x^2 \times x^2), \\ p_2(\mathbb{C}P^2 \times \mathbb{C}P^2) = p_1(\mathbb{C}P^2) \times p_1(\mathbb{C}P^2) = 3x^2 \times 3x^2, \\ p_{11}[\mathbb{C}P^2 \times \mathbb{C}P^2] = 18,\ p_2[\mathbb{C}P^2 \times \mathbb{C}P^2] = 9, \\ \text{and also } \sigma(\mathbb{C}P^2 \times \mathbb{C}P^2) = 1. \end{gathered}$$

EXERCISE 32. Prove that the signature is multiplicative: $\sigma(X \times Y) = \sigma(X)\sigma(Y)$.) Hence, $1 = 10b + 25c, 1 = 9b + 18c$, whence $b = \frac{7}{45}, c = -\frac{1}{45}$. Thus, for $\dim X = 8$,

$$\sigma(X) = \frac{7p_2[X] - p_{11}[X]}{45}. \tag{$**$}$$

(Hence, $7p_2[X] - p_{11}[X]$ is divisible by 45, and if the first Pontryagin class of a closed orientable eight-dimensional manifold is zero, then its signature is divisible by 7.) The formulas ($*$), ($**$) form the beginning of an infinite chain of formulas relating the signature to the Pontryagin numbers. The work of explicitly writing these formulas was done in the 1950s by F. Hirzebruch. He calculated the Pontryagin numbers of manifolds of the form $\mathbb{C}P^{2m_1} \times \cdots \times \mathbb{C}P^{2m_k}$ (which, essentially, we have done) and, using the fact that the signatures of all these manifolds are equal to 1, he found the coefficients of the Pontryagin numbers in the formulas for signatures. The resulting formulas are presented in his book [46].

E: Invariance Problems for Characteristic Classes of Manifolds

As we know, the Euler class of a manifold can be expressed through the Betti numbers of this manifold. It turns out that although the Stiefel–Whitney classes are not determined by either homology groups or even a cohomology ring of this manifold, still they are homotopy invariant; that is, a map of one closed manifold into another one which is a homotopy equivalence takes the Stiefel–Whitney classes into the Stiefel–Whitney classes. This fact will be established (or, at least, discussed) in Chap. 4 (Sect. 31.2). For Pontryagin classes, however, the homotopy invariance fails (the only homotopy invariant nonzero polynomial in Pontryagin classes is the signature). In the 1960s, S. Novikov proved the difficult theorem of topological invariance of rational Pontryagin classes (a homeomorphism between two smooth closed orientable manifolds takes Pontryagin classes into Pontryagin classes modulo elements of finite order; these elements of finite order may be nonzero—there are examples). A decade before that, V. Rokhlin, A. Schwarz, and R. Thom proved this statement for homeomorphisms, establishing a correspondence between some smooth triangulations of two smooth manifolds (see Rokhlin and Schwarz [72], Thom [85]). This result leads naturally to the problem of "combinatorial calculation of Pontryagin classes," that is, their calculation via triangulation (compare to Exercise 29). At present, this problem has been solved only for the first Pontryagin class (see the article by Gabrielov, Gelfand, and Losik [42]).

Homotopical Topology
Fomenko, A.; Fuchs, D.
2016, XI, 627 p. 210 illus., Hardcover
ISBN: 978-3-319-23487-8