Introduction 2

We will now explore the basics of linear programming. In this chapter, we will go through several basic definitions and examples to build our intuition about linear programs. We will also learn the ratio method, which we will find useful in solving certain linear programs.

2.1 Linear Programming

Linear programming is a branch of mathematics just as calculus is a branch of mathematics. Calculus was developed in the seventeenth century to solve problems of physical sciences. Linear programming was developed in the twentieth century to solve problems of social sciences. Today, linear programming or its more generalized version, mathematical programming, has proven to be useful in many diversified fields: economics, management, all branches of engineering, physics, chemistry, and even pure mathematics itself. Linear programming can be viewed as a generalization of solving simultaneous linear equations. If solving a system of simultaneous linear equations can be considered a cornerstone of applied mathematics, then it is not surprising that linear programming has become so prominent in applied mathematics.

In the real world, we may want to maximize profits, minimize costs, maximize speed, minimize the area of a chip, etc. In maximizing profits, we are constrained by the limited resources available or the availability of the market. In minimizing the costs of production, we are constrained to satisfy certain standards. The idea of maximizing or minimizing a function subject to constraints arises naturally in many fields. In linear programming, we assume that the function to be maximized or minimized is a linear function and that the constraints are linear equations or linear inequalities. The assumptions of a linear model may not always be realistic, but it is the first approximate model for understanding a real-world problem. Let us now see the basic concept of a linear program.

T.C. Hu, A.B. Kahng, *Linear and Integer Programming Made Easy*,
DOI 10.1007/978-3-319-24001-5_2

Table 2.1 Simple example of a linear program

	Wood (units)	Iron (units)	Revenue	Output
Product A	12	6	\$4	??
Product B	8	9	\$5	??
Total resource	96	72	??	

Consider the following problem. We have two resources, wood and iron. We can use up to 96 units of wood per day, and we can use up to 72 units of iron per day. With these two resources, we can make two products, A and B. It requires 12 units of wood and 6 units of iron to produce one copy of product A, and it requires 8 units of wood and 9 units of iron to produce one copy of product B. Furthermore, we can sell products A and B for \$4 and \$5 per copy, respectively. How do we allocate resources to the production of A and B in order to maximize our revenue? Table 2.1 summarizes the problem.

With the information in Table 2.1, we can draw Figure 2.1 using A and B to represent the amounts of the two products. Two lines represent upper bounds on utilization of the two resources, wood and iron. For example, the line "wood" shows the maximum usage of wood. That is, the 96 available units of wood could potentially be used toward making eight copies of product A and zero copies of product B. Or, the wood could be used toward making 12 copies of product B and zero copies of product A. An important observation is that even though there is enough wood to make 12 copies of product B, this is not feasible since only 72 units of iron are available. The amounts of A and B produced must satisfy resource constraints on both wood and iron. As such, the polygon $JKLM$ represents our solution space, that is, the set of all feasible solutions to our problem.

Definition A *feasible solution* is a solution to a linear program that satisfies all constraints. The set of all feasible solutions is called the *solution space*. If a linear program has feasible solutions, then we say that it is *feasible*. Otherwise, it is said to be *infeasible*.

To maximize revenue, we have the objective to *maximize* $4A + 5B$. This means that we want to find a feasible solution that has maximum projection on the ray passing from the origin with a slope of $\frac{5}{4}$, shown as the vector $\boldsymbol{c}$ in Figure 2.2. We can draw a line with a slope of $-\frac{4}{5}$ (the dotted line in Figure 2.2), which is perpendicular to the vector $\boldsymbol{c}$. Shifting this dotted line along $\boldsymbol{c}$ allows us to find the feasible point with maximum projection. Figure 2.3 shows that the maximum possible revenue is \$43.20, corresponding to production of 4.8 units each of product A and product B. However, if we are required to produce an integer number of units of each product, the integer solution (3, 6), which achieves revenue of \$42.00, is optimum.

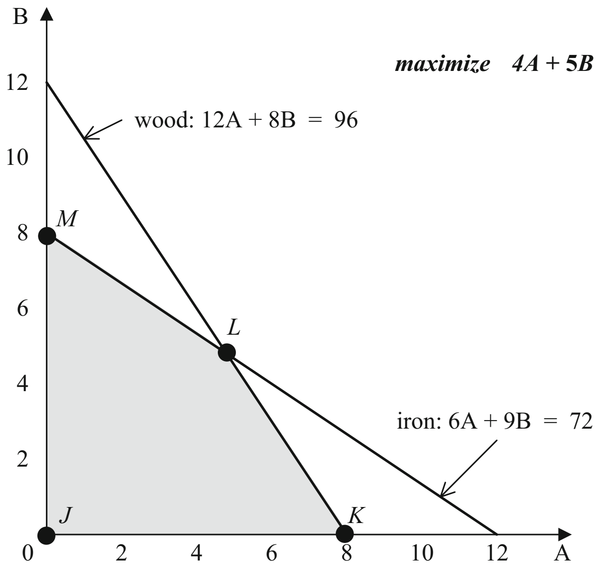

Fig. 2.1 Representation of the feasible regions of products A and B

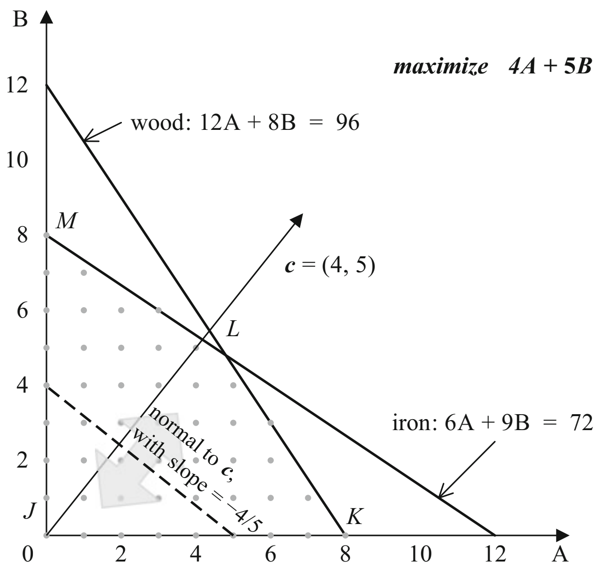

Fig. 2.2 Search for the feasible point with maximum projection by shifting the dotted line along ***c***(4,5)

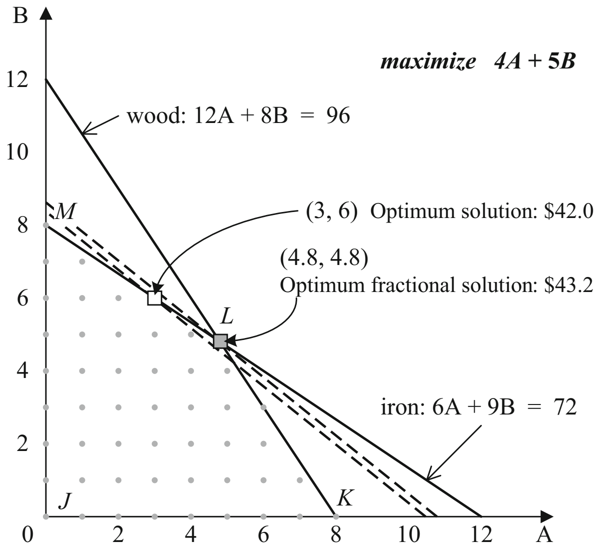

Fig. 2.3 Optimum solution and optimum fractional solution of the revenue

Let us consider one more linear program of two variables:

$$\begin{aligned} \max \quad & z = x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 13 \\ & 0 \leq x_1 \leq 5 \\ & 0 \leq x_2 \leq 5 \end{aligned} \tag{2.1}$$

Notice that our solution space consists of the area bounded by $ABCDE$ in Figure 2.4.

In Figure 2.4, the coordinates of the vertices and their values are $A = (0,\ 0)$ with value $z = 0$, $B = (5,\ 0)$ with value $z = 5$, $C = (5, 3)$ with value $z = 8$, $D = (4, 5)$ with value $z = 9$, and $E = (0, 5)$ with value $z = 5$.

The famous Simplex Method for solving a linear program—which we will present in Chapter 3—starts from vertex A, moves along the x_1 axis to B, and then moves vertically upward until it reaches C. From the vertex C, the Simplex Method goes upward and slightly westward to D. At the vertex D, the gradient of the function z increases toward the northeast, and the gradient is a convex combination of the vectors that are normal to lines ED and DC. This indicates that the corner point D is the vertex that maximizes the value of z[1] (Figure 2.5):

$$\max \quad z = \max(x_1 + x_2) = (4 + 5) = 9$$

[1] A *corner point* is also called an *extreme point*. Such a point is not “between” any other two points in the region.

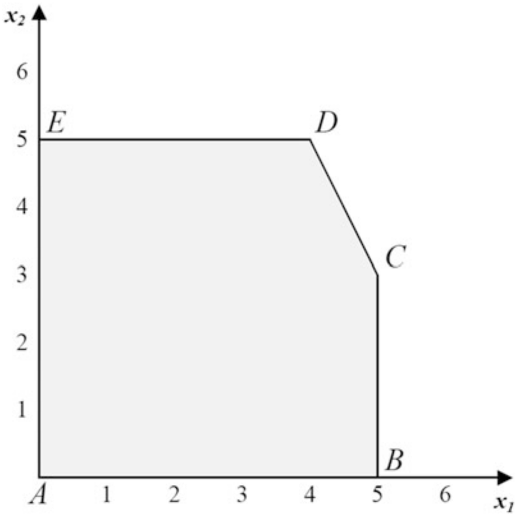

Fig. 2.4 Solution space of (2.1) is bounded by ABCDE

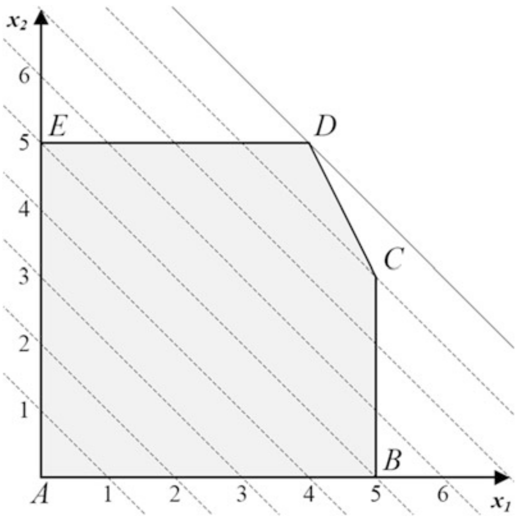

Fig. 2.5 Optimal solution of (2.1) is at the corner point D

For three variables x_1, x_2, and x_3, we can imagine a twisted cube such as a single die, where the minimum is at the lowest corner point and the maximum is at the highest corner point. Then, we can imagine that the Simplex Method would traverse along the boundary of the cube to search for the optimal solution among extreme points.

Definition A *linear program* is the maximization or minimization of a linear function subject to linear constraints (e.g., linear equations or linear inequalities).

For example, the following are two linear programs:

$$\begin{aligned} \max \quad & z = 12x_1 + 10x_2 + x_3 \\ \text{subject to} \quad & 11x_1 + 10x_2 + 9x_3 \le 20 \\ & x_1, x_2, x_3 \ge 0 \end{aligned} \tag{2.2}$$

$$\begin{aligned} \max \quad & z = x_1 + x_2 + x_3 \\ \text{subject to} \quad & 6x_1 + 3x_2 + x_3 \le 15 \\ & 4x_1 + 5x_2 + 6x_3 \le 15 \\ & x_1, x_2, x_3 \ge 0 \end{aligned} \tag{2.3}$$

There are three distinct features of linear programs:

1. The function to be maximized is a linear function.
2. The variables are restricted to be non-negative.
3. The constraints are linear inequalities or linear equations.

Thus, when solving a linear program, the classical tool of "calculus" is not used.

Definition The subject of optimization under constraints is called *mathematical programming*. An optimization problem with a function to be maximized or minimized is called a *mathematical program*.

Definition The function to be maximized or minimized in a mathematical program is called the *objective function*.

Definition If the objective function or the constraints are nonlinear in a mathematical program, then the mathematical program is a *nonlinear program*.

Definition If the variables are restricted to be non-negative integers in a mathematical program, then the mathematical program is called an *integer program*.

2.2 Ratio Method

In this section, we will familiarize ourselves with linear programs and the ratio method. We will proceed by exploring several examples.

Example 1 Consider a bread merchant carrying a knapsack who goes to the farmers market. He can fill his knapsack with three kinds of bread to sell. A loaf of raisin bread can be sold for \$12, a loaf of wheat bread for \$10, and a loaf of white bread for \$1.

Furthermore, the loaves of raisin, wheat, and white bread weigh 11 lbs, 10 lbs, and 9 lbs, respectively. If the bread merchant can carry 20 lbs of bread in the knapsack, what kinds of breads should he carry if he wants to get the most possible cash?

This problem can be formulated as a linear program (2.2) where x_1 denotes the number of loaves of raisin bread, x_2 denotes the number of loaves of wheat bread, and x_3 denotes the number of loaves of white bread in his knapsack.

In this example, there are three "noteworthy" integer solutions: $x_1 = x_3 = 1, x_2 = 0$ or $x_1 = x_3 = 0, x_2 = 2$ or $x_1 = x_2 = 0, x_3 = 2$. So, the merchant would carry two loaves of wheat bread and get \$20, rather than one loaf each of raisin bread and white bread to get \$13 or two loaves of white bread to get \$2. However, notice that if he can cut a loaf of raisin bread into pieces, $\frac{9}{11}$ of a loaf of raisin bread is worth $\$12 \times \frac{9}{11} \approx \$9.82$, and he should therefore carry $\frac{20}{11} = 1 + \frac{9}{11}$ loaves of raisin bread with market value of $\$21.82 = \$12 + \$9.82$.

In a linear program, the variables are not required to be integers, so it is much easier to solve than an integer program which requires the variables to be integers. Let us consider a linear program with only a single constraint. We have

$$\begin{aligned} \max \quad & z = v_1x_1 + v_2x_2 + v_3x_3 \\ \text{subject to} \quad & w_1x_1 + w_2x_2 + w_3x_3 \le b \\ & x_1, x_2, x_3 \ \ge 0 \end{aligned} \tag{2.4}$$

where v_j is the value associated with item j, w_j is the weight associated with item j, and b is the total weight restriction of the knapsack.

Intuitively, we would like to carry an item which is of low weight and high value. In other words, the ratio of the value to weight for such an item should be maximized. Applying this idea to (2.2), we have

$$\frac{12}{11} > \frac{10}{10} > \frac{1}{9}$$
$$(\text{i.e., raisin bread} > \text{wheat bread} > \text{white bread})$$

which indicates that we should fill the knapsack with the first item, that is, carry $\frac{20}{11} = 1 + \frac{9}{11}$ loaves of raisin bread.

Definition The *ratio method* is a method that can be applied to a linear program to get the optimal solution as long as the variables are not restricted to be integers. It operates simply by taking the maximum or minimum ratio between two appropriate parameters (e.g., value to cost, or value to weight).

To formalize this idea of the ratio method, consider the general linear program

$$\begin{aligned} \max \quad & v = v_1x_1 + v_2x_2 + v_3x_3 + \cdots \\ \text{subject to} \quad & w_1x_1 + w_2x_2 + w_3x_3 + \cdots \le b \\ & x_j \ge 0 \end{aligned} \tag{2.5}$$

Furthermore, let $v_k/w_k = \max_j\{v_j/w_j\}$ (where $w_j \neq 0$). Then the feasible solution that maximizes v is $x_k = b/w_k, x_j = 0$ for $j \neq k$, and the maximum profit value of v is obtained by filling the knapsack with a single item. In total, we obtain a profit of $v = b \cdot (v_k/w_k)$ dollars.

It is easy to prove that the max ratio method for selecting the best item is correct for any number of items and any right-hand side (total weight restriction). It should be emphasized that the variables in (2.5) are not restricted to be integers. Normally, when we say a "knapsack problem," we refer to a problem like (2.2) but with integer constraints. To distinguish the difference, we call (2.2) a "fractional knapsack problem."

Example 2 Consider another merchant who goes to the farmers market. His goal is not to get the maximum amount of cash but to minimize the total weight of his knapsack as long as he can receive enough money. He also has the choice of three kinds of bread: raisin, wheat, and white (see (2.6)). Then his problem becomes minimizing the total weight subject to the amount of cash received being at least, say, \$30, as shown in (2.7).

$$\begin{aligned} \min \quad & z = w_1x_1 + w_2x_2 + w_3x_3 \\ \text{subject to} \quad & v_1x_1 + v_2x_2 + v_3x_3 \geq c \\ & x_j \geq 0 \end{aligned} \tag{2.6}$$

where v_j is the value associated with item j, w_j is the weight associated with item j, and c is the minimum amount of cash received.

When the objective function is to minimize the total weight, we also take ratios associated with the items, but we want the minimum ratio of weight to value. We have

$$\begin{aligned} \min \quad & z = 11x_1 + 10x_2 + 9x_3 \\ \text{subject to} \quad & 12x_1 + 10x_2 + 1x_3 \geq 30 \\ & x_j \geq 0 \end{aligned} \tag{2.7}$$

and the minimum ratio is $\frac{11}{12} < \frac{10}{10} < \frac{9}{1}$, which means that we should carry $\frac{30}{12} = 2.5$ loaves of the raisin bread, and the total weight of the knapsack is 11 lbs $\times$ 2.5 = 27.5 lbs.

Thus, in a fractional knapsack problem, we always take a ratio involving weight and value. To maximize profit, we use the maximum ratio of value to weight. To minimize weight, we use the reciprocal, that is, the minimum ratio of weight to value. The ratio method is always correct for a single constraint and no integer restrictions.

Example 3 Consider a merchant selling three kinds of drinks, A, B, and C. The three kinds of drinks are all made by mixing two kinds of juices, apple juice and orange juice. All drinks are sold at the market for one dollar per gallon. The merchant has 15 gallons of apple juice and 15 gallons of orange juice. To mix the juices to form drink A, the ratio of apple juice to orange juice is 4:1. For B, the ratio is 1:4, and for C, the ratio is 3:2. He has to mix the drinks before going to the market, but he is able to carry any combination of the three drinks. To get the maximum amount of cash, he must solve a problem equivalent to (2.8):

$$\begin{aligned} \max \quad & z = x_1 + x_2 + x_3 \\ \text{subject to} \quad & \begin{bmatrix} 4 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 2 \end{bmatrix} x_3 \leq \begin{bmatrix} 15 \\ 15 \end{bmatrix} \\ & x_j \geq 0 \end{aligned} \tag{2.8}$$

where x_1, x_2, and x_3 are the amounts of drinks A, B, and C, respectively.

Definition Mixing the apple juice and orange juice into drink A in a specified proportion is an *activity*. That is, an activity is a column vector in the above linear program, e.g., $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, or $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Also in the example above, the amount of a drink that is produced is called the *activity level*, e.g., x_1 or x_2.

If the activity level is zero, i.e., $x_1 = 0$, it means that we do not select that activity at all. In this example, we have three activities, so we have three activity levels to choose. (In fact, the book of T. C. Koopmans (ed.) in 1951 is called *Activity Analysis of Production and Allocation*.)

Here, x_1, x_2, and x_3 are the amounts of drinks A, B, and C, respectively, that the merchant will carry to the market. By trial and error, we might find that an optimum solution is $x_1 = 3$, $x_2 = 3$, and $x_3 = 0$, or $[x_1, x_2, x_3] = [3, 3, 0]$.

Note that every drink needs five parts of juice, and if we take the ratio of value to weight as before, all the ratios are equal. We have

$$\frac{1}{4+1} = \frac{1}{1+4} = \frac{1}{3+2} = \frac{1}{5} \tag{2.9}$$

The reason that we do not select drinks A and C or B and C in the optimum solution is that drinks A and B are more compatible in the sense that they do not compete for the same kinds of juices heavily.

Definition Two activities are *compatible* if the right-hand side (RHS) of the constraint function can be expressed as a sum of the two activities with non-negative activity level coefficients.

The notion of compatibility is central to a linear program with more than one constraint. To illustrate this point, we let the ratio of apple juice to orange juice for drink A be 2:3. We keep the ratios for drink B and drink C the same. Thus, we have the linear program

$$\begin{aligned} \max \quad & v = x_1 + x_2 + x_3 \\ \text{subject to} \quad & \begin{bmatrix} 2 \\ 3 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 2 \end{bmatrix} x_3 \leq \begin{bmatrix} 15 \\ 15 \end{bmatrix} \\ & x_j \geq 0 \end{aligned} \tag{2.10}$$

Now, the optimum solution is $x_1 = x_3 = 3$ and $x_2 = 0$. Note that we have neither changed the prices of any drinks nor the composition of drinks B and C. Now, drink B is not selected and drink C is selected. The reason is that now drinks A and B are not compatible, whereas drinks A and C are compatible.

If you are the coach of a basketball team, you do not want a team of players who are all only good at shooting. You need a variety of players who can also pass and run fast. The same need for ability in diverse areas is necessary to create compatible teams in most sports.

Similarly, when it comes to linear programming, we are not trying to find just the best column vector, but we want to find the best set of vectors. Just like in the real world, we are not seeking the best single player but the best team of players.

We shall write a linear program of n variables and m constraints as

$$\begin{aligned} \max \quad & v = \sum c_j x_j \\ \text{subject to} \quad & \sum a_{ij} x_j \leq b_i \quad (i = 1, \ldots, m)(j = 1, \ldots, n) \\ & x_j \geq 0 \end{aligned} \tag{2.11}$$

(Strictly speaking, we should count the inequalities $x_j \geq 0$ as constraints, but in most applications, we understand that the requirement of non-negativity of variables is treated separately.)

In matrix notation, we write

$$\begin{aligned} \max \quad & v = \boldsymbol{cx} \\ \text{subject to} \quad & \boldsymbol{Ax} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned} \tag{2.12}$$

where $\boldsymbol{c}$ is a row vector which has c_j as its component $(j = 1, \ldots, n)$, $\boldsymbol{x}$ is a column vector which has x_j as its component $(j = 1, \ldots, n)$, $\boldsymbol{b}$ is a column vector which has b_i as its component $(i = 1, \ldots, m)$, $\boldsymbol{0}$ is a zero vector, and the matrix $\boldsymbol{A}$ is an $m \times n$ matrix with its columns denoted by $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_n$.

One of the earliest applications of linear programming was to find the optimum selection of food items within a budget, i.e., to discover what kinds of food items satisfy the nutritional requirements while minimizing the total cost. We call this problem the Homemaker Problem.

For the purposes of discussion, let us assume that we are concerned with the nutritional requirements for vitamins A and B only. As such, we shall represent every food in the supermarket as a vector with two components, the first component being the amount of vitamin A the food contains and the second component being the amount of vitamin B it contains. For example, we shall represent beef as [3, 1], meaning that a pound of beef contains three units of vitamin A and one unit of vitamin B. Similarly, we may represent wheat as [1, 1]. We may also potentially represent a food as [−1, 2], meaning that that particular food will destroy one unit of vitamin A but provide two units of vitamin B.

Thus, the process of deciding to buy a particular food is equivalent to selecting a column vector $\boldsymbol{a}_j$ in the matrix $\boldsymbol{A}$. There are n column vectors, so we say that there are n activities. The amount of a particular food j to be purchased is called its activity level and is denoted by x_j. For example, if we associate j with beef, then $x_j = 3$ means that we should buy three pounds of beef, and $x_j = 0$ means that we should not buy any beef. Since we can only buy from the supermarket, it is natural to require $x_j \geq 0$. The unit cost of food j is denoted by c_j, so the total bill for all purchases is $\sum c_j x_j$. The total amount of vitamin A in all the foodstuffs purchased is $\sum a_{1j} x_j$, and similarly, the amount of vitamin B is $\sum a_{2j} x_j$. As such, the linear program describing our problem is

$$\begin{aligned} \min \quad & z = \sum c_j x_j \\ \text{subject to} \quad & \sum a_{ij} x_j \geq b_i \quad (i = 1, \ldots, m)(j = 1, \ldots, n) \\ & x_j \geq 0 \end{aligned} \tag{2.13}$$

Example 4 *(Homemaker Problem)* Assume that the supermarket stocks four kinds of food costing \$15, \$7, \$4, and \$6 per pound, and that the relevant nutritional characteristics of the food can be represented as

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

If we know the nutritional requirements for vitamins A and B for the whole family are [3, 5], then we have the following linear program:

$$\begin{aligned} \min \quad & z = 15x_1 + 7x_2 + 4x_3 + 6x_4 \\ \text{subject to} \quad & \begin{bmatrix} 3 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 2 \end{bmatrix} x_4 \geq \begin{bmatrix} 3 \\ 5 \end{bmatrix} \\ & x_j \geq 0 \quad (j = 1, 2, 3, 4) \end{aligned} \tag{2.14}$$

The optimum solution turns out to be $x_1 = 0$, $x_2 = 3$, $x_3 = 2$, and $x_4 = 0$. This means that we should buy three pounds of the second food and two pounds of the third food, and none of the first or the fourth food.

Example 5 *(Pill Salesperson)* Let us twist the Homemaker Problem a little bit and consider a vitamin pill salesperson who wants to compete with the supermarket. Since the taste and other properties of the food are of no concern (by assumption), the salesperson merely wants to provide pills that contain equivalent nutrition at a lower cost than the food. Let us assume that there are two kinds of pill, one for vitamin A and one for vitamin B, and each pill supplies one unit of its vitamin. Suppose that the salesperson sets the price of vitamin A pills at y_1 and vitamin B pills at y_2. If the prices satisfy the constraints

$$
\begin{aligned}
3y_1 + \ \ y_2 &\le 15 \\
y_1 + \ \ y_2 &\le 7 \\
y_2 &\le 4 \\
-y_1 + 2y_2 &\le 6
\end{aligned}
$$

then no matter what combination of food items we select from the supermarket, it is always cheaper to satisfy our nutritional requirements by buying the pills. Since the requirements for vitamins A and B are $[3, 5]$, the total amount that we have to pay the salesperson is $3y_1 + 5y_2$. Of course, in setting prices, the salesperson would like to maximize the amount he receives for his goods. Once again, we can represent his problem as a linear program:

$$
\begin{aligned}
\max \quad & z = 3y_1 + 5y_2 \\
\text{subject to} \quad & 3y_1 + \ \ y_2 \le 15 \\
& \ \ y_1 + \ \ y_2 \le 7 \\
& \qquad\ \ y_2 \le 4 \\
& -y_1 + 2y_2 \le 6 \\
& y_1, y_2 \ge 0
\end{aligned}
\tag{2.15}
$$

Now, let us study a general method to solve the linear program with a single constraint. The following are two linear programs, each of which has a single constraint:

$$
\begin{aligned}
\min \quad & z = x_1 + 2x_2 + 3x_3 \\
\text{subject to} \quad & 4x_1 + 5x_2 + 6x_3 \ge 60 \\
& x_j \ge 0 \quad (j = 1, 2, 3)
\end{aligned}
\tag{2.16}
$$

$$
\begin{aligned}
\max \quad & v = x_1 + 2x_2 + 3x_3 \\
\text{subject to} \quad & 6x_1 + 5x_2 + 4x_3 \le 60 \\
& x_j \ge 0 \quad (j = 1, 2, 3)
\end{aligned}
\tag{2.17}
$$

To solve the minimization problem, let us take one variable at a time. To satisfy (3.15) using x_1, we need

$$x_1 = \frac{60}{4} = 15 \quad \text{and} \quad z = \$15$$

$$\text{For } x_2, \; x_2 = \frac{60}{5} = 12 \quad \text{and} \quad z = \$24$$

$$\text{For } x_3, \; x_3 = \frac{60}{6} = 10 \quad \text{and} \quad z = \$30$$

The intuitive idea is to take the ratios $\left(\frac{1}{4}\right)$, $\left(\frac{2}{5}\right)$, and $\left(\frac{3}{6}\right)$ and select the minimum ratio if we want to minimize. If we want to maximize, the ratios should be $\left(\frac{1}{6}\right)$, $\left(\frac{2}{5}\right)$, and $\left(\frac{3}{4}\right)$, and we select the maximum ratio.

$$\text{For } x_1, \; x_1 = \frac{60}{6} = 10 \quad \text{and} \quad v = \$10$$

$$\text{For } x_2, \; x_2 = \frac{60}{5} = 12 \quad \text{and} \quad v = \$24$$

$$\text{For } x_3, \; x_3 = \frac{60}{4} = 15 \quad \text{and} \quad v = \$45$$

To formalize the idea, let the linear program be

$$\begin{aligned} \min \quad & z = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_1x_1 + a_2x_2 + \cdots + a_nx_n \geq b \\ & x_j \geq 0 \quad (j = 1, \ldots, n) \end{aligned} \tag{2.18}$$

Then we select the minimum ratio

$$\min_j \left(\frac{c_j}{a_j}\right) = \frac{c_k}{a_k} \quad \left(\text{where } a_j \neq 0\right)$$

and let $x_k = \frac{b}{a_k}$ and $x_j = 0$ if $j \neq k$.

If the linear program is

$$\begin{aligned} \max \quad & v = c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_1x_1 + a_2x_2 + \cdots + a_nx_n \leq b \\ & x_j \geq 0 \end{aligned} \tag{2.19}$$

then we select the maximum ratio

$$\max_j \left(\frac{c_j}{a_j}\right) = \frac{c_k}{a_k} \quad \left(\text{where } \; a_j \neq 0\right)$$

and let $x_k = \frac{b}{a_k}$, and $x_j = 0$ if $j \neq k$.

This is a formal restatement of the ratio method.

Congratulations! We have just developed a method to solve any linear program with a single constraint and any number of variables!

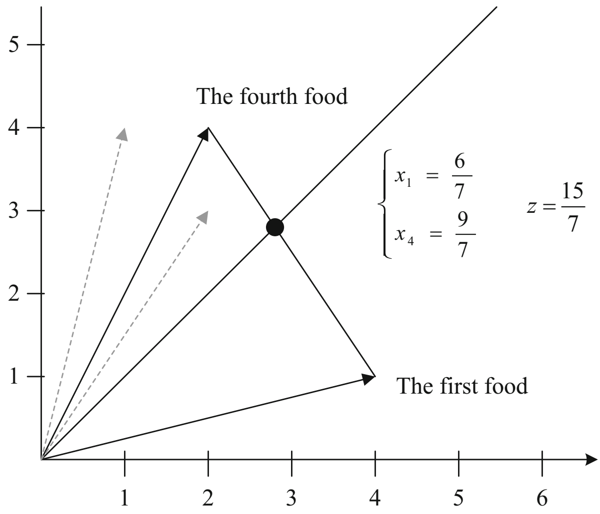

Fig. 2.6 Graphic solution of the Homemaker Problem described in (2.20)

Let us now consider the Homemaker Problem in Example 4 again, but with a different set of data. Assume that the supermarket has four kinds of food, each kind labeled by its vitamin contents of vitamin A and vitamin B, as follows:

$$\begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 8 \end{bmatrix}.$$

All items cost one dollar per pound, except the fourth item which costs two dollars per pound. And the homemaker needs 6 units of vitamin A and 6 units of vitamin B for his family. So the homemaker's linear program becomes

$$\begin{aligned} \min \quad & z = x_1 + x_2 + x_3 + x_4 \\ \text{subject to} \quad & \begin{bmatrix} 4 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} 3 \\ 2 \end{bmatrix} x_3 + \begin{bmatrix} 2 \\ 4 \end{bmatrix} x_4 \geq \begin{bmatrix} 6 \\ 6 \end{bmatrix} \\ & x_j \geq 0 \quad (j = 1, 2, 3, 4) \end{aligned} \tag{2.20}$$

Note that the vector $\begin{bmatrix} 4 \\ 8 \end{bmatrix}$ has been normalized to $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and costs one dollar. Note also that (2.20) has two constraints, and we want to solve the problem by a graphic method. In Figure 2.6, the horizontal axis measures the amount of vitamin A and the vertical axis measures the amount of vitamin B of the item. Also, there is a line from the origin to $\begin{bmatrix} 6 \\ 6 \end{bmatrix}$.

Since we have two constraints, we need to select the best pair of items. The pair which intersects the 45° line furthest from the origin is the cheapest pair. For (2.20), $x_1 = \frac{6}{7}$, $x_4 = \frac{9}{7}$, and $z = \frac{15}{7}$.

The Pill Salesperson Problem corresponding to (2.20) becomes (2.21) where the vitamin A pill costs π_1 dollars and the vitamin B pill costs π_2 dollars:

$$\begin{aligned} \max \quad & v = 6\pi_1 + 6\pi_2 \\ \text{subject to} \quad & 4\pi_1 + \pi_2 \le 1 \\ & \pi_1 + 4\pi_2 \le 1 \\ & 3\pi_1 + 2\pi_2 \le 1 \\ & 2\pi_1 + 4\pi_2 \le 1 \end{aligned} \tag{2.21}$$

The optimum solution is $\pi_1 = \frac{3}{14}$, $\pi_2 = \frac{2}{14}$, and $v = \frac{15}{7}$, same as the value of $z = \frac{15}{7}$. This can be found graphically by finding the intersection point of $4\pi_1 + \pi_2 = 1$ and $2\pi_1 + 4\pi_2 = 1$, where $\pi_1 = \frac{3}{14}$ and $\pi_2 = \frac{2}{14}$.

2.3 Exercises

1. Draw the solution space graphically for the following problems. Then, determine the optimum fractional solutions using your graph.
   (a) Producing one copy of A requires 3 units of water and 4 units of electricity, while producing one copy of B requires 7 units of water and 6 units of electricity. The profits of products A and B are 2 units and 4 units, respectively. Assuming that the factory can use up to 21 units of water and 24 units of electricity per day, how should the resources be allocated in order to maximize profits?
   (b) Suppose there are two brands of vitamin pills. One pill of Brand A costs \$3 and contains 30 units of vitamin X and 10 units of vitamin Y. One pill of Brand B costs \$4 and contains 15 units of vitamin X and 15 units of vitamin Y. If you need 60 units of vitamin X and 30 units of vitamin E and want to minimize your spending, how will you purchase your pills?
2. Why do we use $\max x_0$,

$$x_0 - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0,$$

   and use $\min z$,

$$-z + c_1x_1 + c_2x_2 + \cdots + c_nx_n = 0?$$

3. Use the ratio method to solve the following:
   (a)

$$\begin{aligned} \min \quad & z = x_1 + 3x_2 + 5x_3 \\ \text{subject to} \quad & 4x_1 + 7x_2 + 8x_3 \ge 120 \\ & x_j \ge 0 \end{aligned}$$

   (b)

$$\begin{aligned} \max \quad & z = 3x_1 + 8x_2 + x_3 \\ \text{subject to} \quad & 6x_1 + 15x_2 + 3x_3 \le 90 \\ & x_j \ge 0 \end{aligned}$$

4. Write linear programs for the following two problems.
   (a) A group of 25 adults and 18 children are going to travel. There are two types of vans they can rent. The first type accommodates six adults and six children and costs \$90 to rent for the duration of the trip. The second type accommodates six adults and four children, and it costs \$80. In order to minimize the cost, how should the group rent vans?
   (b) A school wants to create a meal for its students by mixing food A and food B. Each ounce of A contains 2 units of protein, 4 units of carbohydrates, and 2 units of fat. Each ounce of B contains 3 units of protein, 1 unit of carbohydrates, and 4 units of fat. If the meal must provide at least 10 units of protein and no more than 8 units of carbohydrates, how should the school create the meal in order to minimize the amount of fat?
5. Use the ratio method to solve

$$\begin{array}{rl} \text{min} & z = 16x_1 + 9x_2 + 3x_3 \\ \text{subject to} & 8x_1 + 4x_2 + x_3 = 120 \\ & x_j \geq 0 \end{array}$$

6. Use the ratio method to solve

$$\begin{array}{rl} \text{min} & z = 16x_1 + 7x_2 + 3x_3 \\ \text{subject to} & 8x_1 + 4x_2 + x_3 = 120 \\ & x_j \geq 0 \end{array}$$

7. In Exercises 5 and 6, both constraints are equations. Will the solutions change if the constraints are inequalities?
8. Prove that the ratio method works correctly for both maximizing and minimizing the objective function.

This chapter covered the basics of linear programming. You should now be comfortable with the concept of a linear program, how the ratio method works, and the notation that we have been using in order to express linear programs. It is highly advisable for you to be comfortable with the material in this chapter before moving on.

Also, you may have noticed that the Homemaker Problem and the Pill Salesperson Problem in the last section are closely related. It turns out that they are *dual* linear programs. We will be revisiting these problems later in Chapter 4 as we explore duality theory, and you will see why they both yielded the same optimal value for the objective function.

In the next chapter, we will discuss solution spaces and some properties of linear programs. Let's go!

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