

Chapter 2

Noncontextuality Inequalities from Variable Elimination

In this chapter we present a general method for deriving complete sets of Bell and noncontextuality inequalities alternative to the direct solution of the hull problem for correlation polytopes, discussed in the previous chapter, which provides some computational advantages and allows for a complete characterization of the correlation polytope even for scenarios with an arbitrary number of settings. The main results of this chapter have been published in Refs. [1, 2].

Our method is based on the application of Fourier-Motzkin (FM) method of variable elimination for systems of linear inequalities to conditions derived in Ref. [3] as consistency conditions for putting together partial extensions of quantum probabilities in order to obtain a classical probability description. More precisely, such conditions are expressed in terms of a systems of linear inequalities where also correlations between incompatible observables appear as variables: A classical probability space representation exists for a given set of QM predictions if and only if the corresponding system of linear inequalities admits a solution; Bell, or noncontextuality, inequalities are obtained by eliminating the variables associated with correlations between incompatible observables.

Our approach can be seen as a generalization of Fine's derivation of the Clauser-Horne-Shimony-Holt (CHSH) polytope [4]. It provides a generalization of the result obtained by Śliwa [5] and Collins and Gisin [6], namely, the appearance of only a finite number of families of Bell inequalities in measurement scenarios where one experimenter is allowed to choose between an arbitrary number of different measurements, and it allows for a complete characterization of a specific n -setting noncontextuality scenario.

In the following, we shall present the general method and discuss it more in details by means of simple examples. We then proceed to analyze some Bell and noncontextuality scenarios, providing in some cases analytical results, and showing the computational advantage in others.

2.1 Extension of Measures and Consistency Conditions

First we need a result on extension of probability measures which will be the basis of our approach to the computation of $\mathcal{H}$ -representation for correlation polytopes. More details can be found in Ref. [3].

Proposition 1 *Consider any set of probabilities p_i on a set of yes/no (i.e., $\{0, 1\}$ -valued) observables A_i and correlations p_{ij} on a subset of pairs A_i, A_j , defining a probability on each pair A_i, A_j , with $p_i = \langle A_i \rangle$, $p_j = \langle A_j \rangle$, $p_{ij} = \langle A_i A_j \rangle$; describing observables A_i as vertices and the above pairs as edges in a graph, any set of predictions associated with a tree graph, i.e., a graph without closed loops, admits a classical representation.*

We call such a graph representation a *compatibility graph*. Some examples of compatibility graphs are given in Figs. 2.1, 2.2, 2.3, 2.4, and examples of tree graphs are given by Fig. 2.1b, c, d.

Proposition 1 can then be generalized as follows [3].

Proposition 2 *the same holds with yes/no observables A_i substituted by free Boolean algebras $\mathcal{A}_i$, p_i by probabilities on $\mathcal{A}_i$, p_{ij} by probabilities on the Boolean algebra freely generated by the union of the sets of generators of $\mathcal{A}_i$ and $\mathcal{A}_j$.*

We recall that a Boolean algebra is freely generated by n generators $B_1, \dots, B_n$ if such generators are as much unconstrained as possible, i.e., they satisfy no conditions except those necessary conditions defining a Boolean algebra (e.g., distributive law). Since all Boolean algebras freely generated by $n < \infty$ generators are isomorphic, for the sake of simplicity, we can think of the the algebra of subsets 2^X of the set $X = \{0, 1\}^n$, with set theoretic operations ($\cap, \cup, ^c$); then the subsets $B_i = \{(x_1, \dots, x_n) \in X \mid x_i = 1\}$, for $i = 1, \dots, n$, can be taken as free generators. In terms of propositions and truth assignments, the free Boolean algebra assumption amounts to the assumption that each possible $\{0, 1\}$ -valued assignment to propositions is admissible. For more details see [7]. Notice that this is precisely the way how we define the 2^n vertices of the correlation polytope in Eq. (1.30).

The above results suggest a general method for the computation of $\mathcal{H}$ -representation for correlation polytopes, based on exploiting the automatically existing classical representations for subsets of observables with compatibility relations described by tree graphs, as in Propositions 1 and 2 above. Conditions for classical representability arise as consistency (i.e., coincidence on intersections) conditions for putting together partial extensions associated with subgraphs, giving rise to a description of the initial compatibility graph as a tree graph on such extended nodes. Such consistency conditions are expressed in terms of the existence of a solution for a set of linear inequalities. One of the main application of Fourier-Motzkin algorithm is precisely deciding whether a system of inequalities has a solution. It is thus sufficient to solve the hull problem for a smaller polytope, i.e., compute the single partial extensions, then constructing a higher dimensional polytope from that solution, i.e.,

impose the consistency conditions, and finally apply FM algorithm, i.e., derive the conditions for the existence of a solution.

We recall that the Fourier-Motzkin method consists in eliminating a variable from a system of linear inequalities by summing, after a proper normalization, all inequalities where it appears with plus sign with all inequalities where it appears with minus sign. From a geometric point of view, since the system of linear inequalities can represent a polytope (more generally, a cone), the above operation amounts to a projection onto the coordinates associated to the remaining variables; for more details see [8].

Although our method is general, different strategies are possible corresponding to different partitions of the initial graph into subgraphs. We shall introduce the details of our method by means of some simple examples. The first example is the derivation of the CHSH polytope. It is interesting to notice that it is analogous to that presented by Fine [4]; our method can be seen as a generalization of his idea to an arbitrary number of observables. We shall then discuss more complex Bell scenarios and some related computational results, and finally the complete characterization of the correlation polytope for the n -cycle noncontextuality scenario.

2.2 Bell Inequalities

2.2.1 CHSH Polytope from Bell-Wigner Polytope

The CHSH polytope is generated by the following set of vertices

$$u_\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_1\varepsilon_3, \varepsilon_1\varepsilon_4, \varepsilon_2\varepsilon_3, \varepsilon_2\varepsilon_4), \quad \varepsilon_i \in \{0, 1\}. \quad (2.1)$$

As previously discussed, it is associated with a bipartite measurement scenario in which Alice can choose between two measurements, associated with propositions A_1 and A_2 , and Bob can choose between two measurements, associated with propositions A_3 and A_4 .

As already recognized by Fine [4] and discussed also in [3, 9], the existence of a classical description for the four observables is equivalent to the existence of classical descriptions for the two subsystems, $\{A_1, A_2, A_3\}$ and $\{A_1, A_2, A_4\}$, coinciding on $\{A_1, A_2\}$. In fact, the two classical descriptions would give rise to an extension of the probability assignment to the four observables satisfying the hypothesis of Proposition 2 (see Fig. 2.1a, b, c).

The constraints on the subsystem $\{A_1, A_2\}$ imposed by the third observable, A_3 or A_4 , are described by the Bell-Wigner polytope [10], i.e., the correlation polytope associated with three proposition and their pairwise logical conjunctions, which is given by the following inequalities:

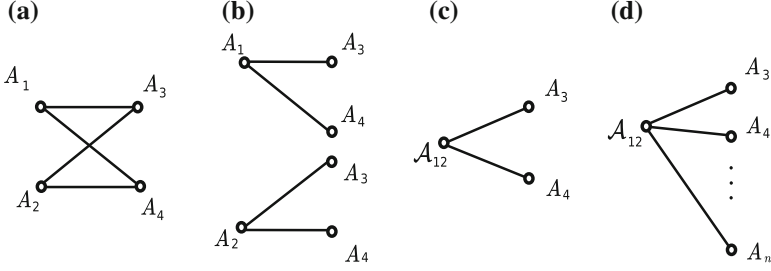


Fig. 2.1 **a** Graph of the compatibility relations between the observables in the CHSH scenario. **b** Partition in two subset of three observables with intersection $\{A_1, A_2\}$. **c** Tree graph obtained by extending the probability measure on the algebra generated by A_1, A_2 . **d** Asymmetric case with additional observables on Bob's side

$$0 \leq p_{ij} \leq p_i, \quad 0 \leq p_{ij} \leq p_j, \quad ij = 12, 1s, 2s, \quad (2.2)$$

$$p_i + p_j - p_{ij} \leq 1, \quad ij = 12, 1s, 2s, \quad (2.3)$$

$$1 - p_1 - p_2 - p_s + p_{12} + p_{1s} + p_{2s} \geq 0, \quad (2.4)$$

$$p_1 - p_{12} - p_{1s} + p_{2s} \geq 0, \quad (2.5)$$

$$p_2 - p_{12} - p_{2s} + p_{1s} \geq 0, \quad (2.6)$$

$$p_s - p_{1s} - p_{2s} + p_{12} \geq 0, \quad (2.7)$$

for fixed $s = 3$ or 4 .

Classical representability for QM prediction in the CHSH scenario amounts, therefore, to the existence of a common solution for the two systems of linear inequalities, namely, it amounts to the existence of a value for p_{12} consistent with the constraints imposed by classical descriptions for the two subsystems of three observables.

As a consequence of general properties of Fourier-Motzkin method (see [8]), a system of inequalities admits a solution if and only if the projected system, i.e., the system obtained by eliminating one or more variables, admits a solution. It follows that the above set of measurements admits a classical description if and only if measured correlations, i.e., correlations between compatible observables, satisfy the system of inequalities obtained by eliminating p_{12} .

In order to eliminate p_{12} , we just apply Fourier-Motzkin method: We sum inequalities where p_{12} appears with opposite sign and keep inequalities where it does not appear. A posteriori, we realize that only redundant inequalities arise from (2.2) and (2.3) and the combination of inequalities with the same index s .

The only interesting, i.e., non redundant, inequalities are those obtained from the combination of (2.4)–(2.7) for different s , namely

$$-1 \leq p_{13} + p_{14} + p_{24} - p_{23} - p_1 - p_4 \leq 0, \quad (2.8)$$

$$-1 \leq p_{23} + p_{24} + p_{14} - p_{13} - p_2 - p_4 \leq 0, \quad (2.9)$$

$$-1 \leq p_{14} + p_{13} + p_{23} - p_{24} - p_1 - p_3 \leq 0, \quad (2.10)$$

$$-1 \leq p_{24} + p_{23} + p_{13} - p_{14} - p_2 - p_3 \leq 0, \quad (2.11)$$

that, together with the Eqs. (2.2), (2.3) in which p_{12} does not appear, give the $\mathcal{H}$ -representation of the CHSH polytope.

2.2.2 Bipartite (2, n) Scenario

An analogous argument applies to the scenario in which Alice can choose between two measurements and Bob can choose among $n > 2$ measurements, associated with propositions $A_3, \dots, A_{n+2}$. The initial system of inequalities is still given by (2.2)–(2.7), but with s taking values in $\{3, 4, \dots, n+2\}$ (see Fig. 2.1d).

Since only one variable (i.e., p_{12}) has to be eliminated, at most two inequalities with different index s can be combined to give a valid inequality. Therefore, the final set of inequalities is given by (2.8)–(2.11) with the pair 3, 4 substituted by any pair i, j with $i, j \in \{3, \dots, n+2\}$ and $i < j$. This is precisely the result obtained in Refs. [5, 6].

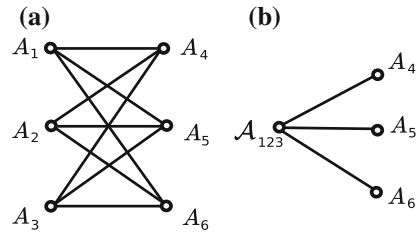
2.2.3 Two Parties, Three Settings

Now consider the bipartite scenario in which Alice can choose among three measurements, associated with propositions A_1, A_2 and A_3 , and Bob can choose among three measurements, associated with propositions A_4, A_5 and A_6 .

Analogously to the previous case, see Fig. 2.2, the existence of a classical description for the six observables is equivalent to the existence of classical descriptions for the three subsystems, $\{A_1, A_2, A_3, A_4\}$, $\{A_1, A_2, A_3, A_5\}$ and $\{A_1, A_2, A_3, A_6\}$, coinciding on $\{A_1, A_2, A_3\}$. A probability on $\{A_1, A_2, A_3\}$ is completely defined once the probabilities $p_1, p_2, p_3, p_{12}, p_{13}, p_{23}, p_{123}$ are given.

It is therefore sufficient to calculate the correlation polytope associated with probabilities $p_1, p_2, p_3, p_s, p_{1s}, p_{2s}, p_{3s}, p_{12}, p_{13}, p_{23}, p_{123}$, then consider the system given by all the above inequalities for $s = 4, 5, 6$ and eliminate the variables $p_{12}, p_{13}, p_{23}, p_{123}$.

Fig. 2.2 **a** Graph of the compatibility relations between the observables in the bipartite (3, 3) scenario. **b** Tree graph obtained by extending the measure on the algebra generated by A_1, A_2, A_3



2.2.4 Bipartite $(3, n)$ Scenario

Again, by adding observables only on Bob's side, one just obtains more copies of the initial system of inequalities, but with different indices s . The situation is analogous to that depicted in Fig. 2.1d, but with $\mathcal{A}_{12}$ substituted by $\mathcal{A}_{123}$.

In particular, since four variables have to be eliminated, at most $2^4 = 16$ inequalities with different indices s can be combined. As a result, all families of valid inequalities for the general case $(3, n)$ already arise in the case in which Alice performs 3 measurement and Bob 16.

2.2.5 Multipartite $(m, \dots, m, n)$ Scenario

The above argument can be extended to the case of p parties in which the first $p - 1$ can choose among m measurements, while the last one can choose among $n > m$ measurements: All families of inequalities can be obtained by studying the case in which the last experimenter performs 2^k measurements, where k is the number of variables to be eliminated.

2.2.6 Computational Results

We have presented an alternative method for the computation of half-space representation for correlation polytopes based on algebraic conditions and variable elimination. Besides the explicit calculations presented above, a reasonable question is: Does our method provide any computational advantage with respect to existing methods? In order to show the advantages of the tree graph method, we have computed the $\mathcal{H}$ -representation for some simple polytopes, with (i) our tree graph method using existing software implementing the Fourier-Motzkin algorithm; specifically, we used `porta` [11], and (ii) using standard software for solving the hull problem; specifically, we used `cdd`.

For simple cases like the $(2, 2)$ (i.e., the CHSH) and $(3, 3)$ scenarios, the computation is equally fast with both methods. However, remarkably, our tree graph method is noticeably faster to compute asymmetric scenarios:

For the $(3, 4)$ scenario, the tree graph method implemented with `porta` completed the calculation in ≈ 11 min, while `cdd` needed ≈ 20 min. The 11 min include the time (seconds) required to calculate the initial polytope (see Sect. 2.2.3).

For the $(3, 5)$ scenario, the tree graph method implemented with `porta` completed the calculation in ≈ 72 min, while `cdd` was still running after a week and we had to stop it.

All computations were performed on the same machine with an Intel Xeon CPU running at 3.20 GHz.

2.3 Noncontextuality Inequalities: The n -Cycle Scenario

The n -cycle contextuality scenario is given by n observables $X_0, \dots, X_{n-1}$ and the set of maximal contexts.

$$\mathcal{C}_n = \{\{X_0, X_1\}, \dots, \{X_{n-2}, X_{n-1}\}, \{X_{n-1}, X_0\}\}. \quad (2.12)$$

This scenario is the natural generalization of CHSH and KCBS scenarios, the most fundamental scenarios for, respectively, nonlocality and contextuality. From the point of view of contextuality scenarios, the n -cycle scenario for odd n has been investigated by Liang, Spekkens, and Wiseman [12], who derived the inequality (2.17) for odd n and discuss the maximal quantum violation. As a bipartite Bell scenario, i.e., for even n , it has been investigated by Braunstein and Caves [13], who derived the inequality (2.17) for even n , later the maximal quantum violation was discussed [14].

In this section we give a complete characterization of the n -cycle polytope for any n , proving that the only facets inequalities of the polytope are given by Eqs. (2.13a)–(2.13d) together with Eq. (2.17).

A compatibility graph representation for the n -cycle scenario is given in Fig. 2.3. Here, it is convenient to use ± 1 -valued variables instead of 0/1-valued. The corresponding correlations, i.e., expectation values of the product of the outcomes, are related to probabilities by the affine invertible transformation defined as

$$4p(++|X_i X_{i+1}) = 1 + \langle X_i \rangle + \langle X_{i+1} \rangle + \langle X_i X_{i+1} \rangle, \quad (2.13a)$$

$$4p(+-|X_i X_{i+1}) = 1 + \langle X_i \rangle - \langle X_{i+1} \rangle - \langle X_i X_{i+1} \rangle, \quad (2.13b)$$

$$4p(-+|X_i X_{i+1}) = 1 - \langle X_i \rangle + \langle X_{i+1} \rangle - \langle X_i X_{i+1} \rangle, \quad (2.13c)$$

$$4p(--|X_i X_{i+1}) = 1 - \langle X_i \rangle - \langle X_{i+1} \rangle + \langle X_i X_{i+1} \rangle. \quad (2.13d)$$

As a consequence, the analysis of the correlation polytopes (i.e., number of vertices and facets, tightness of a given inequality) defined in terms of ± 1 -valued or 0/1-valued variables are equivalent.

The n -cycle correlation polytope is then defined by the following 2^n vertices

$$(x_0, \dots, x_{n-1}, x_0 x_1, \dots, x_{n-1} x_0), \quad x_i \in \{-1, 1\}. \quad (2.14)$$

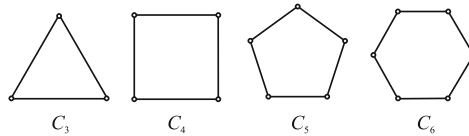


Fig. 2.3 Graphs associated to the compatibility relations among the observables X_i for $n = 3, 4, 5, 6$. C_4 corresponds to CHSH case with the labelling of nodes A_1, B_1, A_2, B_2 , in the usual notation for Alice and Bob observables, and C_5 corresponds to KCBS case with the labelling $X_0, \dots, X_4$

In order to differentiate this with the probability case, we use the following notation for a general vector

$$(\langle X_0 \rangle, \dots, \langle X_{n-1} \rangle, \langle X_0 X_1 \rangle, \dots, \langle X_{n-1} X_0 \rangle). \quad (2.15)$$

A set of tight inequalities is already given by the positivity conditions on the terms in Eq. (2.13), namely

$$1 + \langle X_i \rangle + \langle X_{i+1} \rangle + \langle X_i X_{i+1} \rangle \geq 0 \quad (2.16a)$$

$$1 + \langle X_i \rangle - \langle X_{i+1} \rangle - \langle X_i X_{i+1} \rangle \geq 0 \quad (2.16b)$$

$$1 - \langle X_i \rangle + \langle X_{i+1} \rangle - \langle X_i X_{i+1} \rangle \geq 0 \quad (2.16c)$$

$$1 - \langle X_i \rangle - \langle X_{i+1} \rangle + \langle X_i X_{i+1} \rangle \geq 0. \quad (2.16d)$$

one can easily prove that they are saturated by a set of affinely independent vertices (2.14). Such conditions are trivially satisfied in QM since they involve only pairs of jointly measurable observables.

The remaining facets of the n -cycle polytope are given by the 2^{n-1} inequalities

$$\Omega = \sum_{i=0}^{n-1} \gamma_i \langle X_i X_{i+1} \rangle \stackrel{\text{NCHV}}{\leq} n - 2, \quad (2.17)$$

where $\gamma_i \in \{-1, 1\}$ such that the number of $\gamma_i = -1$ is odd.

Our proof is based on the fact that the existence of a classical probability model for the observables $\{X_0, \dots, X_{n-1}\}$ is equivalent to the existence of classical models for $\{X_0, \dots, X_{n-2}\}$ and $\{X_0, X_{n-1}, X_{n-2}\}$, coinciding on their intersection $\{X_0, X_{n-2}\}$, see Fig. 2.4a, b, c. More precisely, if the two subsets of observables in Fig. 2.4b ($(n-1)$ -cycle and 3-cycle, admit a classical representation, i.e. all the corresponding inequalities are satisfied, then the set of probabilities can be extended, following

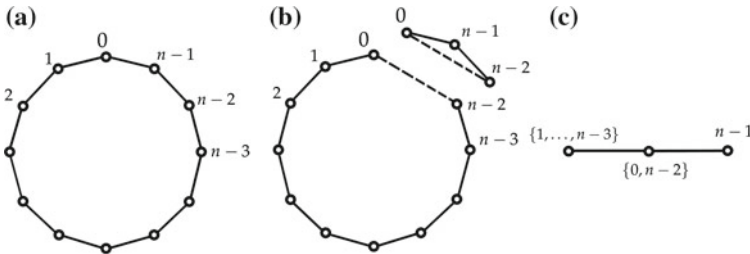


Fig. 2.4 **a** n -cycle scenario. **b** subsets of observables that can be associated with the $(n-1)$ -cycle and 3-cycle scenario by considering the “unmeasurable correlation” $\langle X_0 X_{n-2} \rangle$ (dashed line) **c** Extended classical model that can be obtained if the two subset admits a classical representation coinciding on their intersection. Such a model is automatically classical as it can be depicted as a tree graph

Propositions 1 and 2, as in Fig. 2.4c, i.e. two classical model for $\{0, 1, \dots, n-3, n-2\}$ and for $\{0, n-1, n-2\}$ coinciding on their intersection $\{0, n-2\}$. By Proposition 2, such a set already admits a classical representation.

Such a consistency condition for the intersection is written in terms of the “unmeasurable correlation” $\langle X_0 X_{n-2} \rangle$, i.e., a correlation between observables that are not in a context and therefore cannot be jointly measured, but have nevertheless a well-defined correlation in every classical model [9]. The final set of inequalities must not contain the variable $\langle X_0 X_{n-2} \rangle$, which must be removed by applying Fourier-Motzkin (FM) elimination.

We can now proceed by induction on n . The case $n = 3$ is known. For the inductive step, following the above argument, we calculate the n -cycle inequalities by combining the $(n-1)$ -cycle inequalities for the subset $\{X_0, \dots, X_{n-2}\}$ with the 3-cycle inequalities for $\{X_0, X_{n-1}, X_{n-2}\}$. We apply FM elimination on the variable $\langle X_0 X_{n-2} \rangle$ from the whole set of inequalities. All inequalities in (2.17) are obtained by combining one inequality, of the same form, for the $(n-1)$ -cycle with one for the 3-cycle, and are in the right number. In fact, in half of the $(n-1)$ -cycle inequalities, $\langle X_0 X_{n-2} \rangle$ appears with the $+$ sign, and in half with the $-$ sign, and the same for the 3-cycle. The number of possible combination is, therefore, given by $2^{n-3} \cdot 2 + 2^{n-3} \cdot 2 = 2^{n-1}$.

Combining two inequalities for the $(n-1)$ -cycle, or two for the 3-cycle gives a redundant inequality, as happens for combination of positivity conditions (2.16) with inequalities of the form (2.17), the latter being obtainable as a sum of $n-1$ (or 3) positivity conditions. There are no other inequalities.

Tightness can be proved by showing that inequalities (2.17) correspond to facets of the $2n$ -dimensional correlation polytope, i.e., they are saturated by $2n$ noncontextual vertices which generate an affine subspace of dimension $2n-1$. First, focus on the inequality of the odd n -cycle for which all $\gamma_i = 1$. It is saturated by $2n$ vertices which can be written as $(\pm v_i, w_i)$, for $i = 0, \dots, n-1$, where v_i is a n -dimensional vector given by a cyclic permutation of the components of $v_{n-1} = (+1, -1, +1, \dots, +1)$ and w_i 's components are given by the corresponding products, of v_i 's components, namely, one component equal to 1 and $n-1$ components equal to -1 . Up to a reordering of the vectors, it holds $v_i + v_{i+1} = 2e_{i+1}$, with addition modulo n and $e_0, \dots, e_{n-1}$ the canonical basis of $\mathbb{R}^n$, and $w_i + (1, 1, \dots, 1) = 2e_i$. As a consequence, $\{(\pm v_i, w_i)\}_{i=1, \dots, 2n}$ is a basis of $\mathbb{R}^{2n}$, and, therefore, such vectors generate an affine subspace of dimension $2n-1$. Since all the other vertices and inequalities are obtained from this one via the mapping $X_i \mapsto -X_i$, we are done for the odd case. For the even case, the proof is slightly different: Take the inequality which has $\gamma_0 = -1$ and all the other $\gamma_i = 1$. Again, it is saturated by $2n$ vertices which can be written as $(\pm v_i, w_i)$, for $i = 0, \dots, n-1$, with $v_0 = \sum_{k=0}^{n-1} (-1)^k e_k$, $w_0 = -\sum_{k=0}^{n-1} e_k$ and

$$v_i = e_0 + e_1 + \sum_{2 \leq k \leq i} (-1)^k e_k + \sum_{i < k \leq n-1} (-1)^{k+1} e_k, \quad (2.18)$$

$$w_i = e_0 + e_i - \sum_{k \neq 0, i} e_k, \quad \text{for } i = 1, \dots, n-1. \quad (2.19)$$

The above vectors satisfy $v_{n-1} - v_0 = 2e_0$, $v_1 - v_0 = 2e_1$, and $v_i - v_{i+1} = (-1)^{i+1}e_{i+1}$, for $i = 1, \dots, n-2$, and $w_i + w_j - 2w_0 = 4e_0$, $w_i - w_0 = 2(e_0 + e_i)$ for $i, j = 1, \dots, n-1$. Again, this implies that the vertices are affinely independent, and all the other vertices and inequalities are generated via the relabelling $X_i \mapsto -X_i$.

2.4 Discussion

We developed an alternative method for the computation of the correlation polytope associated with Bell and noncontextuality scenarios based on results on automatic extension of probability measures and Fourier-Motzkin algorithm of variable elimination for systems of linear inequalities.

We applied our method to different Bell and noncontextuality scenarios. For the Bell scenario, we derive some result on the minimal computation needed for a completely characterization of asymmetric scenarios where one experimenter has an arbitrary number of settings, thus generalizing a result by Śliwa [5] and Collins-Gisin [6]. Moreover, we performed some explicit computations to show the advantages with respect to usual algorithms solving directly the hull problem.

For noncontextuality scenario, we are able to give a complete characterization, i.e., the complete set of tight inequalities, for the n -cycle scenario, which is the natural generalization of the two most fundamental scenarios, respectively, for nonlocality and contextuality, namely, the CHSH and KCBS scenario. To complete the discussion on the n -cycle scenario, in Chap. 4, we will compute the quantum bound by means of Cabello-Severini-Winter method [15, 16].

References

1. C. Budroni, A. Cabello, J. Phys. A: Math. Theor. **45**, 385304 (2012), doi:[10.1088/1751-8113/45/38/385304](https://doi.org/10.1088/1751-8113/45/38/385304)
2. M. Araújo, M.T. Quintino, C. Budroni, M.T. Cunha, A. Cabello, Phys. Rev. A **88**, 022118 (2013), doi:[10.1103/PhysRevA.88.022118](https://doi.org/10.1103/PhysRevA.88.022118)
3. C. Budroni, Morchio, J. Math. Phys. **51**, 122205 (2010), doi:[10.1063/1.3523478](https://doi.org/10.1063/1.3523478)
4. A. Fine, Phys. Rev. Lett. **48**, 291 (1982), doi:[10.1103/PhysRevLett.48.291](https://doi.org/10.1103/PhysRevLett.48.291)
5. C. Śliwa, Phys. Lett. A **317** 165 (2003), doi:[10.1016/S0375-9601\(03\)01115-0](https://doi.org/10.1016/S0375-9601(03)01115-0)
6. D. Collins, N. Gisin, J. Phys. A: Math. Gen. **37** 1775 (2004), doi:[10.1088/0305-4470/37/5/021](https://doi.org/10.1088/0305-4470/37/5/021)
7. S. Givant, P. Halmos, *Introduction to Boolean Algebras* (Springer, New York, 2009)
8. G.M. Ziegler, *Lectures on Polytopes* (Springer, New York, 1995)
9. C. Budroni, G. Morchio, Found. Phys. **42**, 544 (2012), doi:[10.1007/s10701-012-9625-0](https://doi.org/10.1007/s10701-012-9625-0)
10. I. Pitowsky, *Quantum Probability—Quantum Logic*, Lecture Notes in Physics (Springer, Berlin, 1989)
11. porta, <http://www.zib.de/Optimization/Software/porta/>
12. Y.-C. Liang, R.W. Spekkens, H.M. Wiseman, Phys. Rep. **506**, 1 (2011), doi:[10.1016/j.physrep.2011.05.001](https://doi.org/10.1016/j.physrep.2011.05.001)

13. S.L. Braunstein, C.M. Caves, Nucl. Phys. B, Proc. Suppl.**6**, 211 (1989), doi:[10.1016/0920-5632\(89\)90441-6](https://doi.org/10.1016/0920-5632(89)90441-6)
14. S. Wehner, Phys. Rev. A **73**, 022110 (2006), doi:[10.1103/PhysRevA.73.022110](https://doi.org/10.1103/PhysRevA.73.022110)
15. A. Cabello, S. Severini, A. Winter (2010). [arXiv:1010.2163](https://arxiv.org/abs/1010.2163)
16. A. Cabello, S. Severini, A. Winter, Phys. Rev. Lett. **112**, 040401 (2014), doi:[10.1103/PhysRevLett.112.040401](https://doi.org/10.1103/PhysRevLett.112.040401)

Temporal Quantum Correlations and Hidden Variable
Models

Budroni, C.

2016, XIII, 114 p., Hardcover

ISBN: 978-3-319-24167-8