Identity in Frege's Shadow

Jaakko Hintikka

Frege's Mistake

One of the crucial figures in early analytic philosophy was Gottlob Frege. This is not due to his direct influence. Frege exerted his influence through his creation, his logic. (The fullest exposition of this logic is in Frege 1893–1903a see also 1893–1903b.) This logic has been used without major changes by the majority of subsequent philosophers. As a consequence its strengths and the weaknesses have been magnified by history. This essay is written not to praise Frege but to bury—or at least to diagnose—one particularly important mistake affecting Frege's logic. The later effects of this virus in Frege's logic are too exstensive to be discussed here. (See here especially Hintikka, forthcoming).

Frege is usually credited with having created the core part of modern symbolic logic, the theory of quantifiers which is essentially what we now call first-order logic. (Other names that have been used include predicate calculus, lower functional calculus and quantification theory). This may be true in some historical sense, but if so, Frege did not finish his task. He made a mistake. He did not fully understand the semantical job description of quantifiers and as a consequence produced a logic that is unnecessarily restricted in its expressive capacities. He thought that the job of quantifiers is exhaustively done by their ranging over a class of values, so as to express exceptionlessness or nonemptiness in this class. He expressed this assumption in so many words in that he characterized quantifiers as higher-order predicates indicating emptiness or exhaustiveness of a given

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lower-order predicate (Frege 1893–1903b). The full range of values of quantified first-order variables is in our days taken to be the domain of a model alias its universe of discourse.

This "ranging over" is admittedly a part of the semantical story of quantification. But there is also a subtler and more interesting task that quantifiers perform. This other task is vitally important in mathematics, science and in everyday life. It is to express dependence relations, that is, dependencies (and by contrast independencies) between variables. On the first-order level of a traditional logical language, the only way of expressing that y depends on x is to make the quantifier (Q_2y) to which y is bound formally dependent (or formally independent) of the quantifier (Q_1x) to which x is bound.

The simplest questions (but not the only ones) that bring out this other function of quantifiers concern quantifier ordering, including the effects of changes in such ordering. "Every man loves some woman" is not equivalent with "Some woman is loved by every man" because in the latter the woman depends on the lover while in the former she does not. Yet the only formal difference between them lies in the (logical) order of the two quantifiers "some" and "every". This order is independent of the range of the quantifiers involved. It matters even if we are conceptualizing the situation in terms of many-sorted logic in which "every man" and "some woman" range over different classes of persons. Questions of quantifier ordering thus form a handy testing ground for problems of quantifier dependence (and independence).

In the received ("Fregean") first-order logic (alias RFO logic), formal dependencies among quantifiers are expressed by the scope brackets (). A quantifier (Q_2y) depends on (Q_1x) if it occurs within its scope, as in

$$(Q_1x)(\ldots(Q_2y)(\ldots)\ldots) \tag{1.1}$$

Scopes are assumed to be nested. There is in RFO logic no such thing as (partial) overlap of scopes. As a result, the dependence structures expressible in RFO logic are finite labeled trees, with dependence relations that exhibit a partial ordering which is linear in one direction. Such dependence structures are nevertheless only a proper subclass of all possible dependence structures. The rest cannot be expressed in RFO logic, which is therefore not a full account of the semantics of quantifiers. Frege's logic, and the logics used by most subsequent logicians and philosophers, are therefore defective. Their expressive power is restricted unnecessarily, seriously handicapping them.

If Logic Corrects Frege's Mistake

This mistake of overlooking the dependence-indicating function of quantifiers will be called in this paper for brevity Frege's Fallacy. It was not unavoidable, and it was not unavoidable in Frege's actual historical situation. Already at Frege's time Charles S. Peirce understood the dependence-indicating aspect of first-order logic better than Frege. The term "Frege's Fallacy" may nevertheless be somewhat unfair in that Frege's failure can be seen as a sin of omission rather than as one of commission.

Frege's mistake can be corrected (see here Hintikka 1996). Systematically the minimal improvement is to introduce a slash notation which makes an existential quantifier (∃y) independent of a universal quantifier (∀x) (in a sentence in a negation normal form) within whose formal scope it occurs by writing it as (∃y/∀x). This results in what is known as independence friendly (IF) logic. This logic is thus not a "nonclassical" variant or a special branch of RFO. It is not an "alternative logic". It replaces RFO by correcting its shortcomings. It is as classical or non-classical as RFO. It contains RFO as a special case, viz. as the logic of those propositions that satisfy the tertium non datur. The status of IF logic as being nothing more and nothing less than an improved version of RFO logic is illustrated by the fact that it could be formulated without any new symbols merely by liberalizing the use of parentheses.

The replacement of RFO by IF first-order logic is nevertheless more than a mere architectonic improvement. Frege's mistake has been recommitted by most subsequent logicians, including the leading ones. It has led to damaging missteps in the development of logic and its applications. On other occasions I will show that Frege's Fallacy has played a role in such important developments as Frege's higher-order logic and Hilbert's epsilon-calculus. It was responsible for the main paradoxes of set theory and hence indirectly for the invention of the theory of types. Less directly, Frege's mistake has played a major role in the (not so new) New Theory of Reference due to Kripke (1963). Here, only some general explanation explanations are offered.

Skolem Functions as Dependence Indicators

The dependence indicating task of quantifiers deserves closer examination. In the abstract eyes of a mathematician, to speak of dependencies is to speak of functions, that is, of the functions that codify those dependencies. Hence, the logic of quantifiers qua dependence indicators is a logic of functions. The functions that are tacitly present in quantificational propositions are the ones that express the dependencies between quantifiers, that is, express how the truth making value of a dependent (embedded) quantified variable is determined by the values of the quantifiers in whose scope it occurs. The functions that serve this purpose in a given quantificational sentence S are in logic called the Skolem functions of S. For instance, the Skolem function *f* of a sentence of the form

$$(\forall x)(\exists \mathrm{y})\mathrm{F}[\mathrm{x}, \mathrm{y}] \tag{3.1}$$

makes true the sentence

$$(\forall x)\mathrm{F}[\mathrm{x}, f(\mathrm{x})]. \tag{3.2}$$

Hence the real logic of quantifiers, the real first-order logic is in effect the logic of Skolem functions, not a "predicate logic" (as it is often called). This might seem to be a mere terminological nuance, but in reality it is a symptom of a deeper difference.

The nature of quantification theory as being virtually but a theory of Skolem functions is highlighted by the intimate connection between Skolem functions and the truth conditions of quantified propositions. The most natural truth predicate of first-order sentences is in fact the existence of (a full set of) their Skolem functions.

Because of this role of Skolem functions, we have to consider first-order logic of functions as being separate from predicate logic. The two have different logical truths, it turns out.

This important fact is not generally recognized. Most logicians treat functions as predicates (relations) of a certain kind. More explicitly expressed, an equation like $\mathrm{y} = f(\mathrm{x})$ is treated as if it were a binary relation F[x, y], with the following special properties

$$(\forall x)(\exists \mathrm{y})\mathrm{F}[\mathrm{x}, \mathrm{y}] \tag{3.3}$$

$$(\forall x)(\forall y)(\forall z)\,((\mathrm{F}[\mathrm{x}, \mathrm{y}]\;\&\;\mathrm{F}[\mathrm{x}, \mathrm{z}]) \supset \mathrm{y} = \mathrm{z}). \tag{3.4}$$

The corresponding function is then defined in terms of F by the equivalence

$$(\forall x)(\forall y)\,((\mathrm{F}[\mathrm{x}, \mathrm{y}] \leftrightarrow \mathrm{y} = f(\mathrm{x})). \tag{3.5}$$

But (3.3) and (3.4) are not logical truths about the predicate F. For each function *f* they have to be introduced as ad hoc premises. If *f* is a function, the assumptions (3.3) and (3.4) should be logical and not merely contingent truths. Hence the logic of functions and the logic of predicates have different logical truths.

The important differences between the logic of functions and the logic of predication are not always appreciated. They can nevertheless be important here. Perhaps the most consequential difference is that even if functions are treated as predicates they cannot be adequately defined, in the sense that the definitions could be treated as conceptual (logical) truths. And if individuals are construed as constant functions, by the same token they cannot be logically defined, either.

This straightforward logical observation has significant implications. For instance, it makes a significant difference to the logical theory of Frege and Wittgenstein (the author of the *Tractatus*) that they did not have functions among their nonlogical primitive constructs (cf. Hintikka 2004).

Quantifiers and Identity

In the dichotomy between the logic of functions and the logic of predication, the notion of identity belongs to the logic of functions and quantifiers. The truth conditions (whatever they may be) of a predication F[a, b] normally turns on what a and b are like on their own or in relation to each other. In contrast, the truth conditions of an equation $y = f(x)$ depend in an obvious sense only on the identity of x and y, that is on which entities x and y are. This is because the function so to speak enables one to find y (whatever object that is) on the basis of merely knowing what x is.

The concrete meaning of this informal observation is illustrated by the fact that quantifiers depend on the mode of identification of the values of the variables. Without trying to spell out the theory of identification here (I have done it elsewhere) suffice it to say as an example that although perspectivally identified and publicly identified objects are not two separated classes of objects existing side by side, they are values of two different pairs of quantifiers.

The same connection between quantifiers and the notion of identity is manifested also in the fact that the substitutivity of identicals must hold for the variables of quantification, other things being equal. The situation is nevertheless complicated by the fact that other things are not always equal. There are different modes of identification, and even apart from them there are different meanings of identity. Comments on these problems can be found in my earlier papers.

Ironically, this feature of the logic of functions may have encouraged Frege's Fallacy. Since what matters in functional equations is apparently only the identity of the terms involved, it might seem that what matters in interpreting quantifiers is merely the class of their possible values.

Substitutional Interpretation of Quantifiers

To return to Frege's Fallacy, most philosophers who commit it do so tacitly, in the privacy of their presuppositions. There is nevertheless a way of committing oneself to the fallacious view publicly. It is to adopt what is usually referred to as the substitutional interpretation of quantifiers. The gist of this so-called interpretation is often taken to lie in formulating truth-conditions for quantificational sentences in terms of names and substitution-instances of formulas using names, and contrasted to what is known as objectual interpretation. However, this is a distinction without much structural difference, as is argued passionately by Kripke (1963). He is right. Whatever differences there may between the two so-called interpretations, they do not need to affect any formal (syntactical) laws or principles. In this sense, the difference lies only in a different way of speaking of the truth-conditions of quantificational sentences. Vienna Circle members might have spoken here of material and formal modes of speech.

In other words, the contrast between the two so-called interpretations is often discussed by asking such questions as to whether there are enough names to capture substitutionally the truth-conditions of quantificational sentences in infinite and perhaps even uncountable domains. Questions like these nevertheless do not cut very deeply. Mathematicians operate routinely with infinite and even uncountable sets. What is so strange about infinite sets of names? Are not ordinary numerals such a set? Furthermore, those names need not be actually present in one's object language. For substitutional interpretation it suffices that we talk about them in a suitable metalanguage.

Kripke [8] has seen correctly where the conceptual gist of the substitutional interpretation lies. It lies in the claim that the truth of a quantificational sentence equals the truth of a certain (possibly infinite) truth-function of atomic sentences and/or of their negations. (For a qualification that does not matter here, see below.) For instance, a universally quantified sentence $(\forall x)\mathrm{F}[x]$ is logically equivalent with the conjunction of its substitution-instances, i.e. equivalent with

$$\mathrm{F}[\mathrm{a}_1]\ \&\ \mathrm{F}[\mathrm{a}_2]\ \&\ldots\&\ \mathrm{F}[\mathrm{a}_\mathrm{i}]\ \&\ \ldots \tag{5.1}$$

where the a_i are all the members of the universe of discourse.

But to maintain this is to take the semantical function of quantifiers to be exhausted by their ranging over their values, in this case over the set $\{\mathrm{a}_\mathrm{i}\}$. In other words, a strictly understood substitutional interpretation is but another form of Frege's Fallacy.

The Inconspicuousness of Frege's Fallacy

The Fregean interpretation of quantifiers has in fact been adopted by some philosophers, for example by the author of *Tractatus Logico-Philosophicus*. Alas, Wittgenstein later came to consider his treatment of quantifiers the "biggest mistake" he made in the Tractatus.

What is wrong with the substitutional interpretation? What are symptoms of the mistakes that Frege's Fallacy causes? One might perhaps expect the problems to show up in the dependence relations between ordinary quantifiers, for instance in the behavior of quantifier ordering. As it happens, such problems are nevertheless all too easily hidden—or, more accurately, brushed under a carpet. The carpet is the usual interpretation of propositional logic. In the substitutional interpretation dependence relations between quantifiers become dependence relations between propositional connectives, which are far less conspicuous than the corresponding relations between quantifiers This does not change the problem, however. The possible patterns of dependencies are analogous in the two cases, and equally in need of liberation. However, the difficulties are much less clearly in evidence in the case of propositional connectives than in the case of quantifiers. Dependence and independence relations between connectives have not attracted much attention by logicans. This is not to say that there is no need of enriching propositional logic, too, by freeing it from the fetters of the conventional scope relation. Indeed, computer scientists are beginning to make use of such liberated patterns of propositional dependence relations. These different patterns are in fact relevant to questions of computer architecture.

Frege's Fallacy therefore rears its ugly head most obviously with quantifiers other than the two standard quantifiers of first-order logic. A prime example is offered by the quantificational character of modal and epistemic notions. In using modal logic we are in effect quantifying over (and into) *possibilia* of some kind or other. For instance the necessity operator employed in possible-world semantics functions in effect as a universal quantifier ranging over possible worlds. One does not have to be Leibniz or Carnap to agree that a sentence is necessary true if and only if it is true in all possible worlds. Likewise, the sentence "it is known that S" (in brief, KS) is true if and only if S is the case in all scenarios compatible with what is known. Hence quantifiers and modal operators can be expected to depend on each other, unless specific assumptions to the contrary are made. One such assumption is the substitutional interpretation of quantifiers. It implies that quantifiers are independent of modal operators in that they range over the same class of values irrespective of the (modal) context. Since that class is the universe of discourse and since modal operators range over possible worlds, it follows that all different possible worlds must have the same domain of individuals (the same universe of discourse).

Logicians' attention has been distracted from the interplay of such quasi-quantifiers with regular quantifiers by their being of a different logical type and hence ranging over different kinds of entities. However, as was pointed out, differences in range do not make any difference to questions of dependence.

Quantifiers Depend on Modal Operators

What creates a new situation is that in modal and intensional logic the dependence relations that various quantifiers serve to express cannot any longer be hidden behind the back of dependence relations among propositional connectives. It is in fact easy to see that the substitutional interpretation fails in contexts involving both quantifier and modal operators (or indeed any notions whose semantics involves a multiplicity of *possibilia*). An example from epistemic logic helps to illustrate what is involved here. We can use the meaning of the order of different operators as an *experimentum crucis*.

For the purpose, consider a sentence in which we "quantify in", for instance a sentence of the form

$$(\forall x)(\mathrm{A}(\mathrm{x}) \supset \mathrm{KB}(\mathrm{x})) \tag{7.1}$$

It might mean "each person in this town in known to be rich." This obviously does not imply "it is known that everybody in this town is rich", i.e.

$$\mathrm{K}\ (\forall x)(\mathrm{A}(\mathrm{x}) \supset \mathrm{B}(\mathrm{x})) \tag{7.2}$$

(Aristotle was already aware that the two are not equivalent; see *Analytica Priora* 67a16 ff. and *Analytica Posteriora* 74a30 ff.)

The reason for the non-equivalence is obvious: the denizens of this town of whom it is known that they are rich need not be known to exhaust its inhabitants. In other words, in some scenarios compatible with what is known, there exist other townsmen. And this is simply an instance of the dependence of the quantifier $(\forall x)$ on the virtual quantifier K. To neglect this dependence is to overlook the obvious difference between (7.1) and (7.2).

The kind of dependence of $(\forall x)$ on K that is involved here is as plain as the proverbial pikestaff (if you know what it is). The range of values of x in $(\forall x)$ is different in the different scenarios ("possible worlds") over which K ranges. The different knowledge worlds (scenarios compatible with what is known) need not have the same objects in them.

This affects the logical laws governing them. For instance, an inference from $(\forall x)$ KF[x] to K$(\forall x)$F[x] fails because there can in alternative knowledge worlds exist individuals that do not exist in the actual world, and an inference from P(∃x)F[x] to (∃x)PF[x] (where P expresses epistemic possibility) fails because possible individuals need not exist in the actual one. In all these cases, the sensitivity of meaning to the relative order of quantifiers and epistemic operators is a symptom of their dependence on each other.

Analogous remarks apply to modal operators in the narrow sense of the word, that is, possibility and necessity. There is no necessity (either logical or natural or nomic) that different worlds must have the same entities existing in them.

It should therefore be perfectly obvious that the substitutional interpretation of quantifiers fails in modal and epistemic contexts. Why then have several astute philosophers adopted it? The answer is that they have had some independent reason to do so. Characteristically, Bertrand Russell the acquaintance theorist and Ludwig Wittgenstein the author of the *Tractatus* believed that we should not assume the existence of any entities not given to us in direct experience. Hence such entities constitute the one and only range for our quantifiers. Alas, when Wittgenstein gave up (on October 11, 1929) the phenomenological languages that have objects of acquaintance as their universe of discourse, his reasons for clinging to the substitutional interpretation disappeared.

References

Frege, G. (1893–1903a). *Grundgesetze der Arithmetik, begriffsschriftlich abgeleitet*. Jena: Herman Pohle.
Frege, G. (1893–1903b) *The basic laws of arithmetic* (M. Furth (Ed.) [partial translation of Frege 1893–1903], Trans.), Berkeley and Los Angeles: University of California Press.
Hintikka, J. (1996). *The principles of mathematics revisited*. Cambridge: Cambridge University Press.
Hintikka, J. (2004). On the different identities of identity: A historical and critical essay. In G. Fløistad (Ed.), *Language, meaning, interpretation* (pp. 117–139). Dordrecht: Kluwer Academic Publishers.
Hintikka, J. (forthcoming) If logic, definitions and the vicious circle principle.
Kripke, S. (1963). Semantical considerations on modal logic. *Acta Philosophica Fennica, 16*, 83–94.

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Jaakko Hintikka was Professor of Philosophy at Boston University. He was born in 1929 and educated in Finland. He defended his dissertation in 1953. In 1956–1959 Hintikka was Junior Fellow at Harvard. From 1959 until 2014 he held professorships at different institutions in Finland and in the US. Hintikka has created game-theoretical semantics and IF logic and made contributions to epistemic logic and epistemology, to inductive logic and the epistemology of induction, to the foundations of mathematics and to formal linguistics. He is also interested in the history of philosophy and has studied Aristotle, Descartes, Kant, Peirce and Wittgenstein among others. Hintikka has received six honorary doctorates. He has been honored by a volume in the series *Library of Living Philosophers* and by Grand Cross of the Order of the Lion of Finland. He died on the 12th of August 2015, when the volume was in print.

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