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**Abstract**

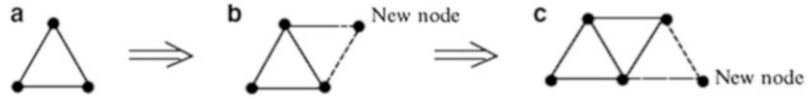
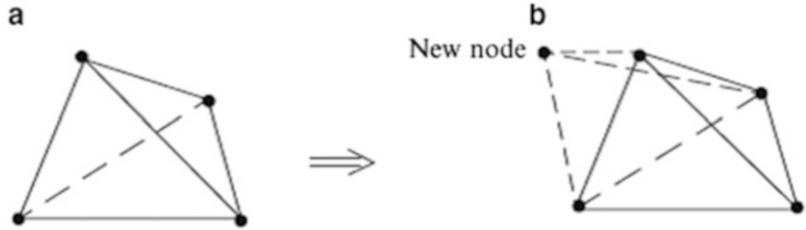
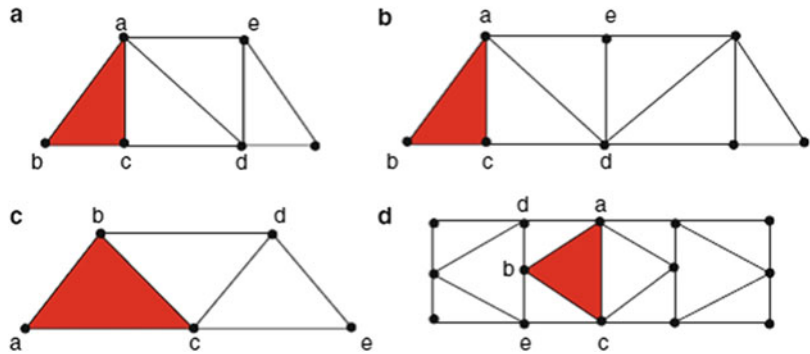
We begin this chapter by reviewing the historical development of truss structures. Trusses have played a key role in the expansion of the highway and railroad systems during the past two centuries. From a mechanics perspective, they are ideal structures for introducing the concepts of equilibrium and displacement. We deal first with the issues of stability and static determinacy, and then move on to describe manual and computer-based techniques for determining the internal forces generated by external loads. A computational scheme for determining the displacements of truss structures is presented next. Given a structure, one needs information concerning how the internal forces vary as the external live load is repositioned on the structure for the design phase. This type of information is provided by an influence line. We introduce influence lines in the last section of this chapter and illustrate how they are constructed for typical trusses. This chapter focuses on linear elastic structural analysis. Nonlinear structural analysis is playing an increasingly more important rule in structural design. However, we believe an understanding of linear analysis is essential before discussing the topic of nonlinear analysis.

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**2.1 Introduction: Types of Truss Structures**

Simple two-dimensional (2-D) truss structures are formed by combining one-dimensional linear members to create a triangular pattern. One starts with a triangular unit, and then adds a pair of members to form an additional triangular unit. This process is repeated until the complete structure is assembled. Figure 2.1 illustrates this process for the case where all the members are contained in a single plane. Such structures are called *plane* trusses; the nodes are also called “*Joints*.”

Three members connected at their ends form a rigid structure in the sense that, when loaded, the change in shape of the structure is due only to the deformation of the members. It follows that a

**Fig. 2.1** Simple planar truss construction**Fig. 2.2** Simple space truss construction**Fig. 2.3** Simple planar trusses

structure constructed in the manner described above is also rigid provided that the structure is suitably supported.

Simple three-dimensional (3-D) space trusses are composed of tetrahedral units. Starting with a tetrahedral unit, one forms an additional tetrahedral unit by adding three linear elements, as illustrated in Fig. 2.2. When the structure is suitably supported to prevent rigid body motion, the assemblage is rigid. The question of suitable supports is addressed later in the chapter.

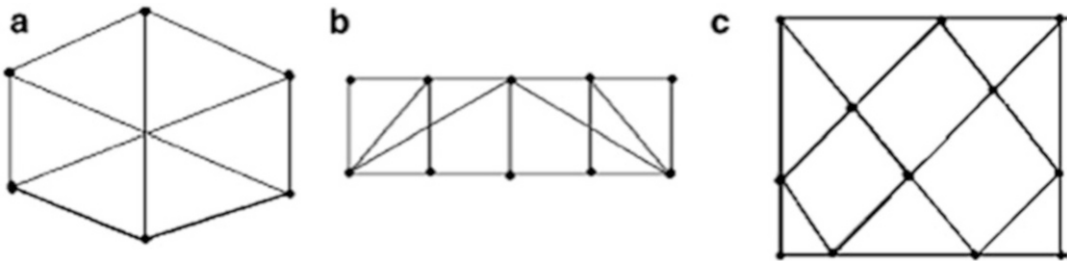
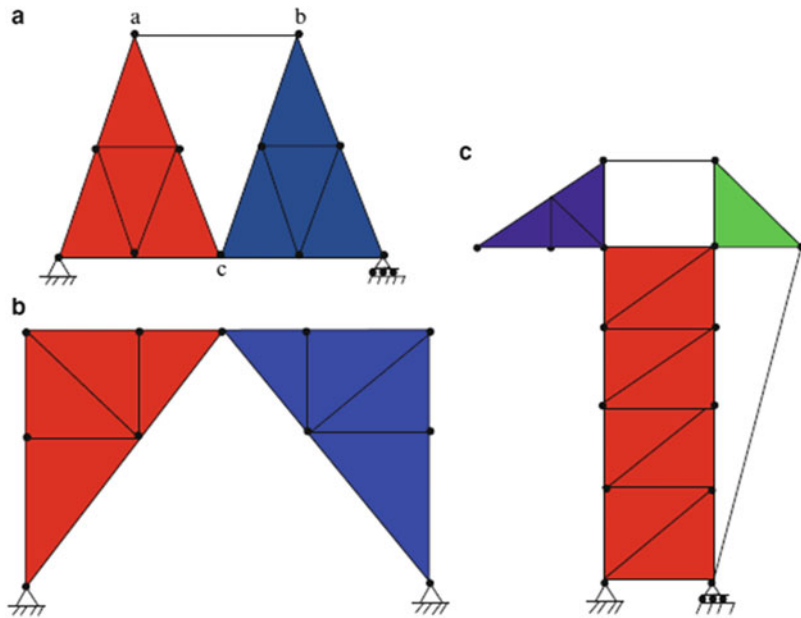
Examples of simple planar trusses are shown in Fig. 2.3. Starting with the initial triangle  $abc$ , one adds the nodes  $d$ ,  $e$ , etc.

Trusses may also be constructed by using simple trusses as the “members,” connected together by additional members or joints. These structures are called *compound trusses*. Figure 2.4 shows several examples of compound trusses, where the simple trusses are shown as shaded areas.

A truss geometry that does not fall in either the simple or compound category is called a *complex truss* [1]. Examples are shown in Fig. 2.5. This type of truss is not used as frequently as either simple or compound trusses.

### 2.1.1 Structural Idealization

Trusses are components of larger structural systems, such as buildings, bridges, and towers. In order to evaluate the behavior under loading, one needs to identify the main structural components of

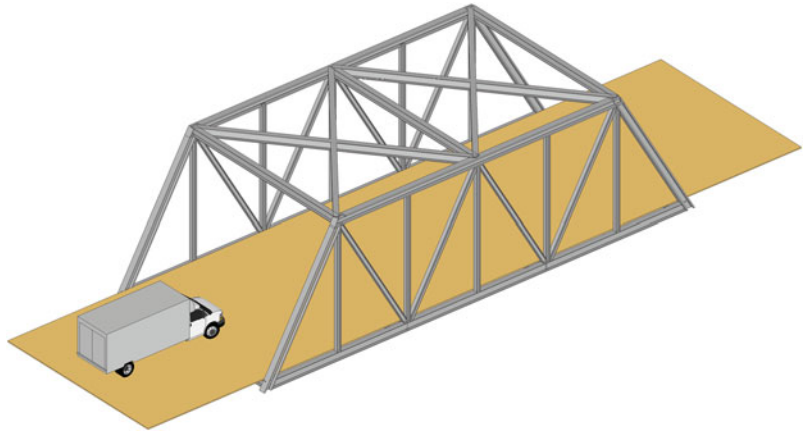
**Fig. 2.4** Compound planar trusses**Fig. 2.5** Complex planar trusses

the system and determine how the external load is transmitted from one component to another. This process is called “structural idealization;” it is a key step in the analysis process. In what follows, we illustrate idealization strategies for typical bridges and roof systems.

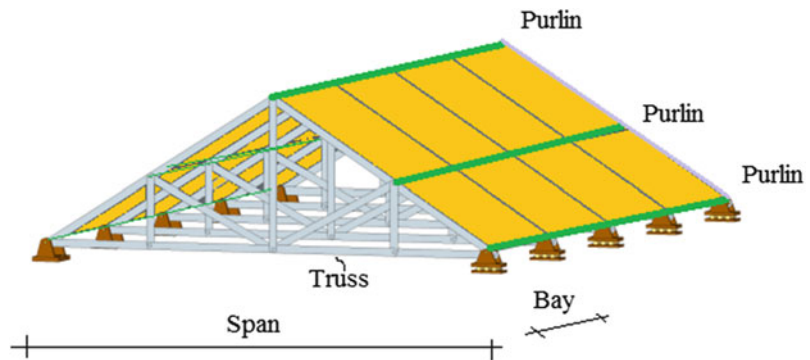
Figure 2.6 shows a typical single span truss bridge system. The key components are the two simple planar trusses, the lateral bracing systems at the top, sides, and bottom levels and the flooring system consisting of floor stringers/beams and deck slab. Loading applied to the deck slab is transmitted through the stringer/beam system to the bottom nodes of the two planar trusses. The major percentage of the analysis effort is concerned with analyzing a simple truss for dead weight, wind, and traffic loading.

Roofing system for buildings such as warehouses, shopping centers, and sports facilities employ trusses to support the roof envelope. Figure 2.7 illustrates a scheme for a typical roofing system for a single-story industrial building. The roof system consists of steel decking attached to purlins which, in turn, are supported at the top nodes of the planar trusses. Loading applied to the roof area in a bay is transmitted through the purlins to the trusses adjacent to the bay, and eventually to the supports. Bracing is incorporated to carry the lateral loading which may act either in the longitudinal or

**Fig. 2.6** Single span truss bridge system



**Fig. 2.7** Typical industrial building roofing system

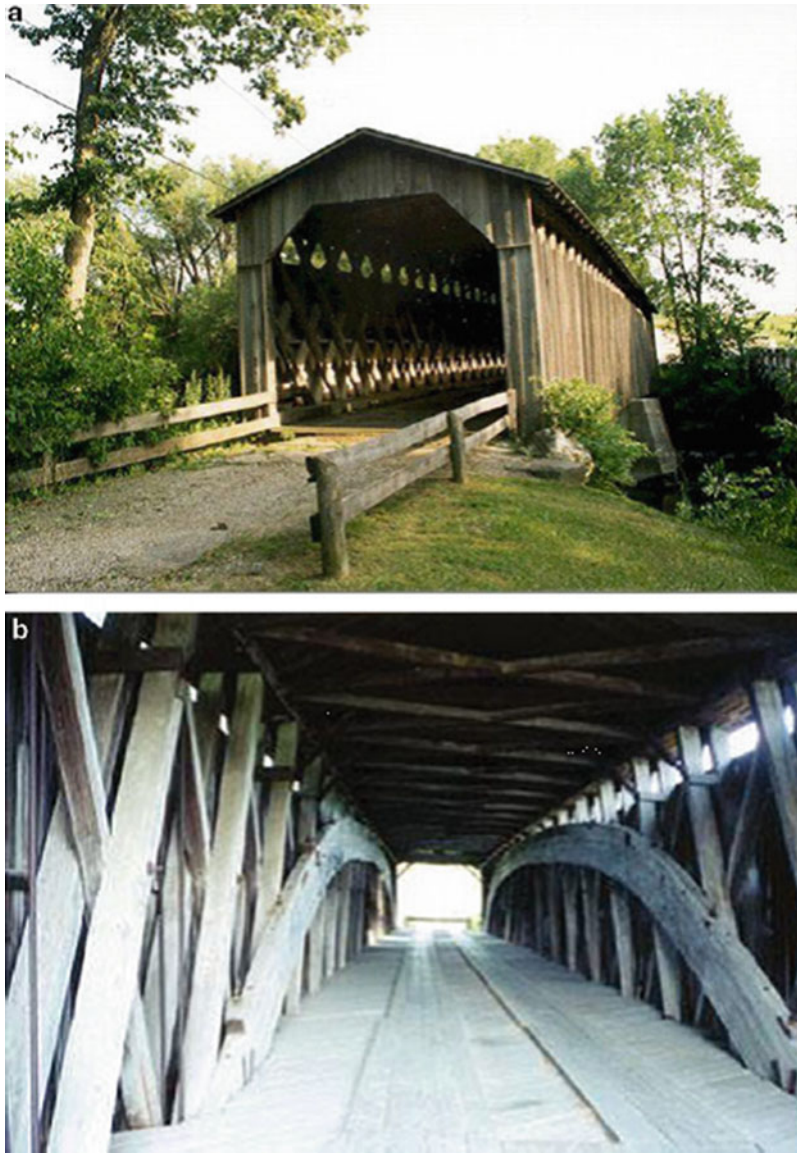


transverse direction. The primary effort for this structural system is concerned with analyzing a single planar roof truss for the tributary area loading applied at the top chord nodes.

### 2.1.2 Historical Background

The first application of truss type structures is believed to be in Egyptian boats built between 3100 and 2700 BC. Egyptian boat builders used trusses constructed by tying the members together with vines to form the sides and attached the outer hull to these structures. The Romans used wood trusses for bridges and roofs. Examples are a bridge over the Danube (circa 106 AD) and the entrance to the Pantheon (circa 120 AD). The next time frame is that of the Middle Ages. Examples of trusses are found in English cathedrals (Salisbury Cathedral, circa 1258 AD) and great halls (Westminster Palace, circa 1400). Deployment of wooden trusses continued through the Gothic and Renaissance periods, mainly to support roofs. The Engineers of these time periods intuitively understood the rigidity provided by the triangular form, but lacked a theory that they could apply to evaluate the response for a given load.

A major contribution to the theory is the work of Leonardo da Vinci (1452–1519), who formulated the concepts of force and moment as vectors and showed that forces can be combined using a graphical construction that is now called the force parallelogram. From the early 1600s to the

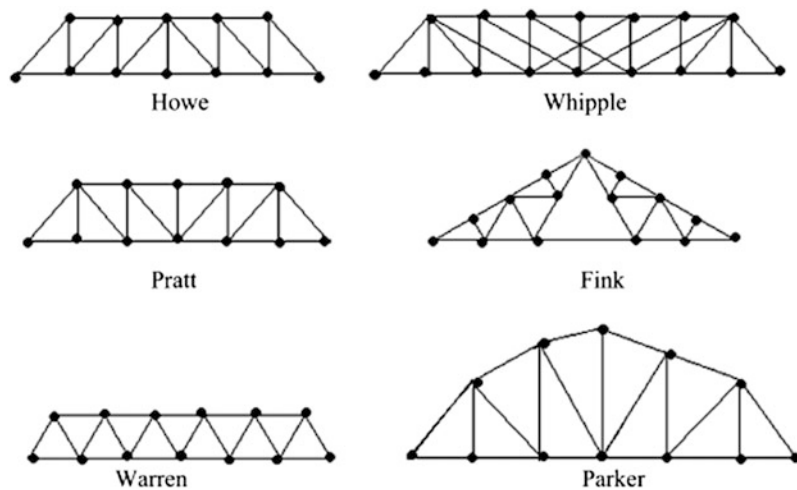


**Fig. 2.8** Covered wood bridges

mid-1800s, many advances in the development of a scientific basis for a theory of structures were made. Key contributors were Newton (1642–1729), Hooke (1635–1703), Galileo Galilei (first useable formula for strength of a cantilever beam-1638), Euler (theory of buckling of columns-1757), Bernoulli (bending deformation of a beam-1741), Navier (produced an integrated theory of Structural Mechanics—1826), and Mobius (published the *Textbook of Statics*—1837).

Wooden bridge truss structures were popular in the early 1800s, especially in the USA. There are many examples of covered wooden bridges in Vermont and New Hampshire. Figure 2.8 illustrates some typical structural schemes.

**Fig. 2.9** Examples of named trusses



There was a flourishing industry in New England producing wooden bridge trusses, many of which were exported to Europe. As with many emerging technologies, competition from other emerging technologies occurred and eventually took over the market for the product. The first impetus for change was the Industrial Revolution which occurred in the early 1800s. The concept of the railroad was invented during this period. This invention created a demand for more robust and more durable bridges to carry the heavier moving loads over longer spans. Cost also became an issue. Up to this time, wooden bridges were designed to carry light pedestrian and horse and carriage traffic over relatively short spans. Their expected life was short, but since they used local materials and local labor, they were not expensive and durability was not an issue. However, they were not adequate for the railroad traffic and other solutions were needed.

Another technology that was evolving in the late 1700s was iron making. Processes for making cast and wrought iron cheaper than existing methods were developed in the 1780s. Methods for shaping wrought iron into shapes that could be used as truss members were also invented simultaneously. These inventions set the stage for the use of iron members in trusses during the early 1800s. Initially, wrought iron was used for tension elements and wood for compression elements. Gradually, cast iron replaced wood for compression elements. The first all iron truss bridge in the USA was built in 1840 by Squire Whipple, a leading bridge designer in the USA at that time. He is also known for his book *Essay on Bridge Building*, published in 1847, the first publication on Structural Theory by a US author. Some other designers active in the 1840s were W. Howe, T. Pratt, A. Fink in North America, and J. Warren in the UK. Trusses of this era were given the name of the individual who designed or constructed them. Examples are shown in Fig. 2.9.

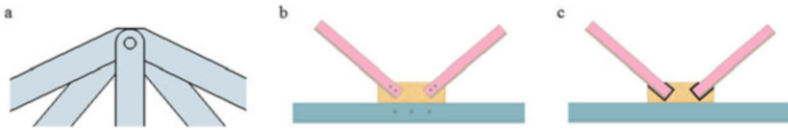
Starting around 1850, iron trusses were used not only for bridges but also for other long-span structures such as market halls, exhibition buildings, and railway stations. Notable examples are the Crystal Palace, the Eiffel tower, and the Saint Pancras station (Fig. 2.10).

During the period from 1850 to 1870, an improved version of iron called steel was invented. This material was much stronger than cast iron; more ductile than wrought iron, and quickly displaced iron as the material of choice. The first all steel truss bridge in the USA was built for the Chicago and Alton Railroad in 1879. The structure consisted of a series of Whipple trusses with a total length of 1500 ft spanning over the Mississippi River at Glasgow, Missouri. The first major long-span steel bridge was the Firth of Forth Bridge built in Scotland in 1890. Another similar



**Fig. 2.10** Examples of structures made of iron trusses. (a) Crystal Palace. (b) Eiffel Tower. (c) Saint Pancras station





**Fig. 2.11** Typical pin joint connections. (a) Frictionless pin. (b) Bolted connection. (c) Welded connection



**Fig. 2.12** Example of early steel bridges—Firth of Forth Bridge, Scotland

cantilever style truss bridge was built over the St. Lawrence River in Quebec, Canada in 1919. The initial steel structures used eyebars and pins. Rivets replaced pins as connectors in the late 1800s.

High-strength bolts and welding are now used to connect the structural members in today's modern steel constructions. Figure 2.11 illustrates typical bolted and welded connections. Connection details are usually designed by the steel fabricator and checked by the structural engineer. The goal in connection design is to minimize steel erection time.

Steel truss bridges were the dominant choice for long-span crossings until the mid-1900s when another structural form, the cable-stayed bridge, emerged as a competitor. Cable-stayed bridges have essentially captured the market for spans up to about 900 m. Segmented concrete girder construction has also emerged as a major competitor for somewhat shorter spans. Plane trusses now are used mainly for prefabricated joists, for gable roof systems, and for spanning long interior distances in buildings and sporting facilities such as convention halls and stadiums. Three-dimensional space trusses are used in dome type structures such as shown in Fig. 2.13, and also for towers. One of the most notable examples of the space truss concept is the Eiffel Tower in Paris, France.





**Fig. 2.13** Three-dimensional truss roof system

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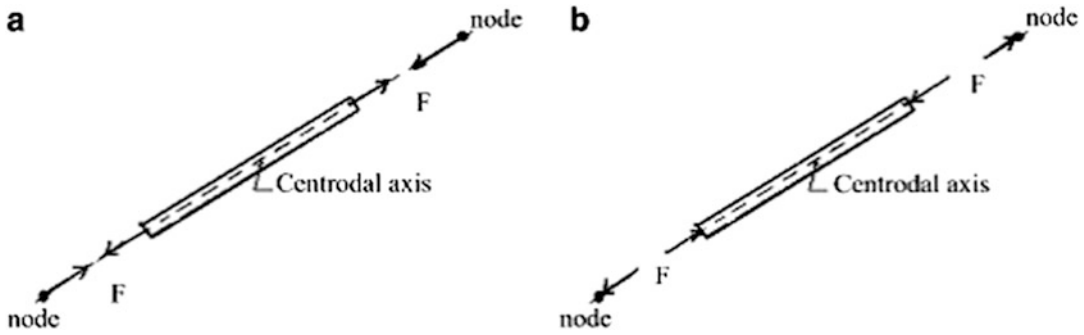
## 2.2 Analysis of Planar Trusses

In this section, we focus on introducing analysis and behavior issues for planar trusses. The discussion is extended in the next section to deal with three-dimensional space structures.

The analysis of trusses is based on the following idealizations that ensure that the forces in the members are purely axial:

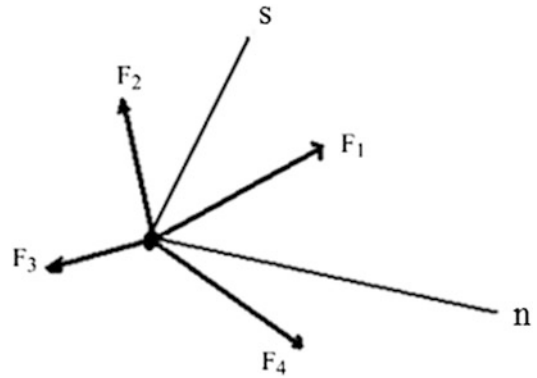
1. The loads and displacement restraints are applied only at the nodes.
2. The members are connected with frictionless pins so that the members can rotate freely and no moment exists at the ends.
3. The stress due to the weight of the members is small in comparison to the stress due to the applied loads.
4. Each member is straight and is arranged such that its centroidal axis coincides with the line connecting the nodal points.

With these restrictions, it follows that a member is subjected only to an axial force at each end. These forces are equal in magnitude and their line of action coincides with the centroidal axis of the member. There is only one unknown per member, the magnitude of the force. The resulting state is uniform axial stress throughout the member. Depending on the loading, the member force may be either tension or compression. Figure 2.14 illustrates free body diagrams for a truss member and its associated nodes.



**Fig. 2.14** Free body diagram of a truss member and its associated nodes. (a) Tension. (b) Compression

**Fig. 2.15** Concurrent force system at a node



### 2.2.1 Equilibrium Considerations

The equilibrium requirements for a body subjected to a non-concurrent force system are specified in Sect. 1.5.2. In general, the resultant force vector and the resultant moment vector with respect to an arbitrary moment center must vanish. One can apply these requirements either to the complete truss or to the individual nodes.

Each node of a plane truss is acted upon by a set of coplanar concurrent forces. There are no moments since the pins are frictionless and the lines of action of the forces intersect at the node. For equilibrium of a node, the resultant force vector must vanish. In Squire Whipple's time (1840s), equilibrium was enforced using Leonardo da Vinci's graphical method based on the force polygon. Now, one applies an analytical approach based on resolving the force vectors into components and summing the components. The corresponding scalar equilibrium equations are

$$\sum F_n = 0 \quad \sum F_s = 0 \quad (2.1)$$

where  $n$  and  $s$  are two arbitrary nonparallel directions in the plane. Figure 2.15 illustrates this notation.

### 2.2.2 Statically Determinate Planar Trusses

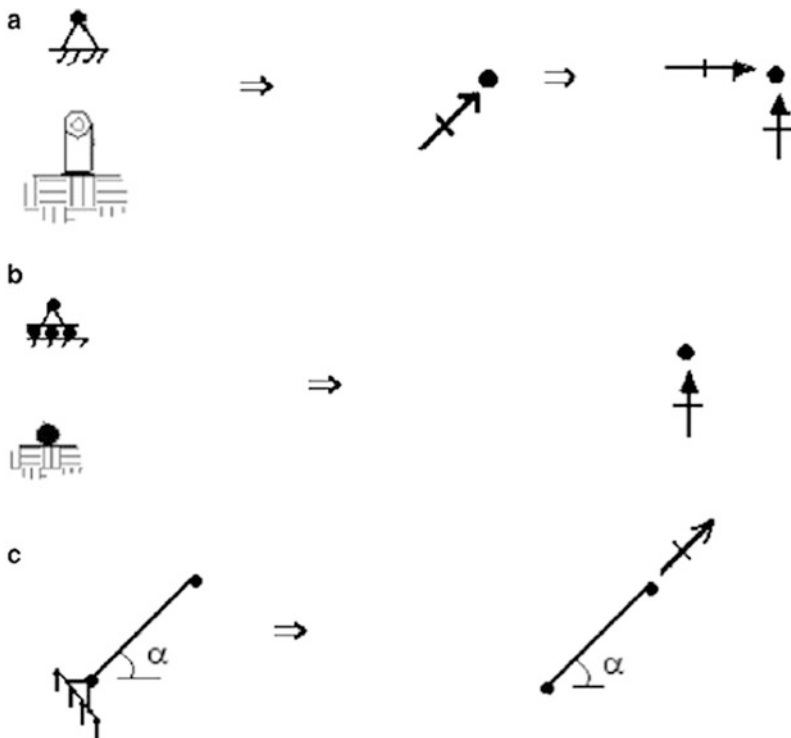
In general, three motion restraints are required to prevent rigid body motion of a planar truss. Two of these restraints may be parallel. However, the third restraint cannot be parallel to the other two restraints since, in this case, the truss could translate in the direction normal to the parallel restraint direction. Each restraint generates an unknown force, called a reaction. Therefore, the minimum number of reactions is 3.

Examples of typical support motion restraints and the corresponding reactions are shown in Fig. 2.16.

There are two scalar equilibrium equations per node for a plane truss. Assuming that there are  $j$  nodes, it follows that there are a total of  $2j$  equilibrium equations available to determine the force unknowns. We suppose there are  $m$  members and  $r$  reactions. Then, since each member and each reaction involves only one unknown force magnitude, the total number of force unknowns is equal to  $m + r$ .

When the number of force unknowns is equal to the number of equilibrium equations, the structure is said to be *statically determinate*. If  $m + r < 2j$ , the truss is unstable since there are an insufficient number of either member forces or reactions or possibly both to equilibrate the applied loads. It follows that a plane truss is statically determinate, unstable, or indeterminate when

$$\begin{aligned} m + r &= 2j && \text{Statically determinate} \\ m + r &< 2j && \text{Unstable} \\ m + r &> 2j && \text{Statically indeterminate} \end{aligned} \quad (2.2)$$



**Fig. 2.16** Types of supports for planar trusses. (a) Hinge support (two restraints  $\Rightarrow$  two reactions). (b) Roller Support (one restraint  $\Rightarrow$  one reaction). (c) Rigid link

A word of caution: a statically determinate truss may also be unstable if the reactions are not properly aligned so as to prevent rigid body motion of the truss. We discuss this point in more detail in the following section.

### 2.2.3 Stability Criterion

In this section, we examine in more detail the question of whether a planar truss structure is initially stable. Assuming a plane truss has  $m$  members,  $r$  reactions, and  $j$  joints, there are  $2j$  force equilibrium equations that relate the known (given) joint loads and the  $(m + r)$  unknown forces. If  $m + r < 2j$ , the number of force unknowns is less than the number of equilibrium equations that the forces must satisfy. Mathematically, the problem is said to be underdetermined or inconsistent. One cannot find the exact solution for an arbitrary loading, except in the trivial case where the magnitude of all the loads is zero, and consequently the forces are zero. Once a nontrivial load is applied, the structure cannot resist it, and motion ensues. The descriptor “initial instability” is used to denote this condition.

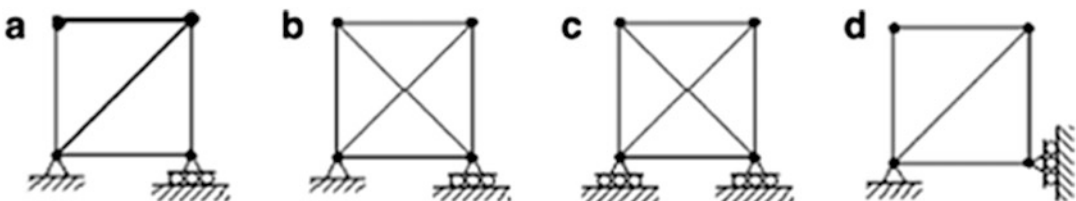
Even when  $m + r = 2j$ , a truss may still be unstable if the motion. restraints are not properly arranged to prevent rigid body motion of the structure. There may be an insufficient number of restraints or the restraints may be aligned in such a way that rotation of a segment is possible. The stability of a complex truss depends on the geometrical arrangements of the members. Even though the truss satisfies the condition,  $m + r = 2j$ , and has sufficient restraints, it still may be unstable. The instability condition becomes evident when one attempts to determine the member forces using the  $2j$  force equilibrium equations. The solution is not unique when the structure is unstable.

When  $m + r > 2j$  and the structure is suitably restrained against rigid body motion, the structure is said to be statically indeterminate. This terminology follows from the fact that now there are more force unknowns than equilibrium equations, and not all the forces can be determined with only equilibrium considerations. One needs some additional equations. We address this type of problem in Part II.

In what follows, we illustrate the initial stability criteria with typical examples. Stability can be defined in a more rigorous way using certain concepts of linear algebra, a branch of mathematics that deals with linear algebraic equations. This approach is discussed in Sect. 2.6

#### Example 2.1 Simple Trusses

**Given:** The structures. shown in Fig. E2.1a–d



**Fig. E2.1**

**Determine:** Whether the structures are initially stable, determinate, or indeterminate.

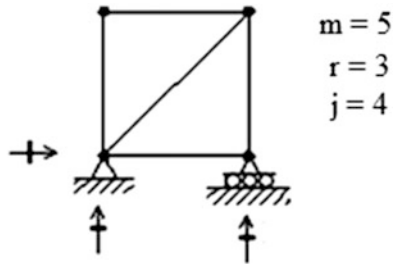
**Solutions:**

*Case (a):* There are five members, three reactions, and four nodes. Then applying (2.2)

$$m + r = 8$$

$$2j = 8$$

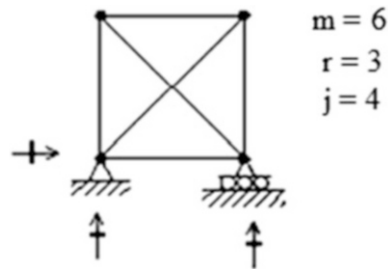
The structure is determinate and initially stable.



Case (b):

$$m + r = 9$$

$$2j = 8$$

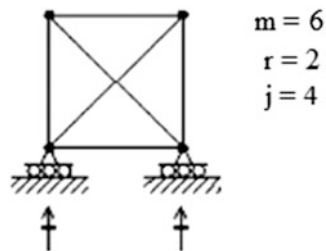


There is one extra force and therefore the structure is initially stable and indeterminate to the first degree.

Case(c): The stability criterion appears to be satisfied.

$$m + r = 8$$

$$2j = 8$$



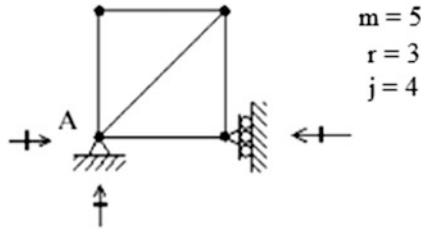
However, the number of supports is insufficient to prevent rigid body motion in the horizontal direction. Therefore, the structure is initially unstable.



Case (d): The stability criterion appears to be satisfied.

$$m + r = 8$$

$$2j = 8$$



However, the three displacement restraints are concurrent (point A), and therefore the structure can rotate at point A. It follows that the structure is initially unstable.

### Example 2.2 A Compound Truss

**Given:** The structure shown in Fig. E2.2a. This compound truss is actually a combination of two simple trusses.

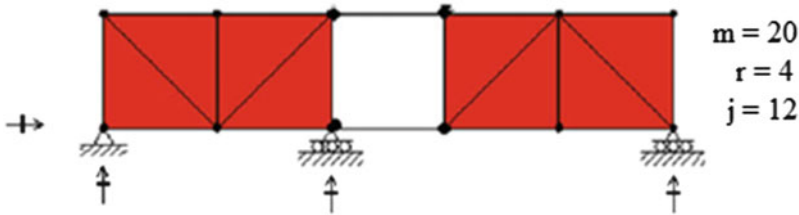


Fig. E2.2a

**Determine:** Is the structure statically determinate?

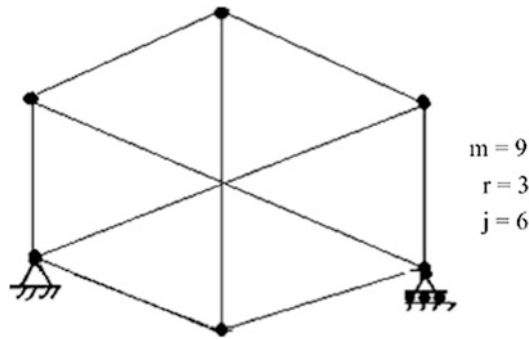
**Solution:** The structure is statically determinate and stable.

$$m + r = 24$$

$$2j = 24$$

### Example 2.3 A Complex Truss

**Given:** The complex truss defined in Fig. E2.3a.



**Fig. E2.3a**

**Determine:** Is the truss statically determinate?

**Solution:** There are three restraints, six joints, and nine members.

$$\begin{aligned} m + r &= 12 \\ 2j &= 12 \end{aligned}$$

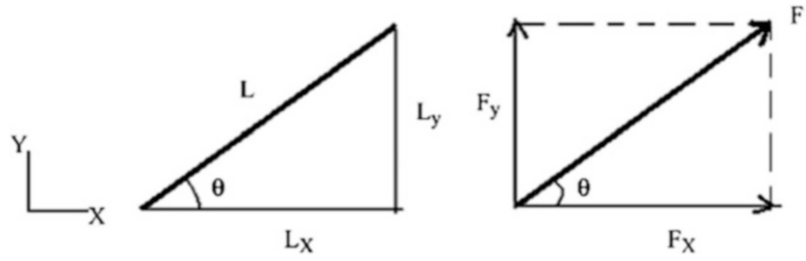
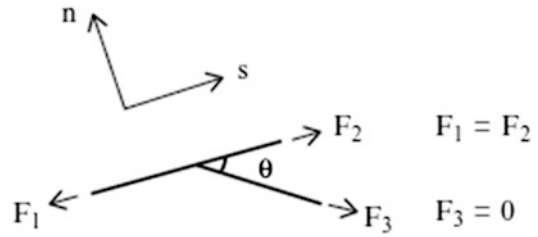
The truss appears to be stable. Note that the condition,  $m + r = 2j$  is *necessary* but not *sufficient* to ensure stability of this truss. Sufficient conditions are discussed further in Sect. 2.6.

In what follows, we describe two classical hand computation-based procedures for finding the member forces in *simple* and *compound trusses* due to an applied loading. These approaches are useful for gaining insight as to how loads are carried by structures. That is the most important aspect of structural engineering that one needs to master in order to be a successful practitioner. Also, although most current structural analysis is computer based, one still needs to be able to assess the computer-generated results with a simple independent hand computation.

### 2.2.4 Method of Joints: Planar Trusses

Each joint of a plane truss is subjected to a concurrent force system. Since there are two equilibrium equations for a 2-D concurrent force system, one can solve for at most two force unknowns at a particular joint. The strategy for the method of joints is to proceed from joint to joint, starting with the free body diagram of a joint that has only two unknowns, solving for these unknowns, and then using this newly acquired force information to identify another eligible joint. One continues until equilibrium has been enforced at all the joints. When all the joints are in equilibrium, the total structure will be in equilibrium. This analysis procedure was first described in Squire Whipple's 1840 Essay on Bridge Building.

Enforcing the equilibrium conditions is simplified if one works with the force components referred to a common reference frame. Once one component is known, it is a simple step to determine the magnitude of the other component and the force. As an illustration, we consider the member shown in Fig. 2.17. The ratio of force components is equal to the ratio of the projected lengths. This equality follows from the fact that the direction of the force is the same as the direction of the line. Here, we are taking the horizontal and vertical directions as the common reference frame.

**Fig. 2.17** Force components**Fig. 2.18** Zero force member

$$\frac{F_y}{F_x} = \frac{L_y}{L_x} = \tan \theta$$

Similarly, the force is determined using

$$F = \frac{F_x}{\cos \theta} = \frac{F_y}{\sin \theta}$$

Another simplification is possible when the joint has only three members, two of which are colinear, and there is *no* applied load at the joint. Figure 2.18 illustrates this case. There is only one force acting at an angle to the direction of the two common members, and equilibrium in the normal direction ( $n$ ) requires the magnitude of this force to be zero. The other two forces must have the same magnitude.

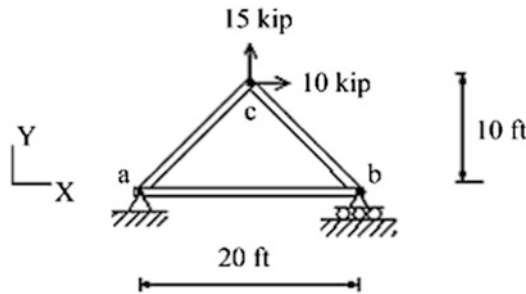
When applying the method of joints, it is convenient to first determine the reactions by enforcing global equilibrium on the total structure. With the reactions known, it may be easier to locate a joint having only two unknown member forces.

In what follows, we present a set of examples that illustrate how the method of joints is efficiently applied.

#### Example 2.4 Three-Member Truss Analyzed by the Methods of Joints

**Given:** The truss and loading defined by Fig. E2.4a.

**Determine:** The reactions and member forces.



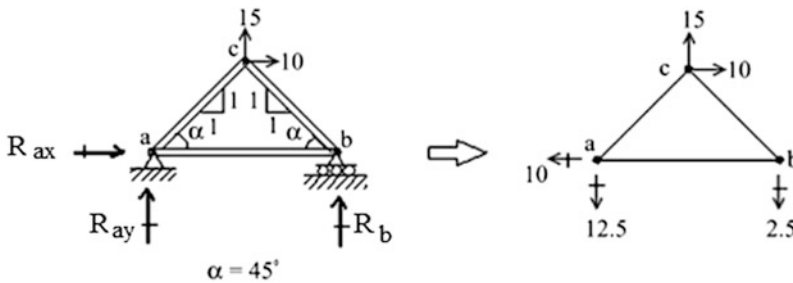
**Fig. E2.4a**

**Solution:** We first find the reactions at joints a and b. Moment summation about joint a leads to the y reaction at b. Force summations provide the remaining two reaction forces. The results are shown in Fig. E2.4b.

$$\sum M_a = 0 \quad \curvearrowright \quad 10(10) - 15(10) - R_b(20) = 0 \quad \Rightarrow \quad R_b = -2.5 \quad \therefore R_b = 2.5 \text{ kip} \downarrow$$

$$\sum F_X = 0 \rightarrow + \quad R_{ax} + 10 = 0 \quad \Rightarrow \quad R_{ax} = -10 \quad \therefore R_{ax} = 10 \text{ kip} \leftarrow$$

$$\sum F_Y = 0 \uparrow + \quad R_{ay} + 15 - 2.5 = 0 \quad \Rightarrow \quad R_{ay} = -12.5 \quad \therefore R_{ay} = 12.5 \text{ kip} \downarrow$$

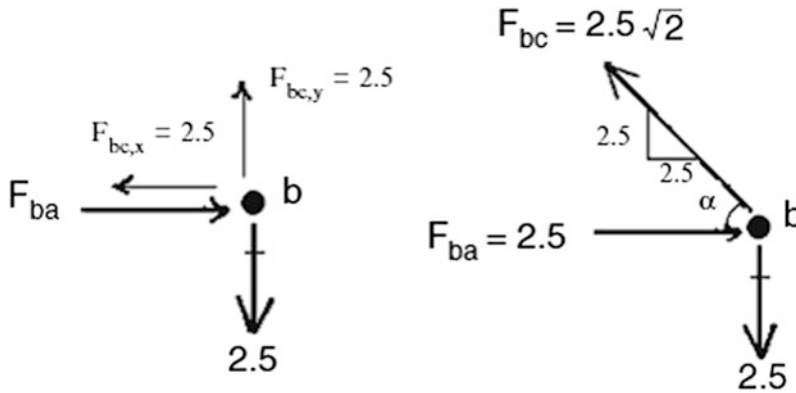


**Fig. E2.4b** Reactions

One can start at any joint since they all have just two unknown member forces. We pick joint b (Fig. E2.4c). Summation of forces in the y direction gives  $F_{bc,y} = 2.5$  kip. Then, summing forces in the x direction requires  $F_{ba}$  being compressive and equal to 2.5 kip. We indicate a tensile member force with an arrow pointing away from the joint. The opposite sense is used for compression. One converts the force components to the force magnitude using the Pythagorean Theorem,  $F = \sqrt{F_x^2 + F_y^2}$ .

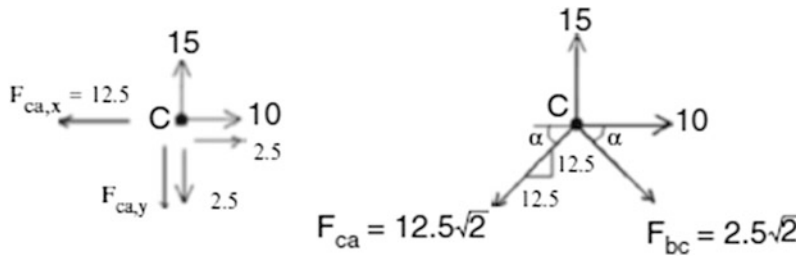
$$\sum F_y = 0 \quad F_{bc,y} = 2.5 \uparrow \quad \text{Then } F_{bc,x} = 2.5 \leftarrow \quad \therefore F_{bc} = 2.5\sqrt{2} \text{ kip (tension)}$$

$$\sum F_x = 0 \quad F_{ba} = 2.5 \text{ kip (compression)}$$

**Fig. E2.4c** Joint b

There is only one unknown member force left,  $F_{ca}$ . One can work with either joint a or joint c. We pick joint c. The free body diagram for joint c is shown in Fig. E2.4d. Equilibrium in the  $x$  direction requires  $F_{ca,x} = 12.5$  kip.

$$\sum F_x = 0 \quad F_{ca,x} = 12.5 \leftarrow \therefore F_{ca} = 12.5\sqrt{2} \text{ kip (tension)}$$

**Fig. E2.4d** Joint C

Note that in this example we do not need to use the remaining equilibrium equations (one for joint c and two for joint a) since we used instead three global equilibrium equations to calculate the reactions. The total number of joint equilibrium equations is equal to six (two per joint  $\times$  three joints). If we use three equations for global equilibrium, there are only three independent equations left to apply to the joints. The final results are shown on the sketch below. Tensile forces are denoted with a + sign, and compressive forces with a - sign.



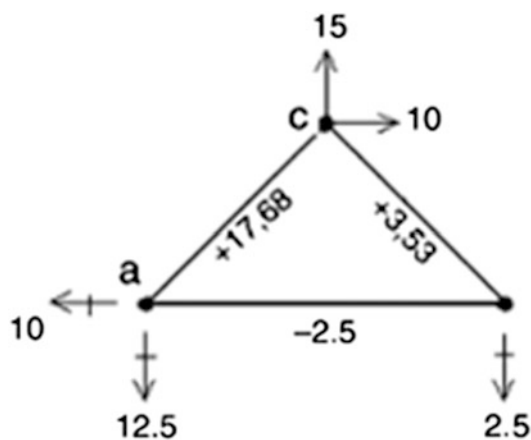


Fig. E2.4e

*Example 2.5* Five-Member Truss Analyzed by the Methods of Joints

**Given:** The truss defined in Fig. E2.5a.

**Determine:** The reactions and member forces for the loading shown.

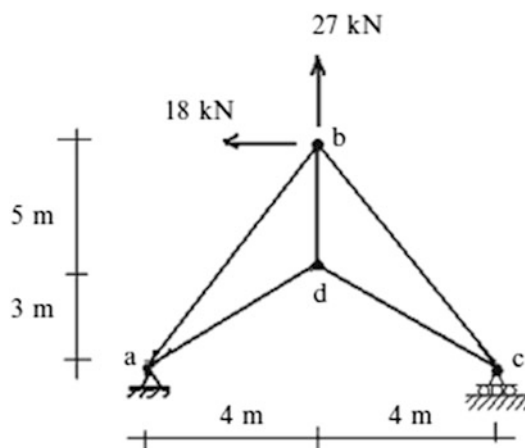


Fig. E2.5a

**Solution:** We first find the reactions and then proceed starting with joint a, and then moving to joints c and d.

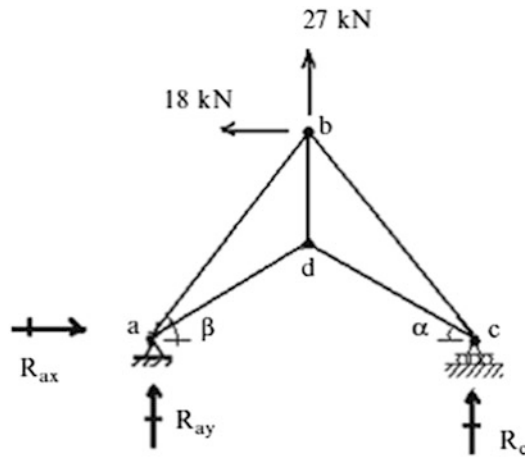


Fig. E2.5b

$$\Sigma M_a = 0 \quad +\curvearrowright \quad -27(4) - 18(8) - R_c(8) = 0 \Rightarrow R_c = -31.5 \quad \therefore R_c = 31.5 \text{ kip} \downarrow$$

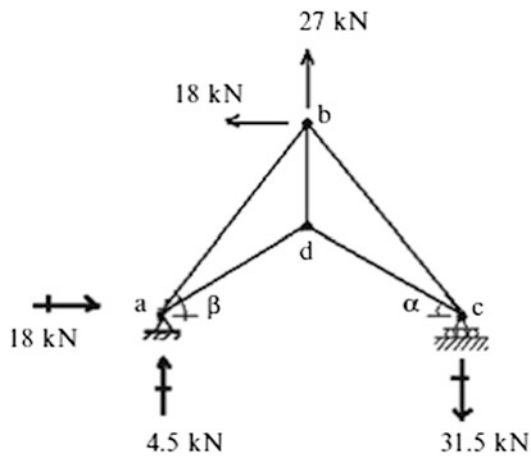
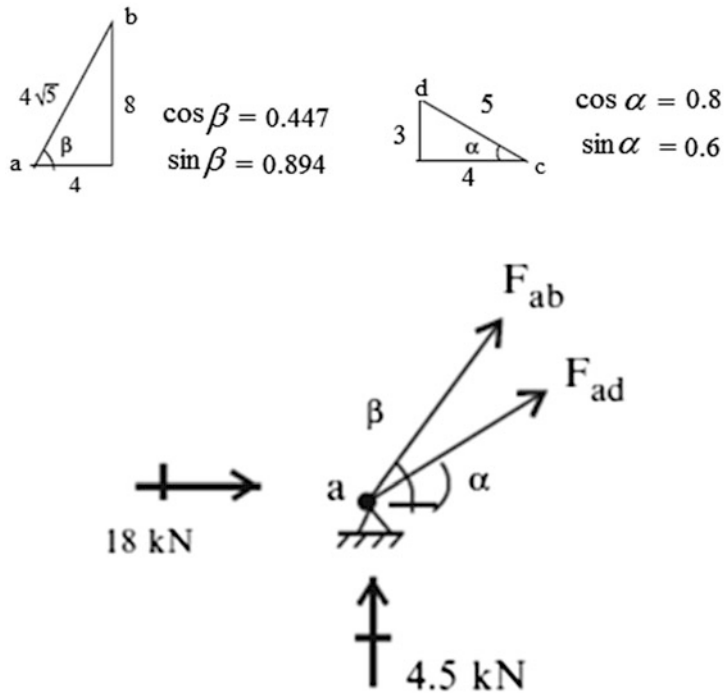
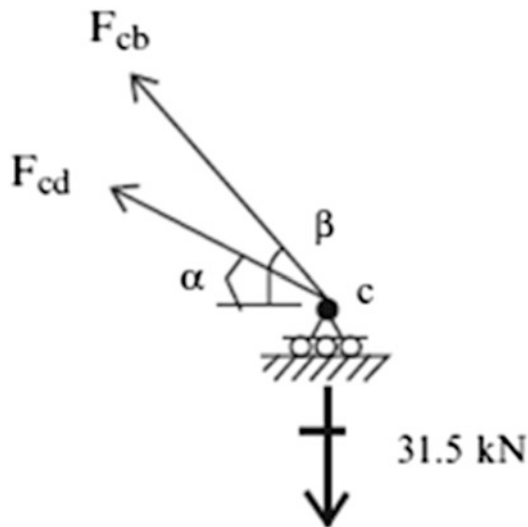


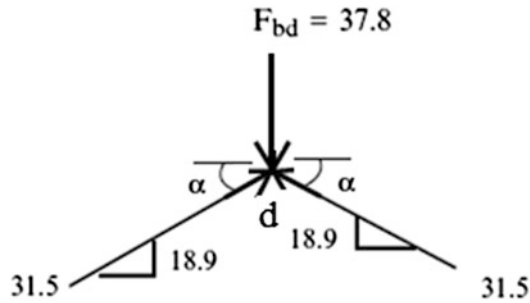
Fig. E2.5c

**Fig. E2.5d** Joint a

$$\begin{cases} \sum F_x = 0 & F_{ad} \cos \alpha + F_{ab} \cos \beta + 18 = 0 \\ \sum F_y = 0 & F_{ad} \sin \alpha + F_{ab} \sin \beta + 4.5 = 0 \end{cases} \Rightarrow \begin{cases} F_{ad} = -31.5 \text{ kN} \\ F_{ab} = 16.1 \text{ kN} \end{cases}$$

**Fig. E2.5e** Joint c

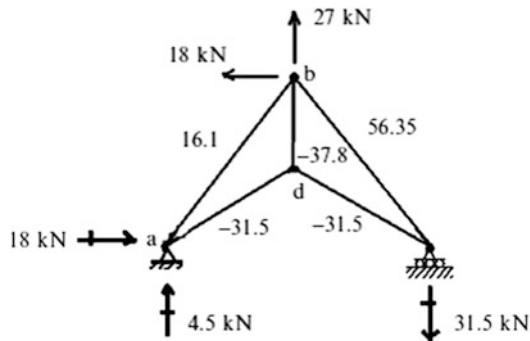
$$\begin{cases} \sum F_x = 0 & F_{cd} \cos \alpha + F_{cb} \cos \beta = 0 \\ \sum F_y = 0 & F_{cd} \sin \alpha + F_{cb} \sin \beta - 31.5 = 0 \end{cases} \Rightarrow \begin{cases} F_{cd} = -31.5 \text{ kN} \\ F_{cb} = 56.35 \text{ kN} \end{cases}$$



**Fig. E2.5f** Joint d

$$\sum F_y = 0 \quad F_{bd} = 37.8 \text{ kN (compression)}$$

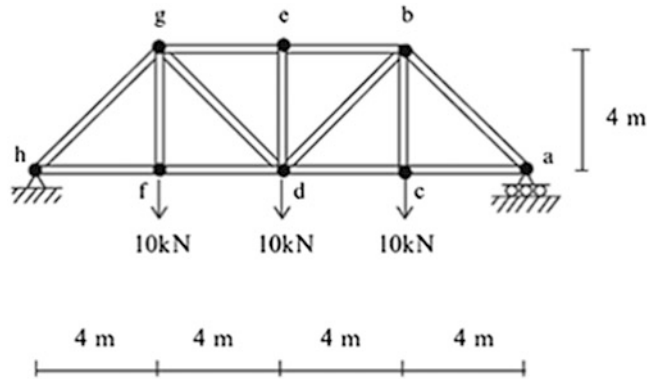
The final forces are listed below.



### Example 2.6 Five-Member Truss Analyzed by the Methods of Joints

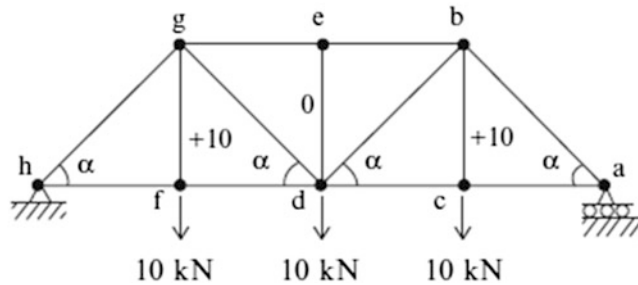
**Given:** The truss defined in Fig. E2.6a.

**Determine:** The reactions and member forces for the loading shown.

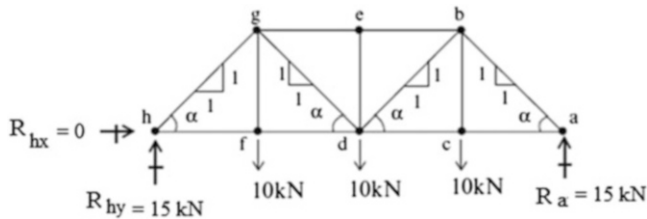
**Fig. E2.6a** Bridge truss

**Solution:** We note that the structure and loading are symmetrical with respect to a vertical axis through points e and d. It follows that the forces in symmetrically located members are equal, and therefore we need to find the forces in only  $\frac{1}{2}$  of the structure. Joints c, e, and f are special in the sense that two incident members are colinear. Then, noting Fig. 2.18,

$$F_{cb} = 10 \text{ kN (tension)} \quad F_{ed} = 0 \quad F_{fg} = 10 \text{ kN (tension)}$$

**Fig. E2.6b**

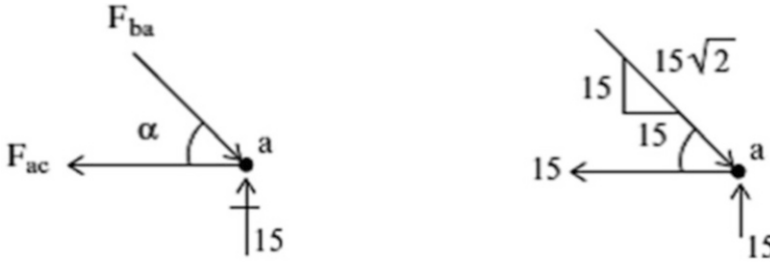
There are multiple options. We can first find the reactions and then proceed inward, starting with joint a, and then moving to joints c and b. An alternate approach would be to start at joint d, find the y component of  $F_{bd}$ , and then move to joint b.

**Fig. E2.6c** Reactions

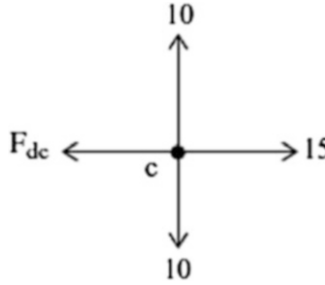
We list the results for the first approach below. We first find  $F_{ba,y}$  with the vertical equilibrium condition at joint a. Then, we find  $F_{ac}$  from the horizontal component of  $F_{ba}$ .



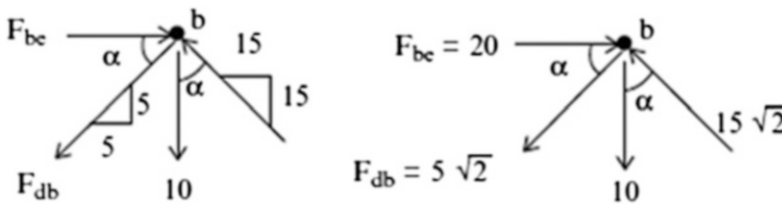
$$\begin{aligned}\sum F_y = 0 \quad F_{ba,y} = 15 \text{ kN} \downarrow \quad \text{Then } F_{ba,x} = 15 \text{ kN} \rightarrow \\ \therefore F_{ba} = 15\sqrt{2} \text{ kN (compression)} \\ \sum F_x = 0 \quad F_{ac} = 15 \text{ kN (tension)}\end{aligned}$$

**Fig. E2.6d** Joint a

At joint c, we note from the sketch that  $F_{dc} = 15 \text{ kN}$  (tension).

**Fig. E2.6e** Joint c

At joint b, we note from the sketch that  $F_{db}$  must be in tension and  $F_{be}$  must be in compression.

**Fig. E2.6f** Joint b

We first find  $F_{db,y}$  with the vertical equilibrium condition at joint b.

$$\sum F_y = 0 \quad F_{db,y} = 5 \downarrow$$

Then,  $F_{db,x} = 5 \leftarrow \therefore F_{db} = 5\sqrt{2} \text{ kN (tension)}$ .

Then, we apply the horizontal equilibrium equation at joint b.

$$\sum F_x = 0 \quad F_{be} = 20 \text{ kN (compression)}$$

The resultant member forces are shown below. Note that, for this loading, the members in the top zone (the top chord) are in compression and the bottom chord members are in tension. The interior vertical and diagonal members are in tension. When iron was used as a structural material, cast iron, which is relatively weak in tension, was employed for the top chord members and wrought iron, which is relatively strong in tension, for the verticals, diagonals, and bottom chord members.

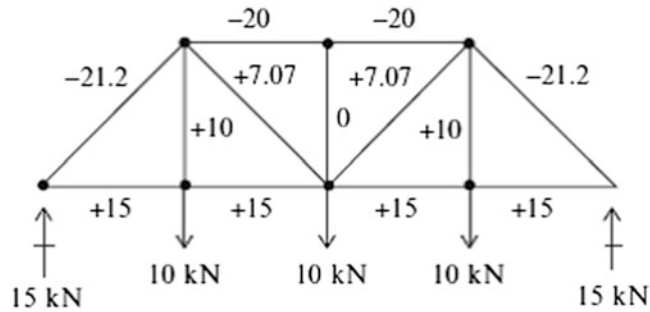


Fig. E2.6g

If the truss structure is inverted as shown below, the sense of the member forces is also reversed. This geometric arrangement is preferred for bridge crossings when the clearance below the structure is not a problem.

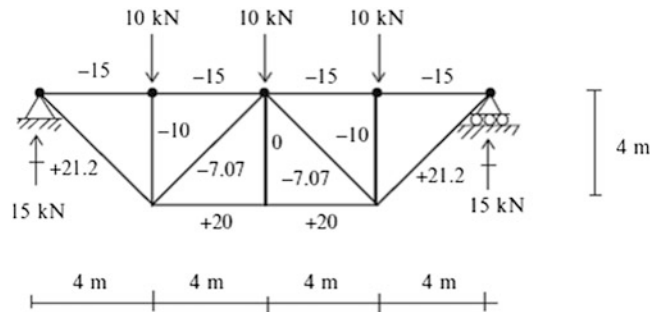
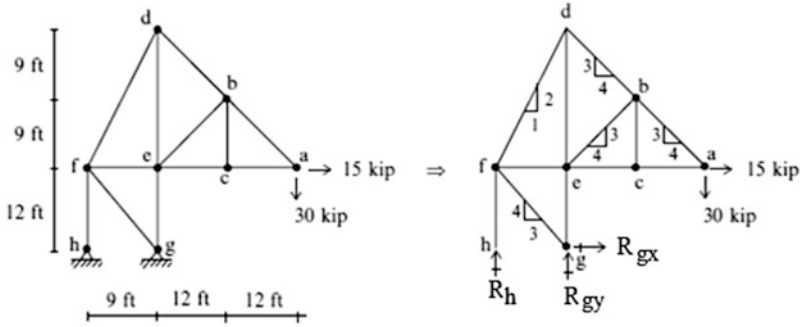


Fig. E2.6h

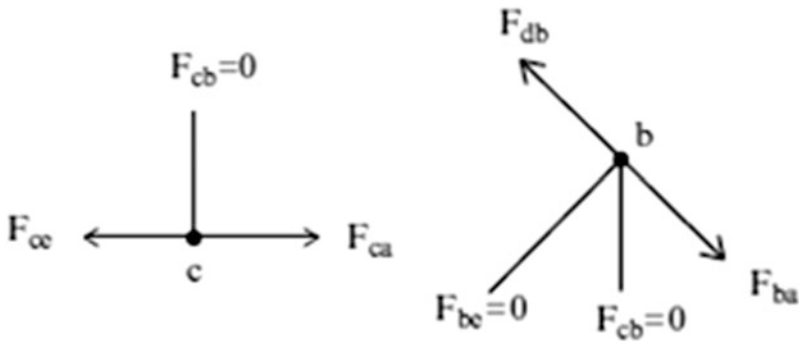
### Example 2.7 A Cantilever Truss Analyzed by Methods of Joints

**Given:** The truss and loading defined by Fig. E2.7a.

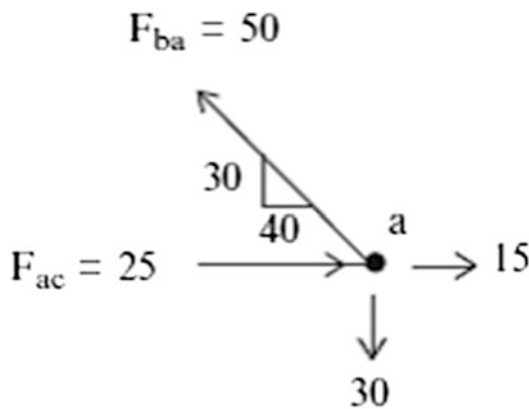
**Determine:** The member forces for the loading shown.

**Fig. E2.7a**

**Solution:** First, we determine the zero force members. Starting at joint c, we observe that  $F_{cb} = 0$ . Then, moving to joint b, it follows that  $F_{be} = 0$ .

**Fig. E2.7b** Zero force members

In this case, we do not need to first find the reactions. We can start at joint a.

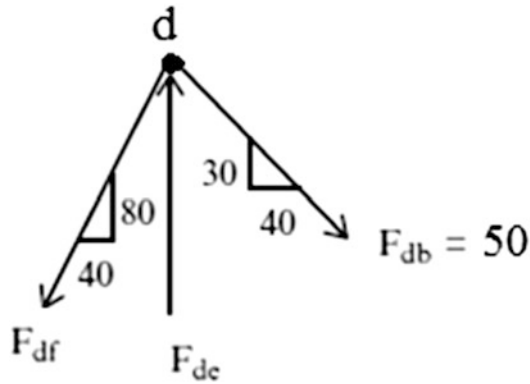
**Fig. E2.7c** Joint a

$$\sum F_y = 0 \quad F_{ba,y} = 30 \rightarrow F_{ba,x} = 40 \quad \therefore F_{ba} = 50 \text{ kip (tension)}$$

Given  $F_{ba}$ , we determine  $F_{ac}$

$$\sum F_x = 0 \quad F_{ac} = 25 \text{ kip (compression)}$$

Next, we move to joint d and determine  $F_{df}$



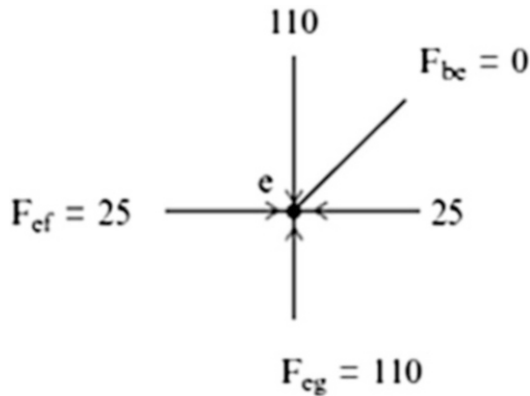
**Fig. E2.7d** Joint d

$$\sum F_x = 0 \quad F_{df,x} = 40 \quad \therefore F_{df,y} = 80 \quad F_{df} = 40\sqrt{5} \text{ kip (tension)}$$

With  $F_{df}$  known, we can determine  $F_{de}$

$$\sum F_y = 0 \quad F_{de} = 110 \text{ kip (compression)}$$

At joint e, we determine  $F_{ef}$  and  $F_{eg}$ .

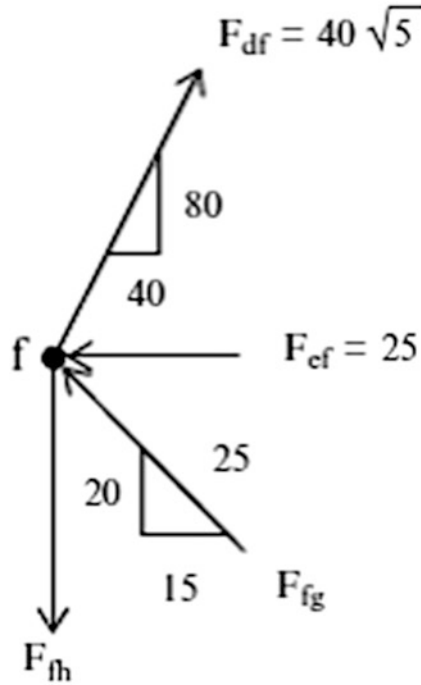


**Fig. E2.7e** Joint e

The last joint is joint f. We first determine  $F_{fg,x}$

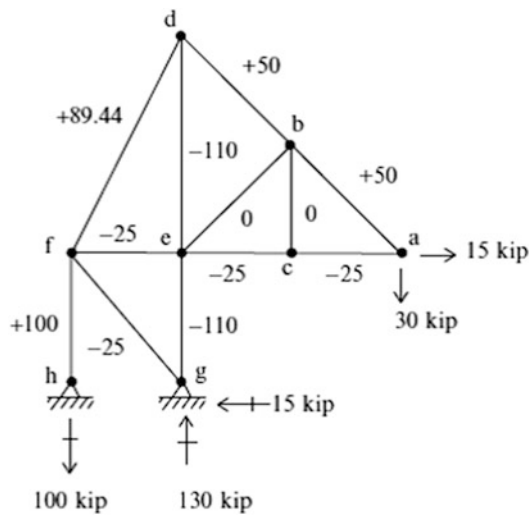
$$\sum F_x = 0 \quad F_{fg,x} = 15 \quad \therefore F_{fg,y} = 20 \quad F_{fg} = 25 \text{ kip (compression)}$$

Then,  $\sum F_y = 0 \quad F_{fh} = 100 \text{ kip (tension)}$



**Fig. E2.7f** Joint f

The final forces are listed below.



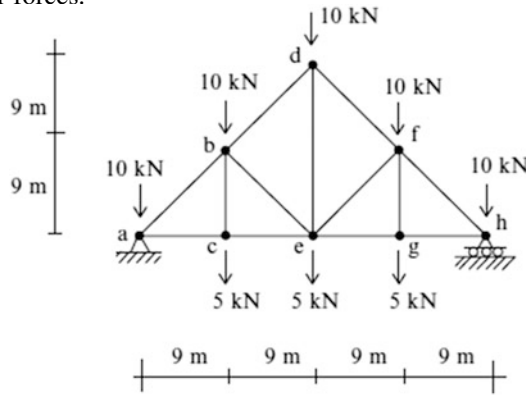
**Fig. E2.7g**



*Example 2.8 Gable Roof Truss Analyzed by the Method of Joints*

**Given:** The truss and loading defined by Fig. E2.8a.

**Determine:** The member forces.

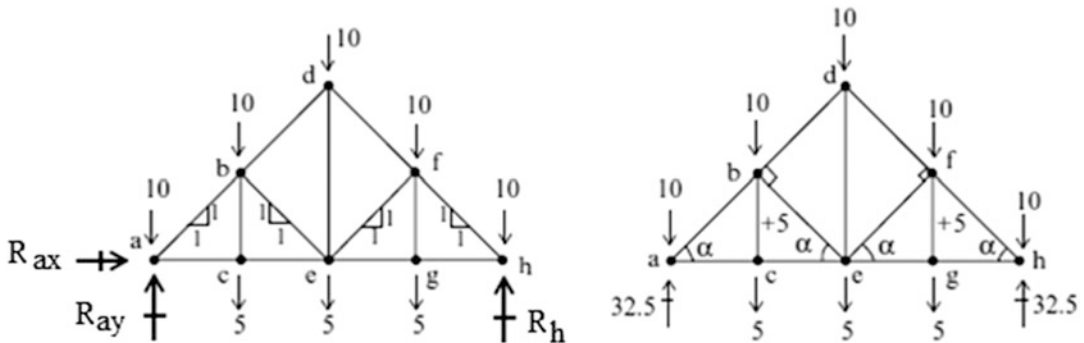


**Fig. E2.8a**

**Solution:** Fig. E2.8a shows a typical truss structure for supporting roof (top joints) and ceiling (bottom joints) loads. Members cb and gf function to transfer loads to the top joints (b and f). Their force magnitudes are

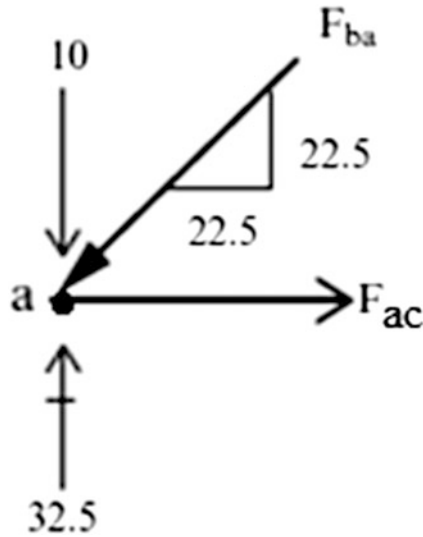
$$F_{bc} = 5 \text{ kN (tension)} \quad F_{gf} = 5 \text{ kN (tension)}$$

All the remaining joints have at least three unknown member forces and reactions. Therefore, we start the analysis by first finding the reactions.



**Fig. E2.8b** Reactions

Given the reactions, we start at joint a. Force  $F_{ba}$  must be compression and  $F_{ba,y} = 22.5 \downarrow$ . Then,  $F_{ba,x} = 22.5 \leftarrow$  and  $F_{ba} = 22.5\sqrt{2} \text{ kN (compression)}$ . It follows that,  $F_{ac}$  is in tension and equal to 22.5 kN.

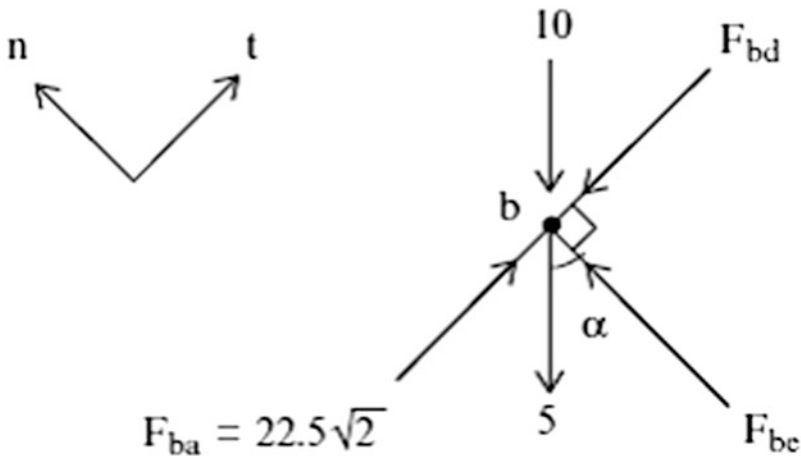
**Fig. E2.8c** Joint a

We then move on to joint b. Members ab and bd are colinear, and member be is normal to this common direction. Summing forces in the normal direction results in

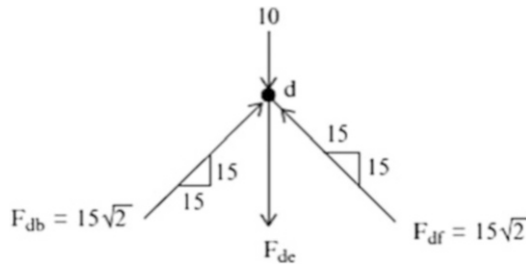
$$\sum F_n = 0 \quad F_{be} = (10 + 5) \cos \alpha = 15 \frac{\sqrt{2}}{2} \text{ kN (compression)}$$

Next, summing forces in the tangential direction leads to  $F_{bd}$ .

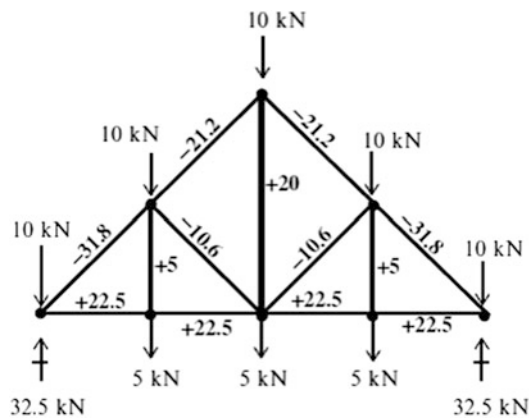
$$\sum F_t = 0 \quad F_{bd} = 22.5\sqrt{2} - (10 + 5) \cos \alpha = 15\sqrt{2} \text{ kN (compression)}$$

**Fig. E2.8d** Joint b

The last force is  $F_{de}$ . We use joint d shown in Fig. E2.8e. Summing forces in the y direction leads to  $F_{de} = 20 \text{ kN (tension)}$

**Fig. E2.8e** Joint d

The final forces are listed below.

**Fig. E2.8f**

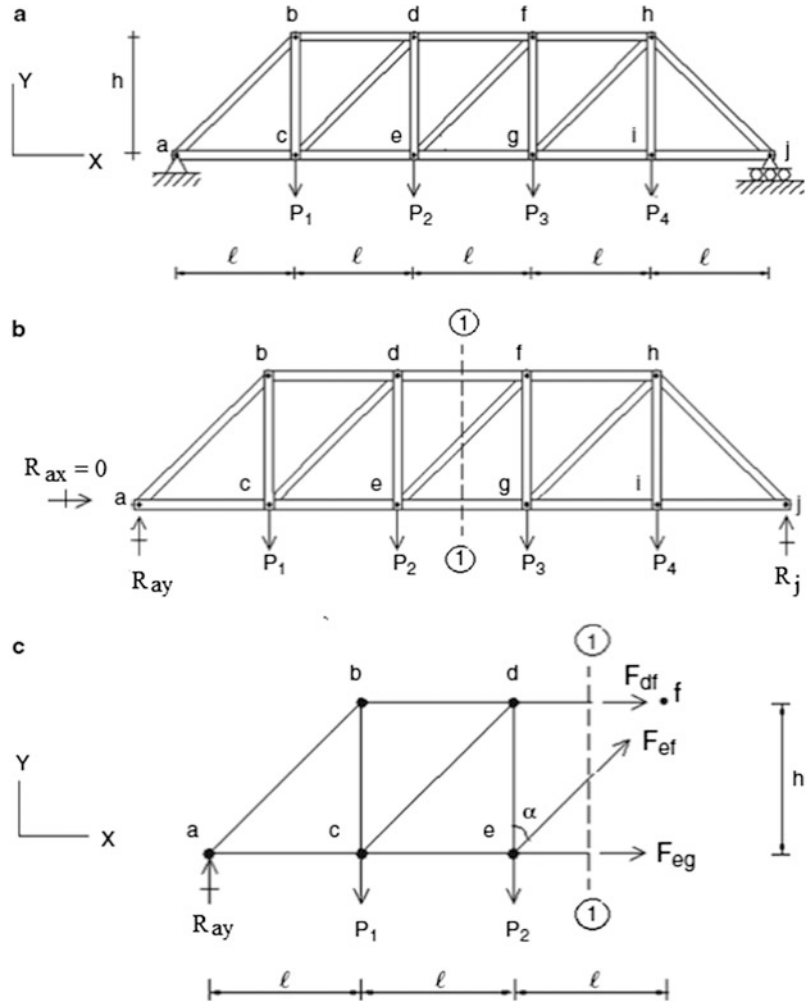
### 2.2.5 Method of Sections

If one wants to determine only the force in a particular member, applying the method of joints might not be convenient since in general it involves first finding the force in other members. For example, consider the truss shown in Fig. 2.19a. Suppose the force in member  $ef$  is desired. One possible strategy is to first determine the reactions at joint  $a$ , then proceed to joints  $b$ ,  $c$ ,  $d$ , and lastly  $e$ , where the  $Y$  component of  $F_{ef}$  can be determined once  $F_{ed}$  is known. Another possible strategy is to start at joint  $j$ , and then precede to joints  $i$ ,  $h$ ,  $g$ , and  $f$ . Either approach requires some preliminary computation that provides information on forces that may or may not be of interest.

The method of sections is an analysis procedure that avoids this preliminary computation. One passes a cutting plane through the truss, isolates either the left or right segment, and applies the equilibrium equations for a rigid body to the segment. The choice of cutting plane is critical. It must cut the particular member whose force is desired, and other members that are concurrent. This restriction is necessary since there are only three equilibrium equations for planar loading, and therefore, one can only determine three unknowns.

We illustrate this method for the truss defined in Fig. 2.19a. We start by determining the reaction at  $a$ . To determine  $F_{ef}$ , we use the vertical cutting plane 1-1 and consider the left segment shown in Fig. 2.19c. Summing forces in the  $Y$  direction leads to:

**Fig. 2.19** (a) An example of a truss. (b) Cutting vertical plane. (c) Truss segment for method of sections



$$\sum F_y = 0 \quad \uparrow^+ \quad F_{ef} \cos \alpha = P_1 + P_2 - R_{ay} \quad (2.3)$$

We point out here that the function of the diagonal members is to equilibrate the unbalanced vertical forces at the sections along the longitudinal axis. These forces are called “shear” forces.

If the force in member  $df$  is desired, one can use the moment equilibrium condition with respect to joint  $e$  which is the point of concurrency for members  $ef$  and  $eg$ .

$$\sum M_{\text{about } e} = 0 \quad hF_{df} = lP_1 - 2lR_{ay} \quad (2.4)$$

Similarly, for member  $eg$ , we use moment equilibrium about joint  $f$ :

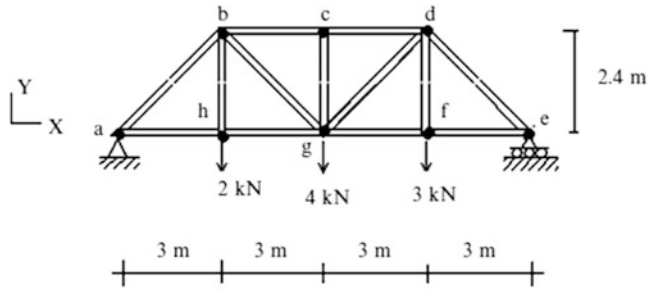
$$\sum M_{\text{about } f} = 0 \quad hF_{eg} = 3lR_{ay} - 2lP_1 - lP_2 \quad (2.5)$$

For parallel chord trusses (top and bottom chords are parallel), the function of the chords is to equilibrate the unbalanced moments at the various sections. One chord force is compressive, the other force is tensile. For downward vertical loading, the top chord is generally in compression, and the bottom is in tension. The method of section is convenient in the sense that it allows one to easily identify the sense of a particular member force.

*Example 2.9 Application of the Method of Sections to a Parallel Chord Truss*

**Given:** The structure and loading shown in Fig. E2.9a

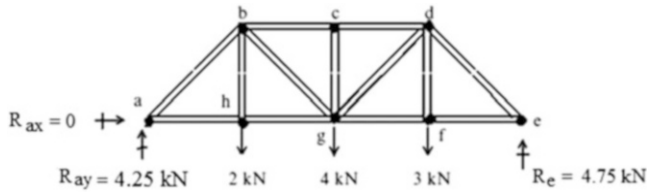
**Determine:** The force in members  $F_{gd}$ ,  $F_{gf}$ , and  $F_{dc}$ .



**Fig. E2.9a**

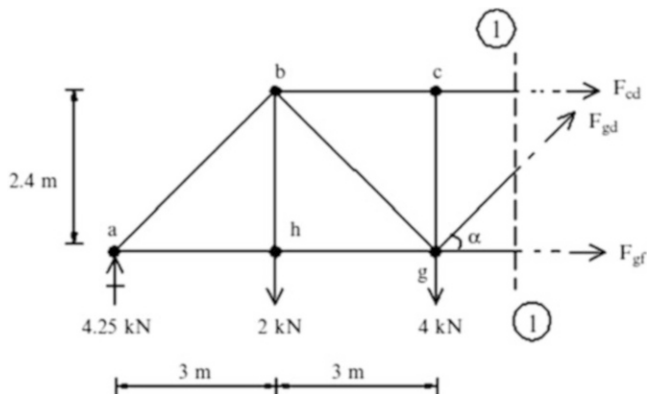
**Solution:** We start by determining the reactions.

$$\Sigma M_a = 0 \quad \curvearrowright \quad 2(3) + 4(6) + 3(9) - R_e(12) = 0 \quad \Rightarrow \quad R_e = 4.75 \text{ kN} \uparrow$$



**Fig. E2.9b**

Then, we pass a vertical cutting plane through the panel between joints d and c and consider the left segment. Enforcing equilibrium leads to:



**Fig. E2.9c**

$$\sum F_y = 0 \quad F_{gd,y} = 1.75 \uparrow$$

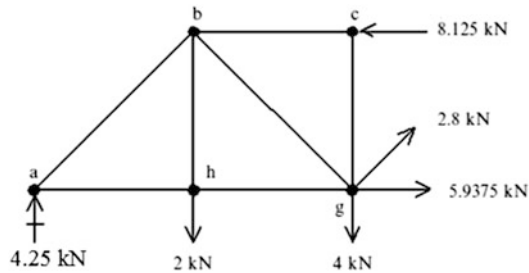
Therefore,  $F_{gd,x} = 2.1875$  and  $F_{gd} = 2.8$  kN (tension)

$$\sum M_{atg} = 0 \quad F_{cd}(2.4) - 2(3) + 4.25(6) = 0 \quad F_{cd} = -8.125$$

Therefore,  $F_{cd} = 8.125$  kN (compression)

$$\sum F_x = 0 \quad F_{gf} - 8.125 + 2.1875 = 0 \quad F_{gf} = +5.9375$$

Therefore,  $F_{gf} = 5.9375$  kN (tension).

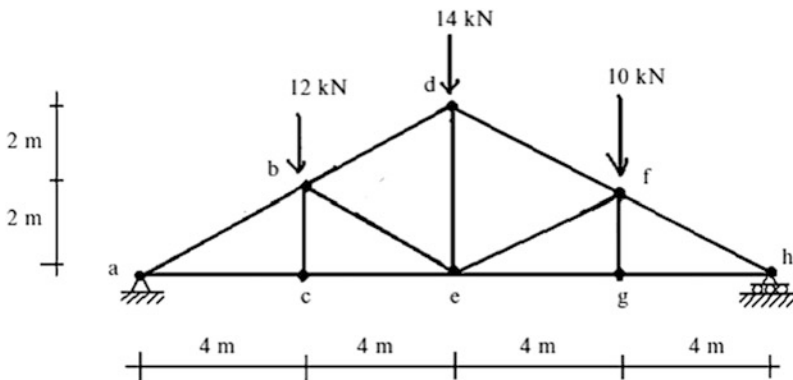


**Fig. E2.9d**

*Example 2.10* The Method of Sections Applied to a Roof Truss

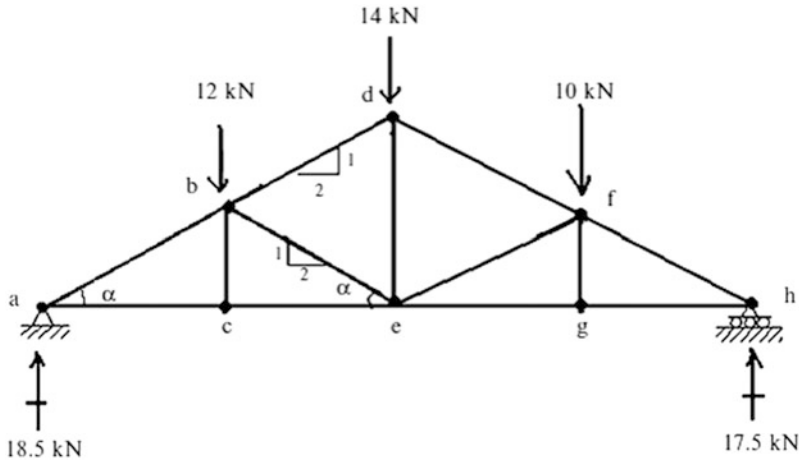
**Given:** The structure shown in Fig. E2.10a.

**Determine:** The member forces  $F_{db}$ ,  $F_{be}$ , and  $F_{ce}$ .

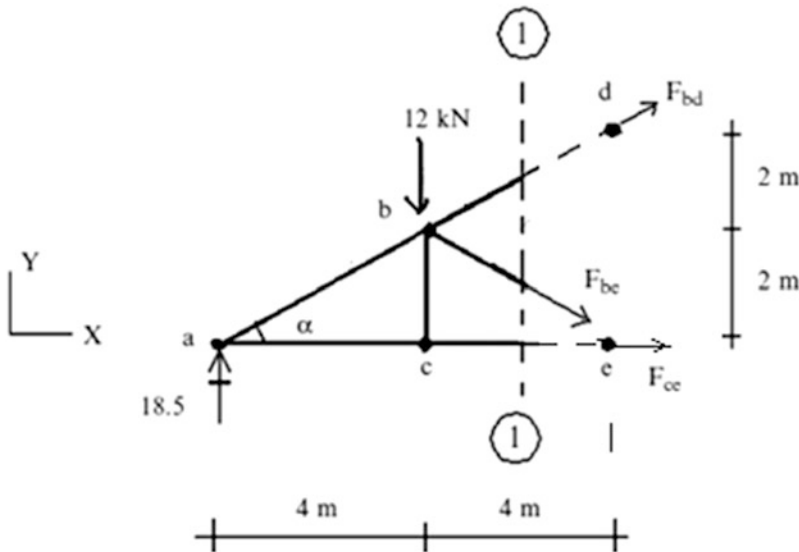


**Fig. E2.10a**

**Solution:** We determine the reactions first.

**Fig. E2.10b**

To determine the member forces  $F_{db}$ ,  $F_{be}$ , and  $F_{ce}$ , we use a vertical cutting plane. The appropriate segment is shown in Fig. E2.10c. Various options are possible. We choose first to determine  $F_{db}$  by summing moments about e. Then, summing moments about b leads to  $F_{ce}$ . Lastly, we can find  $F_{be}$  by summing either  $X$  or  $Y$  forces.

**Fig. E2.10c**

The calculations for this analysis procedure are listed below:

$$\begin{aligned}\sum M_e &= 0 \quad \curvearrowright \quad 4 F_{bd,x} + (18.5) 8 - (12) 4 = 0 \\ F_{bd,x} &= -25 \quad F_{bd} = \frac{F_{bd,x}}{\cos \alpha} = 27.95 \text{ kN (Compression)}\end{aligned}$$

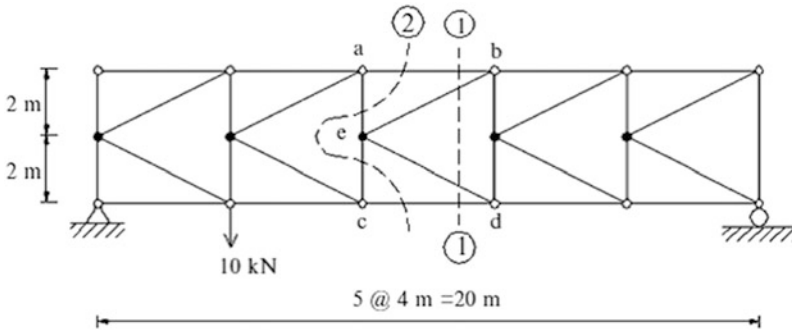
$$\begin{aligned}\Sigma M_b = 0 \quad \curvearrowright \quad & -2F_{ce} + (18.5)4 = 0 \\ & F_{ce} = 37 \text{ kN (Tension)}\end{aligned}$$

$$\begin{aligned}\Sigma F_Y = 0 \quad \uparrow + \quad & -F_{be,y} - 12 - 12.5 + 18.5 = 0 \\ F_{be,y} = -6 \quad \Rightarrow \quad & F_{be} = \frac{F_{be,y}}{\sin \alpha} = 13.41 \text{ kN (compression)}\end{aligned}$$

**Example 2.11** Analysis of K-Type Trusses with the Method of Sections

**Given:** The truss defined in Fig. E2.11a.

**Determine:** The member forces  $F_{ab}$ ,  $F_{be}$ ,  $F_{cd}$ , and  $F_{cd}$ .

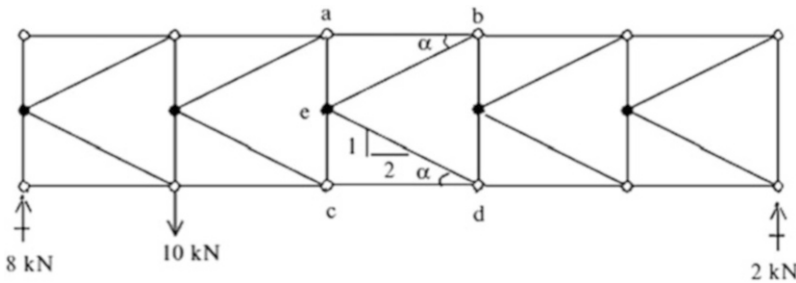


**Fig. E2.11a**

**Solution:** We determine reactions first.

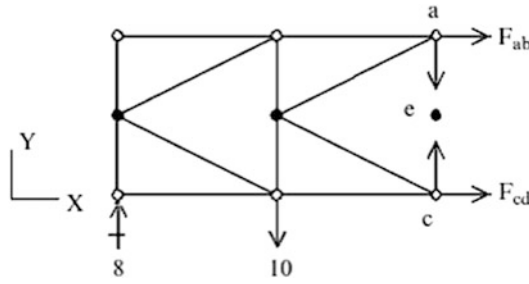
A vertical section such as ①—① cuts four unknown forces and does not lead to a solution. There are no vertical cutting planes that involve only three unknown forces. Therefore, one has to be more creative with the choice of planes. For this type of truss, plane ②—② is the appropriate choice. Isolating the left segment and summing moments about joint c results in  $F_{ab}$ :

$$\Sigma M_c = 0 \quad \curvearrowright \quad 4F_{ab} - 10(4) + 8(8) = 0 \rightarrow F_{ab} = -6 \quad \therefore F_{ab} = 6 \text{ kN (Compression)}$$



**Fig. E2.11b**



**Fig. E2.11c**

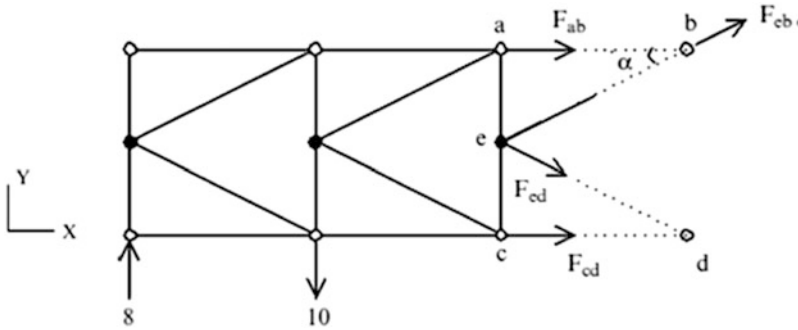
Then summing  $X$  forces,

$$\sum F_x = 0 \quad F_{cd} = -F_{ab} = +6 \quad \therefore F_{cd} = 6 \text{ kN (tension)}$$

The diagonal forces  $F_{eb}$  and  $F_{ed}$  are found using section ①—①. Summing moments about joint  $d$  leads to  $F_{eb}$ :

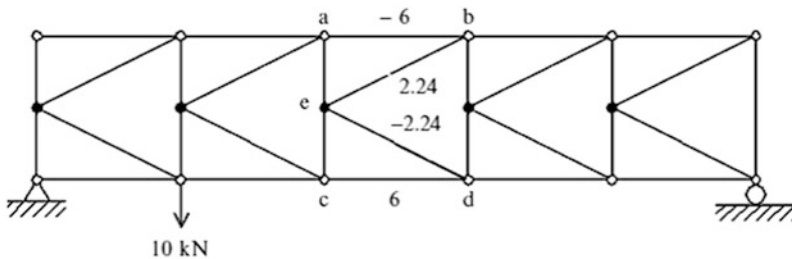
$$\sum M_d = 0 \quad \curvearrowright \quad 4(F_{ab} + F_{eb,x}) - 10(8) + 8(12) = 0 \quad 4(-6 + F_{eb,x}) + 16 = 0$$

$$F_{eb,x} = +2 \quad \therefore F_{eb} = \frac{F_{eb,x}}{\cos \alpha} = 2.24 \text{ kN (tension)}$$

**Fig. E2.11d**

We find  $F_{ed}$  by summing  $x$  forces, and noting that the horizontal components of the chord forces must cancel.

$$\sum F_x = 0 \quad F_{ed,x} = -F_{cd,x} \quad \therefore F_{ed} = 2.24 \text{ kN (compression)}$$

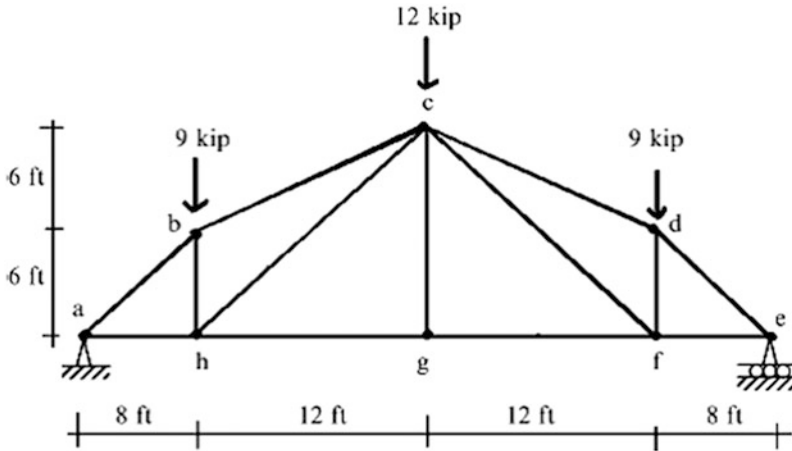
**Fig. E2.11e**

If one wants all the member forces, one can apply multiple cutting planes or combinations of the method of joints and method of sections. How one proceeds is a matter of personal preference.

*Example 2.12: A Hybrid Analysis Strategy*

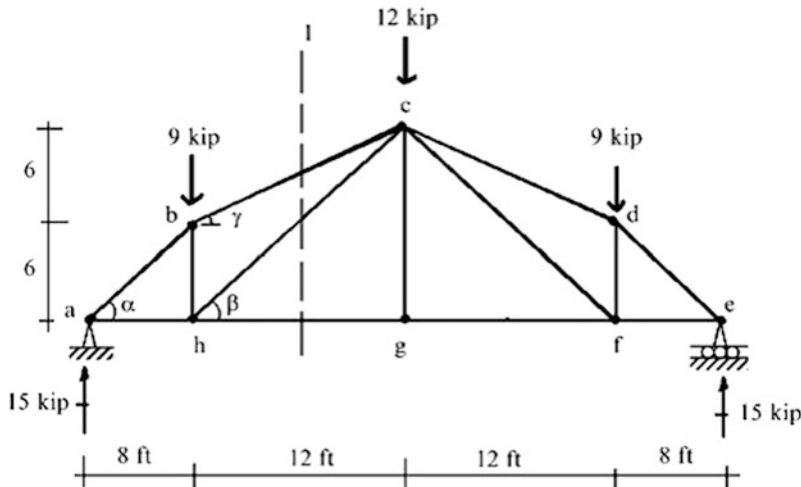
**Given:** The truss defined in Fig. E2.12a

**Determine:** All the member forces using a combination of the method of joints and the method of sections.



**Fig. E2.12a**

**Solution:** We note that the structure and loading are symmetrical with respect to a vertical axis through points  $c$  and  $g$ . It follows that the forces in symmetrically located members are equal, and therefore we need to find the forces in only  $\frac{1}{2}$  of the structure. We start by determining the reactions. The member forces  $F_{bc}$ ,  $F_{hc}$ , and  $F_{hg}$  can be determined by passing vertical cutting plane 1-1 and enforcing the equilibrium equations.



**Fig. E2.12b**

Considering the left segment and enforcing equilibrium leads to:

*Section I-I:*

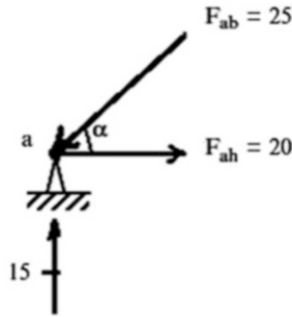
$$\sum M_{at h} = 0 \quad (F_{bc} \cos \gamma)(6) + 15(8) = 0 \quad F_{bc} = 10\sqrt{5} \text{ kip (compression)}$$

$$\sum F_y = 0 \quad F_{hc} = 4\sqrt{2} \text{ kip (tension)}$$

$$\sum F_x = 0 \quad F_{hg} = 16 \text{ kip (tension)}$$

We then enforce equilibrium at joints a and h.

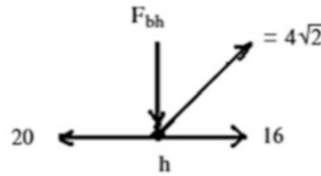
*Equilibrium at joint a:*



$$\sum F_y = 0 \quad F_{ab,y} = 15 \downarrow \quad \therefore F_{ab} = 25 \text{ kip (compression)}$$

$$\sum F_x = 0 \quad F_{ah} = -F_{ab,x} = 20 \text{ kip (tension)}$$

*Equilibrium at joint h:*



$$\sum F_y = 0 \quad F_{bh} = -F_{ch,y} = 4 \text{ kip (compression)}$$

The final member forces are listed below.

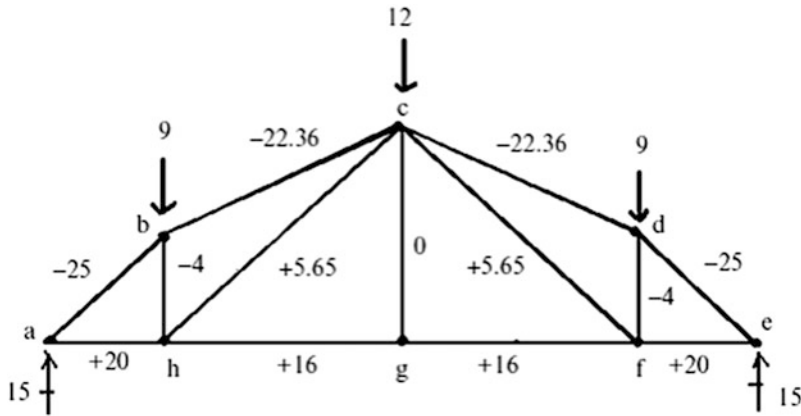


Fig. E2.12c

### 2.2.6 Complex Trusses

Complex trusses are defined as truss structures that cannot be classified as either simple or compound trusses. In order to determine the member forces, one has to establish the complete set of nodal force equilibrium equations expressed in terms of the member forces. If the truss is statically determinate, the number of equations will be equal to the number of force unknowns, and theoretically one can solve these equations for the force unknowns. However, if one cannot determine the member forces, the statically determinate truss is said to be geometrically unstable. In what follows, we expand on this point.

Consider the planar truss shown in Fig. 2.20a. There are nine members, three reactions, and six nodes. Then,

$$2j = 12$$

$$m + r = 9 + 3 = 12$$

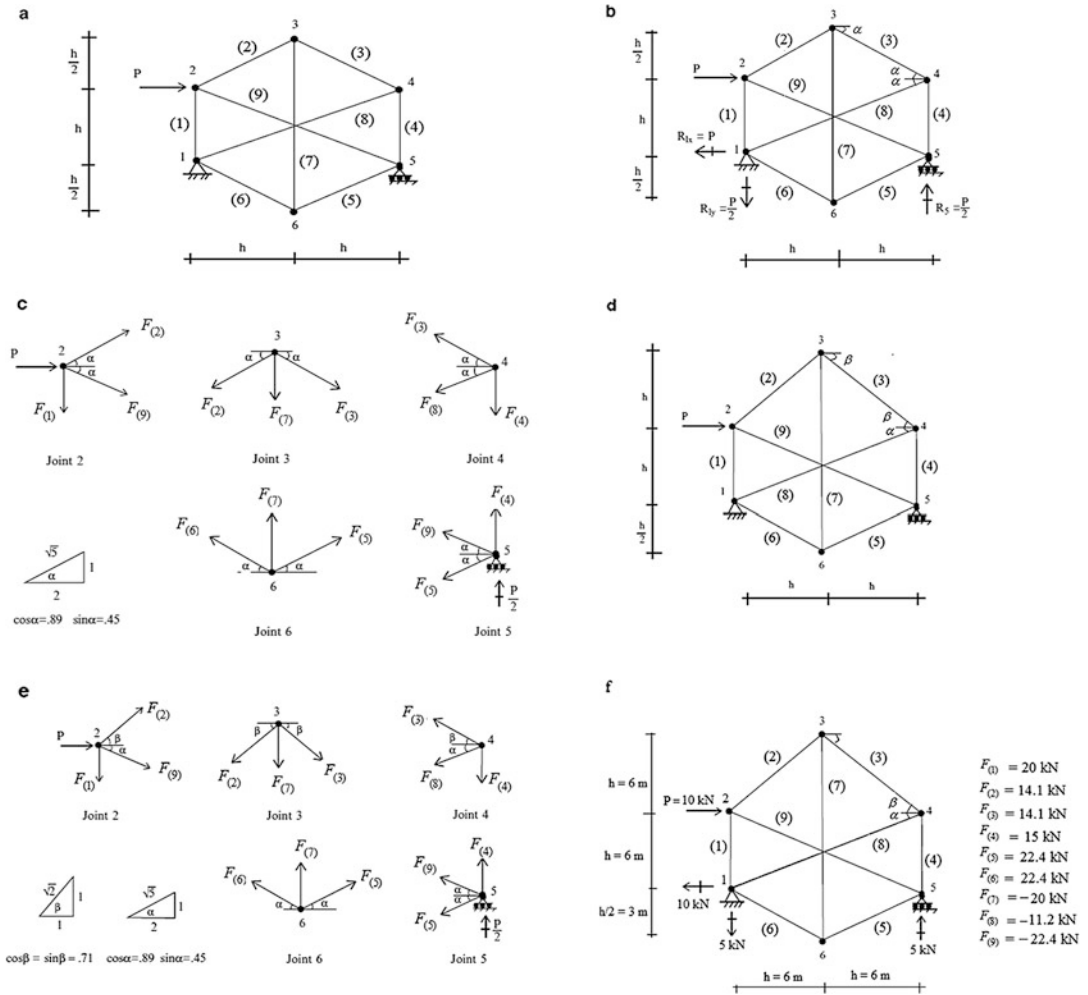
and the truss is statically determinate. It also has a sufficient number of reactions to prevent rigid body motions.

We use 3 of the 12 equilibrium equations to determine the reactions, leaving 9 equations available to solve for the 9 member forces.

$$\sum F_x = 0 \quad R_{1x} = P \leftarrow$$

$$\sum M_{at 1} = 0 \quad R_5 = \frac{P}{2} \uparrow$$

$$\sum F_y = 0 \quad R_{1y} = \frac{P}{2} \downarrow$$



**Fig. 2.20** (a) Planar truss geometry. (b) Reactions. (c) Joint equilibrium. (d) Modified geometry. (e) Joint equilibrium. (f) Member forces

Enforcing equilibrium at joints 2–6 results in the following nine equations:

$$\begin{aligned}
 \text{Joint 2} \quad & \begin{cases} \sum F_x = 0 & (\cos \alpha)F_{(2)} + (\cos \alpha)F_{(9)} = -P \\ \sum F_y = 0 & -F_{(1)} + (\sin \alpha)F_{(2)} - (\sin \alpha)F_{(9)} = 0 \end{cases} \\
 \text{Joint 3} \quad & \begin{cases} \sum F_x = 0 & (\cos \alpha)F_{(2)} - (\cos \alpha)F_{(3)} = 0 \\ \sum F_y = 0 & (\sin \alpha)F_{(2)} + (\sin \alpha)F_{(3)} + F_{(7)} = 0 \end{cases} \\
 \text{Joint 4} \quad & \begin{cases} \sum F_x = 0 & (\cos \alpha)F_{(3)} + (\cos \alpha)F_{(8)} = 0 \\ \sum F_y = 0 & (\sin \alpha)F_{(3)} - F_{(4)} - (\sin \alpha)F_{(8)} = 0 \end{cases} \\
 \text{Joint 5} \quad & \begin{cases} \sum F_x = 0 & (\cos \alpha)F_{(5)} + (\cos \alpha)F_{(9)} = 0 \\ \sum F_y = 0 & F_{(4)} - (\sin \alpha)F_{(5)} + (\sin \alpha)F_{(9)} = -\frac{P}{2} \end{cases} \\
 \text{Joint 6} \quad & \begin{cases} \sum F_x = 0 & (\cos \alpha)F_{(5)} - (\cos \alpha)F_{(6)} = 0 \end{cases}
 \end{aligned} \tag{2.6}$$

We express (2.6) in matrix form

$$\mathbf{BF} = \mathbf{C} \quad (2.7)$$

where

$$\mathbf{F} = \begin{Bmatrix} F_{(1)} \\ F_{(2)} \\ F_{(3)} \\ F_{(4)} \\ F_{(5)} \\ F_{(6)} \\ F_{(7)} \\ F_{(8)} \\ F_{(9)} \end{Bmatrix} \quad \mathbf{C} = \begin{Bmatrix} -P \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{P}{2} \\ 0 \end{Bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 0 & \cos \alpha & 0 & 0 & 0 & 0 & 0 & 0 & \cos \alpha \\ -1 & \sin \alpha & 0 & 0 & 0 & 0 & 0 & 0 & -\sin \alpha \\ 0 & -\cos \alpha & \cos \alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin \alpha & \sin \alpha & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \alpha & 0 & 0 & 0 & 0 & \cos \alpha & 0 \\ 0 & 0 & \sin \alpha & -1 & 0 & 0 & 0 & -\sin \alpha & 0 \\ 0 & 0 & 0 & 0 & \cos \alpha & 0 & 0 & 0 & \cos \alpha \\ 0 & 0 & 0 & 1 & -\sin \alpha & 0 & 0 & 0 & \sin \alpha \\ 0 & 0 & 0 & 0 & \cos \alpha & -\cos \alpha & 0 & 0 & 0 \end{bmatrix}$$

The coefficient matrix,  $\mathbf{B}$  is singular (the determinate of  $\mathbf{B}$  equals 0). Therefore, a unique solution for the unknown forces does not exist for an arbitrary nodal load. The truss is said to be geometrically unstable since the elements of  $\mathbf{B}$  depend only on the geometric pattern.

In order to eliminate the instability, one needs to change the geometry. We modify the truss by changing the vertical position of node 3 as shown in Fig. 2.20d. The individual nodal force systems are defined in Fig. 2.20e and the corresponding nodal force equilibrium equations are listed in (2.8). Note the change in the coefficients.

$$\begin{aligned}
\text{Joint 2} \quad & \begin{cases} \sum F_x = 0 & (\cos \beta)F_{(2)} + (\cos \alpha)F_{(9)} = -P \\ \sum F_y = 0 & -F_{(1)} + (\sin \beta)F_{(2)} - (\sin \alpha)F_{(9)} = 0 \end{cases} \\
\text{Joint 3} \quad & \begin{cases} \sum F_x = 0 & -(\cos \beta)F_{(2)} + (\cos \beta)F_{(3)} = 0 \\ \sum F_y = 0 & (\sin \beta)F_{(2)} + (\sin \beta)F_{(3)} + F_{(7)} = 0 \end{cases} \\
\text{Joint 4} \quad & \begin{cases} \sum F_x = 0 & (\cos \beta)F_{(3)} + (\cos \alpha)F_{(8)} = 0 \\ \sum F_y = 0 & (\sin \beta)F_{(3)} - F_{(4)} - (\sin \alpha)F_{(8)} = 0 \end{cases} \\
\text{Joint 5} \quad & \begin{cases} \sum F_x = 0 & (\cos \alpha)F_{(5)} + (\cos \alpha)F_{(9)} = 0 \\ \sum F_y = 0 & F_{(4)} - (\sin \alpha)F_{(5)} + (\sin \alpha)F_{(9)} = -\frac{P}{2} \end{cases} \\
\text{Joint 6} \quad & \begin{cases} \sum F_x = 0 & (\cos \alpha)F_{(5)} - (\cos \alpha)F_{(6)} = 0 \end{cases}
\end{aligned} \tag{2.8}$$

In this case, the coefficient matrix  $\mathbf{B}$  is nonsingular ( $\det \mathbf{B} \neq 0$ ), and it follows that the structure is geometrically stable:

$$\mathbf{B} = \begin{bmatrix} 0 & \cos \beta & 0 & 0 & 0 & 0 & 0 & 0 & \cos \alpha \\ -1 & \sin \beta & 0 & 0 & 0 & 0 & 0 & 0 & -\sin \alpha \\ 0 & -\cos \beta & \cos \beta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sin \beta & \sin \beta & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \beta & 0 & 0 & 0 & 0 & \cos \alpha & 0 \\ 0 & 0 & \sin \beta & -1 & 0 & 0 & 0 & -\sin \alpha & 0 \\ 0 & 0 & 0 & 0 & \cos \alpha & 0 & 0 & 0 & \cos \alpha \\ 0 & 0 & 0 & 1 & -\sin \alpha & 0 & 0 & 0 & \sin \alpha \\ 0 & 0 & 0 & 0 & \cos \alpha & -\cos \alpha & 0 & 0 & 0 \end{bmatrix}$$

Solving (2.8) using a computer software system [2] leads to the member forces listed below.

$$\mathbf{F} = \mathbf{B}^{-1}\mathbf{C} = \begin{Bmatrix} 2P \\ 1.41P \\ 1.41P \\ 1.5P \\ 2.24P \\ 2.24P \\ -2P \\ -1.12P \\ -2.24P \end{Bmatrix} \Rightarrow \begin{Bmatrix} F_{(1)} \\ F_{(2)} \\ F_{(3)} \\ F_{(4)} \\ F_{(5)} \\ F_{(6)} \\ F_{(7)} \\ F_{(8)} \\ F_{(9)} \end{Bmatrix} = \begin{Bmatrix} 2P \\ 1.41P \\ 1.41P \\ 1.5P \\ 2.24P \\ 2.24P \\ -2P \\ -1.12P \\ -2.24P \end{Bmatrix}$$

For  $P = 10$  kN and  $h = 6$  m, the member forces are listed in Fig. 2.20f.

Assembling the nodal force equilibrium equations usually is a tedious operation, especially for three-dimensional space structures. The process can be automated by using matrix operations. We will describe one approach later in Sect. 2.6.

## 2.3 Computation of Deflections

### 2.3.1 Introduction

The deflections of the joints are due to the change in length of the members that make up the truss. Each member is subjected to an axial force which produces, depending on the sense, either an extension or a contraction along the member. We call these movements “axial deformation.” The study of deflection involves two steps. Firstly, we determine the axial deformation due to the applied loading. This step involves introducing the material properties for the members. Secondly, we need to relate the deflections to the axial deformations. This step is purely geometric. In what follows, we develop procedures for determining the axial deformation due to an axial force, and the joint deflections resulting from a set of axial deformations. The latter procedure is carried out here using a manual computation scheme. A computer-based scheme is described in the next section.

### 2.3.2 Force–Deformation Relationship

Consider the axially loaded member shown in Fig. 2.21. We suppose an axial force,  $F$ , is applied, and the member extends an amount  $e$ . Assuming the material is linear elastic,  $e$  is a linear function of  $F$ . We estimate the proportionality factor by first determining the stress, then the strain, and lastly the extension. We discussed this approach in Chap. 1. The steps are briefly reviewed here.

1. Stress

$$\sigma = \frac{F}{A}$$

where  $A$  is the cross-sectional area

2. Strain

$$\varepsilon = \frac{\sigma}{E} = \frac{F}{AE}$$

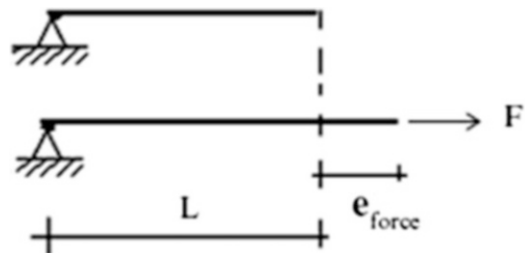
where  $E$  is young’s modulus

3. Extension

$$e_{\text{force}} = L\varepsilon = \frac{FL}{AE}$$

where  $L$  is the member length

**Fig. 2.21** Axially loaded member





The member may also experience an extension due to a temperature change or a fabrication error. Introducing these additional terms, the total extension is expressed as

$$e = e_{\text{force}} + e_{\text{temperature}} + e_{\text{fabrication error}} \quad (2.9)$$

where

$$e_{\text{force}} = \frac{FL}{AE}$$

$$e_{\text{temperature}} = \alpha \Delta T L$$

$$e_{\text{fabrication error}} = e_0$$

$\alpha$  is the coefficient of thermal expansion,  $\Delta T$  is the temperature change, and  $e_0$  represents the fabrication error. The *total extension*,  $e$ , is the quantity that produces the displacement of the node.

### 2.3.3 Deformation–Displacement Relations

Consider the planar truss structure shown in Fig. 2.22. Suppose the members experience deformation and one wants to determine the final position of node B. Our approach is based on first temporarily disconnecting the members at B, allowing the member deformations to occur, and then rotating the members such that they are reconnected. The movements of the nodes from the original configuration to the new configuration are defined as the displacements. These quantities are usually referred to a global reference frame having axes  $X$  and  $Y$  and corresponding displacement components  $u$  and  $v$ .

For structural materials such as steel, the extensions are small in comparison to the original length. Then, the member rotations will also be small. Noting Fig. 2.22b, and the above assumptions, it follows that the displacements are related to the deformations by

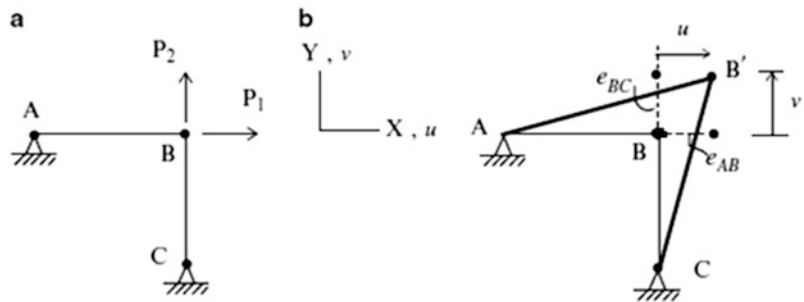
$$u \approx e_{AB}$$

$$v \approx e_{BC} \quad (2.10)$$

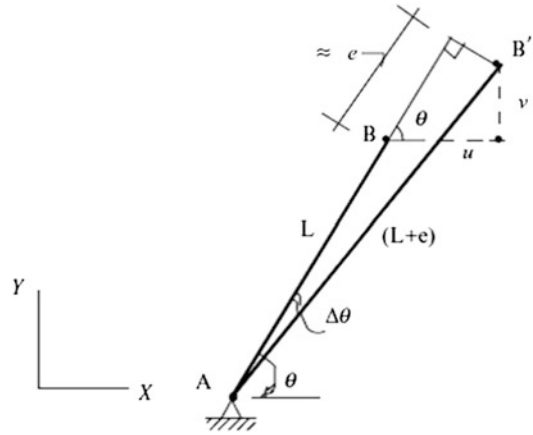
The simplicity of this results is due to the fact that the structure's geometry is simple (the members are orthogonal to the coordinate axes).

We consider the single member AB defined in Fig. 2.23. Our strategy is to track the motion of the end B as it experiences an extension,  $e$ . The final length is  $(L + e)$  where  $e$  is the extension. We assume  $\Delta\theta$  is small and project the final length onto the original direction. This step provides a first-order estimate for the extension in terms of the nodal displacements.

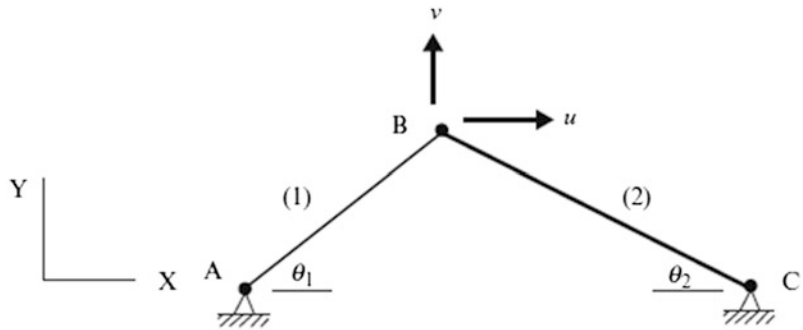
**Fig. 2.22** Initial and deformed geometries. (a) Initial geometry. (b) Deformed configuration



**Fig. 2.23** Extension displacement relationships



**Fig. 2.24** Geometry—two-member truss



$$e \approx u \cos \theta + v \sin \theta \quad (2.11)$$

We consider next a two-member planar truss shown in Fig. 2.24. Since the member orientations are arbitrary, the deformation–displacement relations will involve all the displacement components.

Applying (2.11) to the above structure leads to

$$\begin{aligned} e_1 &= u \cos \theta_1 + v \sin \theta_1 \\ e_2 &= -u \cos \theta_2 + v \sin \theta_2 \end{aligned} \quad (2.12)$$

Given the member forces, one computes the extensions  $e_1$  and  $e_2$  and finally determines the displacements by solving (2.12).

$$\begin{aligned} u &= e_1 \frac{\sin \theta_2}{\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2} - e_2 \frac{\sin \theta_1}{\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2} \\ v &= e_1 \frac{\cos \theta_2}{\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2} + e_2 \frac{\cos \theta_1}{\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2} \end{aligned}$$

### 2.3.4 Method of Virtual Forces

The formulation described in the previous section is not convenient for manual computation, even for fairly simple trusses. However, there is an alternative procedure called the Virtual Force Method, which avoids the need to solve simultaneous equations. Engineers prefer this approach since it is based on executing a set of force equilibrium analyses, a task that they are more familiar with.

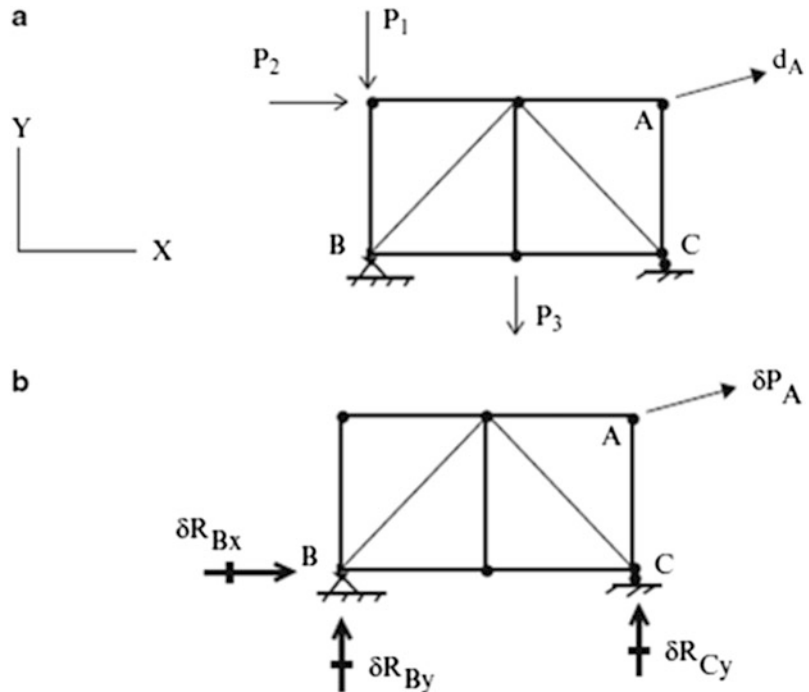
The Method of Virtual Forces is a procedure for determining the deflection at a particular point in a structure given that the member forces are known. A general proof of the method can be found in [3]. We apply the method here for truss type structures. Later in the following chapters, we apply the procedure to beam and frame type structures. The method is restricted to static loading and geometrically linear behavior, i.e., where the displacements are small. This is not a serious restriction for civil structures such as building and bridges.

Consider a typical truss shown in Fig. 2.25a. Suppose the deflection,  $d_A$ , in a specified direction at point A is desired. One applies a virtual force,  $\delta P_A$ , at A in the specified desired direction and computes the corresponding member forces,  $\delta F$ , and reactions,  $\delta R$ , using only the static equilibrium equations. Usually, one takes  $\delta P_A$  to be a unit load. Note that this virtual force system is “specialized” for the particular displacement that one is seeking. The displacement is determined using the following expression:

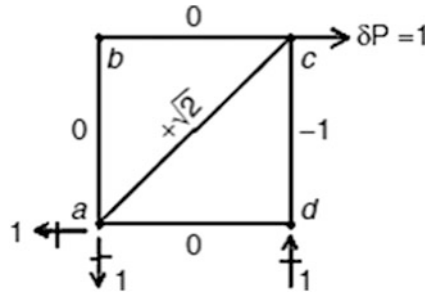
$$d_A \delta P_A = \sum_{\text{members}} e \delta F - \sum_{\text{reactions}} \bar{d} \delta R \quad (2.13)$$

where  $e$  is the total extension defined by (2.9),  $\bar{d}$  is the support movement, and  $\delta R$  the corresponding reaction. When the supports are unyielding,  $\bar{d} = 0$ , and the statement simplifies to

**Fig. 2.25** (a) Desired deflection—actual force system  $F$ . (b) Virtual force system  $\delta F$







**Fig. E2.13c** Virtual forces,  $\delta F$

Using this data, and assuming  $AE$  is constant, the computation proceeds as follows:

Member	$L$	$F$	$\delta F_u$	$e_{\text{force}} = \frac{FL}{AE}$	$e \delta F_u$
ab	$l$	$-10$	$0$	$-10 \frac{l}{AE}$	$0$
bc	$l$	$-40$	$0$	$-40 \frac{l}{AE}$	$0$
cd	$l$	$-50$	$-1$	$-50 \frac{l}{AE}$	$50 \frac{l}{AE}$
da	$l$	$0$	$0$	$0$	$0$
ac	$l\sqrt{2}$	$40\sqrt{2}$	$\sqrt{2}$	$80 \frac{l}{AE}$	$80\sqrt{2} \frac{l}{AE}$

$$u_c = \sum e_{\text{force}} \delta F_u = \frac{l}{AE} (80\sqrt{2} + 50)$$

The plus sign indicates the deflection is in the direction of the unit load. For  $A = 1300 \text{ mm}^2$ ,  $E = 200 \text{ GPa}$ , and  $l = 3 \text{ m}$ , the displacement is

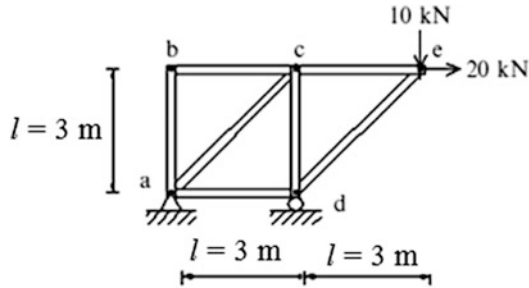
$$u_c = \frac{3(10^3)}{1300(200)} (80\sqrt{2} + 50) = 1.88 \text{ mm} \rightarrow$$

We point out that the virtual force ( $\delta F$ ) results identify which member deformations contribute to the corresponding deflection. In this case, only two-member deformations contribute to the horizontal displacement.

*Example 2.14* Computation of Deflection—Virtual Force Method

**Given:** The plane truss shown in Fig. E2.14a. Assume  $E = 200 \text{ GPa}$ .

**Determine:** The value of  $A$  required to limit the vertical displacement at  $e$  ( $v_e$ ) to be equal to 10 mm. Assume  $AE$  is constant for all members.

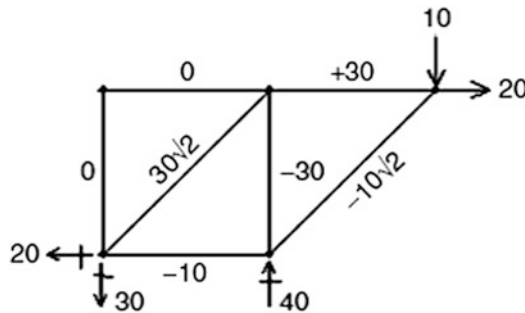


**Fig. E2.14a** Geometry and loading

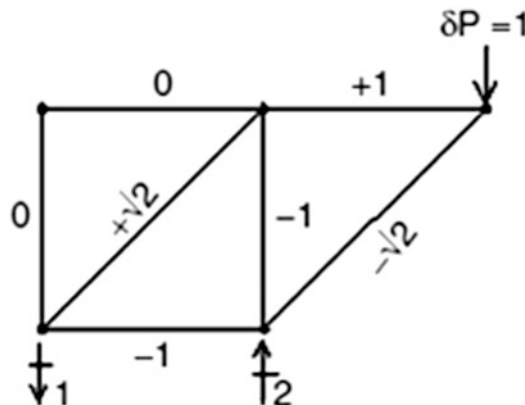
**Solution:** Using (2.14) the vertical displacement at node  $e$  ( $v_e$ ) is determined with

$$v_e \delta P = \sum e_{\text{force}} \delta F_v = \sum \left( \frac{FL}{AE} \right) \delta F_v$$

The actual and virtual forces are listed below.



**Fig. E2.14b** Actual forces,  $F$



**Fig. E2.14c** Virtual forces,  $\delta F$

Using this data, and assuming  $AE$  is constant, the following computations are carried out:

Member	$L$	$F$	$\delta F_v$	$e_{\text{force}} = \frac{F L}{AE}$	$e_{\text{force}} \delta F_v$
ab	$l$	0	0	0	0
bc	$l$	0	0	0	0
cd	$l$	-30	-1	$-30 \frac{l}{AE}$	$30 \frac{l}{AE}$
da	$l$	-10	-1	$-10 \frac{l}{AE}$	$10 \frac{l}{AE}$
ac	$l\sqrt{2}$	$30\sqrt{2}$	$\sqrt{2}$	$60 \frac{l}{AE}$	$60\sqrt{2} \frac{l}{AE}$
ce	$l$	30	1	$30 \frac{l}{AE}$	$30 \frac{l}{AE}$
ed	$l\sqrt{2}$	$-10\sqrt{2}$	$-\sqrt{2}$	$-20 \frac{l}{AE}$	$20\sqrt{2} \frac{l}{AE}$
					$\sum e_{\text{force}} \delta F_v = \frac{l}{AE}(80\sqrt{2} + 70)$

$$v_c = \sum e_{\text{force}} \delta F_v = \frac{l}{AE} (80\sqrt{2} + 70)$$

The plus sign indicates the deflection is in the direction of the unit load. For  $E = 200$  GPa, and  $l = 3000$  mm, the required area is

$$A_{\text{required}} = \frac{l}{v_c E} (80\sqrt{2} + 70) = \frac{(3000)}{10(200)} (80\sqrt{2} + 70) = 275 \text{ mm}^2$$

### Example 2.15 Computation of Deflection—Virtual Force Method

**Given:** The plane truss shown in Fig. E2.15a. Assume  $A = 3000 \text{ mm}^2$  and  $E = 200$  GPa for all members.

**Determine:** The vertical displacement at  $c$  ( $v_c$ ) due to the loading shown and a settlement of 10 mm at support  $a$ .

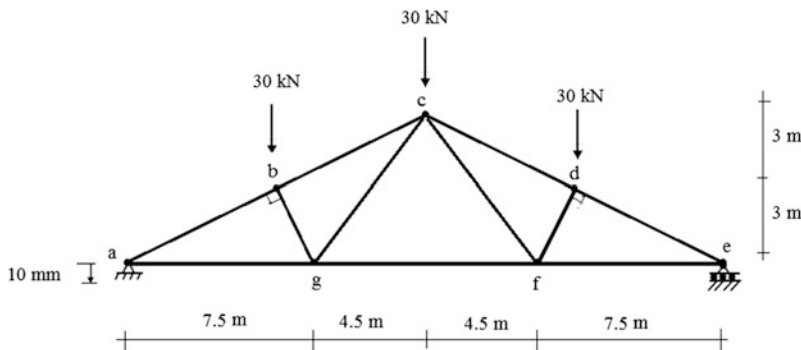


Fig. E2.15a

**Solution:** Using (2.13), the vertical displacement at  $c$  ( $v_c$ ) is determined with  $v_c$

$$v_c = \sum_{\text{members}} e \delta F_v - \sum_{\text{reactions}} \bar{d} \delta R$$

The actual and virtual forces are listed below.

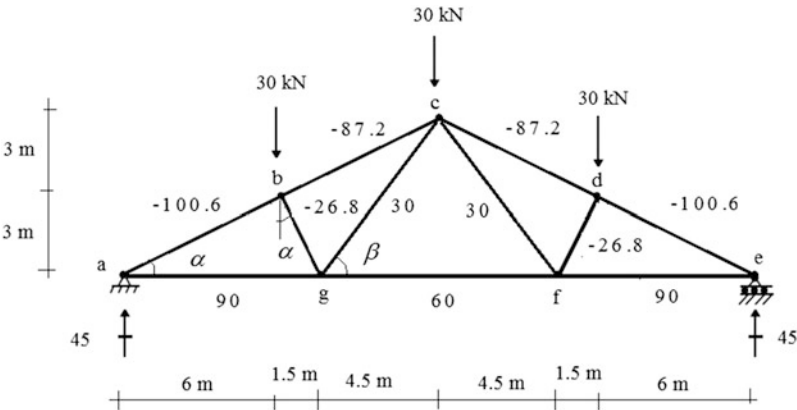


Fig. E2.15b Actual forces,  $F$

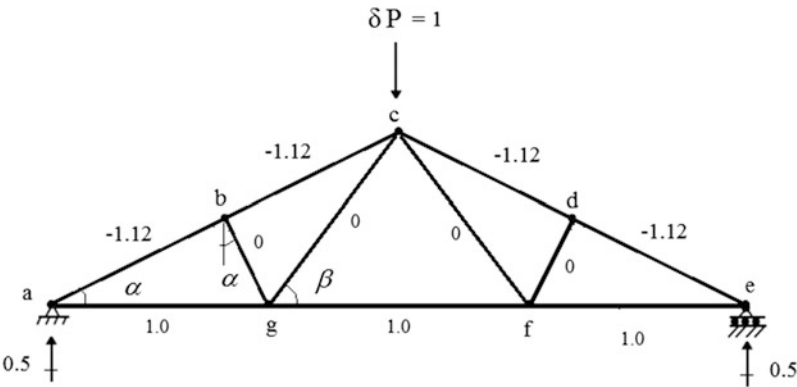


Fig. E2.15c Virtual forces,  $\delta F$

Using this data, and assuming  $AE$  is constant, the computation proceeds as follows:

Member	$L$ (mm)	$F$	$\delta F_v$	$L F \delta F_v$
ab	6708	-100.6	-1.12	$756(10)^3$
bc	6708	-87.2	-1.12	$655(10)^3$
cd	6708	-87.2	-1.12	$655(10)^3$
de	6708	-100.6	-1.12	$756(10)^3$
ef	7500	90	1.0	$675(10)^3$
fg	9000	60	1.0	$540(10)^3$
ga	7500	90	1.0	$675(10)^3$
bg	3354	-26.8	0	0
gc	7500	30	0	0
cf	7500	30	0	0
fd	3354	-26.8	0	0



$$\sum LF \delta F_v = 4712(10)^3$$

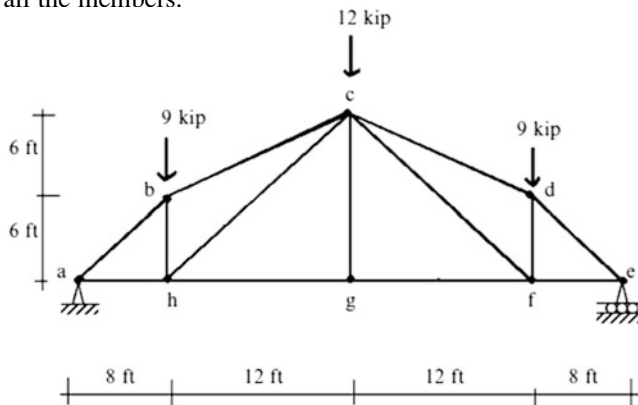
$$v_c = \frac{1}{AE} \left( \sum LF \delta F_v \right) - \delta R_a(v_a) = \frac{1}{3000(200)} (4,712,000) - (0.5)(-10) = +12.85 \text{ mm}$$

$$\therefore v_c = 12.85 \text{ mm} \downarrow$$

**Example 2.16** Computation of Deflection—Virtual Force Method

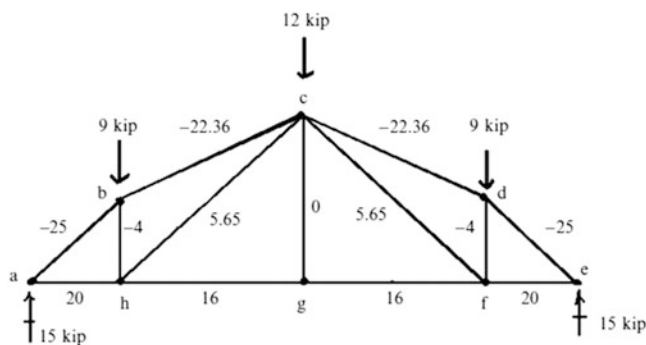
**Given:** The plane truss shown in Fig. E2.16a. Member bc and cf also have a fabrication error of +0.5 in.

**Determine:** The vertical component of the displacement at joint g ( $v_g$ ). Take  $A = 2 \text{ in.}^2$  and  $E = 29,000 \text{ ksi}$  for all the members.

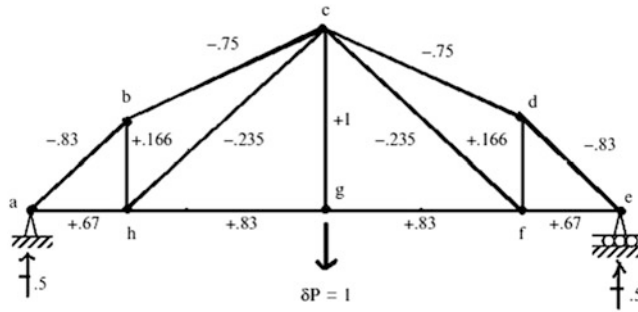


**Fig. E2.16a**

**Solution:** The actual and virtual forces are listed below.



**Fig. E2.16b** Actual forces,  $F$



**Fig. E2.16c** Virtual forces,  $\delta F$

Using this data, the following computations are carried out:

Member	$L$ (in.)	$L/A$	$F$	$\delta F_v$	$\frac{L}{A} \delta F_v$	$e_0$ in.	$e_0 \delta F_v$
ab	120	60	-25	-0.83	1245	0	0
bc	161	80.5	-22.36	-0.75	1350	+0.5	-0.375
cd	161	80.5	-22.36	-0.75	1350	0	0
de	120	60	-25	-0.83	1245	0	0
ef	96	48	20	0.67	643	0	0
fg	144	72	16	0.83	960	0	0
gh	144	72	16	0.83	960	0	0
ha	96	48	20	0.67	643	0	0
bh	72	36	-4	0.166	-24	0	0
cg	144	72	0	1	0	0	0
df	72	36	-4	0.166	-24	0	0
ch	203.6	101.8	5.65	-0.235	-135.7	0	0
cf	203.6	101.8	5.65	-0.235	-135.7	+0.5	-0.1175
					$\sum \frac{L}{A} \delta F_v = 8077$		$\sum e_0 \delta F_v = -0.49$

$$v_{g_{\text{load}}} = \sum e_{\text{force}} \delta F_v = \sum \left( \frac{L}{AE} F \right) \delta F_v = \frac{8077}{29,000} = +0.278 \text{ in.} \Rightarrow v_{g_{\text{load}}} = 0.28 \text{ in.} \downarrow$$

$$v_{g \text{ fabrication error}} = \sum e_0 \delta F_v = -0.49 \text{ in.} \Rightarrow v_{g \text{ fabrication error}} = 0.49 \text{ in.} \uparrow$$

$$v_{g(\text{load}+\text{fabrication})} = +0.278 - 0.49 = -0.21 \text{ in.} \Rightarrow v_{g(\text{load}+\text{fabrication})} = 0.21 \text{ in.} \uparrow$$

### Example 2.17 Deflection of a Gable Truss

**Given:** The plane truss shown in Fig. E2.17a. The truss has variable cross sections, such that  $A = 6500 \text{ mm}^2$  for top chord members,  $A = 3900 \text{ mm}^2$  for bottom chord members,  $A = 1300 \text{ mm}^2$  for diagonal members, and  $A = 650 \text{ mm}^2$  for vertical members,  $l = 3 \text{ m}$ ,  $P = 10 \text{ kN}$ , and  $E = 200 \text{ GPa}$ .



Member	$L$ (mm)	$A$ (mm <sup>2</sup> )	$L/A$	$F$ (kN)	$\delta F_u$	$\delta F_v$	$(L/A)F \delta F_u$	$(L/A)F \delta F_v$
la	3000	3900	0.77	125.0	1.0	2.5	96.2	240.6
bl	600	650	0.92	0.0	0.0	0.0	0	0
ck	1200	650	1.85	5.0	0.0	0.0	0	0
dj	1800	650	2.77	20.0	0.0	1.0	0	55.4
ei	1200	650	1.85	5.0	0.0	0.0	0	0
fh	600	650	0.92	0.0	0.0	0.0	0	0
bk	3059	1300	2.35	-25.5	0.0	0.0	0	0
cj	3231	1300	2.48	-26.9	0.0	0.0	0	0
ej	3231	1300	2.48	-26.9	0.0	0.0	0	0
fi	3059	1300	4.71	-25.5	0.0	0.0	0	0

The remaining computations involve dividing by  $E$ .

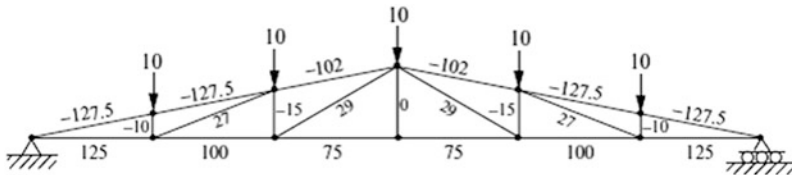
$$\sum \left( \left( \frac{L}{A} \right) F \delta F_u \right) = 538.8 \text{ kN/mm}$$

$$\therefore u_g = \sum \Sigma e_{\text{force}} \delta F_u = \frac{1}{E} \sum \Sigma \left( \left( \frac{L}{A} \right) F \delta F_u \right) = 538.8/200 = 2.69 \text{ mm} \rightarrow$$

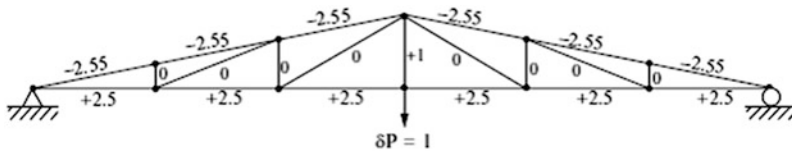
$$\sum \left( \left( \frac{L}{A} \right) F \delta F_v \right) = 2136.2 \text{ kN/mm}$$

$$\therefore v_j = \sum \Sigma e_{\text{force}} \delta F_v = \frac{1}{E} \sum \Sigma \left( \left( \frac{L}{A} \right) F \delta F_v \right) = 2136.2/200 = 10.7 \text{ mm} \downarrow$$

We pointed out earlier that the distribution of member forces depends on the orientation of the diagonal members. We illustrate this behavior by reversing the diagonal pattern for the truss defined in Fig. E2.17a. The member forces corresponding to the same loading are listed in Fig. E2.17e. Suppose the vertical deflection at mid-span is desired. The corresponding virtual force system is shown in Fig. E2.17f.



**Fig. E2.17e** Diagonal pattern reversed—actual forces  $F$



**Fig. E2.17f** Diagonal pattern reversed—virtual forces  $\delta F_v$

Member	$L$ (mm)	$A$ (mm <sup>2</sup> )	$L/A$	$F$	$\delta F_v$	$(L/A)F \delta F_v$
ab	3059	6500	0.47	-127.5	-2.55	153
bc	3059	6500	0.47	-127.5	-2.55	153
cd	3059	6500	0.47	-102	-2.55	122.2

Member	$L$ (mm)	$A$ (mm <sup>2</sup> )	$L/A$	$F$	$\delta F_v$	$(L/A)F \delta F_v$
de	3059	6500	0.47	−102	−2.55	122.2
ef	3059	6500	0.47	−127.5	−2.55	153
fg	3059	6500	0.47	−127.5	−2.55	153
gh	3000	3900	0.77	125.0	2.5	240.6
hi	3000	3900	0.77	100	2.5	192.5
ij	3000	3900	0.77	75	2.5	144.4
jk	3000	3900	0.77	75	2.5	144.4
kl	3000	3900	0.77	100	2.5	192.5
la	3000	3900	0.77	125	2.5	240.6
bl	600	650	0.92	−10	0.0	0
ck	1200	650	1.85	−15	0.0	0
dj	1800	650	2.77	0	1.0	0
ei	1200	650	1.85	−15	0.0	0
fh	600	650	0.92	−10	0.0	0
bk	3059	1300	2.35	27	0.0	0
dk	3498	1300	2.69	29	0.0	0
di	3498	1300	2.69	29	0.0	0
fi	3059	1300	2.35	27	0.0	0

Using the data listed above, the mid-span deflection calculations are

$$\sum \left( \left( \frac{L}{A} \right) F \delta F_v \right) = 2011 \text{ kN/mm}$$

$$\therefore v_j = \sum \Sigma e_{\text{force}} \delta F_v = \frac{1}{E} \sum \Sigma \left( \left( \frac{L}{A} \right) F \delta F_v \right) = 2011/200 = 10 \text{ mm} \downarrow$$

The examples presented to this point have been concerned with loads. Structures are also subjected to seasonal (and daily) temperature changes and it is of interest to determine the corresponding nodal displacements. A unique feature of statically determinate structures is their ability to accommodate temperature changes without experiencing member forces. When subjected to a temperature change, a statically determinate structure adjusts its geometry in such a way that there are *no forces introduced in the members*. From a design perspective, this behavior is very desirable since member forces, i.e., stresses, are due only to the loads. However, one may need to compute the deflected shape due to temperature change from some initial state. The effect of temperature change is to produce an additional extension in a truss member given by:

$$e_{\text{temperature}} = \alpha \Delta T L$$

where  $\alpha$  is a material property, defined as the coefficient of thermal expansion, and  $\Delta T$  is the temperature change from the initial state. Then, the form of the Principle of Virtual force specialized for only temperature and unyielding supports reduces to

$$d\delta P = \sum (e_{\text{temperature}})(\delta F) \quad (2.15)$$

The computational procedure is similar to the approach discussed earlier. We evaluate  $(\alpha \Delta T L)$  for the members. Then, given a desired deflection, we apply the appropriate virtual loading and compute  $\delta F$  for the members. Lastly, we evaluate the summation. The following example illustrates the details. This discussion applies only for statically determinate trusses. A temperature change introduces internal forces in statically indeterminate trusses. Analysis procedures for this case are discussed in Chaps. 9 and 10.

Example 2.18 Computation of Deflection Due to Temperature

**Given:** The plane truss shown in Fig. E2.18a.

**Determine:** The vertical displacement at joint d due to temperature increase of  $\Delta T = 65^\circ\text{F}$  for all members. Assume  $A = 2\text{ in.}^2$ ,  $E = 29,000\text{ ksi}$ , and  $\alpha = 6.5 (10^{-6})/^\circ\text{F}$ .

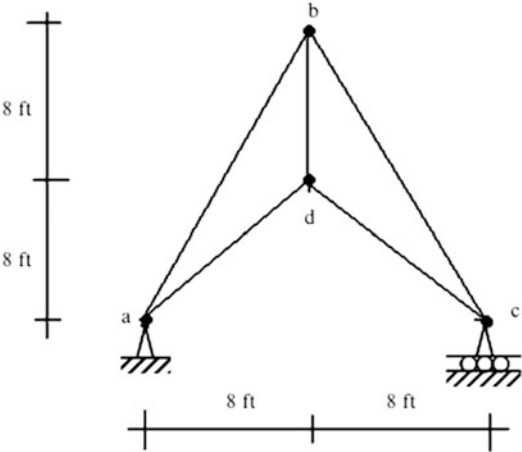


Fig. E2.18a

**Solution:** The corresponding virtual force system is listed below.

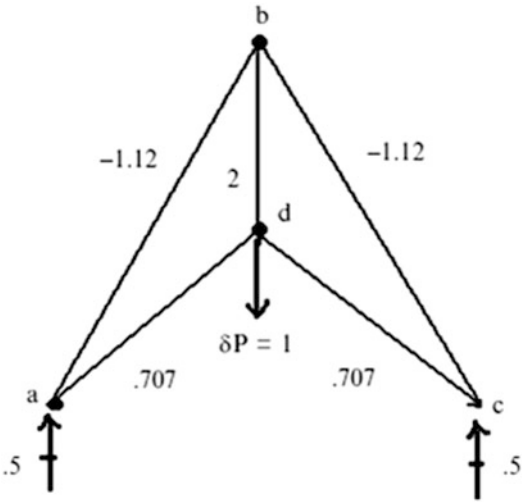


Fig. E2.18b Virtual forces  $\delta F$

Member	$L\text{ (in.)}$	$e = \alpha\Delta TL$	$\delta F_v$	$e\delta F_v$
ab	214.7	0.091	-1.12	-0.102
bc	214.7	0.091	-1.12	-0.102
cd	135.7	0.057	0.707	0.04

Member	$L$ (in.)	$e = \alpha \Delta T L$	$\delta F_v$	$e \delta F_v$
da	135.7	0.057	0.707	0.04
bd	96	0.041	2	0.082
				$\sum e \delta F = -0.042$

$$v_d = \sum e_{\text{temperature}} \delta F_v = (\alpha \Delta T L) \delta F_v = -0.042 \Rightarrow v_d = 0.042 \text{ in. } \uparrow$$

## 2.4 Influence Lines

Consider the plane bridge truss shown in Fig. 2.26a. To design a particular member, one needs to know the maximum force in the member due to the design loading. The dead loading generally acts over the entire structure, i.e., on all the nodes. For this loading component, one places all the dead load on the structure and carries out a single analysis for the member forces. The live loading, by definition, can act anywhere on the structure and therefore one needs to determine the location of the live loading that produces the maximum force in the member that is being designed. A systematic approach to locating the critical position of the live loading is based on first constructing an influence line for the member force. This construction involves a series of analyses, one for each possible location of live loading. The live load is usually taken as a single force, of unit magnitude, which is moved from node to node across the structure. The resulting influence line is a plot of the member force as a function of the location of the applied load. Figure 2.26b illustrates the possible nodal positions of a vertical load applied to the bottom chord, and the corresponding member forces. Given this data, one can construct an influence line for any of the member forces.

The process described above assumes the loading is a concentrated load applied at the nodes. For bridge structures, the live loading is actually applied to the deck which transmits the load to the transverse beams, and finally to the nodes. The deck is usually simply supported on the transverse beams, so the complete deck-beam system is statically determinate and one can determine the reactions at the nodes using only the equations of statics. We illustrate this computation using the structure shown in Fig. 2.27a. We suppose a truck loading is passing over the span.

Consider the position shown in Fig. 2.27b. The wheel loads act on the deck segments  $gf$  and  $fe$ . The live load vehicle analysis reduces to just applying loads to the nodes adjacent to the vehicle since the deck segments ( $gf$  and  $fe$ ) are simply supported. Noting Fig. 2.27c, the equivalent nodal loads are

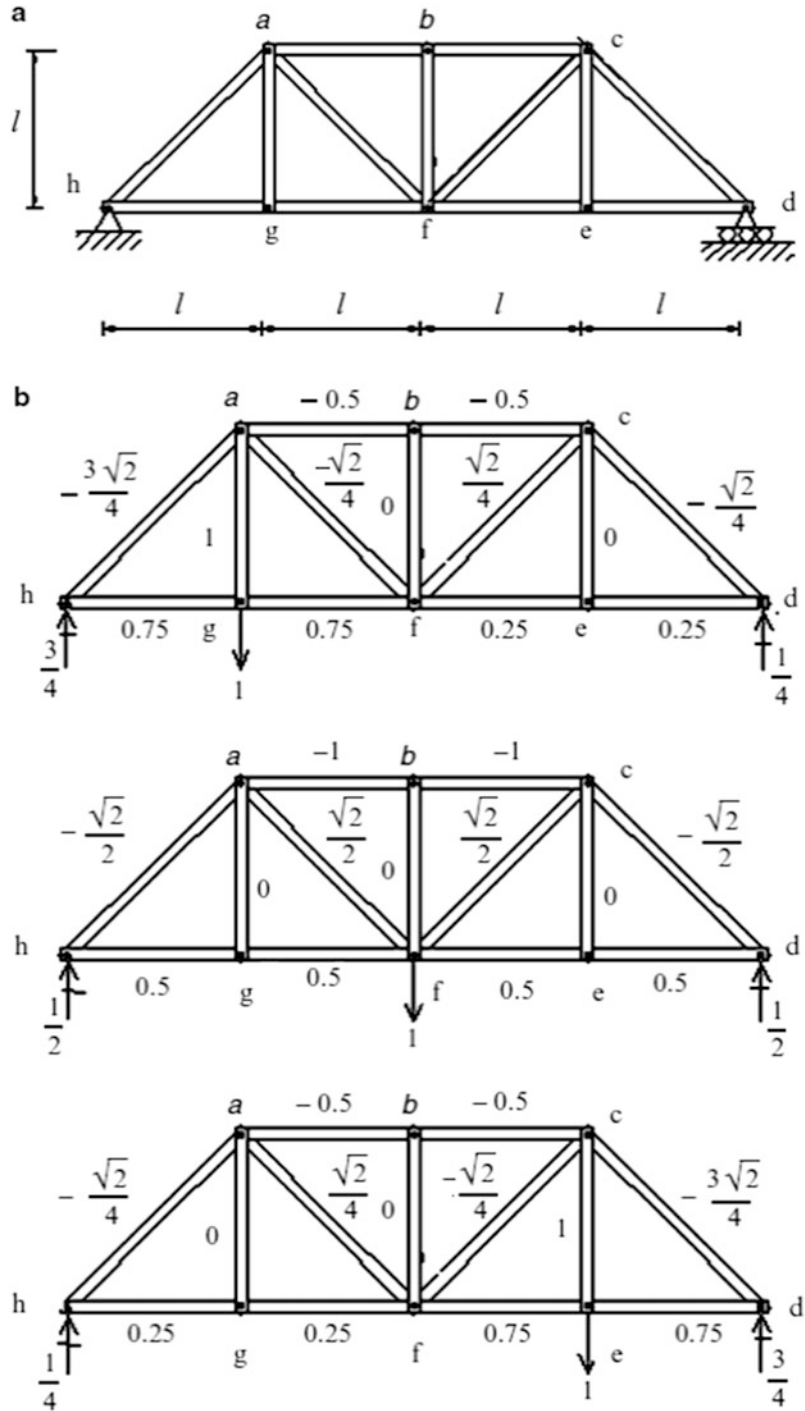
$$\begin{aligned} R_1 &= \left(1 - \frac{x}{l}\right) P_1 \\ R_2 &= \left(\frac{x}{l}\right) P_1 + \left(2 - \frac{x}{l} - \frac{h}{l}\right) P_2 \\ R_3 &= \left(\frac{x}{l} + \frac{h}{l} - 1\right) P_2 \end{aligned}$$

Note that the reactions are linear functions of  $x$ , the position coordinate for the truck.

We define  $F_j$  as the force in member  $j$ . Applying separate unit loads at nodes  $g$  and  $f$  leads to  $F_j|_g$  and  $F_j|_f$ . Then, according to the equations listed above, the force due to a unit load at  $x$  is

$$F_j|_x = \left(1 - \frac{x}{l}\right) F_j|_g + \left(\frac{x}{l}\right) F_j|_f$$

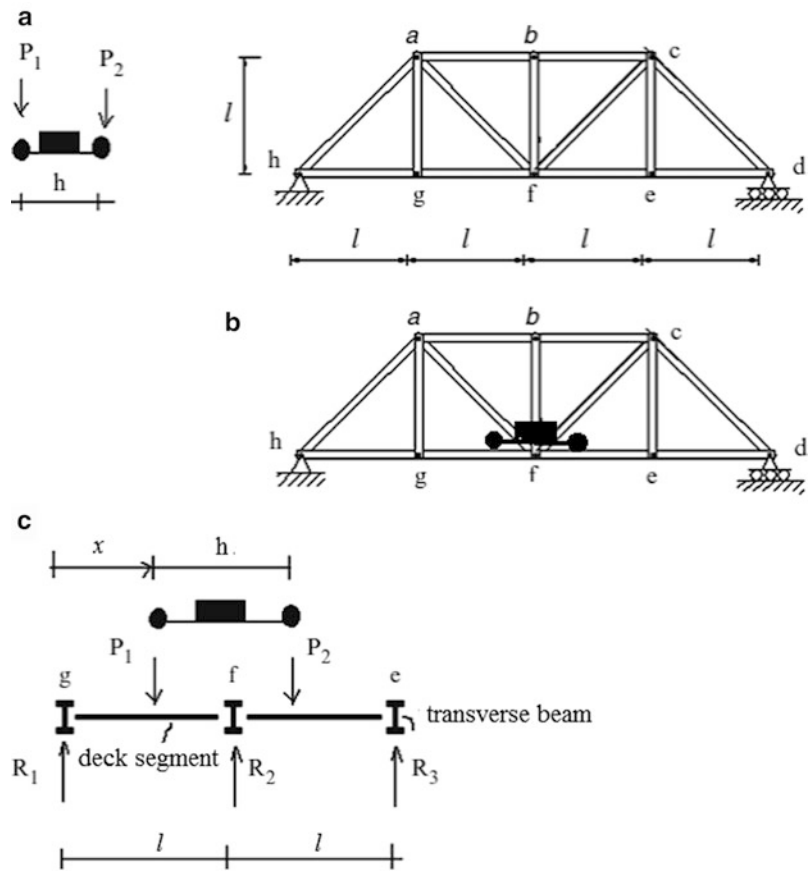
**Fig. 2.26** (a) Truss geometry. (b) Load positions and corresponding member forces



The most convenient way to present these results is to construct a plot of  $F_j$  vs.  $x$ , where  $F_j$  is the force in member  $j$  due to a unit load at  $x$ , and  $x$  is taken to range over the nodes on the bottom chord. We need to apply these loads *only at the nodes* since the plot is *linear* between adjacent nodes. Plots of this type are called influence lines. Figure 2.28a shows the influence line for chord member  $ab$ . This



**Fig. 2.27** (a) Truss geometry. (b). Loaded position. (c) Free body diagram-transverse beam

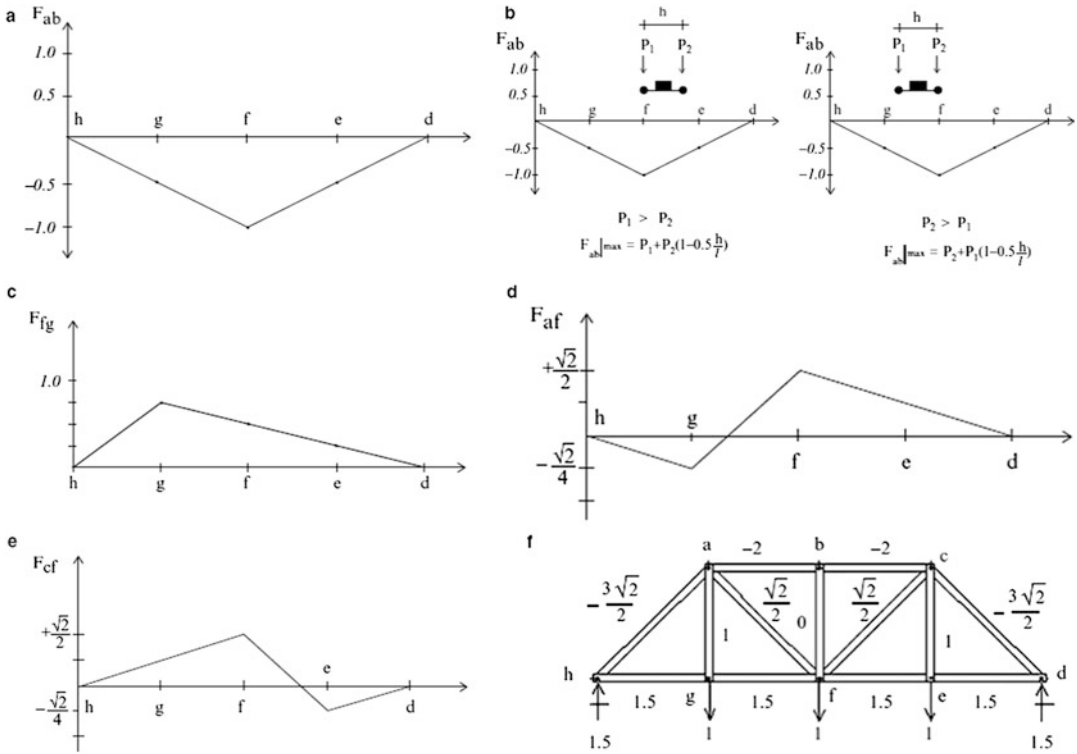


visual representation is convenient since one can immediately identify the critical location of the load. For the chord member,  $ab$ , the maximum magnitude occurs when the load is applied at mid-span. Also, we note that the force is compression for all locations.

Given an actual loading distribution, one evaluates the contribution of each load, and then sums the contributions. If the actual live load consisted of a uniform loading, then it follows that one would load the entire span. The maximum force due to the truck loading is determined by positioning the truck loads as indicated in Fig. 2.28b. In general, one positions the vehicle such that the maximum vehicle load acts on node  $f$ .

The influence line for member  $fg$  is plotted in Fig. 2.28c. In this case, the member force is always tension.

The function of the diagonal members is to transmit the vertical forces from node to node along the span. This action is called “shear.” The influence line for a diagonal is different than the influence lines for upper and lower chord members, in that it has both positive and negative values. Figure 2.28d shows the result for diagonal  $af$ . A load applied at node  $g$  generates compression, whereas loads at nodes  $f$  and  $e$  produce tension. Lastly, a symmetrically located diagonal with opposite orientation, such as  $cf$  vs.  $af$ , has an influence line that is a rotated version of its corresponding member (see Fig. 2.28d vs. Fig. 2.28e).



**Fig. 2.28** (a) Influence line for cord member ab. (b) Vehicle positioning for  $F_{ab}|_{\max}$ . (c) Influence line for chord member fg. (d) Influence line for diagonal member af. (e) Influence line for diagonal member cf. (f) Uniform unit load

Because the influence lines for diagonals have both positive and negative values, one needs to consider two patterns of live load in order to establish the peak value of the member force.

For member af, the extreme values are

$$\text{Load at node f } F = \sqrt{2}/2$$

$$\text{Load at node g } F = \sqrt{2}/4$$

For member cf, the extreme values are

$$\text{Loads at node e } F = \sqrt{2}/4$$

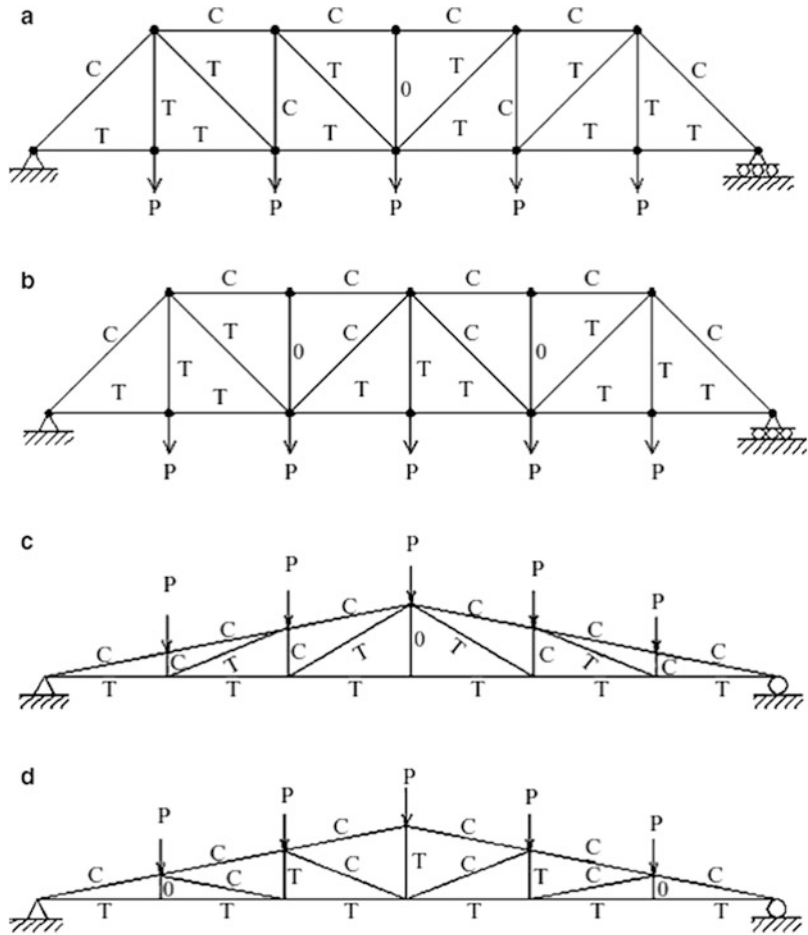
$$\text{Loads at node f } F = \sqrt{2}/2$$

If a uniform load is applied (see Fig. 2.28f), the peak force values for both members will be:

$$F_{\max} = +\sqrt{2}/2$$

As mentioned earlier, diagonal members function to transmit vertical loads to the end supports. We showed above that the sense of the diagonal force depends on the orientation of the member. The sense of the diagonal force is important since slender members behave differently under compression vs. tension. A slender member subjected to compressive load will fail by buckling rather than by yielding since the buckling load is considerably less than the yield force. Therefore, from a design perspective one should avoid using slender compression members. For truss type structures, this problem can be avoided by selecting an appropriate diagonal orientation pattern.

**Fig. 2.29** Force pattern for various truss geometries. (a) Pratt truss. (b) Warren truss. (c) Pratt truss. (d) Howe truss

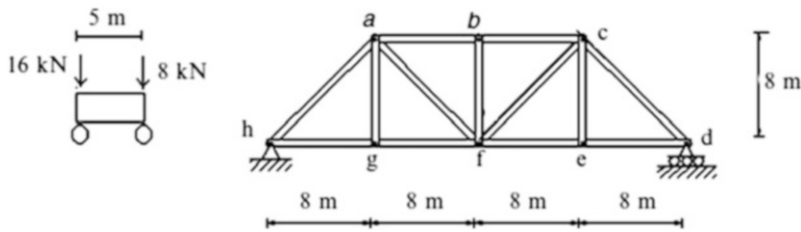


As an example, consider the two diagonal patterns shown in Fig. 2.29a, b. The sense of the member forces due to a uniform live load is indicated by C (compression) and T (tension).

Pattern (a) is more desirable since all the interior diagonals are in tension. However, some of the vertical members are in compression. Pattern (b) has alternating sense for the diagonals; the vertical hangers are all in tension. In general, for both truss types the top chord forces are compression and the bottom chord forces are tension. Figure 2.29c, d show similar results for inclined chord trusses. The designators “Pratt,” “Warren,” and “Howe” refer to the individuals who invented these geometrical forms.

### Example 2.19

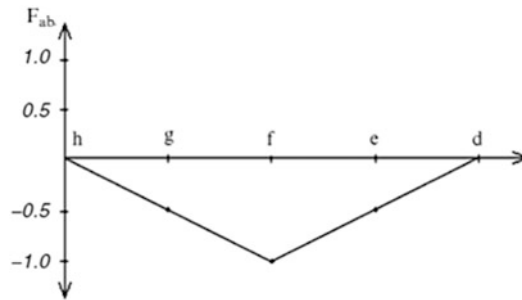
**Given:** The structure and truck loading shown in Fig. E2.19a.



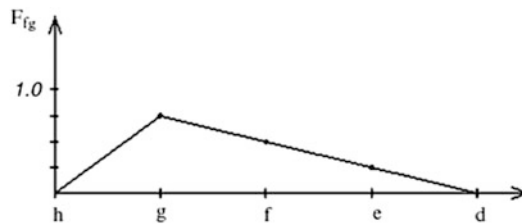
**Fig. E2.19a**

**Determine:** The maximum force in members ab and fg due to the truck loading.

**Solution:** We first determine the influence lines for a unit vertical force applied along the bottom nodes.



**Fig. E2.19b** Influence line for member ab



**Fig. E2.19c** Influence line for member fg

Then, we position the truck loading as indicated in Figs. E2.19d and E2.19e

$$F_{ab_{\max}} = 16(-1) + 8(-0.69) = -21.52 \quad \therefore F_{ab_{\max}} = 21.52 \text{ kN compression}$$

$$F_{fg_{\max}} = 16(0.75) + 8(0.59) = +16.72 \quad \therefore F_{fg_{\max}} = 16.72 \text{ kN tension}$$

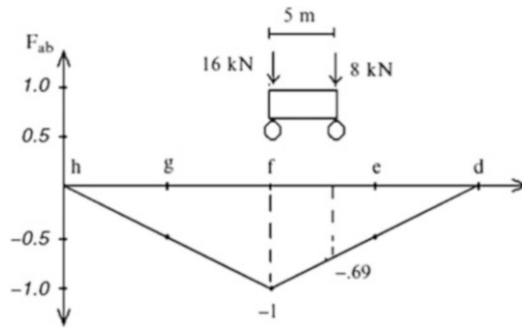


Fig. E2.19d

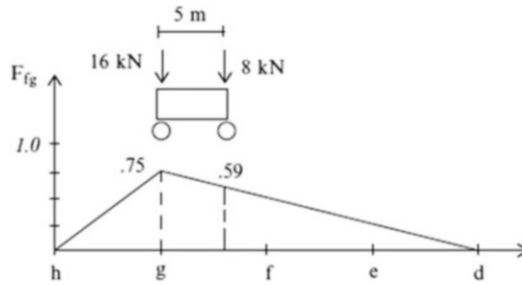


Fig. E2.19e

*Example 2.20* Live Load Analysis for a Gable Roof Structure

**Given:** The gable roof structure shown in Fig. E2.20a.

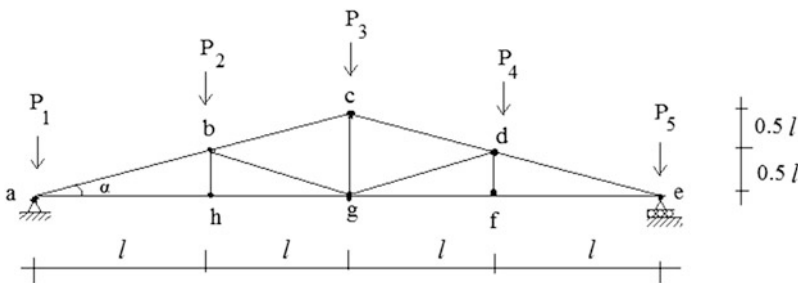


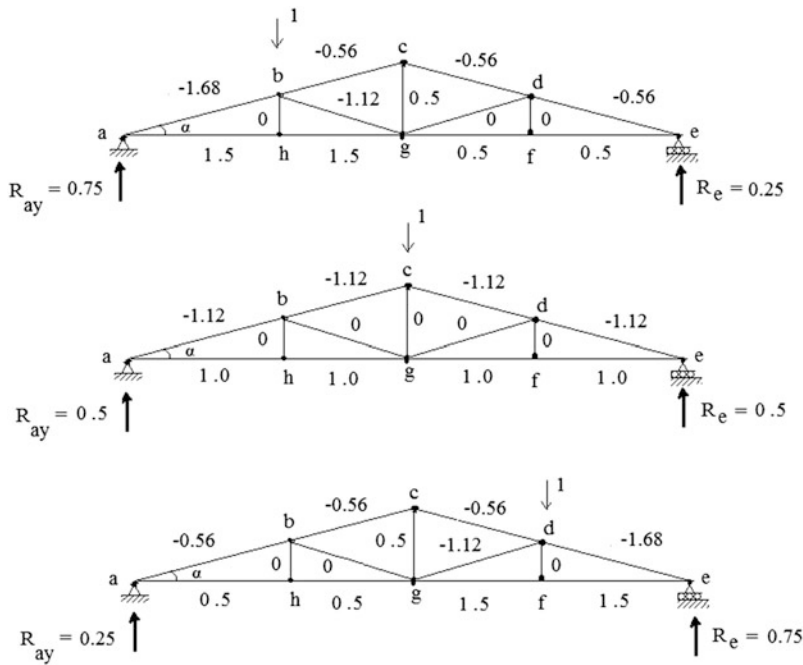
Fig. E2.20a Structural geometry and nodal loads

**Determine:**

- Tabulate all the member forces due to the individual unit nodal forces applied to the top chord. We refer to this type of table as a force influence table.
- Use the force influence table to draw the influence lines for member cd and fg.
- Calculate the reactions and member forces for members cd and fg for  $P_1 = P_5 = 7.5\text{ kN}$ , and  $P_2 = P_3 = P_4 = 15\text{ kN}$ .

**Solution:**

**Part (i)** the member forces due to the individual unit nodal loads are listed in Fig. E2.20b.

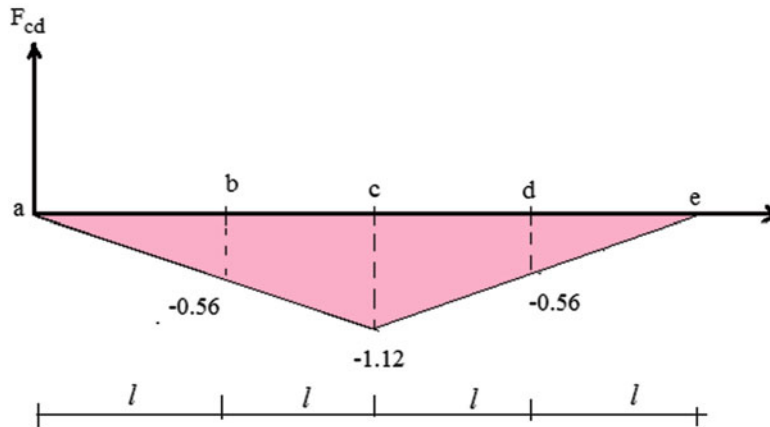


**Fig. E2.20b** Force results for different unit load positions

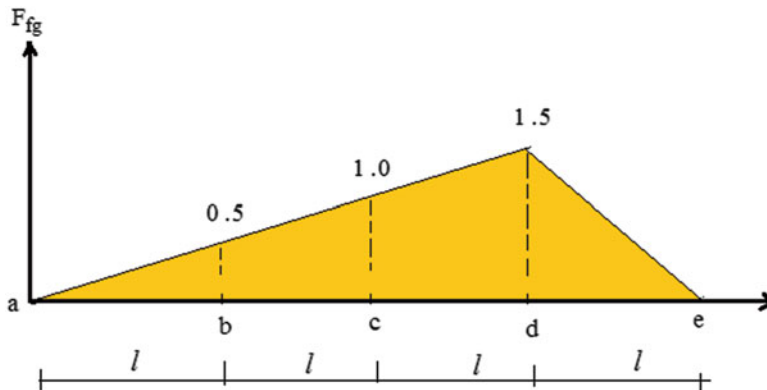
The complete set of member force results are listed in the following Table. One uses this table in two ways. Firstly, scanning down a column shows the member which is most highly stressed by the loading acting at the position corresponding to the column. Scanning across a row identifies the loading which has the maximum contribution to the member force.

Force influence table			
Member	$P_2 = 1$	$P_3 = 1$	$P_4 = 1$
ab	-1.68	-1.12	-0.56
bc	-0.56	-1.12	-0.56
cd	-0.56	-1.12	-0.56
de	-0.56	-1.12	-1.68
ef	0.5	1.0	1.5
fg	0.5	1.0	1.5
gh	1.5	1.0	0.5
ha	1.5	1.0	0.5
bh	0	0	0
cg	0.5	0	0.5
df	0	0	0
bg	-1.12	0	0
gd	0	0	-1.12
$R_{ay}$	0.25	0.5	0.75
$R_e$	0.75	0.5	0.25

**Part (ii)** One can interpret the force influence table as representing the complete set of influence lines for the individual members. We use this data to draw influence line for member cd and fg.



**Fig. E2.20c** Influence line for member cd



**Fig. E2.20d** Influence line for member fg

**Part (iii)** By using the force influence table, the corresponding forces in members cd and fg and reactions are determined as follows:

$$F_{cd} = 15(-0.56) + 15(-1.12) + 15(-0.56) = -33.6 \quad \therefore F_{cd} = 33.6 \text{ kN compression}$$

$$F_{fg} = 15(0.5) + 15(1.0) + 15(1.5) = 45 \quad \therefore F_{fg} = 45 \text{ kN tension}$$

$$R_{ay} = 7.5 + 15(0.25) + 15(0.5) + 15(0.75) = 30 \quad \therefore R_{ay} = 30 \text{ kN } \uparrow$$

$$R_e = 7.5 + 15(0.75) + 15(0.5) + 15(0.25) = 30 \quad \therefore R_e = 30 \text{ kN } \uparrow$$

## 2.5 Analysis of Three-Dimensional Trusses

### 2.5.1 Introduction

Most structural systems such as highway bridges and roof systems can be considered to be composed of a set of planar trusses. However, there are exceptions, such as towers and domed structures, which cannot be decomposed into planar components and consequently one needs to deal with three-dimensional combinations of members. These structural types are called space structures.

The basic unit for a 3-D space truss is the tetrahedron, a geometrical object composed of six members that form four triangular faces. Figure 2.30 illustrates this object. We form a 3-D structure by attaching members to existing nodes. Each new node requires three members. Provided that the structure is suitably supported with respect to rigid body motion, the displacements that the structure experiences when loaded are due only to deformation of the members.

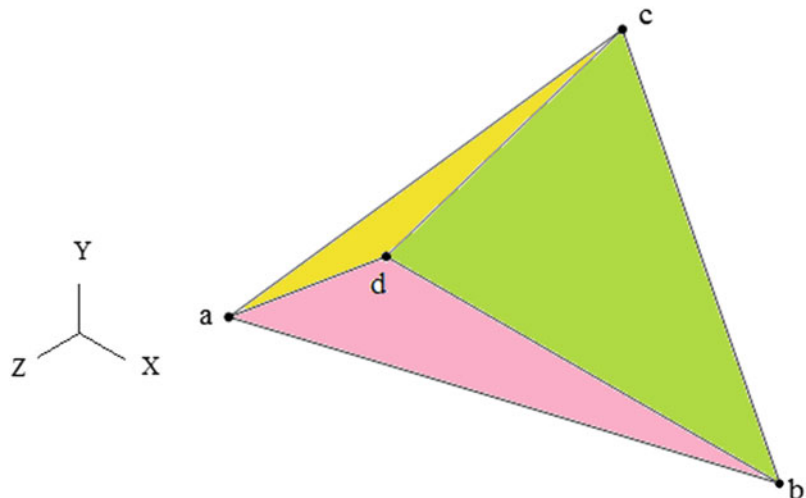
Space truss structures are used for vertical structures such as towers and long-span horizontal structures covering areas such as exhibition halls and covered stadiums. They usually are much more complex than simple plane trusses, and therefore more difficult to analyze.

The equilibrium analysis for three-dimensional trusses is similar to that for planar structures except that now there are three force equilibrium equations per node instead of two equations. One can apply either the method of joints or the method of sections. Manual analysis techniques are difficult to apply for large-scale space structures, and one usually resorts to computer-based analysis procedures. Our immediate objectives in this section are to discuss how a space structure needs to be restrained in order to prevent rigid body motion and to illustrate some manual calculations using the methods of joints. We present a computer-based method in the next section.

### 2.5.2 Restraining Rigid Body Motion

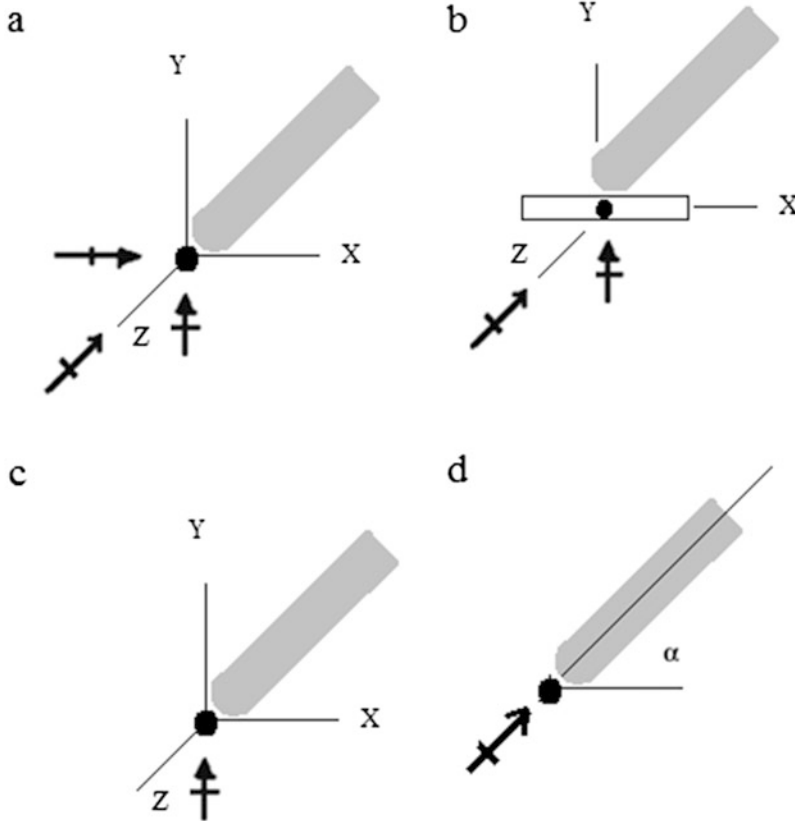
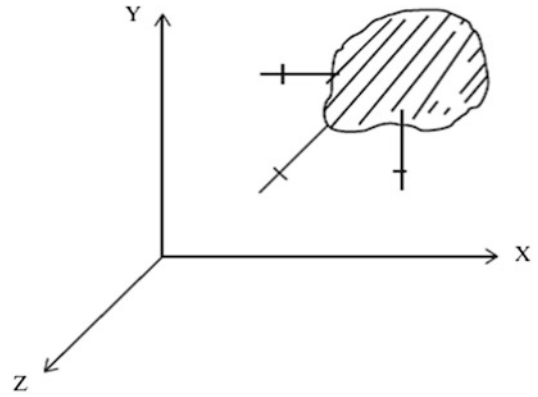
A rigid three-dimensional body requires six motion constraints to be fully constrained; three with respect to translation, and three with respect to rotation. We select an orthogonal reference frame having directions  $X$ ,  $Y$ , and  $Z$ . Preventing translation is achieved by constraining motion in the  $X$ ,  $Y$ , and  $Z$  directions as illustrated in Fig. 2.31. Even when suitably restrained against translation, the body

**Fig. 2.30** Tetrahedron units for 3-D trusses





**Fig. 2.31** Restraints for a 3-D rigid object



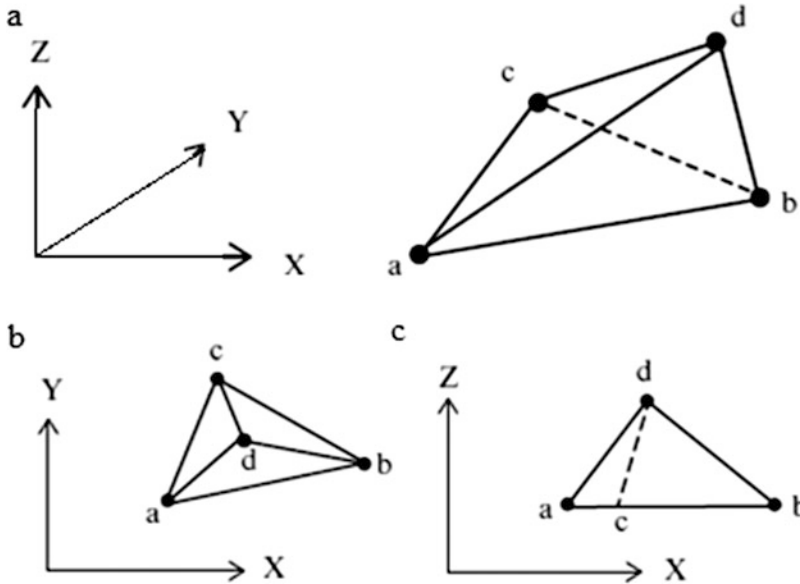
**Fig. 2.32** Types of supports for space trusses. (a) Hinge joint. (b) Slotted roller. (c) Roller. (d) Rigid link

can rotate and we need to provide additional constraints which eliminate rotation about the  $X$ ,  $Y$ , and  $Z$  axes. To prevent rotation about an axis, say the  $X$  axis, one applies a translational constraint in a direction which does not pass through  $X$ . This rule is used to select three additional constraint directions, making a total of six restraints. If one introduces more than six restraints, the structure is said to be statically indeterminate with respect to the reactions. Various examples illustrating the selection of restraints are listed below (Fig. 2.32).

*Example 2.21* Various Restraint Schemes

**Given:** The 3-D truss shown in Fig. E2.21a, b.

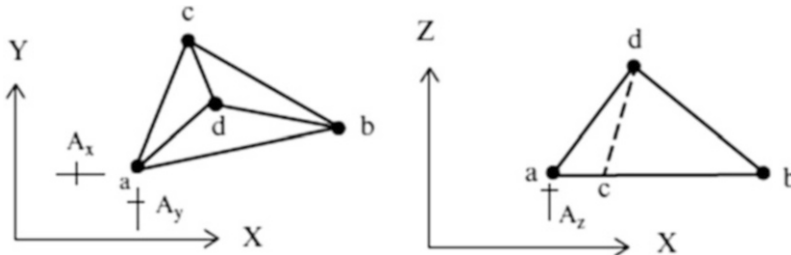
**Determine:** Possible restraint schemes.



**Fig. E2.21** (a) 3-D truss. (b)  $x - y$  plan view. (c)  $x - z$  plan view

**Solution:** The preferred way of displaying 3-D objects is to work with projections on the  $X - Y$  and  $X - Z$  planes, referred to as the “plan” and “elevation” views. The projections corresponding to the object defined in Fig. E2.21a are shown in Fig. E2.21b, c.

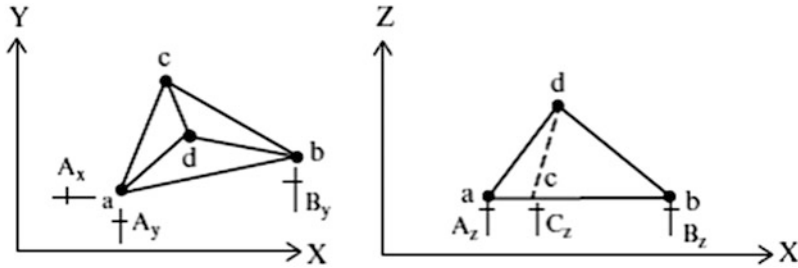
The choice of restraints is not unique. One can employ a 3-D hinge which provides full restraint against translation, or roller type supports which provide restraint against motion in a particular direction. Suppose we place a 3-D hinge at joint a. Then, a is “fixed” with respect to translation in the  $X$ ,  $Y$ , and  $Z$  directions.



**Fig. E2.21d** 3-D hinge at a

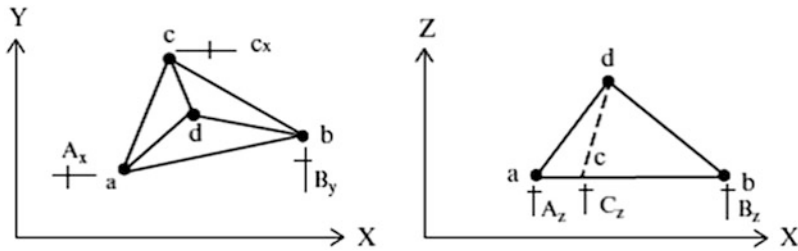
With these restraints, the body can still rotate about either line a-b or line a-c, or a line parallel to the  $Z$  axis through a. The first two modes are controlled with  $Z$  restraints applied at b and c. The third

mode is controlled with either an  $X$  or  $Y$  restraint applied at either  $b$  or  $c$ . Figure E2.21e shows the complete set of displacement restraints chosen.

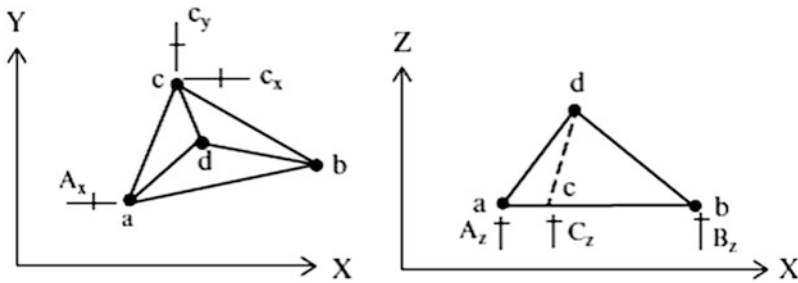


**Fig. E2.21e** Complete set of restraints

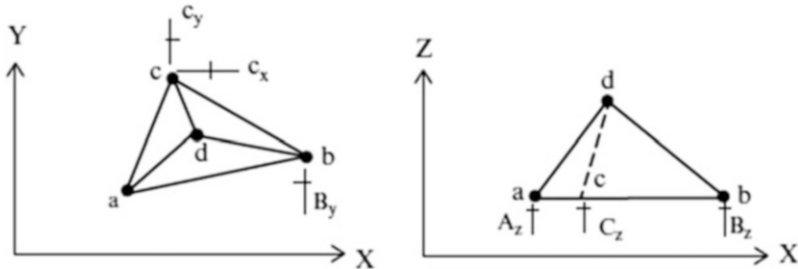
Other possible restraint schemes are shown in Figs. E2.21f, E2.21g, and E2.21h. Our strategy is to first restrain translation and then deal with the rotation modes.



**Fig. E2.21f** Alternative restraint scheme #1



**Fig. E2.21g** Alternative restraint scheme #2



**Fig. E2.21h** Alternative restraint scheme #3

### 2.5.3 Static Determinacy

The approach we followed in Sect. 2.2.2 for 2-D Plane trusses is also applicable for 3-D trusses. One just has to include the additional variables associated with shifting from two to three dimensions. Each member of a truss structure has a single force measure, the magnitude of the axial force. However, for 3-D trusses, there are three equilibrium equations per node instead of two for a plane truss. Defining  $m$  as the number of members,  $r$  as the number of reactions, and  $j$  as the number of nodes, it follows that the number of force unknowns and the number of force equilibrium equations available are

$$\text{Force unknowns} = m + r$$

$$\text{Force equilibrium equations} = 3j$$

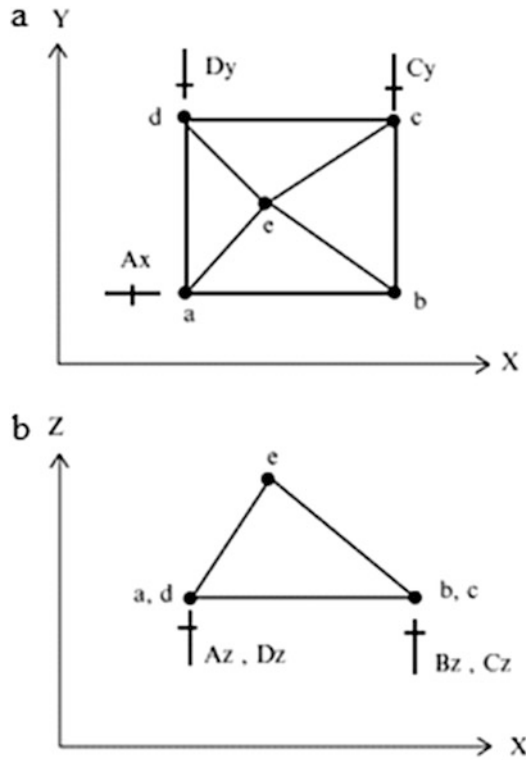
The structure is statically determinate when  $m + r = 3j$ . If  $m + r > 3j$ , there are more force unknowns than available equilibrium equations and the structure is designated as statically indeterminate. Lastly, if  $m + r < 3j$ , there are less force unknowns than required to withstand an arbitrary nodal loading, and the structure is unstable, i.e., it is incapable of supporting an arbitrarily small loading.

$$m + r \begin{cases} < 3J & \text{unstable} \\ = 3J & \text{determinate} \\ > 3J & \text{indeterminate} \end{cases}$$

In addition to these criteria, the structure must be suitably restrained against rigid body motion.

#### *Example 2.22* A Stable Determinate Truss

**Given:** The truss defined in Fig. E2.22a, b.



**Fig. E2.22** 3-D truss. (a)  $x - y$  plan view. (b)  $x - z$  plan view

**Determine:** The stability

**Solution:** For the structure shown above, there are eight members, seven reactions, and five joints.

$$m = 8 \quad r = 7 \quad j = 5$$

$$m + r = 3j$$

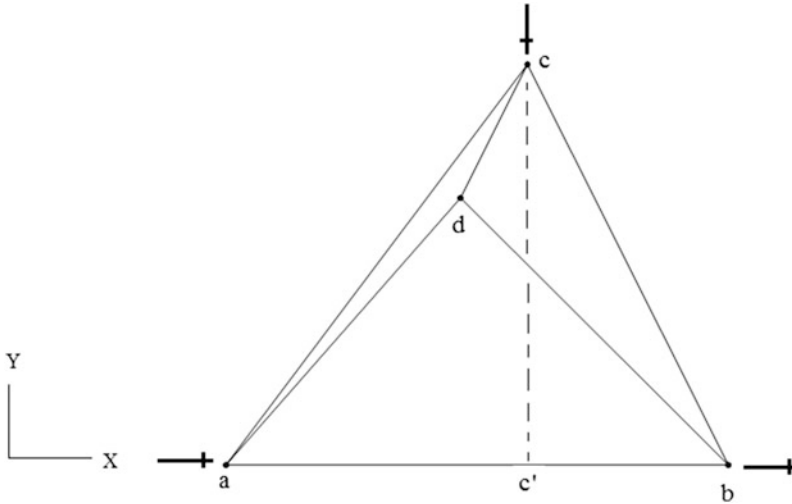
The structure is initially stable.

### Example 2.23 An Unstable Structure

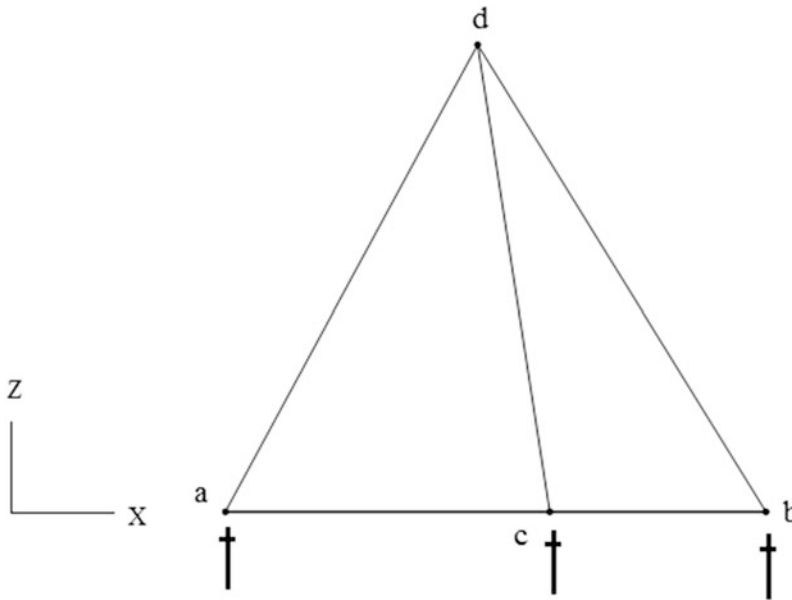
**Given:** The truss defined in Figs. E2.23a and E2.23b

**Determine:** The stability

**Solution:** The number of force unknowns is equal to the number of available force equilibrium equations but the structure has a fundamental flaw. The translation restraints in the  $X - Y$  plane are concurrent, i.e., they intersect at a common point,  $c'$ , shown in Fig. E2.23a. As a result, the structure cannot resist rotation about a  $Z$  axis through  $c'$ .



**Fig. E2.23a**  $x - y$  plan view

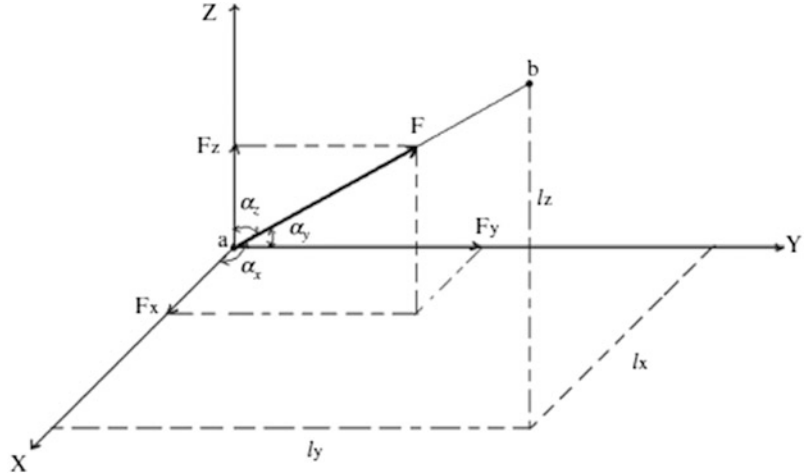


**Fig. E2.23b**  $x - z$  plan view

### 2.5.4 Method of Joints for 3-D Trusses

Each member of a space truss is assumed to be pinned at its ends to nodes in such a way that there is no bending in the member, only an axial force whose direction coincides with the centroidal axis. The direction of the force is determined by the geometry of the member, so one needs only to determine the magnitude. We find these quantities using force equilibrium equations. Our overall strategy is to first determine the reactions with the global equilibrium conditions. Once the reactions are known, we

**Fig. 2.33** Resolution of a force into its components



range over the nodes and establish the nodal force equilibrium equations for each node. This process is similar to the method of joints for Planar Trusses except that now there are three equilibrium equations per node. The member forces are computed by solving the set of nodal force equilibrium equations.

Consider the force vector shown in Fig. 2.33. Since the force vector orientation coincides with the direction of the centroidal axis for member  $ab$ , the force components are related to the geometric projections of the member length. We resolve the force vector into  $X$ ,  $Y$ , and  $Z$  components, and label the components as  $F_x$ ,  $F_y$ , and  $F_z$ . Noting the commonality of directions, the force components are related to the force magnitude and geometric projections by

$$\begin{aligned}\frac{F_x}{F} &= \frac{\ell_x}{\ell} = \cos \alpha_x = \beta_x \\ \frac{F_y}{F} &= \frac{\ell_y}{\ell} = \cos \alpha_y = \beta_y \\ \frac{F_z}{F} &= \frac{\ell_z}{\ell} = \cos \alpha_z = \beta_z\end{aligned}\quad (2.16)$$

The coefficients,  $\beta_x$ ,  $\beta_y$ , and  $\beta_z$ , are called direction cosines. Given the coordinates of the nodes at each end ( $a$ ,  $b$ ), one determines the projection and length using

$$\begin{aligned}l_x &= x_b - x_a \\ l_y &= y_b - y_a \\ l_z &= z_b - z_a \\ l &= \sqrt{l_x^2 + l_y^2 + l_z^2}\end{aligned}\quad (2.17)$$

We are assuming the positive sense of the member is from node  $a$  toward node  $b$ . These relationships allow one to carry out the equilibrium analysis working initially with the components and then evaluate the force magnitude.

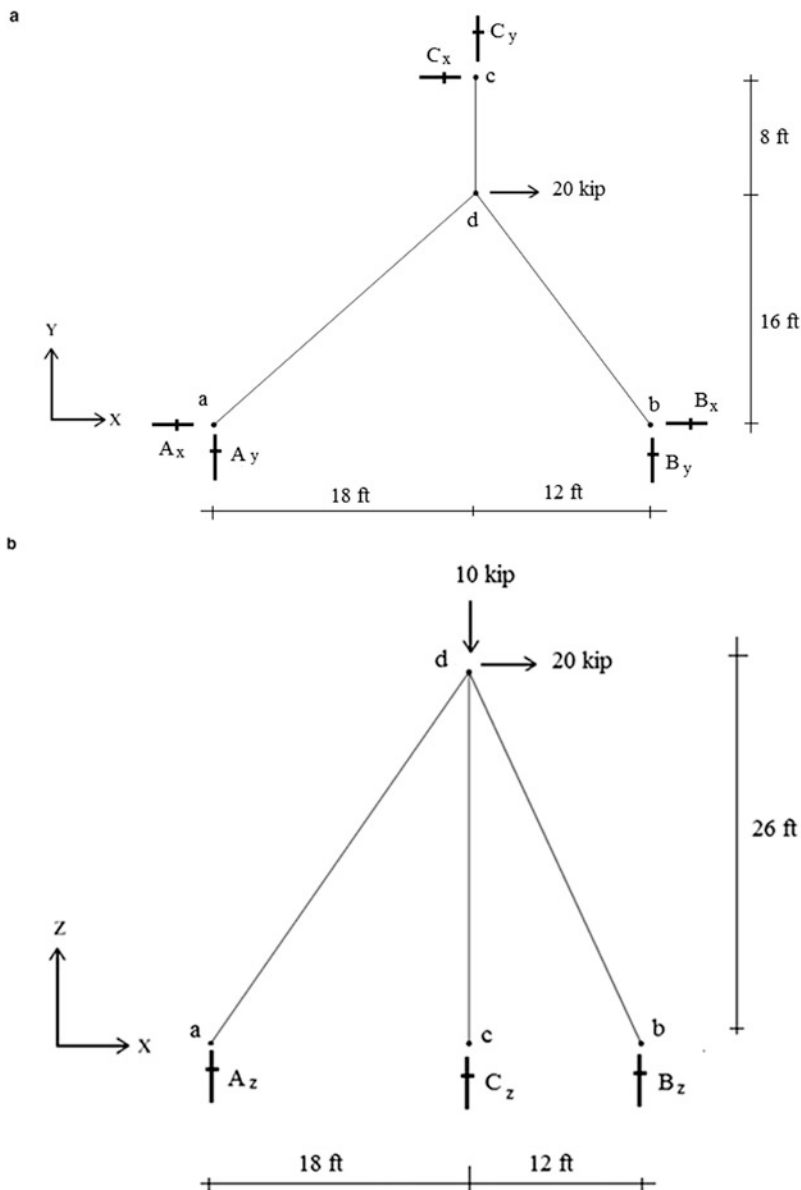
$$F = \sqrt{F_x^2 + F_y^2 + F_z^2}\quad (2.18)$$

We illustrate the analysis process with the following examples. There are many ways to carry out the analysis. Our approach here is based primarily on trying to avoid solving sets of simultaneous equations relating the force magnitudes. However, there are cases where this strategy is not possible.

*Example 2.24* Analysis of a Tripod Structure

**Given:** The tripod structure shown in Fig. E2.24a, b. The supports at a, b, and c are fully restrained against translation with 3-D hinges.

**Determine:** The force in each member.



**Fig. E2.24** Tripod geometry and supports. (a)  $x - y$  plan view. (b)  $x - z$  plan view



**Solution:** There are three reactions per support, making a total of nine reaction unknowns. Adding the three unknown member forces raises the total number of force unknowns to 12. Each joint has three force equilibrium equations and there are four joints, so the structure is statically determinate.

The first step is to determine the direction cosines for the members. This data is listed in Table E2.24.1 below.

**Table E2.24.1**

Member	$l_x$	$l_y$	$l_z$	$l$	$\beta_x$	$\beta_y$	$\beta_z$
ad	18	16	26	35.4	0.508	0.452	0.734
bd	12	16	26	32.8	0.366	0.488	0.793
cd	0	8	26	27.2	0.000	0.294	0.956

We first determine the  $Z$  reaction at c by enforcing moment equilibrium about an  $X$  axis through a-b.

$$10(16) - C_z(24) = 0$$

$$C_z = 6.67 \text{ kip } \uparrow$$

The reaction force at c is equal to the  $z$  component of the force in member cd. Therefore,

$$F_{cd,z} = -C_z = -6.67 \Rightarrow F_{cd} = \frac{6.67}{0.956} = 6.98 \text{ (compression)}$$

Then,

$$C_x = F_{cd,x} = 0$$

and

$$C_y = -F_{cd,y} = -6.67(0.294) = 2.05 \text{ kip } \downarrow$$

We determine the  $Y$  reaction at b by summing moments about the  $Z$  axis through a.

$$20(16) + 2.05(18) = 30B_y$$

$$B_y = 11.90 \text{ kip } \uparrow$$

Then,

$$F_{bd,y} = -B_y = -11.90 \Rightarrow F_{bd} = \frac{11.9}{0.488} = 24.39 \text{ (compression)}$$

Therefore,

$$B_x = -F_{bd,x} = 8.92 \leftarrow$$

and

$$F_{bd,z} = 0.793(24.39) = 19.33$$

$$B_z = -F_{bd,z} = 19.33 \uparrow$$

Lastly, we sum forces in the Z direction and determine the reaction at A.

$$B_z + C_z + A_z = 10$$

$$A_z = 10 - 6.67 - 19.33 = -16$$

$$A_z = 16 \text{ kip } \downarrow$$

Then,

$$F_{ad,z} = -A_z = 16 \Rightarrow F_{ad} = \frac{16}{0.734} = 21.8 \text{ (tension)}$$

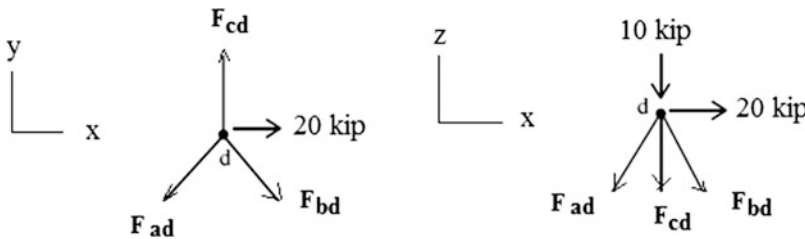
and

$$A_x = -F_{ad,x} = 0.508(21.8) = 11.07 \text{ kip } \leftarrow$$

$$A_y = -F_{ad,y} = 0.452(21.8) = 9.85 \text{ kip } \downarrow$$

We were able to find the member forces working at any time with no more than a single unknown. A more direct but also more computationally intensive approach would be to work with joint d and generate the three force equilibrium equations expressed in terms of the magnitudes of the three-member forces. In this approach, we use the direction cosine information listed in Table E2.24.1 and assume all the member forces are tension. Noting (2.16), the corresponding force equilibrium equations are

$$\text{Joint d} \begin{cases} \sum F_x = 0 & 20 + 0.366F_{bd} - 0.508F_{ad} = 0 \\ \sum F_y = 0 & 0.294F_{cd} - 0.452F_{ad} - 0.488F_{bd} = 0 \\ \sum F_z = 0 & 10 + 0.734F_{ad} + 0.793F_{bd} + 0.956F_{cd} = 0 \end{cases} \Rightarrow \begin{cases} F_{ad} = 21.81 \text{ kip} \\ F_{bd} = -24.39 \text{ kip} \\ F_{cd} = -6.97 \text{ kip} \end{cases}$$



**Fig. E2.24c** Joint d

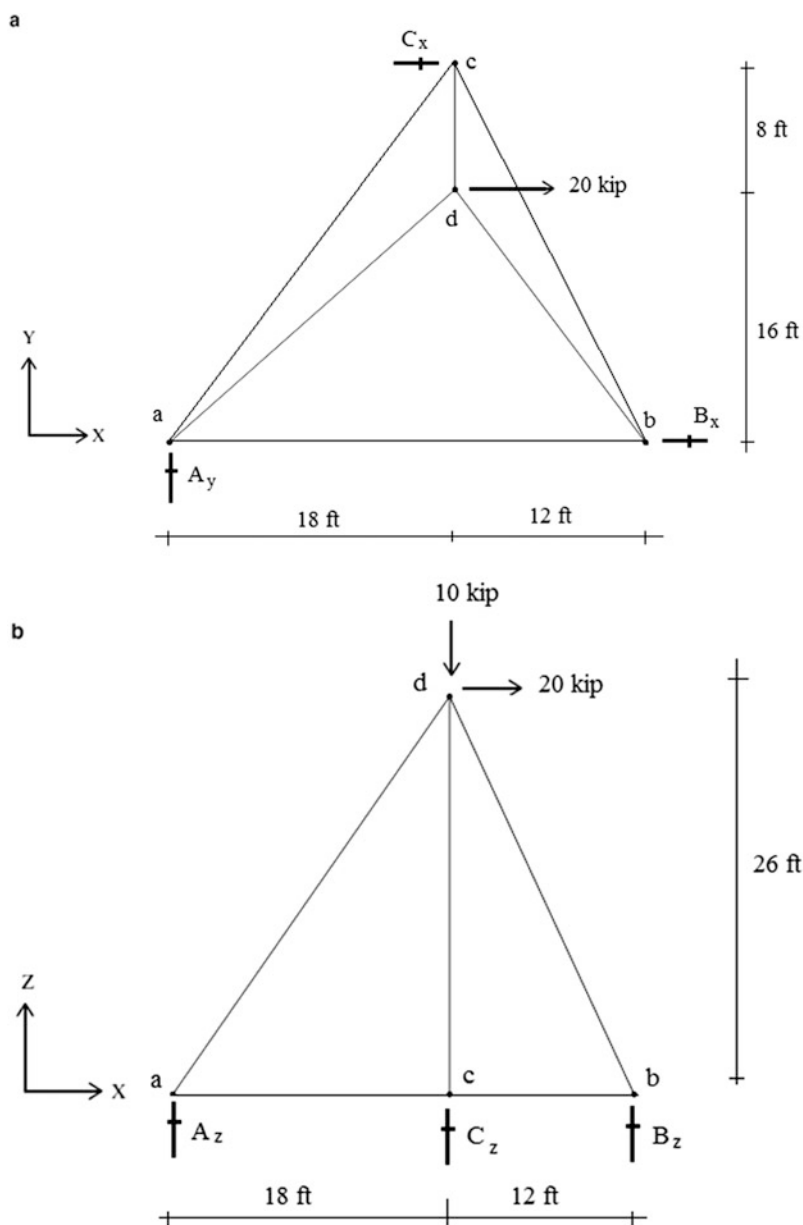
**Table E2.24.2**

Member	Force	Force <sub>x</sub>	Force <sub>y</sub>	Force <sub>z</sub>	
ad	21.81(tension)	11.07	9.85	16.00	
bd	24.39 (compression)	8.93	11.90	19.33	
cd	6.97(compression)	0.00	2.05	6.67	

*Example 2.25 Analysis of a Tetrahedron*

**Given:** The tetrahedron structure defined in Fig. E2.25a, b.

**Determine:** The member forces.



**Fig. E2.25** Tetrahedron geometry and support. (a)  $x-y$  plan view. (b)  $x-z$  plan view

**Solution:** There are six reactions (three  $Z$  forces, two  $X$  forces, and one  $Y$  force), six members, and four joints. The determinacy criteria,

$$3j = m + r \rightarrow 3(4) = 6 + 6$$

is satisfied, so the structure is statically determinate.

We first determine the direction cosines for the members listed in Table E2.25.1

**Table E2.25.1**

Member	$l_x$	$l_y$	$l_z$	$l$	$\beta_x$	$\beta_y$	$\beta_z$
ac	18	24	0	30.0	0.600	0.800	0.000
ab	30	0	0	30.0	1.000	0.000	0.000
bc	12	24	0	26.8	0.447	0.895	0.000
ad	18	16	26	25.4	0.508	0.452	0.734
cd	0	8	26	27.2	0.000	0.290	0.960
bd	12	16	26	32.8	0.366	0.488	0.793

Next, we determine the  $Z$  reactions at a, b, and c.

$$\begin{aligned}\sum M_x \quad \text{at } a &= 0 \\ 10(16) &= 24C_z \\ C_z &= 6.67 \uparrow\end{aligned}$$

$$\begin{aligned}\sum M_y \quad \text{at } a &= 0 \\ 20(26) + 10(18) &= 6.67(18) + 30B_z \\ B_z &= 19.33 \uparrow\end{aligned}$$

$$\begin{aligned}\sum F_z &= 0 \uparrow^+ \\ 19.33 + 6.67 + A_z &= 10 \\ A_z &= 16 \downarrow\end{aligned}$$

The  $Y$  component at a is determined with:  $\sum F_y = 0 \therefore A_y = 0$ .  
Then, we enforce  $\sum M_z = 0$  with respect to a  $Z$  axis through a.

$$\begin{aligned}24C_x &= 20(16) \\ C_x &= 13.34 \leftarrow\end{aligned}$$

Lastly, we evaluate  $B_x$

$$\begin{aligned}\sum F_x &= 0 \quad C_x + B_x = 20 \\ B_x &= 6.66 \leftarrow\end{aligned}$$

With the reactions known, each of the joints involves only three unknowns, and we can start with any joint. It is most convenient to start with joint b and enforce  $Z$  equilibrium.

$$\sum F_z = 0 \quad F_{bd,z} = B_z = 19.33$$

Then,

$$F_{bd} = -24.4 \text{ (compression)}$$

We find  $F_{cb}$  by summing  $Y$  forces at b.

$$\begin{aligned}\sum F_y &= 0 + F_{cb,y} + F_{bd,y} = 0 \\ F_{cb,y} &= 11.91 \\ F_{cb} &= +13.3 \text{ (tension)}\end{aligned}$$

Then, we find  $F_{ab}$  by summing  $X$  forces at b.

$$\begin{aligned}F_{ab} + F_{cb,x} + B_x - F_{bd,x} &= 0 \\ F_{ab} &= -3.69 \text{ (compression)}\end{aligned}$$

We move on to joint c. Summing  $Z$  forces yields  $F_{cd}$

$$\begin{aligned}\sum F_z &= 0 \quad C_z + F_{cd,z} = 0 \\ F_{cd,z} &= -6.67 \\ F_{cd} &= -6.95 \text{ (compression)}\end{aligned}$$

Summing  $X$  (or  $Y$ ) forces leads to  $F_{ab}$

$$\begin{aligned}\sum F_x &= 0 \quad B_x + F_{ab,x} - F_{cb,x} = 0 \\ F_{ab,x} &= -13.34 + 5.94 = -7.40 \\ F_{ab} &= -12.33 \text{ (compression)}\end{aligned}$$

The last step is to determine  $F_{ad}$  by enforcing  $Z$  force equilibrium at a.

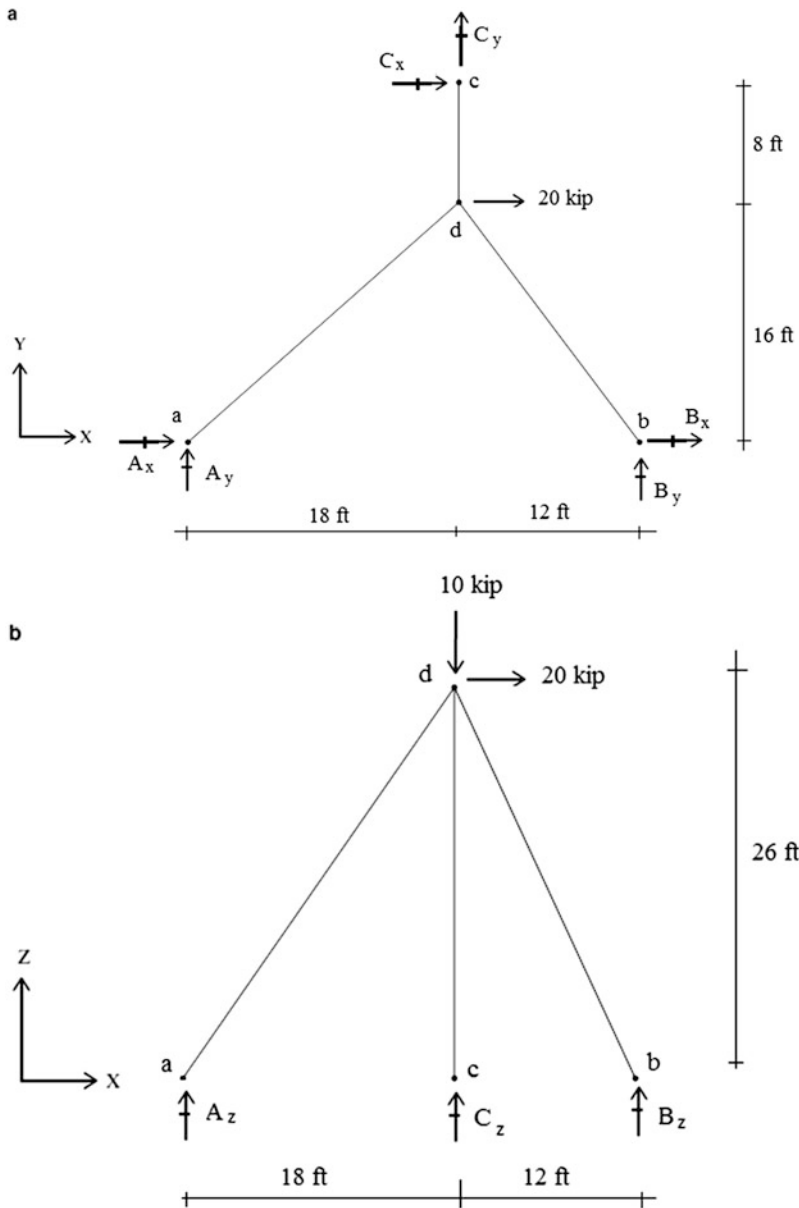
$$\begin{aligned}\sum F_z &= 0 \quad F_{ad,z} + A_z = 0 \\ F_{ad,z} &= 16 \\ F_{ad} &= +21.8 \text{ (tension)}\end{aligned}$$

We could have solved this problem by establishing the three force equilibrium equations for joint d, and finding  $F_{ad}$ ,  $F_{cd}$ ,  $F_{bd}$ . Once the reactions are known, we could set up the equations for joints c and b, and solve for the member forces  $F_{ac}$ ,  $F_{bc}$ , and  $F_{ab}$ . We followed a different approach to illustrate how one applies the method of joints in a selective manner to a 3-D space truss.

### Example 2.26 Displacement Computation—3-D Truss

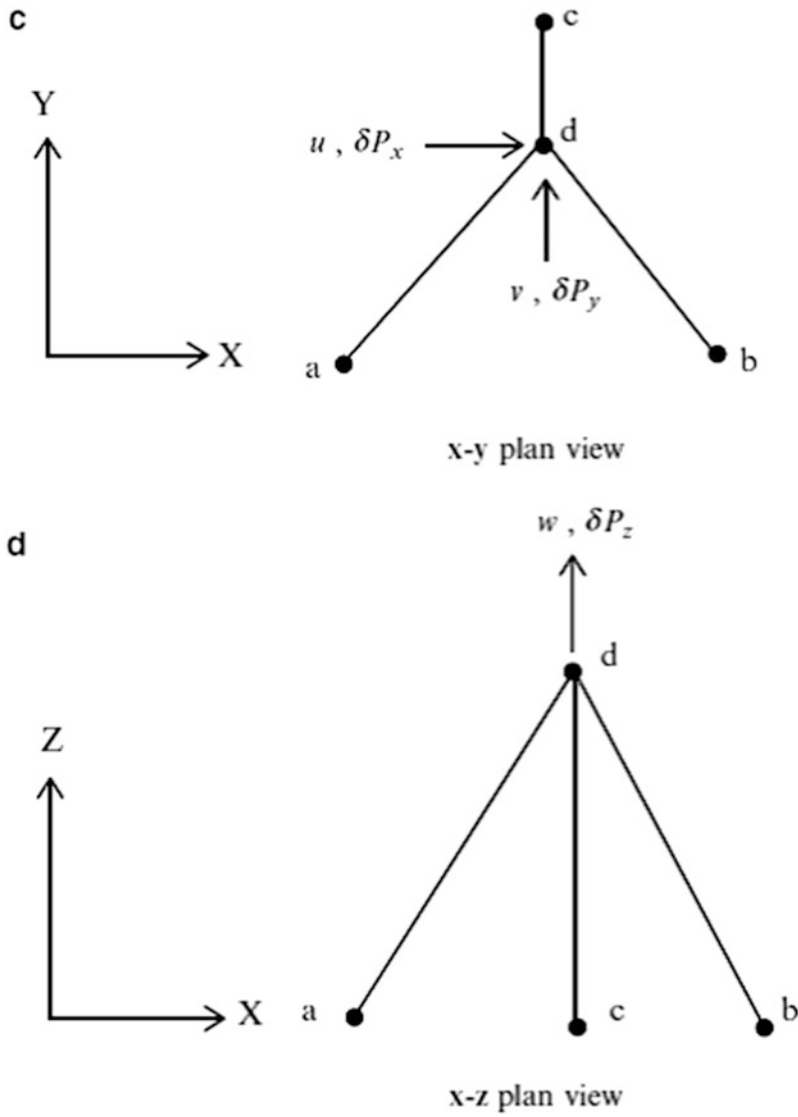
**Given:** The tripod structure defined in Fig. E2.26a, b.

**Determine:** The displacements at joint d due to loading shown and a temperature increase of  $\Delta T = 80^\circ\text{F}$  for all the members. Assume  $A = 2.0 \text{ in.}^2$ ,  $E = 29 \times 10^3 \text{ ksi}$ , and  $\alpha = 6.6 \times 10^{-6}/^\circ\text{F}$ .



**Fig. E2.26** Tripod geometry and supports. (a) *x* – *y* plan view. (b) *x* – *z* plan view

**Solution:** We apply the virtual loads  $\delta P_x$ ,  $\delta P_y$ , and  $\delta P_z$ , (see Fig. E2.26c, d) at joint *d* and determine the corresponding virtual member forces,  $\delta F_u$ ,  $\delta F_v$ , and  $\delta F_w$ . The individual displacement components due to loading are determined with:

**Fig. E2.26c** Virtual forces

$$u \delta P_x = \sum \left( \frac{FL}{AE} \right) \delta F_u$$

$$v \delta P_y = \sum \left( \frac{FL}{AE} \right) \delta F_v$$

$$w \delta P_z = \sum \left( \frac{FL}{AE} \right) \delta F_w$$

For temperature change, we use

$$\begin{aligned} u \delta P_x &= \sum (\alpha \Delta T L) \delta F_u \\ v \delta P_y &= \sum (\alpha \Delta T L) \delta F_v \\ w \delta P_z &= \sum (\alpha \Delta T L) \delta F_w \end{aligned}$$

The relevant data needed to evaluate displacements is listed in Table E2.26.1. Note that we need to shift length units over to inches when computing  $(\frac{FL}{AE})$  and  $(\alpha \Delta T L)$ .

We use the member forces determined in Example 2.24.

$$\text{Joint d} \begin{cases} \sum F_x = 0 & 20 + 0.366F_{bd} - 0.508F_{ad} = 0 \\ \sum F_y = 0 & 0.294F_{cd} - 0.452F_{ad} - 0.488F_{bd} = 0 \\ \sum F_z = 0 & 10 + 0.734F_{ad} + 0.793F_{bd} + 0.956F_{cd} = 0 \end{cases} \Rightarrow \begin{cases} F_{ad} = 21.81 \text{ kip} \\ F_{bd} = -24.39 \text{ kip} \\ F_{cd} = -6.97 \text{ kip} \end{cases}$$

For  $\delta P_u = 1$ :

$$\text{Joint d} \begin{cases} \sum F_x = 0 & 0.366\delta F_{u bd} - 0.508\delta F_{u ad} = -1 \\ \sum F_y = 0 & 0.294\delta F_{u cd} - 0.452\delta F_{u ad} - 0.488\delta F_{u bd} = 0 \\ \sum F_z = 0 & 0.734\delta F_{u ad} + 0.793\delta F_{u bd} + 0.956\delta F_{u cd} = 0 \end{cases} \Rightarrow \begin{cases} \delta F_{u ad} = 1.18 \\ \delta F_{u bd} = -1.09 \\ \delta F_{u cd} = 0 \end{cases}$$

For  $\delta P_v = 1$

$$\text{Joint d} \begin{cases} \sum F_x = 0 & 0.366\delta F_{v bd} - 0.508\delta F_{v ad} = 0 \\ \sum F_y = 0 & 0.294\delta F_{v cd} - 0.452\delta F_{v ad} - 0.488\delta F_{v bd} = -1 \\ \sum F_z = 0 & 0.734\delta F_{v ad} + 0.793\delta F_{v bd} + 0.956\delta F_{v cd} = 0 \end{cases} \Rightarrow \begin{cases} \delta F_{v ad} = 0.59 \\ \delta F_{v bd} = 0.82 \\ \delta F_{v cd} = -1.13 \end{cases}$$

For  $\delta P_w = 1$ :

$$\text{Joint d} \begin{cases} \sum F_x = 0 & 0.366\delta F_{w bd} - 0.508\delta F_{w ad} = 0 \\ \sum F_y = 0 & 0.294\delta F_{w cd} - 0.452\delta F_{w ad} - 0.488\delta F_{w bd} = 0 \\ \sum F_z = 0 & 0.734\delta F_{w ad} + 0.793\delta F_{w bd} + 0.956\delta F_{w cd} = -1.0 \end{cases} \Rightarrow \begin{cases} \delta F_{w ad} = 0.18 \\ \delta F_{w bd} = 0.25 \\ \delta F_{w cd} = 0.70 \end{cases}$$

The relevant data needed to evaluate displacements is listed in Table E2.26.1.

**Table E2.26.1**

				For $\delta P = 1.0$				
Member	$l$ (ft)	$A$ (in. <sup>2</sup> )	$F$	$\delta F_u$	$\delta F_v$	$\delta F_w$	$\frac{FL}{AE}$ (in.)	$\alpha L \Delta T$ (in)
ad	35.4	2.0	21.81	1.18	0.59	0.18	0.160	0.224
bd	32.8	2.0	-24.39	-1.09	0.82	0.25	-0.165	0.208
cd	27.2	2.0	-6.97	0.00	-1.13	0.70	-0.039	0.172



The displacements due to loads are

$$u_{\text{load}} = \sum \left( \frac{FL}{AE} \right) \delta F_u = (0.160)(1.18) + (-0.165)(-1.09) = 0.369 \text{ in.}$$

$$v_{\text{load}} = \sum \left( \frac{FL}{AE} \right) \delta F_v = (0.160)(0.59) + (-0.165)(0.82) + (-0.039)(-1.13) = 0.003 \text{ in.}$$

$$w_{\text{load}} = \sum \left( \frac{FL}{AE} \right) \delta F_w = (0.160)(0.18) + (-0.165)(0.25) + (-0.039)(0.70) = -0.039 \text{ in.}$$

A 80 °F temperature increase produces the following displacements:

$$u_{\text{temp}} = \sum (\alpha \Delta T L) \delta F_u = (0.224)(1.18) + (0.208)(-1.09) = 0.038 \text{ in.}$$

$$v_{\text{temp}} = \sum (\alpha \Delta T L) \delta F_v = (0.224)(0.59) + (0.208)(0.82) + (0.172)(-1.13) = 0.108 \text{ in.}$$

$$w_{\text{temp}} = \sum (\alpha \Delta T L) \delta F_w = (0.224)(0.18) + (0.208)(0.25) + (0.172)(0.70) = 0.213 \text{ in.}$$

The total displacements are

$$u_{(\text{load}+\text{temp})} = 0.369 + 0.038 = 0.407 \text{ in.}$$

$$v_{(\text{load}+\text{temp})} = 0.003 + 0.108 = 0.111 \text{ in.}$$

$$w_{(\text{load}+\text{temp})} = -0.039 + 0.213 = 0.174 \text{ in.}$$

## 2.6 Matrix Formulation: Equilibrium Analysis of Statically Determinate 3-D Trusses

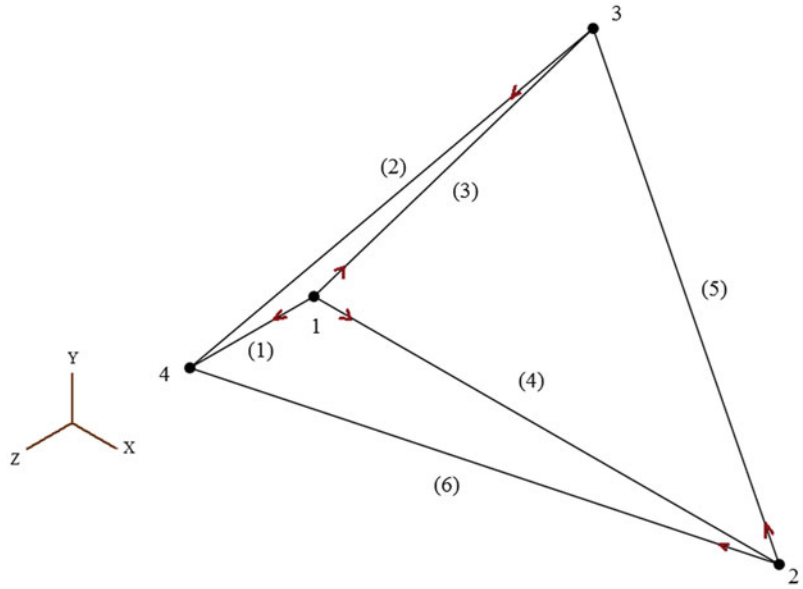
Manual techniques are easy to apply for simple geometries, but become more difficult with increasing geometric complexity. The equilibrium analysis approaches described in the previous sections can be formulated as a sequence of matrix operations which can be readily automated for computer-based analysis. In what follows, we describe one approach for the equilibrium analysis of statically determinate 3-D trusses. We present a more general matrix formulation later in Chap. 12.

### 2.6.1 Notation

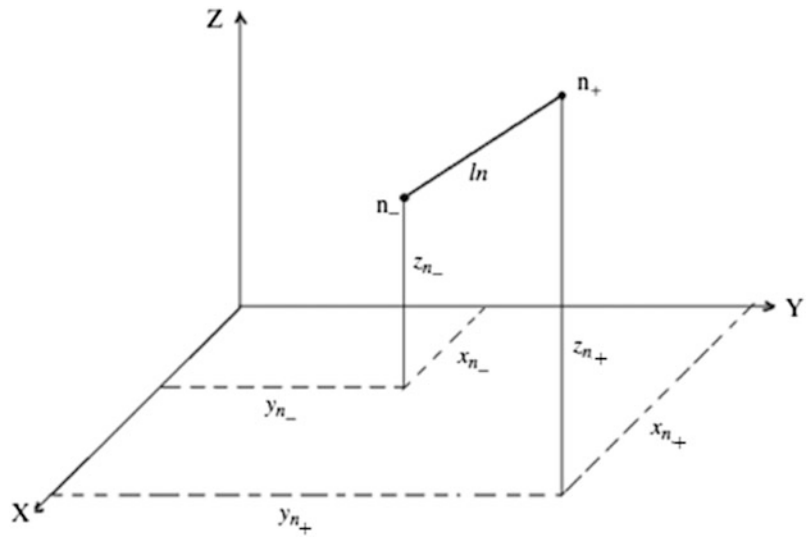
A truss is an assembly of nodes that are interconnected with members. It is convenient to define the geometry with respect to a global Cartesian coordinate system,  $XYZ$ , and number the nodes and members sequentially. Figure 2.34 illustrates this scheme. The structure has four nodes and six members.

We assume a positive sense for each member and define the direction cosines consistent with the assumed sense. The positive and negative nodes for member  $n$  are denoted as  $n_+$  and  $n_-$ . Noting (2.16) and (2.17), the direction cosines for member  $n$  are determined using

**Fig. 2.34** Numbering scheme



**Fig. 2.35** Geometry-member  $n$



$$\begin{aligned}\frac{x_{n+} - x_{n-}}{l_n} &= \beta_{n_x} \\ \frac{y_{n+} - y_{n-}}{l_n} &= \beta_{n_y} \\ \frac{z_{n+} - z_{n-}}{l_n} &= \beta_{n_z}\end{aligned}\tag{2.19}$$

It is convenient to introduce matrix notation at this point (Fig. 2.35). We define the nodal coordinate matrix for node  $j$  as

$$\mathbf{x}_j = \begin{Bmatrix} x_j \\ y_j \\ z_j \end{Bmatrix} \quad (2.20)$$

and the direction cosine matrix for member  $n$  as

$$\boldsymbol{\beta}_n = \begin{Bmatrix} \beta_{nx} \\ \beta_{ny} \\ \beta_{nz} \end{Bmatrix} \quad (2.21)$$

With this notation, the matrix form of (2.19) is

$$\boldsymbol{\beta}_n = \frac{1}{l_n} (\mathbf{x}_{n+} - \mathbf{x}_{n-}) \quad (2.22)$$

where

$$l_n^2 = (\mathbf{x}_{n+} - \mathbf{x}_{n-})^T (\mathbf{x}_{n+} - \mathbf{x}_{n-})$$

### 2.6.2 Member–Node Incidence

The computation of the direction cosines can be automated using the topological data for the members and nodes. This data is represented in tabular form. One lists, for each member, the node numbers for the positive and negative ends of the member. It is commonly referred to as the member–node incidence table. The table corresponding to the structure defined in Fig. 2.34 is listed below. One loops over the members, extracts the nodal coordinates from the global coordinate vector, executes the operation defined by (2.22), and obtains the member direction cosine matrix,  $\boldsymbol{\beta}$ .

Member	Negative node	Positive node
(1)	1	4
(2)	3	4
(3)	1	3
(4)	1	2
(5)	2	3
(6)	2	4

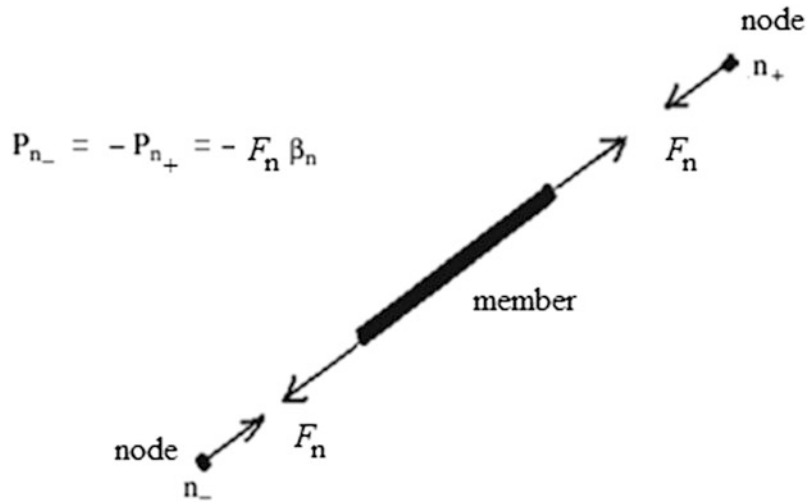
### 2.6.3 Force Equilibrium Equations

The force vector for a member points in the positive direction of the member, i.e., from the negative end toward the positive end. Noting (2.16), the set of Cartesian components for member  $n$  are listed in the matrix,  $\mathbf{P}_n$ , which is related to  $\boldsymbol{\beta}_n$  by

$$\mathbf{P}_n = \begin{Bmatrix} F_x \\ F_y \\ F_z \end{Bmatrix} = F_n \boldsymbol{\beta}_n \quad (2.23)$$

The force components acting on the nodes at the ends of the member are equal to  $\pm \mathbf{P}_n$ . Figure 2.36 illustrates this distribution.

**Fig. 2.36** Member end forces



We generate the set of force equilibrium equations for a node by summing the force matrices acting on the node. Consider node  $l$ . Let  $\mathbf{P}_l$  be the external force matrix for node  $l$ . The matrix equation for node  $l$  involves the member force matrices for those members which are positive incident and negative incident on node  $l$ .

$$\mathbf{P}_l = \sum_{n_+} (F_n \beta_n) + \sum_{n_-} (-F_n \beta_n) \quad (2.24)$$

This step is carried out for each node. Equation (2.24) represents  $i$  scalar equations, where  $i = 2$  for a plane truss and  $i = 3$  for a space truss. We assemble the complete set of equations in partitioned form, taking blocks of  $i$  rows. Assuming  $j$  nodes and  $m$  members, the equations are written as.

$$\mathbf{P}' = \mathbf{B}' \mathbf{F} \quad (2.25)$$

where the dimensions of the global matrices are

$$\mathbf{B}' = (i \text{ times } j) \times m, \quad \mathbf{F} = m \times 1, \quad \mathbf{P}' = (i \text{ times } j) \times 1$$

The algorithms for generating  $\mathbf{P}'$  and  $\mathbf{B}'$  are

$$\left. \begin{array}{l} \text{For member } n \quad (n = 1, 2, \dots, m) \\ \left. \begin{array}{l} +\beta_n \text{ in partitioned row } n_+, \text{ column } n \\ -\beta_n \text{ in partitioned row } n_-, \text{ column } n \end{array} \right\} \text{ of } \mathbf{B}' \end{array} \right\} \quad (2.26)$$

For node  $l \quad (l = 1, 2, \dots, j)$   
 External load  $\mathbf{P}_l$  in partitioned row  $l$  of  $\mathbf{P}'$

These operations can be easily implemented using spreadsheet software. The required size of the spreadsheet is  $i$  times  $j$  rows and  $m + 1$  columns, ( $m$  columns for the member forces and one column for the external nodal loads). Applying (2.26) to the structure shown in Fig. 2.34 and noting the incidence table leads to the following form of  $\mathbf{B}'$ .

	$F_1$	$F_2$	$F_3$	$F_4$	$F_5$	$F_6$	
Node 1	$-\beta_1$		$-\beta_3$	$-\beta_4$			$\left. \begin{array}{c} \uparrow \\ \downarrow \end{array} \right\} i \text{ row}$
Node 2				$+\beta_4$	$-\beta_5$	$-\beta_6$	
Node 3		$-\beta_2$	$+\beta_3$		$+\beta_5$		
Node 4	$+\beta_1$	$+\beta_2$				$+\beta_6$	

Certain joint loads correspond to the  $r$  reactions which are not initially known. We separate out the rows in  $\mathbf{B}'$  and  $\mathbf{P}'$  corresponding to the  $r$  reactions, resulting in  $i$  times  $(j - r)$  rows relating the  $m$  force unknowns. The reduced set of equations is expressed as (we drop the prime superscript on  $\mathbf{B}$  and  $\mathbf{P}$  to simplify the equation)

$$\mathbf{P} = \mathbf{B}\mathbf{F} \quad (2.27)$$

When the structure is statically determinate,  $m = i$  times  $(j - r)$ , and since the coefficient matrix  $\mathbf{B}$  is now square, one can solve for  $\mathbf{F}$ . We used a similar approach when discussing complex planar trusses in Sect. 2.2.6.

### 2.6.4 Stability

A structure is said to be stable when a unique solution for the member forces exists for a given set of external loads. The relationship between the loading and the resulting member forces is defined by the linear matrix equation, (2.27). Noting Cramer's rule [4], the stability requirement can be expressed as

$$\text{determinant}(\mathbf{B}) \neq 0 \quad (2.28)$$

which is equivalent to requiring  $\mathbf{B}$  to be nonsingular. Singularity can be due to an insufficient number or improper orientation of the restraints. It may also arise due to the geometrical pattern of the members. Complex trusses, such as the example discussed in Sect. 2.2, may exhibit this deficiency even though they appear to be stable.

### 2.6.5 Matrix Formulation: Computation of Displacements

The manual process described in the previous section for computing displacements is not suited for large-scale structures. We faced a similar problem with the analysis of space trusses, and in that case, we resorted to a computer-based scheme. We follow a similar strategy here. We utilize the matrix notation introduced earlier, and just have to define some additional terms related to deformation and nodal displacement.

Noting (2.11), we see that  $e$  involves the direction cosines for the member, and the nodal displacements. Using the notation for the direction cosine matrix defined by (2.15) and also defining  $\mathbf{u}$  as the nodal displacement matrix,

$$\begin{aligned}\boldsymbol{\beta} &= \{\beta_x, \beta_y, \beta_z\} \\ \mathbf{u} &= \{u, v, w\}\end{aligned}\tag{2.29}$$

we express the extension  $e$  as a matrix product.

$$e = \boldsymbol{\beta}^T \mathbf{u}\tag{2.30}$$

We generalize (2.30) for a member  $n$  connected to nodes  $n_+$  and  $n_-$

$$e_n = \boldsymbol{\beta}_n^T (\mathbf{u}_{n_+} - \mathbf{u}_{n_-})\tag{2.31}$$

Note that this matrix expression applies to both 2-D and 3-D structures.

Following the strategy used to assemble the matrix force equilibrium equations, we assemble the complete set of deformation–displacement relations for the structure. They have the following form

$$\mathbf{e} = (\mathbf{B}')^T \mathbf{U}'\tag{2.32}$$

where

$$\mathbf{U}' = \{u_1, u_2, \dots, u_j\}, \quad \mathbf{e} = \{e_1, e_2, \dots, e_m\}$$

and  $\mathbf{B}'$  is defined by (2.20). Note that  $\mathbf{B}'$  is the matrix associated with the matrix force equilibrium equations (2.19). Some of the nodal displacements correspond to locations, where constraints are applied and their magnitudes are known. When the structure is statically determinate, support movement introduces no deformation, and we can delete these terms from  $\mathbf{U}'$ . We also delete the corresponding rows of  $\mathbf{B}'$ . These operations lead to the modified equation

$$\mathbf{e} = \mathbf{B}^T \mathbf{U}\tag{2.33}$$

Note that the corresponding modified equilibrium equations have the form  $\mathbf{P} = \mathbf{B}\mathbf{F}$ .

The duality between these equations is called the “Static-Geometric” analogy.

Once  $\mathbf{F}$  is known, one determines the extension of a member using

$$e = \left(\frac{L}{AE}\right)F + e_1$$

where  $e_1$  contains terms due to temperature and fabrication error. We express the set of deformations in matrix form

$$\mathbf{e} = \mathbf{f}\mathbf{F} + \mathbf{e}_1\tag{2.34}$$

where  $\mathbf{f}$  is a diagonal matrix containing the flexibility coefficients for the members,

$$\mathbf{f} = \begin{bmatrix} \left(\frac{L}{AE}\right)_1 & & & \\ & \left(\frac{L}{AE}\right)_2 & & \\ & & \ddots & \\ & & & \left(\frac{L}{AE}\right)_m \end{bmatrix}\tag{2.35}$$

Given  $\mathbf{P}$ , one generates  $\mathbf{B}$  and, solves for  $\mathbf{F}$ ,

$$\mathbf{F} = \mathbf{B}^{-1} \mathbf{P} \quad (2.36)$$

Then, we compute  $\mathbf{e}$  with (2.34) and lastly solve for  $\mathbf{U}$  using.

$$\mathbf{U} = (\mathbf{B}^{-1})^T \mathbf{e} \quad (2.37)$$

This approach can be represented as a series of computer operations. The major computational effort is in assembling and inverting  $\mathbf{B}$ . The deflection computation requires minimal additional effort since one needs to compute  $\mathbf{B}^{-1}$  in order to determine the member forces.

Using matrix notation, it is relatively straightforward to prove the validity of the Method of Virtual Forces. We apply a virtual force  $\delta \mathbf{P}'$  and find the corresponding virtual forces using the matrix equilibrium equations.

$$\delta \mathbf{P}' = \mathbf{B}' \delta \mathbf{F} \quad (2.38)$$

Member forces which satisfy the force equilibrium equations are said to be statically permissible. Note that  $\delta \mathbf{P}'$  includes both the external nodal loads and the reactions. The extensions are related to the nodal displacements by (2.32)

$$\mathbf{e} = (\mathbf{B}')^T \mathbf{U}' \quad (2.38a)$$

where  $\mathbf{U}'$  contains both the nodal displacements and support movements. We multiply (2.38a) by  $\delta \mathbf{F}^T$ ,

$$\delta \mathbf{F}^T \mathbf{e} = \delta \mathbf{F}^T \left[ (\mathbf{B}')^T \mathbf{U}' \right] \quad (2.38b)$$

and note the identity,

$$\delta \mathbf{F}^T (\mathbf{B}')^T \equiv [\mathbf{B}' \delta \mathbf{F}]^T \equiv (\delta \mathbf{P}')^T \quad (2.39)$$

Then, (2.38b) takes the form

$$\delta \mathbf{F}^T \mathbf{e} = (\delta \mathbf{P}')^T \mathbf{U}' \quad (2.40)$$

Separating out the prescribed support displacements and reactions, and expanding the matrix products leads to the scalar equation

$$\sum \delta F \cdot e = \sum \delta P \cdot u + \sum \delta R \cdot \bar{u} \quad (2.41)$$

The final form follows when  $\delta P$  is specialized as a single force.

### Example 2.27 Planar Complex Truss

**Given:** The planar structure shown in Fig. E2.27. Assume equal cross-sectional areas.

**Determine:**

- The displacements at the nodes. Take  $A = 10 \text{ in.}^2$  and  $E = 29,000 \text{ ksi}$ .
- The value of  $A$  required to limiting the horizontal displacement to 1.5 in.

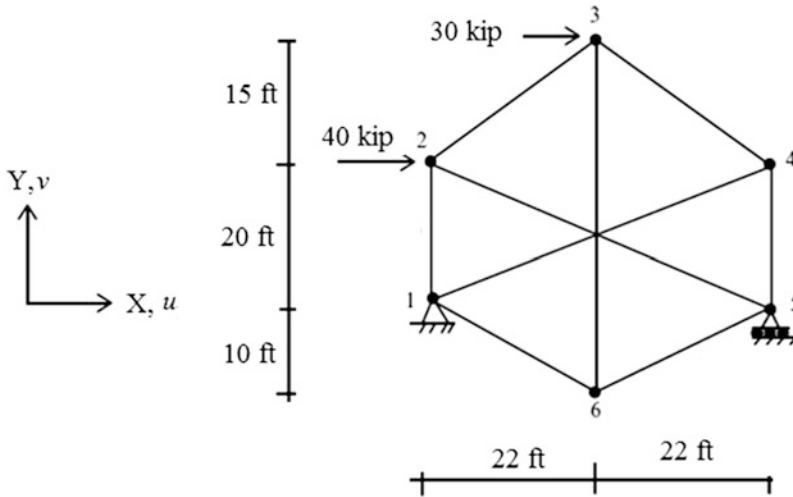


Fig. E2.27

**Solution:** This truss is a complex truss similar to example discussed in Sect. 2.2.5. One needs to solve the complete set of force equilibrium equations to find the member forces. Therefore, applying the Method of Virtual Forces is not computationally advantageous in this case, so we use a computer-based scheme. The computer method presented above is applicable for both planar and 3-D trusses. We just need to take  $i = 2$  for the planar case. The results for the nodal displacements are listed below.

$$\begin{aligned} &\begin{cases} u_1 = 0 \\ v_1 = 0 \end{cases} \\ &\begin{cases} u_2 = 4.88 \text{ in.} \\ v_2 = 0.13 \text{ in.} \end{cases} \\ &\begin{cases} u_3 = 2.34 \text{ in.} \\ v_3 = 4.15 \text{ in.} \end{cases} \\ &\begin{cases} u_4 = -0.27 \text{ in.} \\ v_4 = 0.09 \text{ in.} \end{cases} \\ &\begin{cases} u_5 = 4.42 \text{ in.} \\ v_5 = 0 \end{cases} \\ &\begin{cases} u_6 = 2.2 \text{ in.} \\ v_6 = 4.43 \text{ in.} \end{cases} \end{aligned}$$

The area required to limit the horizontal displacement to 1.5 in. is

$$A_{\text{required}} = (10) \frac{4.88}{1.5} = 32.53 \text{ in.}^2$$



The revised nodal displacements for  $A = 32.53 \text{ in.}^2$  will be

$$\begin{cases} u_1 = 0 \\ v_1 = 0 \end{cases}$$

$$\begin{cases} u_2 = 1.5 \text{ in.} \\ v_2 = 0.04 \text{ in.} \end{cases}$$

$$\begin{cases} u_3 = 0.72 \text{ in.} \\ v_3 = 1.27 \text{ in.} \end{cases}$$

$$\begin{cases} u_4 = -0.08 \text{ in.} \\ v_4 = 0.03 \text{ in.} \end{cases}$$

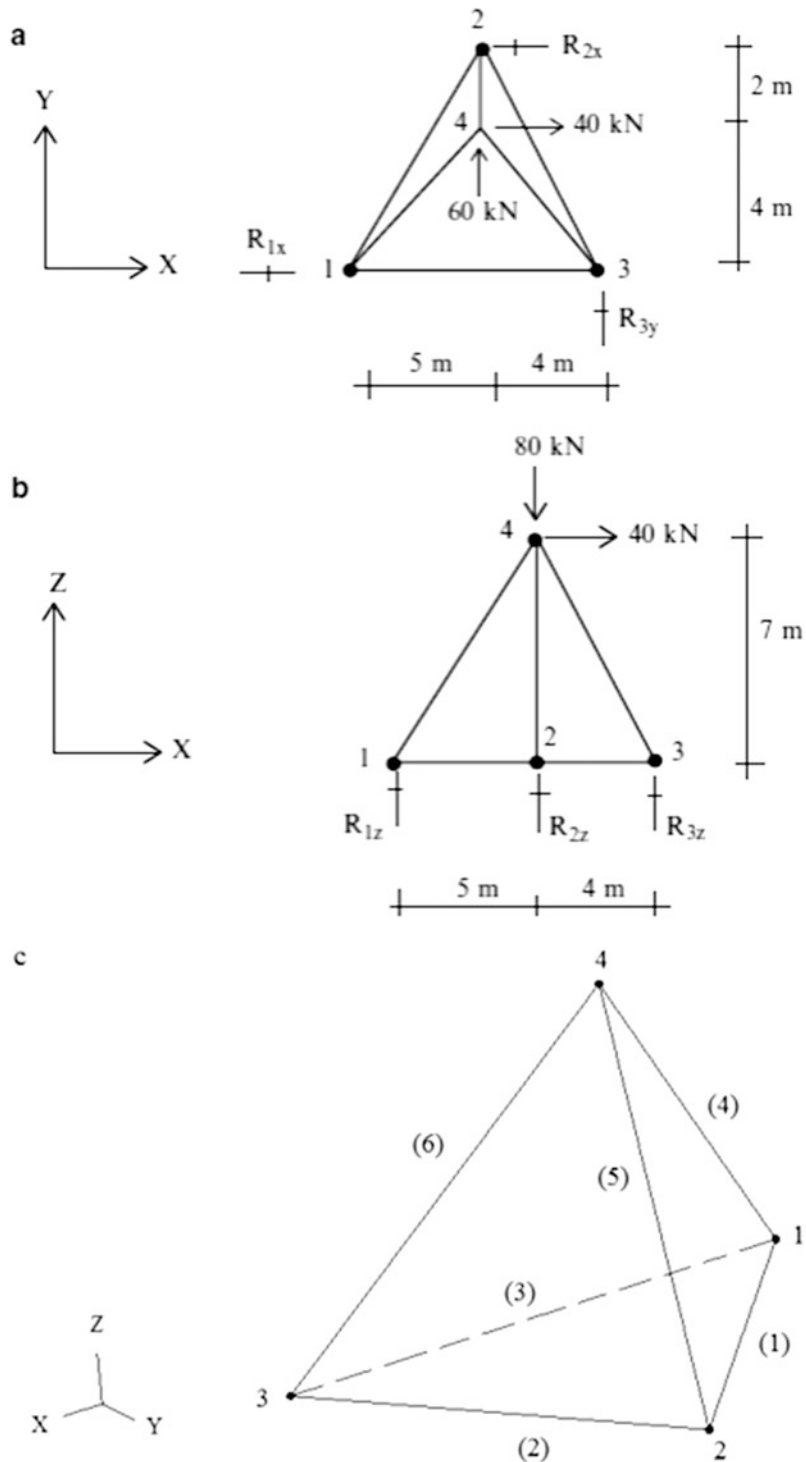
$$\begin{cases} u_5 = 1.36 \text{ in.} \\ v_5 = 0 \end{cases}$$

$$\begin{cases} u_6 = 0.68 \text{ in.} \\ v_6 = 1.36 \text{ in.} \end{cases}$$

### *Example 2.28* Space Truss

**Given:** The space structure shown in Fig. E2.28. Assume equal cross-sectional areas. Take  $A = 1300 \text{ mm}^2$  and  $E = 200 \text{ GPa}$ .

**Determine:** The member forces, the reactions, and the nodal displacements. Use computer-based scheme.



**Fig. E2.28** 3-D Truss. (a)  $x - y$  plan view. (b)  $x - z$  plan view. (c) Isometric view

**Solution:** The joint displacements, the member forces, and the reactions are listed below.

**Joint displacements:**

$$\text{Joint 1} \begin{cases} u_1 = 0 \\ v_1 = 4.9 \text{ mm} \\ w_1 = 0 \end{cases}$$

$$\text{Joint 2} \begin{cases} u_2 = 0 \\ v_2 = 3.5 \text{ mm} \\ w_2 = 0 \end{cases}$$

$$\text{Joint 3} \begin{cases} u_3 = -1.3 \text{ mm} \\ v_3 = 0 \\ w_3 = 0 \end{cases}$$

$$\text{Joint 4} \begin{cases} u_4 = 4.5 \text{ mm} \\ v_4 = 8.9 \text{ mm} \\ w_4 = -2.2 \text{ mm} \end{cases}$$

**Member forces and reactions:**

$$\begin{cases} F_{(1)} = -37.47 \text{ kN} \\ F_{(2)} = 76.94 \text{ kN} \\ F_{(3)} = -38.66 \text{ kN} \\ F_{(4)} = 68.26 \text{ kN} \\ F_{(5)} = -128.27 \text{ kN} \\ F_{(6)} = -9.05 \text{ kN} \end{cases} \quad \begin{cases} R_{1x} = 26.67 \text{ kN} \\ R_{1z} = -50.37 \text{ kN} \\ R_{2x} = -66.67 \text{ kN} \\ R_{2z} = 123.33 \text{ kN} \\ R_{3y} = -60.00 \text{ kN} \\ R_{3z} = 7.04 \text{ kN} \end{cases}$$

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**2.7 Summary****2.7.1 Objectives of the Chapter**

- To develop a criteria for assessing the initial stability of truss type structures
- To present methods for determining the axial forces in the members of statically determinate trusses
- To present a matrix-based formulation for the analyses of arbitrary statically determinate trusses
- To present methods for computing the displaced configuration of a truss
- To introduce the concept of an influence line and illustrate its application to trusses

### 2.7.2 Key Facts and Concepts

- The statical determinacy of a plane truss is determined by comparing the number of unknown forces vs. the number of available force equilibrium equations.
- The forces in the members of a statically determinate truss are independent of the member properties such as area and material modulus and support movements.
- The two force analysis procedures are the method of joints and the method of sections. The method of joints strategy proceeds from joint to joint, always working with a joint having a statically determinate force system. This approach generates all the member forces. The method of sections is designed to allow one to determine the force in a particular member. One passes a cutting plane through the structure, selects either segment, and applies the equilibrium conditions. This method requires less computation and generally is easier to apply.
- Given the external loads, one can determine the internal member forces using force equilibrium equations when the truss is statically determinate. The displacements due to the loading can be computed manually using the method of virtual forces. To determine the displacement at a point A in a particular direction,  $d_a$ , one applies a virtual force  $\delta P_a$  at point A in the same direction as the desired displacement and computes, using static equilibrium equations, the internal forces  $\delta F$ , and reactions,  $\delta R$ , due to  $\delta P_a$ . The displacement is given by

$$d_a \delta P_a = \sum_{\text{members}} e \delta F - \sum_{\text{reactions}} \bar{d} \delta R$$

where  $\bar{d}$  is the prescribed support movement and  $e$  is the elongation of the member due to force, temperature change, and initial fabrication error.

$$e = \left( \frac{FL}{AE} \right) + (\alpha \Delta T L) + e_0$$

This method is restricted to static loading and small displacements. It is also applicable for statically indeterminate trusses when the member forces are known.

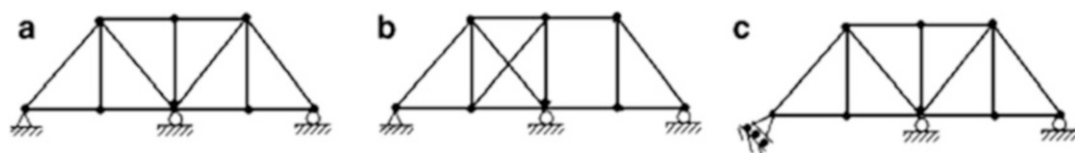
- The concept of influence lines is very useful for dealing with the live loading which can act anywhere on the structure. Given a particular member force and a particular type of live loading, usually a unit vertical loading, the influence line displays graphically the magnitude of the force for various locations of the load. By viewing the plot, one can immediately determine the position of the load that produces the peak magnitude of the member force.

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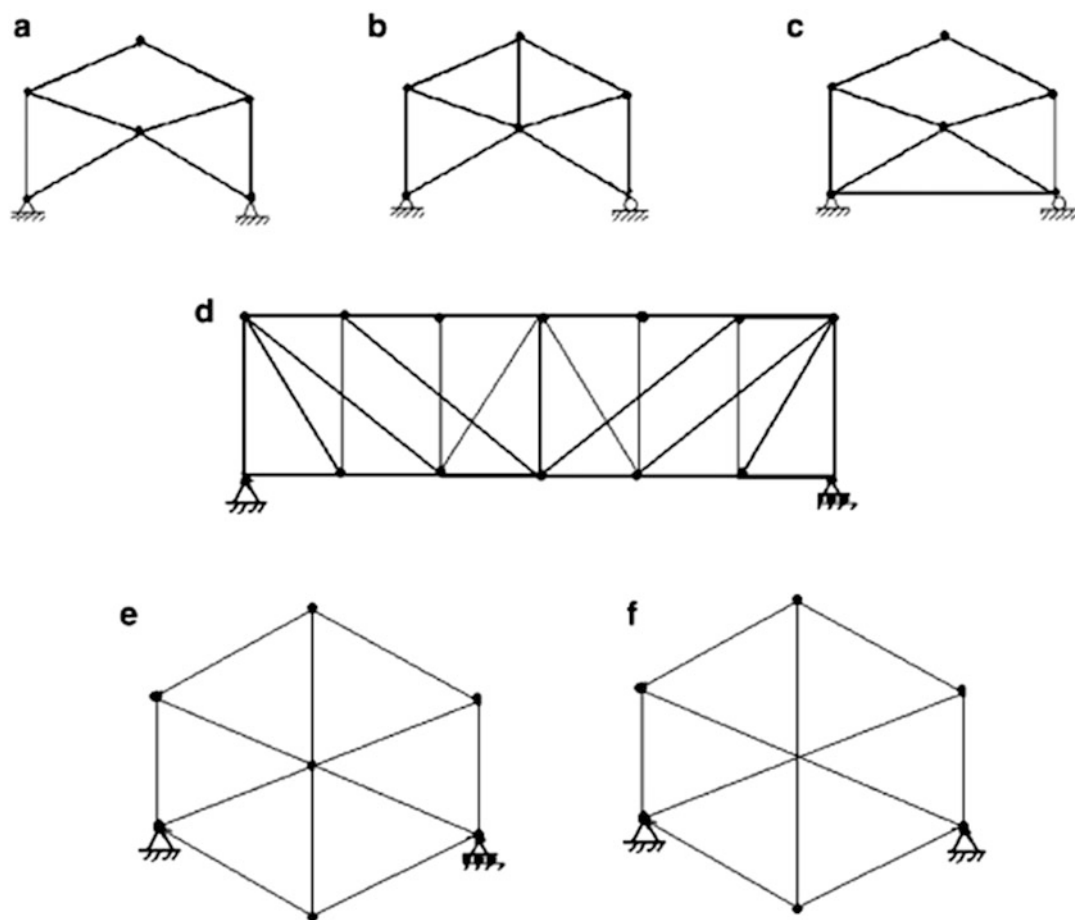
## 2.8 Problems

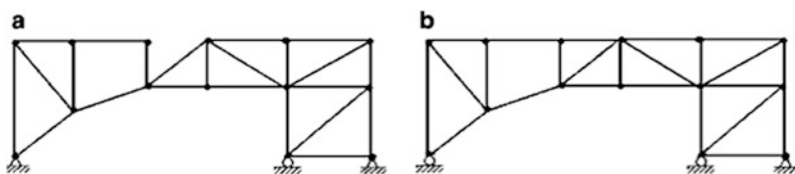
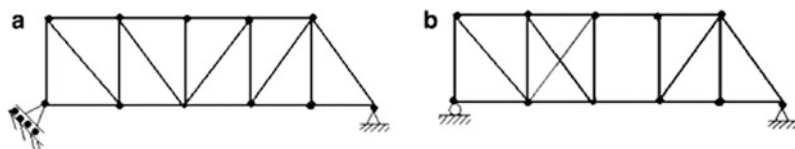
Classify each of the following plane trusses defined in Problems 2.1–2.4 as initially stable or unstable. If stable, then classify them as statically determinate or indeterminate. For indeterminate trusses, determine the degree of static indeterminacy.

## Problem 2.1

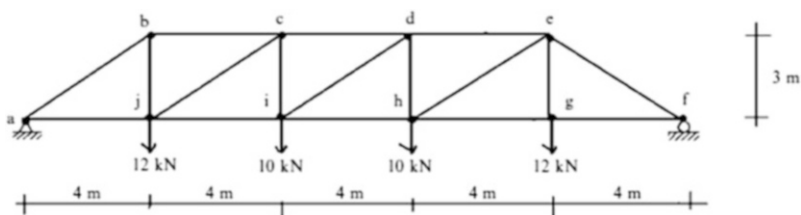


## Problem 2.2

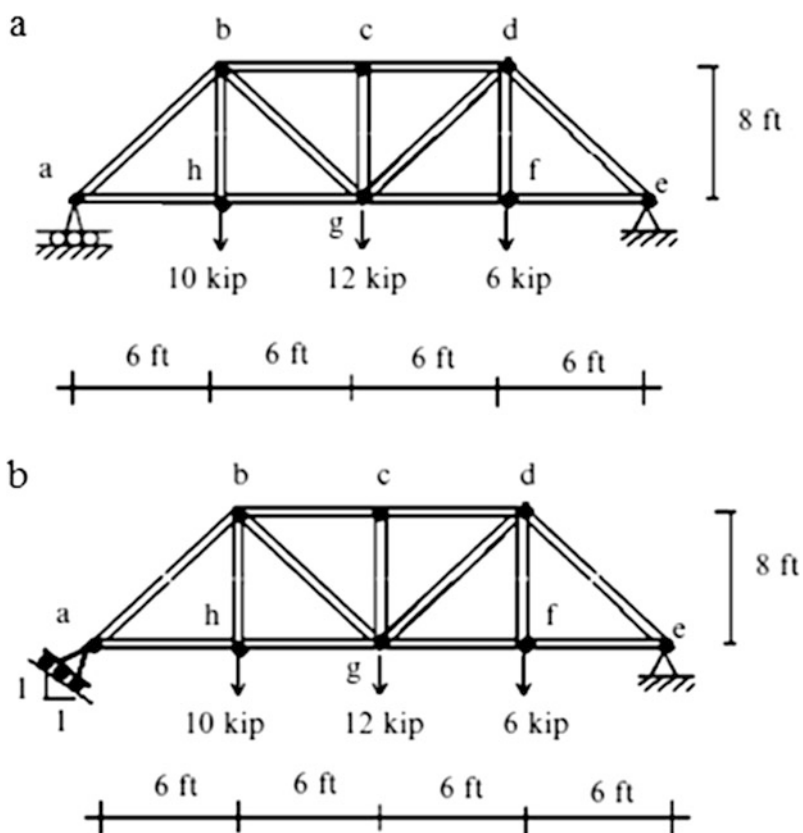


**Problem 2.3****Problem 2.4**

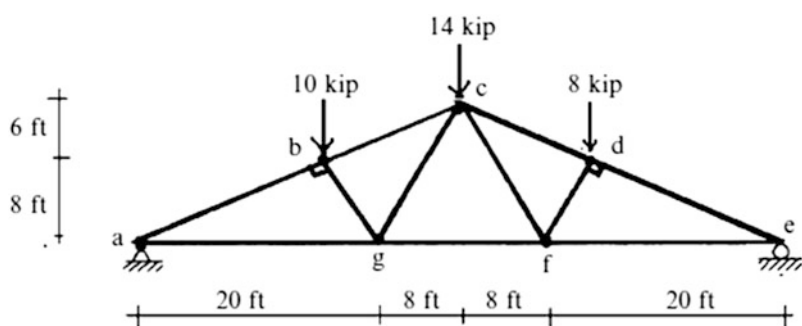
Determine all the member forces for the plane trusses defined in Problems 2.5–2.12 using the method of joints.

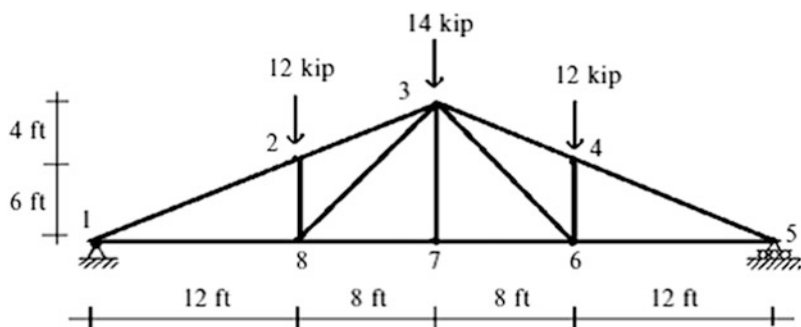
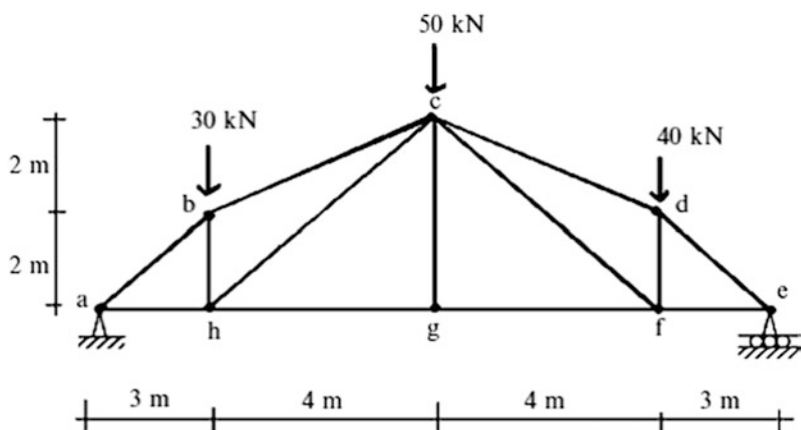
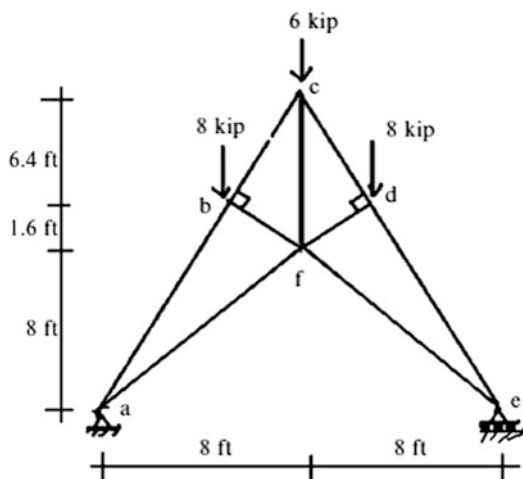
**Problem 2.5**

## Problem 2.6

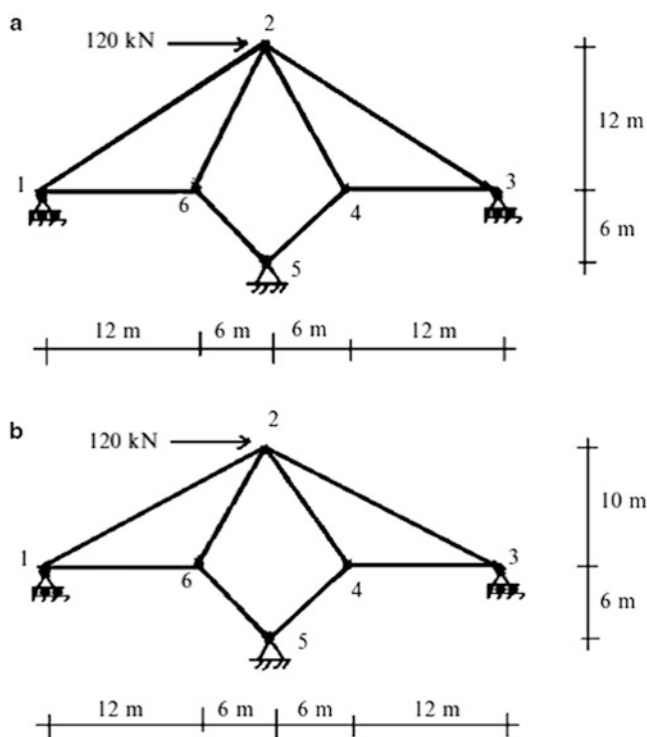
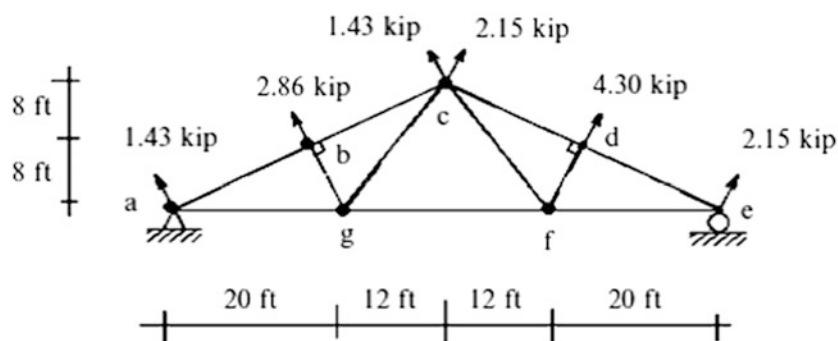


## Problem 2.7

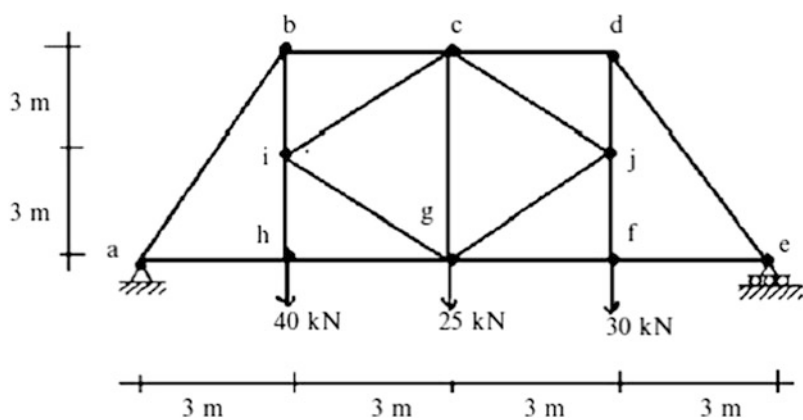
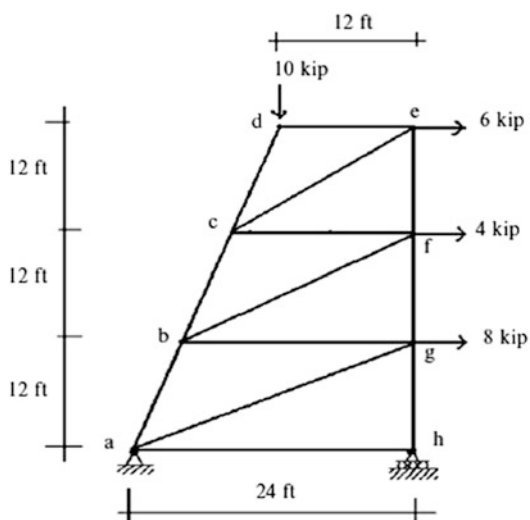
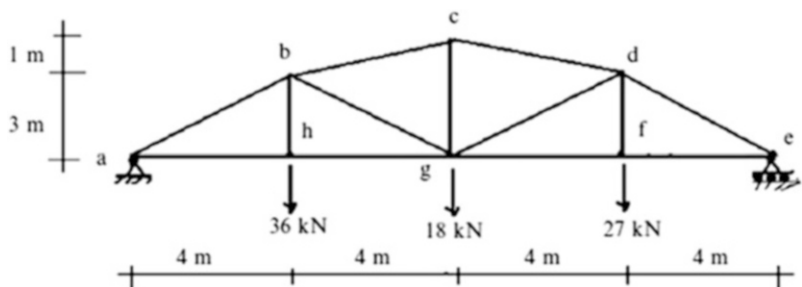


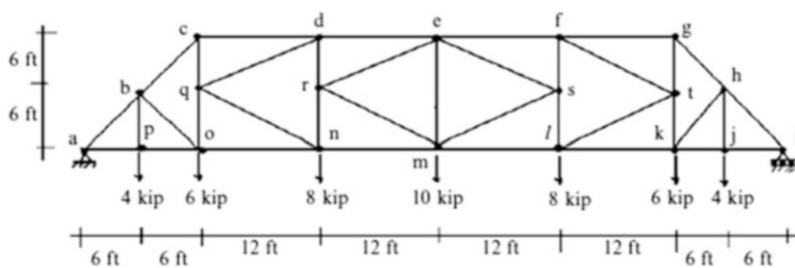
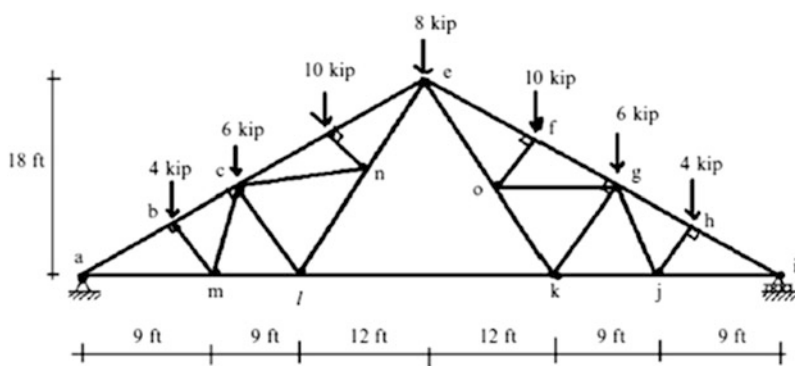
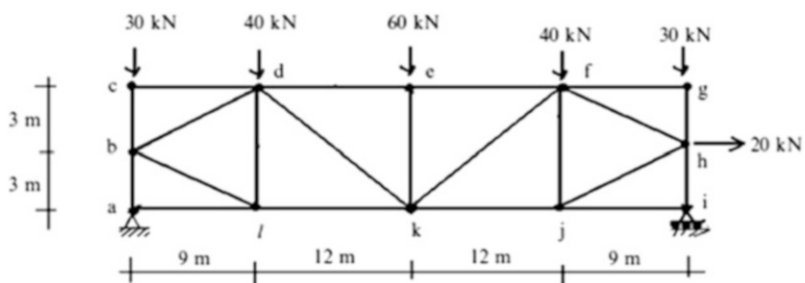
**Problem 2.8****Problem 2.9****Problem 2.10**



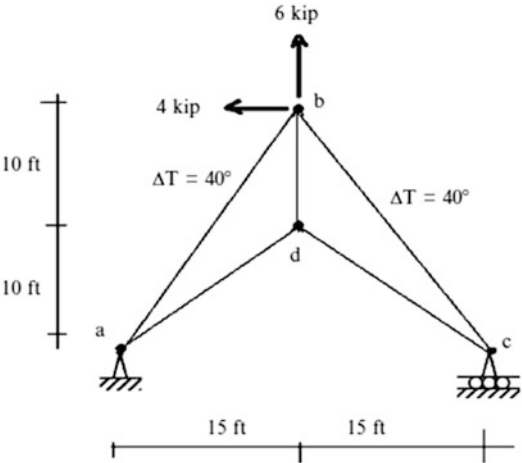
**Problem 2.11****Problem 2.12**

Determine all the member forces for the plane trusses defined in Problems 2.13–2.18 using a combination of the method of joints and the method of sections.

**Problem 2.13****Problem 2.14****Problem 2.15**

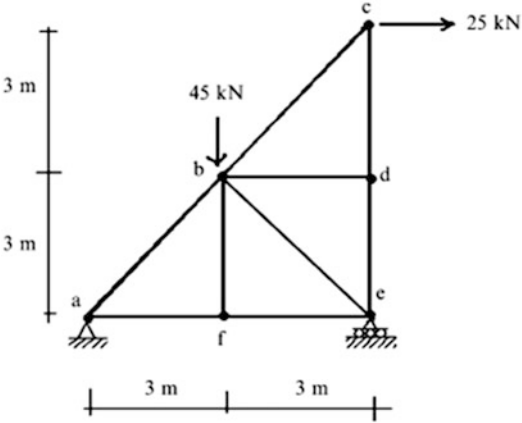
**Problem 2.16****Problem 2.17****Problem 2.18**

**Problem 2.19** Use the principle of virtual forces to determine the horizontal and vertical displacement at joint b due to loading shown and temperature increase of  $\Delta T = 40^\circ\text{F}$  for members ab and bc. Assume  $A = 1.4\text{ in.}^2$ ,  $E = 29,000\text{ ksi}$ , and  $\alpha = 6.5 (10^{-6})/^\circ\text{F}$

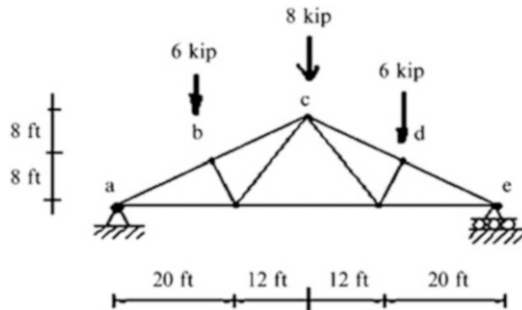


**Problem 2.20** For the plane truss shown, use the principle of virtual forces to determine the vertical displacement at joint b and the horizontal displacement at joint c.  $E = 200\text{ GPa}$ . The areas of the members are as follow:

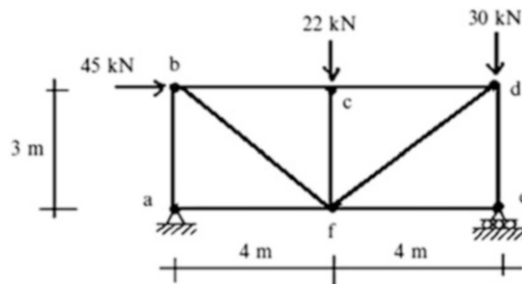
$$\begin{aligned} A_{ab} &= A_{bc} = A_{be} = 1290\text{ mm}^2 \\ A_{bf} &= A_{bd} = 645\text{ mm}^2 \\ A_{cd} &= A_{de} = 1935\text{ mm}^2 \\ A_{af} &= A_{fe} = 2580\text{ mm}^2 \end{aligned}$$



**Problem 2.21** For the plane truss shown, use the principle of virtual forces to determine the vertical displacement at joint C due to the loading shown and a settlement of 0.5 inch at support a. Assume  $A = 2 \text{ in.}^2$  and  $E = 29,000 \text{ ksi}$ .



**Problem 2.22** For the plane truss shown, use the principle of virtual forces to determine the vertical and horizontal displacement at joint d.



$$A = 1300 \text{ mm}^2$$

$$E = 200 \text{ GPa}$$

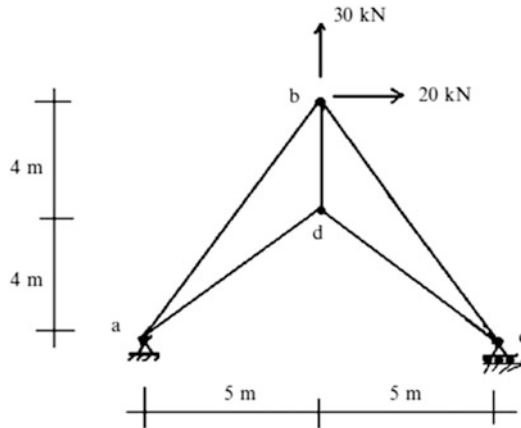
**Problem 2.23** Use the principle of virtual forces to determine the horizontal and vertical displacement at joint b due to:

- (a) Loading shown.
- (b) Temperature increase of  $\Delta T = 16^\circ\text{C}$  for members ab and bc.

$$A = 900 \text{ mm}^2$$

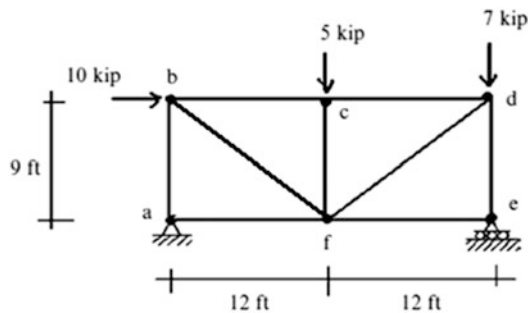
$$E = 200 \text{ GPa}$$

$$\alpha = 12 \times 10^{-6} / ^\circ\text{C}$$



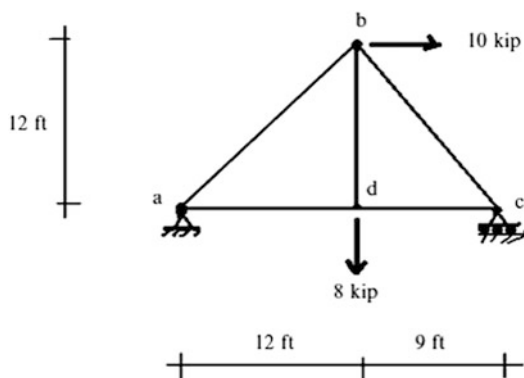
**Problem 2.24** Use the principle of virtual force method to determine the horizontal component of the displacement at joint d. Assume  $A = 0.5 \text{ in.}^2$  and  $E = 29,000 \text{ ksi}$ .

- (i) For the loading shown
- (ii) For a fabrication error of  $-0.25 \text{ in.}$  for members ac and df
- (iii) For the summation of Case (i) and Case (ii) loadings.



**Problem 2.25** Use the principle of virtual forces method to determine the horizontal component of the displacement at joint b. Assume  $A = 0.5 \text{ in.}^2$ ,  $E = 30,000 \text{ ksi}$ ,  $\alpha = 6.5 \times 10^{-6}/^\circ\text{F}$

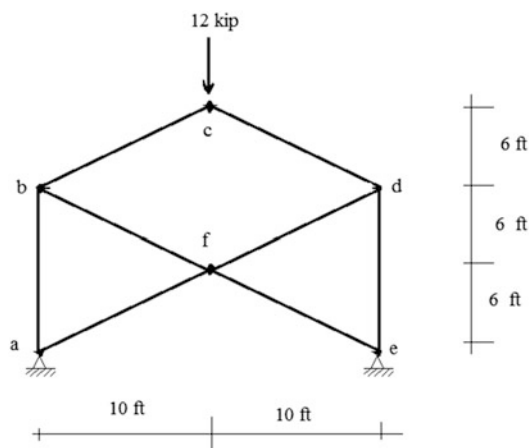
- (i) For the loading shown
- (ii) For a temperature increase of  $\Delta T = 60^\circ\text{F}$  for all members
- (iii) For the summation of Case (i) and Case (ii) loadings.



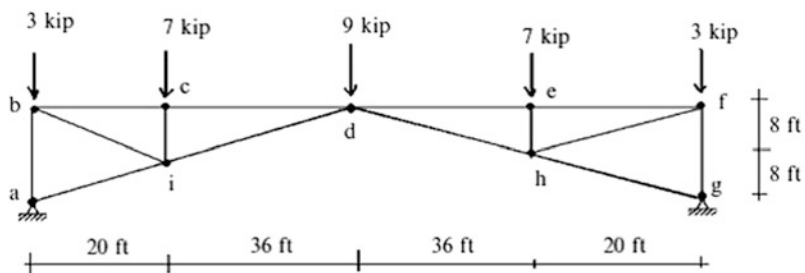
**Problem 2.26** For the plane truss shown below, use the principle of virtual forces to determine the vertical displacement at joint f.

$$A = 2 \text{ in.}^2$$

$$E = 29,000 \text{ ksi}$$

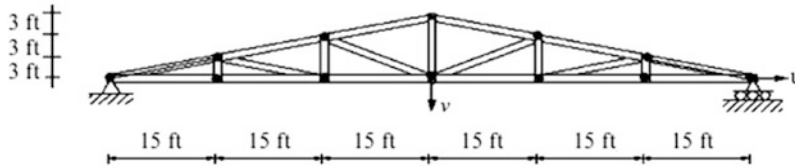


**Problem 2.27** For the plane truss shown below, determine the required cross-sectional area for the truss members to limit the vertical deflection at d to 0.56 in. Assume equal cross-sectional areas.  $E = 29,000 \text{ ksi}$ .



**Problem 2.28** For the plane truss shown in Problem 2.12, use the principle of virtual forces to determine the vertical displacement at joint g. The areas are  $4 \text{ in.}^2$  for top chord members,  $3 \text{ in.}^2$  for bottom chord members, and  $2 \text{ in.}^2$  for other members.  $E = 29,000 \text{ ksi}$ .

**Problem 2.29**



Suppose the top chord members in the truss defined above experience a temperature decrease of  $60^\circ\text{F}$ . Determine the resulting displacements,  $u$  and  $v$ .  $A = 2 \text{ in.}^2$ ,  $E = 29,000 \text{ ksi}$  and  $\alpha = 6.5 \times 10^{-6}/^\circ\text{F}$ .

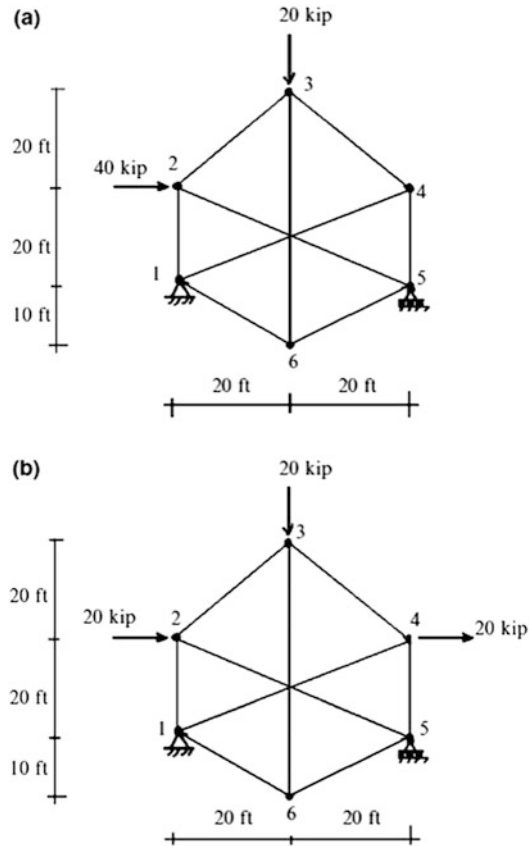
**Problem 2.30** Solve Problem 2.15 using computer software. Assume the cross-sectional areas are equal to  $A$ .

- Demonstrate that the member forces are independent of  $A$  by generating solutions for different values of  $A$ .
- Determine the value of  $A$  required to limit the vertical displacement to 50 mm.

**Problem 2.31** Consider the complex truss defined below in Figure (a). Use computer software to determine the member forces for the loading shown in Figure (a).

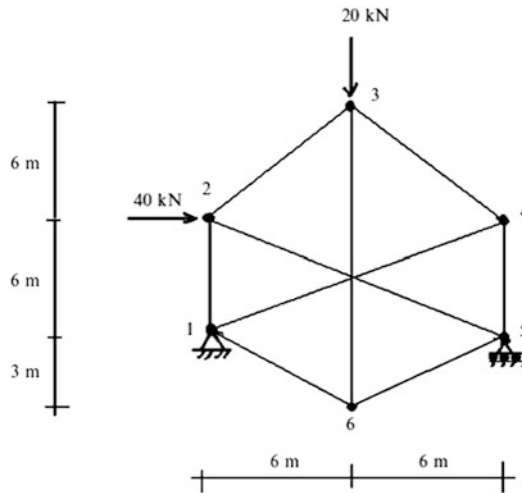
- Assume equal areas
- Take an arbitrary set of areas
- Determine the member forces corresponding to the loading shown in Figure (b). Are the forces similar to the results of part (a). Discuss.



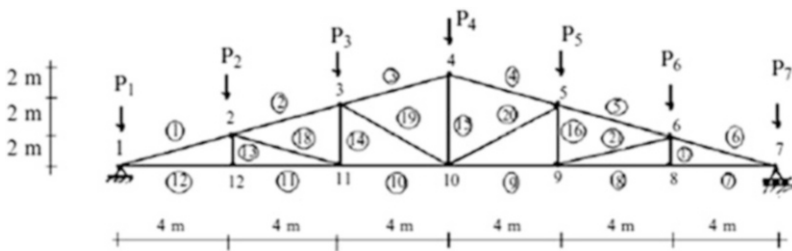


**Problem 2.32** Solve Problem 2.11(a) using computer software. Assuming the cross-sectional areas are equal to  $A$ . Demonstrate that the member forces are independent of  $A$  by generating solution of different values of  $A$ .

**Problem 2.33** Consider the complex truss defined below. Assume equal areas. Use computer software to determine the member forces and joint displacements. Determine the area for which the maximum displacement equals 30 mm.  $E = 200$  GPa.

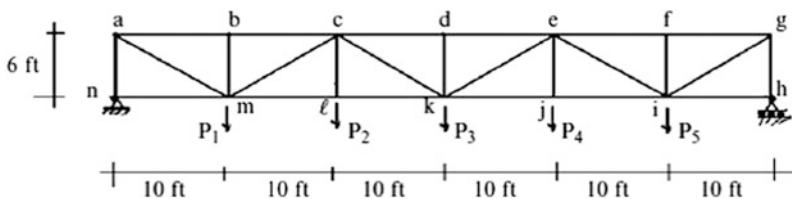


**Problem 2.34** For the truss and the loading shown:



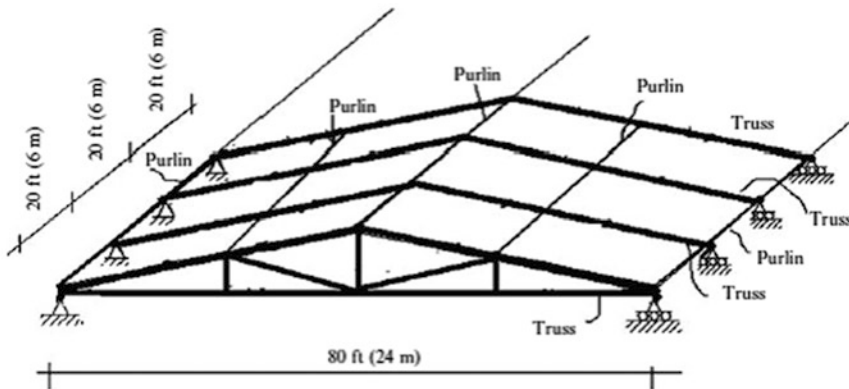
- Tabulate all the member forces due to the individual unit vertical nodal forces applied to the top chord (force influence table). Use computer software.
- Use the force influence table in part (a) to
  - Draw influence lines for members 15, 4, and 20.
  - Calculate the member forces in members 3, 19, 10, and 14 for the following loading:  
 $P_2 = 10 \text{ kN}$ ,  $P_4 = 6 \text{ kN}$ , and  $P_6 = 8 \text{ kN}$ .

**Problem 2.35** For the truss and the loading shown:

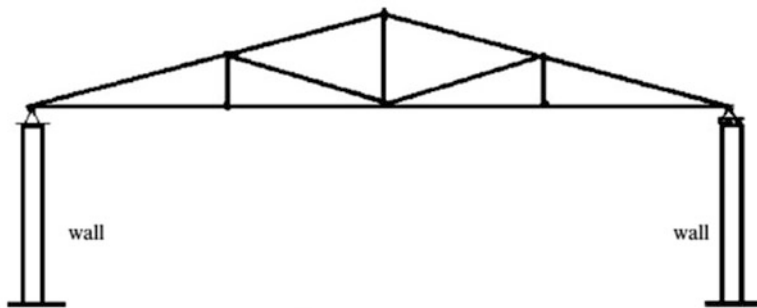


- (a) Tabulate all the member forces due to the individual unit vertical nodal forces applied to the bottom chord (force influence table). Use computer software.
- (b) Use the force influence table in part (a) to
  - (i) Draw influence lines for members bc, cm, and ji.
  - (ii) Calculate the member forces in members ke, de, kl, and ei for the loading  $P_1 = 8$  kip,  $P_3 = 10$  kip, and  $P_4 = 4$  kip.

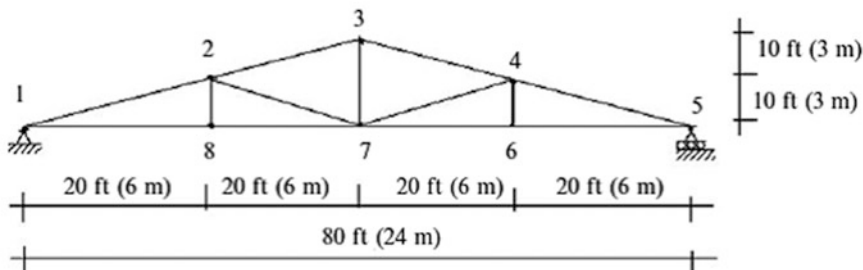
**Problem 2.36** The roof structure shown below consists of trusses spaced uniformly, 20 ft (6 m) on center, along the length of the building and tied together by purlins and x-bracing. The roofing materials are supported by the purlins which span between trusses at the truss joints.



(a) Roof structural make-up



(b) Elevation-typical truss



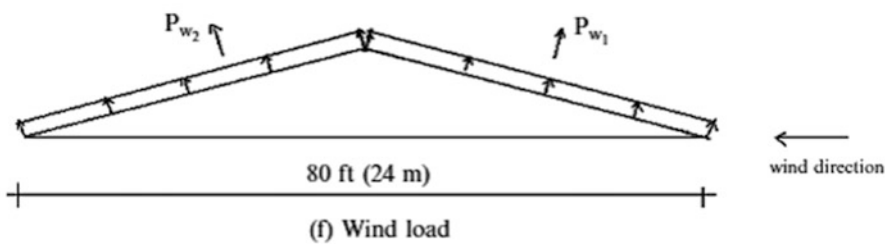
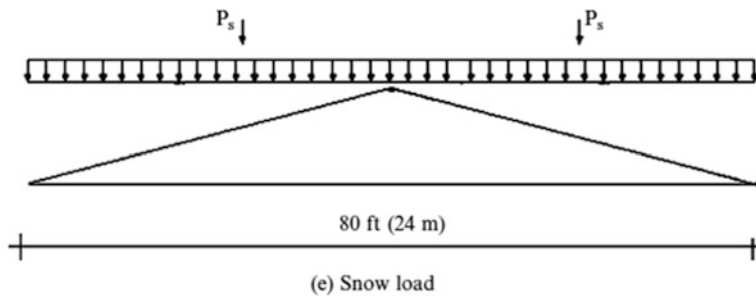
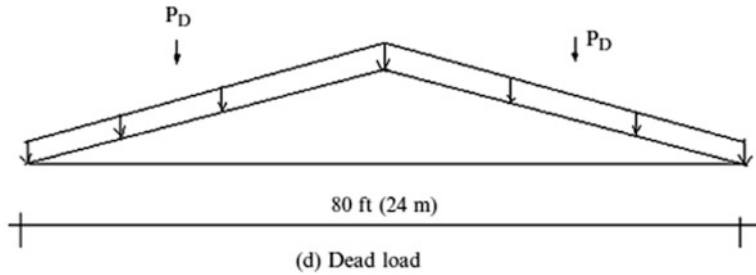
(c) Truss geometry

Consider the following loadings:

Dead load: roof material, purlins, truss members, estimated at 15 psf (720 Pa) of roof surface

Snow load: 20 psf (960 Pa) of horizontal projection of the roof surface

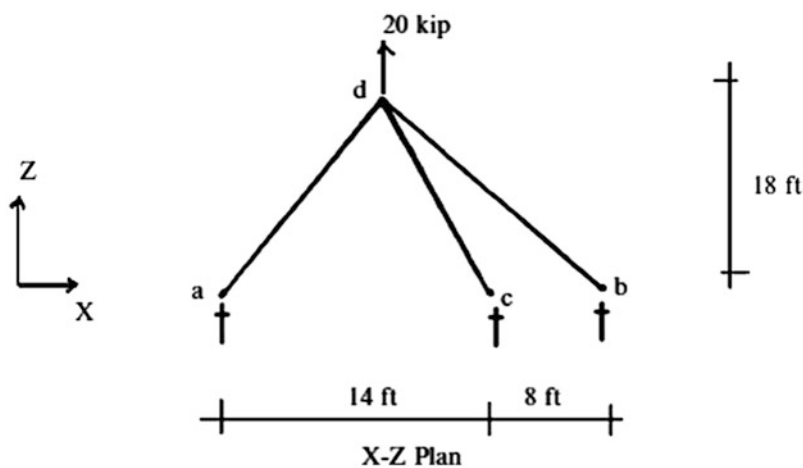
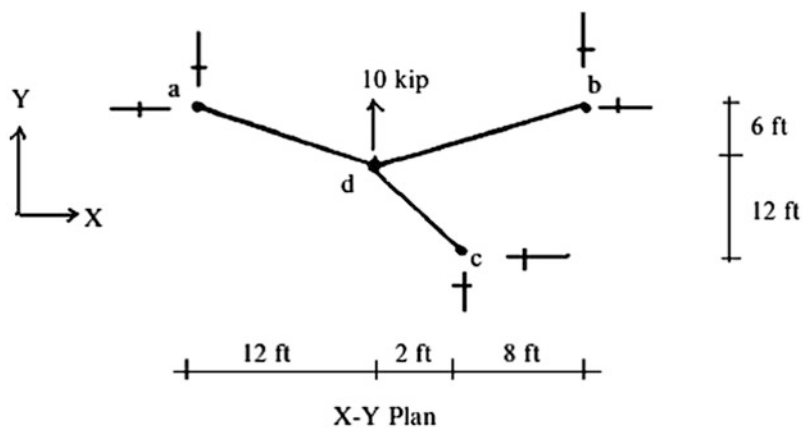
Wind load: windward face 12 psf (575 Pa), leeward face 8 psf (385 Pa) normal to roof surface



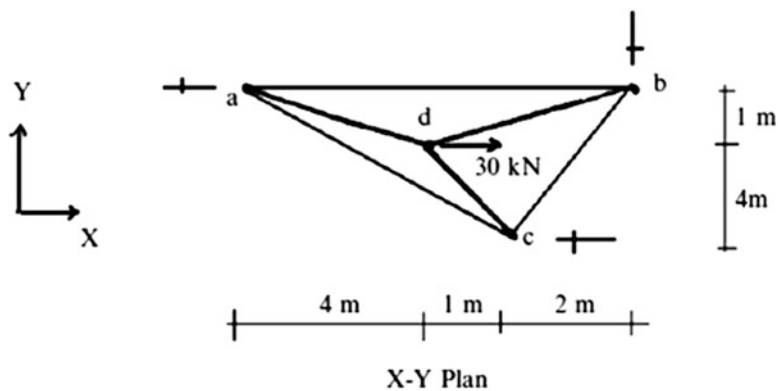
Determine the following quantities for the typical interior truss:

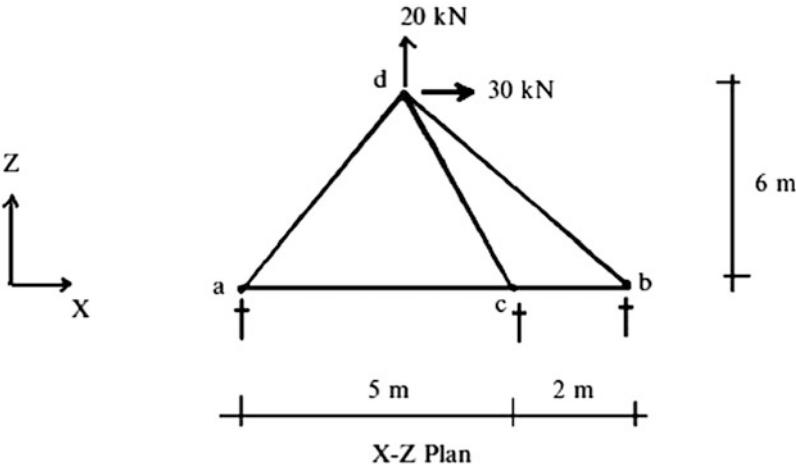
- Compute the truss nodal loads associated with gravity, snow, and wind.
- Use computer software to determine the member forces due to dead load, snow load, and wind. Tabulate the member force results.

**Problem 2.37** Determine the member forces for the space truss shown.

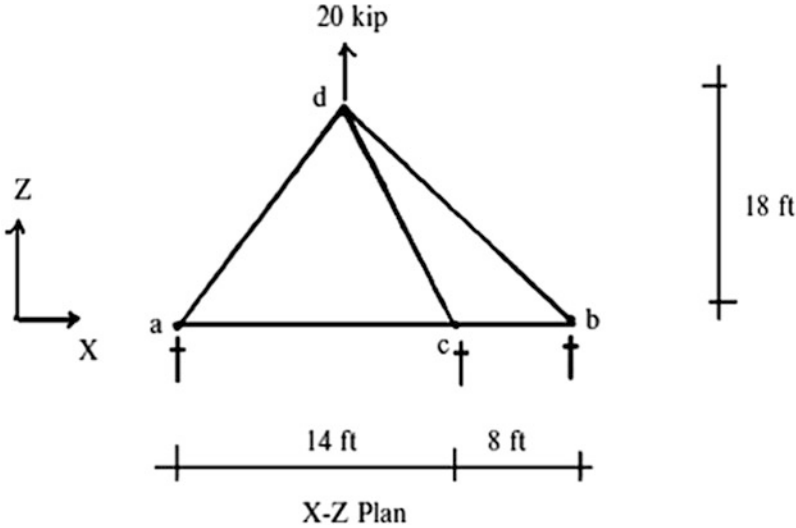
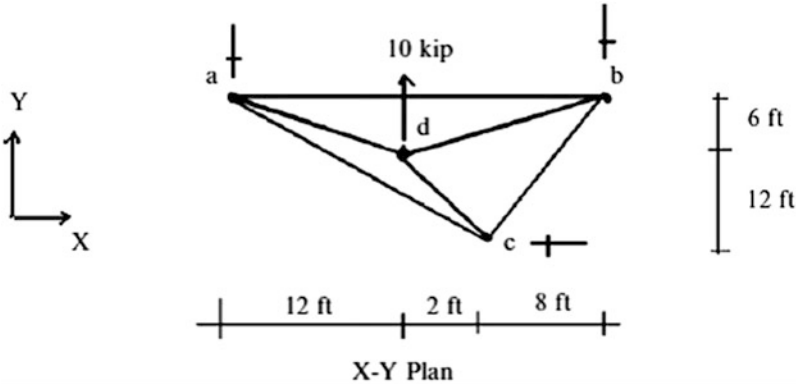


**Problem 2.38** Determine the member forces for the space truss shown.

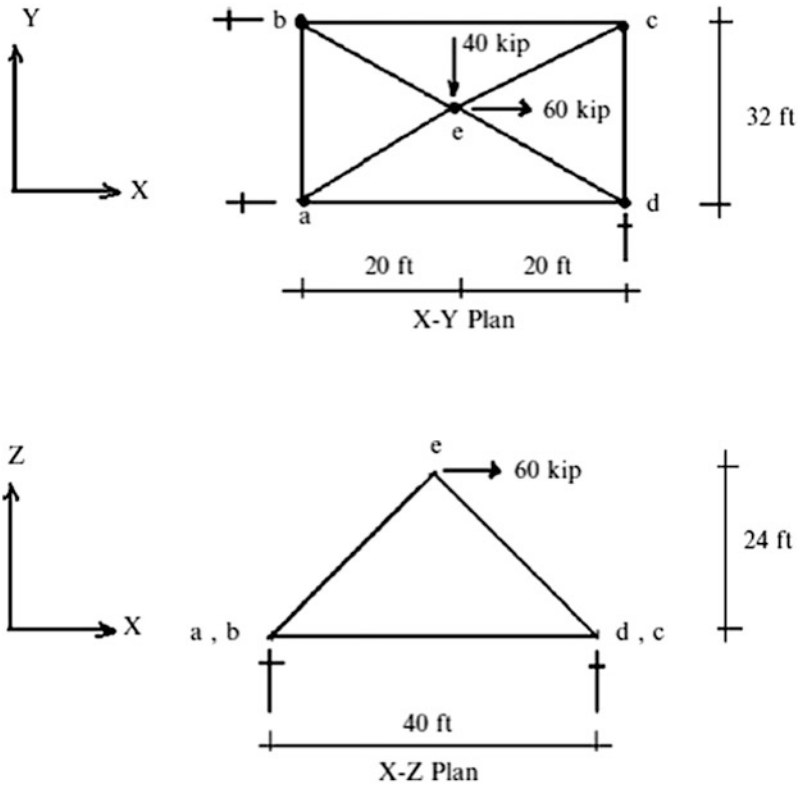




**Problem 2.39** Determine the member forces for the space truss shown.



**Problem 2.40** Determine the member forces for the space truss shown.



**Problem 2.41** For the space truss shown in Problem 2.37, use the principle of virtual forces to determine the displacements  $u$ ,  $v$ , and  $w$  at joint *d*.  $E = 29,000$  ksi and  $A = 3.0$  in.<sup>2</sup>

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