

Preface

Averaging is ubiquitous in many sciences, engineering, and everyday practice. The notions of the arithmetic, geometric, and harmonic means developed by the ancient Greeks are in widespread use today. When thinking of an average, most people would use arithmetic mean, “the average”, or perhaps its weighted version in order to associate the inputs with the degrees of importance. While this is certainly the simplest and most intuitive averaging function, its use is often not warranted. For example, when averaging the interest rates, it is the geometric and not the arithmetic mean which is the right method. On the other hand, the arithmetic mean can also be biased for a few extreme inputs, and hence can convey false meaning. This is the reason why real estate markets report the median and not the average prices (which could be biased by one or a few outliers), and why judges’ marks in some Olympic sports are trimmed of the smallest and the largest values.

The world of averages (also called means) is very rich in both mathematical and practical senses. There exist averages that allow one to incorporate not only the weights of importance but also various interactions among the inputs; averages that are robust to a few, or even many outlying values; averages that model various types of majority, necessary, desirable, and sufficient inputs; averages that model a representative (in various senses) input, and so on.

The theory of aggregation functions, which includes most averages, became an established area of research in the last 30 years. Theoretical advances are complemented by numerous applications in decision sciences, artificial intelligence, fuzzy systems, and image processing. Several monographs and edited volumes dedicated to this topic provide a comprehensive analysis of both theory and applications, and regular conferences and special sessions on aggregation provide a forum for presenting the latest achievements.

However, it has been 7 years since the publication of the most recent monograph in the field of aggregation, and we think it is time to provide an update on the most recent developments in this area. Our specific focus is on averaging functions. These functions, whose prototypical example is the arithmetic mean, are most often used in decision sciences as they provide compensatory properties: low values of

some inputs are compensated by high values of the others, and the output is always bounded by the smallest and the largest input. The result of the averaging is a value representative of the inputs.

The target audience of this book is computer scientists, system architects, knowledge engineers, and programmers, as well as decision scientists and mathematicians, who face a problem of combining various inputs into a single output. Our intent is to provide these people with an easy-to-use guide about possible ways of averaging input values given on a numerical scale, and ways of choosing/constructing aggregation functions for their specific applications. All relevant mathematical notions are explained in the book (in the introduction or as footnotes).

Chapter 1 gives a broad introduction to the topic of aggregation functions. It covers important general properties and lists the most important prototypical examples: means, ordered weighted averaging (OWA) functions, and Choquet integrals, as well as non-averaging aggregation functions such as triangular norms and conorms, uninorms and nullnorms. It addresses the problem of choosing the right aggregation function, and also introduces a number of basic numerical tools: methods of interpolation and smoothing, linear and nonlinear optimization, which will be used to construct aggregation functions from empirical data. Prototypical applications are presented.

The remaining chapters focus on different types of averaging functions. Chapter 2 presents classical averages, which include the arithmetic mean, power means, quasi-arithmetic means, medians, and several other functions. Some less known averages such as Gini, Bonferroni, Heronian, and logarithmic means are also presented. Chapter 3 discusses the ordered weighted averaging and focuses on the issue of OWA weights. Chapter 4 is dedicated to fuzzy integrals, in particular the Choquet and Sugeno integrals. Each class of these functions has many distinct families which are treated in separate sections. We give formal definitions, discuss important properties of the averaging functions and their interpretation, and also present specific methods for fitting a particular family to the empirically collected data. We also provide various generalizations, advanced constructions, and pointers to specific literature.

Chapters 5–8 focus on more advanced and recently developed methods of averaging. In Chap. 5 we present a view of averaging functions from the perspective of minimizing some sort of a penalty. Such a penalty can be interpreted from the point of view of inputs disagreement. If all inputs coincide, then the average is that common value, whereas when the inputs differ we impose a price on the difference between the output and each input. The value that minimizes the total penalty is the output of the averaging function.

It turns out that all averaging functions allow such an interpretation through a properly chosen penalty, and that the classical averages such as the arithmetic means and the median are the minimizers of the sum of squared and absolute differences between the inputs and the output, respectively. One can define new averages by specifying a particular suitable penalty function.

In Chap. 6 we present several construction methods for averaging functions and discuss some of their useful properties. The construction methods include

idempotization, composition, fitting averaging functions to empirical data, and graph-based construction. We discuss overlap and grouping functions, and present in detail generalizations of the Bonferroni mean. We also treat the issues of consistency and stability of the families of averaging functions.

In Chap. 7 we extend the class of averaging aggregation functions, which are by definition monotone increasing, to include averaging functions that are not always monotone. We present a weaker notion of monotonicity, which requires the output not to decrease if all the inputs are increased by the same amount. Weak monotonicity makes sense when averaging data containing significant noise, the outlying values. The outliers should not contribute to the average, or even worse, drag the average in their direction. On the contrary, the average must be robust to often unavoidable gross errors in the data, and the price for such robustness is lack of monotonicity. We study various robust averages (also called robust estimators of location) and establish weak monotonicity of several classes of averaging functions. We also develop some new weakly monotone averaging methods using penalty-based approach and extend the mode operation. We present directionally monotone functions and define pre-aggregation functions.

The final Chap. 8 presents some topics related to averages on lattices, in particular product lattices. We focus on averaging functions for Atanassov intuitionistic fuzzy sets and interval-valued fuzzy sets. We present general construction methods and then treat some special cases, such as the weighted means, OWA, and Bonferroni means. A special attention is paid to the medians on lattices, and several alternative constructions are explored.

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