

Chapter 2

The Boltzmann Transport Equation

2.1 Derivation of the Boltzmann Transport Equation

The non-relativistic Boltzmann equation is given by

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{r}} f + \frac{\mathbf{F}}{m} \cdot \nabla_{\mathbf{v}} f = \left(\frac{\delta f}{\delta t} \right)_{coll}, \quad (2.1)$$

where $\mathbf{r}$, $\mathbf{v}$, and t are *independent* variables.

Problem 2.1 Show that the Boltzmann equation (2.1) is invariant with respect to Galilean transformations.

Solution For simplicity let us consider a Cartesian inertial system K with (orthogonal) axes x, y, z and an inertial system K' with axes x', y', z' that is moving with speed w in positive x -direction with respect to K . The Galilean transformation is then given by

$$t' = t \quad \mathbf{r}'(t) = \mathbf{r} - wt\mathbf{e}_x \quad \mathbf{v}' = \mathbf{v} - w\mathbf{e}_x \quad \mathbf{a}' = \mathbf{a},$$

where $\mathbf{e}_x$ is the unit vector in x -direction. In particular, for each coordinate we find

$$\begin{aligned} t' &= t & x'(t) &= x - wt & v'_x &= v_x - w \\ & & y' &= y & v'_y &= v_y \\ & & z' &= z & v'_z &= v_z. \end{aligned}$$

Note that $\mathbf{r}$, $\mathbf{v}$, and t are independent variables, while the new coordinate $\mathbf{r}'$ depends on time t . More precisely, $\mathbf{r}'$ depends on time t .

Consider now the change in variables, i.e., $f(\mathbf{r}, \mathbf{v}, t) \rightarrow f(\mathbf{r}'(t), \mathbf{v}', t)$. Thus, we have the following transformations:

- A. Since the new spatial coordinate depends on time we have to rewrite the time derivative and obtain

$$\frac{\partial f}{\partial t} \Rightarrow \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{r}'} \cdot \frac{\partial \mathbf{r}'}{\partial t} = \frac{\partial f}{\partial t} - w\mathbf{e}_x \cdot \nabla_{\mathbf{r}'} f,$$

where $\partial f / \partial \mathbf{r}' = \nabla_{\mathbf{r}'}$ and $\partial \mathbf{r}' / \partial t = -w\mathbf{e}_x$.

- B. For the gradient of f we find

$$\begin{aligned} \nabla_{\mathbf{r}} f &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial f}{\partial y'} \frac{\partial y'}{\partial y} + \frac{\partial f}{\partial z'} \frac{\partial z'}{\partial z} \\ &= \frac{\partial f}{\partial x'} + \frac{\partial f}{\partial y'} + \frac{\partial f}{\partial z'} = \nabla_{\mathbf{r}'} f. \end{aligned}$$

Therefore, $\nabla_{\mathbf{r}} = \nabla_{\mathbf{r}'}$.

- C. Similarly we find

$$\begin{aligned} \nabla_{\mathbf{v}} f &= \frac{\partial f}{\partial v_x} + \frac{\partial f}{\partial v_y} + \frac{\partial f}{\partial v_z} = \frac{\partial f}{\partial v'_x} \frac{\partial v'_x}{\partial v_x} + \frac{\partial f}{\partial v'_y} \frac{\partial v'_y}{\partial v_y} + \frac{\partial f}{\partial v'_z} \frac{\partial v'_z}{\partial v_z} \\ &= \frac{\partial f}{\partial v'_x} + \frac{\partial f}{\partial v'_y} + \frac{\partial f}{\partial v'_z} = \nabla_{\mathbf{v}'} f. \end{aligned}$$

It follows that $\nabla_{\mathbf{v}} = \nabla_{\mathbf{v}'}$. Moreover, since $\mathbf{a}' = \mathbf{a}$ we have $\mathbf{F}' = \mathbf{F}$.

Substituting the above results, the Boltzmann equation reads

$$\begin{aligned} -w\mathbf{e}_x \nabla_{\mathbf{r}'} f + \frac{\partial f}{\partial t} + [\mathbf{v}' + w\mathbf{e}_x] \cdot \nabla_{\mathbf{r}'} f + \frac{\mathbf{F}'}{m} \cdot \nabla_{\mathbf{v}'} f &= \left(\frac{\delta f}{\delta t} \right)'_{coll} \\ \Rightarrow \frac{\partial f}{\partial t} + \mathbf{v}' \cdot \nabla_{\mathbf{r}'} f + \frac{\mathbf{F}'}{m} \cdot \nabla_{\mathbf{v}'} f &= \left(\frac{\delta f}{\delta t} \right)'_{coll}. \end{aligned}$$

Note that $\mathbf{v}$ was replaced by $\mathbf{v}' + w\mathbf{e}_x$. Therefore, the Boltzmann equation is invariant under Galilean transformation, i.e., it retains its form.

Problem 2.2 Show that the Boltzmann equation (2.1) transforms into the mixed phase space coordinate form

$$\frac{\partial f}{\partial t} + (u_i + c_i) \frac{\partial f}{\partial r_i} - \left(\frac{\partial u_i}{\partial t} + (u_j + c_j) \frac{\partial u_i}{\partial r_j} - \frac{F_i}{m} \right) \frac{\partial f}{\partial c_i} = \left(\frac{\delta f}{\delta t} \right)'_{coll}, \quad (2.2)$$

where we used Einstein's summation convention, i.e., we sum over double indices.

Solution We rewrite the Boltzmann equation as

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial r_i} + \frac{F_i}{m} \frac{\partial f}{\partial v_i} = \left(\frac{\delta f}{\delta t} \right)_{coll}.$$

The velocity $\mathbf{v}$ of a particle can also be described with respect to the bulk flow velocity $\mathbf{u}(\mathbf{r}, t)$, through $\mathbf{v} = \mathbf{c}(\mathbf{r}, t) + \mathbf{u}(\mathbf{r}, t)$, where $\langle \mathbf{v} \rangle = \mathbf{u}(\mathbf{r}, t)$ and $\langle \mathbf{c}(\mathbf{r}, t) \rangle = 0$. Here, $\mathbf{c}$ is the random velocity and sometimes also called the peculiar or thermal velocity. The random velocity can then be written as $\mathbf{c}(\mathbf{r}, t) = \mathbf{v} - \mathbf{u}(\mathbf{r}, t)$, where each component can be described by

$$c_i(\mathbf{r}, t) = v_i - u_i(\mathbf{r}, t). \quad (2.3)$$

Note that each component c_i depends on all spatial coordinates and time.

Sometimes it might be more convenient to express the phase space distribution f in terms of $\mathbf{r}$, $\mathbf{c}$, and t instead of $\mathbf{r}$, $\mathbf{v}$, and t . This means, that one has to introduce a *mixed* phase space, where the configuration space coordinate is inertial, but the velocity-space coordinate system is an accelerated system because it is tied to the instantaneous local bulk velocity. (Taken from [1].)

In this case one has to transfer the Boltzmann equation into the new *mixed* phase space coordinate system. The pdf $f(\mathbf{r}, \mathbf{v}, t)$ transforms then into $f(\mathbf{r}, \mathbf{c}, t)$. By replacing the independent variable $\mathbf{v}$ with the random velocity $\mathbf{c}$, one has to take into account that the new variable $\mathbf{c}(\mathbf{r}, t)$ depends on $\mathbf{r}$ and t . Similar to the preceding problem we have to transform the derivatives accordingly, which is done in the following.

- The time derivative transforms into

$$\frac{\partial f}{\partial t} \implies \frac{\partial f}{\partial t} + \frac{\partial f}{\partial c_i} \frac{\partial c_i}{\partial t}. \quad (2.4)$$

The time derivative of the new variable $c_i(\mathbf{r}, t)$ can be written as (because it depends on $\mathbf{u}$, which depends on time t)

$$\frac{\partial c_i}{\partial t} = \frac{\partial c_i}{\partial u_i} \frac{\partial u_i}{\partial t} = -\frac{\partial u_i}{\partial t},$$

where we used $\partial c_i / \partial u_i = -1$. In this case Eq. (2.4) becomes

$$\frac{\partial f}{\partial t} \implies \frac{\partial f}{\partial t} - \frac{\partial u_i}{\partial t} \frac{\partial f}{\partial c_i}. \quad (2.5)$$

- The derivative with respect to the spatial coordinates can be written as

$$v_i \frac{\partial f}{\partial r_i} \implies v_i \frac{\partial f}{\partial r_i} + v_i \frac{\partial f}{\partial c_j} \frac{\partial c_j}{\partial r_i}. \quad (2.6)$$

Note that we use a different index for c , since each component c_j depends on all spatial coordinates r_i . By using Eq. (2.3) we find

$$\frac{\partial c_j}{\partial r_i} = \frac{\partial c_j}{\partial u_j} \frac{\partial u_j}{\partial r_i} = -\frac{\partial u_j}{\partial r_i},$$

where we again used $\partial c_j / \partial u_j = -1$. In this case we find for relation (2.6)

$$v_i \frac{\partial f}{\partial r_i} \implies v_i \frac{\partial f}{\partial r_i} - v_i \frac{\partial f}{\partial c_j} \frac{\partial u_j}{\partial r_i} = v_i \frac{\partial f}{\partial r_i} - v_j \frac{\partial f}{\partial c_i} \frac{\partial u_i}{\partial r_j}. \quad (2.7)$$

Note that we swapped the indices i and j for the velocity v and the random velocity c in the last term on the right side. We do that, because we want to summarize the result in terms of $\partial f / \partial c_i$, and not $\partial f / \partial c_j$. According to Einstein's summation convention, swapping the indices has no influence on the summation. Substituting v_{ij} by Eq. (2.3) we find

$$v_i \frac{\partial f}{\partial r_i} \implies (u_i + c_i) \frac{\partial f}{\partial r_i} - (u_j + c_j) \frac{\partial u_i}{\partial r_j} \frac{\partial f}{\partial c_i}.$$

- Finally, we transform

$$\frac{\partial f}{\partial v_i} \implies \frac{\partial f}{\partial c_i} \frac{\partial c_i}{\partial v_i} = \frac{\partial f}{\partial c_i},$$

where we used $\partial c_i / \partial v_i = 1$ according to Eq. (2.3).

By summarizing all transformations we obtain Eq. (2.2).

Problem 2.3 Find the general solution to the Boltzmann equation (2.1) in the absence of collisions, i.e., $(\delta f / \delta t)_{coll} = 0$. Derive the general solution for the case that the force $\mathbf{F} = 0$.

Solution We consider first the force-free case with $\mathbf{F} = 0$ and then the case where $F = \text{const.} \neq 0$.

- For simplicity we use the one-dimensional Boltzmann equation

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} = 0$$

and deduce the three-dimensional case from that result. Note that the function f depends on the independent variables x and t , i.e., $f = f(x, t)$. We solve this partial differential equation by using the method of characteristics. Therefore, we parameterize x and t through the parameter s , so that $f = f(t(s), x(s))$.

The *system of characteristics* (system of ordinary differential equations, ODE) and the initial conditions (for $s = 0$) are given by

$$\frac{dt(s)}{ds} = 1 \qquad t(0) = 0 \qquad (2.8a)$$

$$\frac{dx(s)}{ds} = v_x \qquad x(0) = x_0 \qquad (2.8b)$$

$$\frac{df(s)}{ds} = 0 \qquad f(0) = f(x_0). \qquad (2.8c)$$

The solutions to the differential equations (2.8a)–(2.8c) are given by

$$\begin{array}{lll} t(s) = s + c_1 & c_1 = 0 & t(s) = s \\ x(s) = v_x s + c_2 & c_2 = x_0 & \Rightarrow \quad x(s) = v_x s + x_0 \\ f(s) = c_3 & c_3 = f(x_0) & f(s) = f(x_0), \end{array}$$

and are referred to as the characteristic curve or simply the *characteristic*. In particular, we find immediately

$$\begin{aligned} s &= t \\ x(t) &= v_x t + x_0 & x_0 &= x - v_x t. \end{aligned}$$

The characteristics are therefore curves which go through the point x_0 at time $t = 0$ into the direction of v_x . The solution of the Boltzmann equation with initial condition $f(x, t = 0) = f(x_0)$ can, therefore, be written as

$$f(x, t) = f(x_0) = G(x - v_x t),$$

which means that the solution is constant along the characteristic. One can interpret this solution in the way, that the initial profile $f(x_0)$ is transported with velocity v_x without changing the form of that profile. Going back to the three-dimensional case the solution is simply given by

$$f(\mathbf{x}, t) = f(\mathbf{x}_0) = G(\mathbf{x} - \mathbf{v}t).$$

- For the (1D) collisionless Boltzmann equation with $F = \text{const.} \neq 0$ we have

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + a_x \frac{\partial f}{\partial v_x} = 0.$$

Here, the function f depends on x , v_x and t , i.e., $f = f(x, v_x, t)$. The system of characteristics with the initial conditions is given by

$$\frac{dt(s)}{ds} = 1 \quad t(0) = 0 \quad (2.9a)$$

$$\frac{dx(s)}{ds} = v_x \quad x(0) = x_0 \quad (2.9b)$$

$$\frac{dv_x(s)}{ds} = a_x \quad v_x(0) = v_{x0} \quad (2.9c)$$

$$\frac{df(s)}{ds} = 0 \quad f(0) = f(x_0, v_{x0}), \quad (2.9d)$$

where a_x is the acceleration in x direction. We start by solving the characteristic equation (2.9c)

$$\frac{dv_x(s)}{ds} = a_x \quad \implies \quad v_x(s) = a_x s + c_1 \quad \implies \quad v_x(s) = a_x s + v_{x0},$$

where we used the initial condition $v_x(s = 0) = v_{x0}$. The second differential equation (2.9b) can then be written as

$$\frac{dx(s)}{ds} = v_x = a_x s + v_{x0},$$

where we substituted the solution of the first differential equation. The solution to this differential equation is given by

$$x(s) = \frac{a_x}{2} s^2 + v_{x0} s + c_2 \quad \implies \quad x(s) = \frac{a_x}{2} s^2 + v_{x0} s + x_0,$$

where we used the above initial condition. Obviously, we also find

$$t(s) = s + c_3 \quad \implies \quad t(s) = s.$$

Therefore, the characteristics are

$$\begin{aligned} x(t) &= \frac{a_x}{2} t^2 + v_{x0} t + x_0 & \implies & \quad x_0 = x - \frac{a_x}{2} t^2 - v_{x0} t \\ v_x(t) &= a_x t + v_{x0} & \implies & \quad v_{x0} = v_x - a_x t. \end{aligned} \quad (2.10)$$

The function f is constant along the characteristic described by Eq. (2.10),

$$f(x, v_x, t) = f(x_0, v_{x0}) = G \left[\left(x - \frac{a_x}{2} t^2 - v_{x0} t \right), (v_x - a_x t) \right].$$

We can easily test, that the function G solves the Boltzmann equation. We use

$$f(x, v_x, t) = G(\Phi, \Theta),$$

where

$$\Phi = x - \frac{a_x}{2}t^2 - v_{x0}t, \quad \Theta = v_x - a_xt.$$

We find for the derivatives (for simplicity we abbreviate derivatives as $\partial G / \partial t = G_t$ and $\partial G / \partial \Phi = G_\Phi$):

$$\begin{aligned} \frac{\partial f}{\partial t} &= G_\Phi \Phi_t + G_\Theta \Theta_t = G_\Phi (-a_xt - v_{x0}) + G_\Theta (-a_x) \\ v_x \frac{\partial f}{\partial x} &= v_x G_\Phi \Phi_x = v_x G_\Phi \\ a_x \frac{\partial f}{\partial v_x} &= a_x G_\Theta \Theta_{v_x} = a_x G_\Theta. \end{aligned}$$

Adding all results up we find (since $-a_xt - v_{x0} + v_x = 0$, see Eq. (2.10))

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + a_x \frac{\partial f}{\partial v_x} = G_\Phi (-a_xt - v_{x0} + v_x) + G_\Theta (-a_x + a_x) = 0.$$

2.2 The Boltzmann Collision Operator

Problem 2.4 Show that for $a \neq b$,

$$\delta((x-a)(x-b)) = \frac{1}{|a-b|} [\delta(x-a) + \delta(x-b)].$$

Solution For an arbitrary function $g(x)$ with roots¹ x_i and $g'(x_i) \neq 0$ the delta function is given by

$$\delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}.$$

¹A root or zero, x_i , of a function $g(x)$ is defined such that $g(x_i) = 0$, i.e., the function vanishes at x_i .

For the above function $g(x) = (x - a)(x - b) = x^2 - ax - bx + ab$ the roots are $x_1 = a$ and $x_2 = b$ and the derivative is given by $g'(x) = 2x - a - b$. We find

$$\begin{aligned} \delta((x - a)(x - b)) &= \frac{1}{|2a - a - b|} \delta(x - a) + \frac{1}{|2b - a - b|} \delta(x - b) \\ &= \frac{1}{|a - b|} [\delta(x - a) + \delta(x - b)]. \end{aligned}$$

Problem 2.5 Consider the relative motion of two particles P_1 and P_2 moving in each other's field of force with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, and with masses m_1 and m_2 , see Fig. 2.1. The particles are subject to the central forces $\mathbf{F}_1$ and $\mathbf{F}_2$, which are parallel to $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and depend, therefore, only on $r = |\mathbf{r}_1 - \mathbf{r}_2|$. Starting from the reduced mass equation of motion in polar coordinates,

$$M(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + M(r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta = -\frac{\partial V(r)}{\partial r}\mathbf{e}_r, \quad (2.11)$$

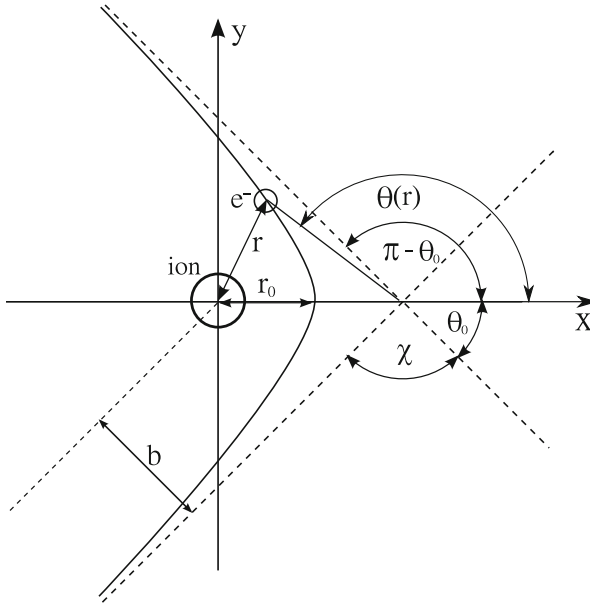


Fig. 2.1 Schematic of an electron (with charge eZ_1 , where $Z_1 = -1$) scattering in the Coulomb field of an ion (with charge eZ_2 , where $Z_2 > 0$). The trajectory of the electron is hyperbolic with eccentricity ϵ

complete the steps in the derivation of

$$\frac{d\theta}{dr} = \pm \frac{b}{r^2} \left[1 - \frac{b^2}{r^2} - \frac{2V(r)}{Mg^2} \right], \quad (2.12)$$

where θ is the angle of the incoming particle, b is the *impact parameter*, g is the constant relative speed, $M = m_1 m_2 / (m_1 + m_2)$ is the relative (or reduced) mass, and $V(r)$ is the potential energy with $V(r = \infty) = 0$.

Solution We consider first the conservation laws for angular momentum and total energy. In the second part we derive the expression given by Eq. (2.12).

A. **Conservation of Angular Momentum.** Here we describe two alternatives. The first alternative is based on the *Lagrangian* and the conservation law for angular momentum is derived by considering the angular component (coordinate) of the Lagrangian. The second alternative is based on the *definition of the angular momentum*.

- a. *Alternative 1*—Equation (2.11) describes the equations of motion in both directions, r and θ . To verify this equation we start with the Lagrangian, which is given by

$$\mathcal{L} = T - V,$$

where T is the kinetic energy and $V = V(r)$ is the potential energy. Note that in this particular case (polar coordinates) the kinetic energy $T = Mv^2/2$ consists of a radial and angular component, $T = T_r + T_\theta$, so that

$$\mathcal{L} = \frac{M}{2}v_r^2 + \frac{M}{2}v_\theta^2 - V(r) = \frac{M}{2}\dot{r}^2 + \frac{M}{2}r^2\dot{\theta}^2 - V(r),$$

where the radial and angular components of the velocity are given by $v_r = \dot{r}$ and $v_\theta = r\dot{\theta}$, with $v^2 = v_r^2 + v_\theta^2$. The equations of motion are then given by

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0,$$

where the coordinate $q_i = r, \theta$. Let us consider first the radial component, where

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = M(\ddot{r} - r\dot{\theta}^2) + \frac{\partial V(r)}{\partial r} = 0.$$

Note that this corresponds exactly to the radial component in Eq. (2.11). Similarly we obtain for the angular component

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} = M \frac{d}{dt} (r^2 \dot{\theta}) = Mr(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0, \quad (2.13)$$

since $\partial L / \partial \theta = 0$. For $r \neq 0$ we can divide by r . The remaining expression corresponds to the θ -component in Eq. (2.11). From Eq. (2.13) we can deduce that

$$r^2 \dot{\theta} = \text{const.} = gb,$$

since its time derivative is zero. However, the latter relation can also be derived from the conservation law for the angular momentum L , which is described briefly in the following.

- b. *Alternative 2*—The conservation law for angular momentum L is $dL/dt = 0$. With $\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta$ and $\mathbf{r} = r \mathbf{e}_r$ we find

$$\begin{aligned} \mathbf{L} = \mathbf{r} \times \mathbf{p} &= M \mathbf{r} \times \mathbf{v} = M r v_r \mathbf{e}_r \times \mathbf{e}_r + M r v_\theta \mathbf{e}_r \times \mathbf{e}_\theta \\ &= M r^2 \dot{\theta} \mathbf{e}_r \times \mathbf{e}_\theta, \end{aligned}$$

since $\mathbf{e}_r \times \mathbf{e}_r = 0$ and $v_\theta = r \dot{\theta}$. Since $\mathbf{e}_r \times \mathbf{e}_\theta$ is perpendicular to the plane of the particle motion, we simply write $L = m r^2 \dot{\theta}$. Since the particle mass M is a constant we find

$$\frac{dL}{dt} = M \frac{d}{dt} (r^2 \dot{\theta}) = 0 \quad \Rightarrow \quad r^2 \dot{\theta} = \text{const.} = gb. \quad (2.14)$$

- B. **Conservation of Total Energy.** The *conservation law for the total energy* can be derived by using the Hamiltonian,

$$\mathcal{H} = T + V.$$

With the above considerations we obtain

$$\mathcal{H} = \frac{M}{2} \dot{r}^2 + \frac{M}{2} r^2 \dot{\theta}^2 + V(r). \quad (2.15)$$

The Hamiltonian $\mathcal{H}$ describes the total energy of the system. It can be shown that the total time derivative of the Hamiltonian equals the partial time derivative of $\mathcal{H}$,

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}.$$

Further, if the Hamiltonian does not explicitly depend on time, the total time derivative vanishes and the total energy is conserved (constant). According to Eq. (2.15) the Hamiltonian is independent of time, i.e., $\partial \mathcal{H} / \partial t = 0$, and hence

$$\frac{d\mathcal{H}}{dt} = 0.$$

Therefore, the total energy is constant and given by

$$\frac{M}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + V(r) = \text{const.} = \frac{M}{2} g^2. \quad (2.16)$$

We have now established that Eq. (2.11) describes the equations of motion for both the r and θ direction, and that the angular momentum is conserved. After deriving the conservation laws for the angular momentum and the total energy we proceed now with the derivation of Eq. (2.12).

Derivation of Eq. (2.12) We consider two particles P_1 and P_2 . When particle P_1 approaches particle P_2 both the radial distance r and angle θ change with time t . Therefore, the time derivative of r is given by

$$\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta} \frac{d\theta}{dt} = \frac{dr}{d\theta} \dot{\theta},$$

where $\dot{\theta} = d\theta/dt$. Starting with Eq. (2.16) we substitute $\dot{r}$, multiply by 2, and divide by M and obtain

$$\left[\left(\frac{dr}{d\theta} \right)^2 \dot{\theta}^2 + r^2 \dot{\theta}^2 \right] = g^2 - \frac{2V(r)}{M},$$

where we moved the potential energy $V(r)$ to the right side. Substituting now $\dot{\theta}^2$ by Eq. (2.14) we obtain

$$\begin{aligned} \left(\frac{dr}{d\theta} \right)^2 \frac{g^2 b^2}{r^4} &= g^2 - \frac{2V(r)}{M} - \frac{g^2 b^2}{r^2}, \\ \left(\frac{dr}{d\theta} \right)^2 &= \frac{r^4}{b^2} \left(1 - \frac{2V(r)}{Mg^2} - \frac{b^2}{r^2} \right). \end{aligned}$$

By inverting and taking the square root, we find

$$\frac{d\theta}{dr} = \pm \frac{b}{r^2} \left[1 - \frac{2V(r)}{Mg^2} - \frac{b^2}{r^2} \right]^{-1/2}. \quad (2.17)$$

The negative root corresponds to an incoming particle since the radial coordinate decreases with time until reaching the point of closest approach. The positive root corresponds to an outgoing particle.

Problem 2.6 Consider the scattering of an electron with charge $q_1 = eZ_1$ (where $Z_1 = -1$) in the Coulomb field of an ion of charge $q_2 = eZ_2$ (where $Z_2 > 0$),

$$\mathbf{E} = \frac{eZ_2}{4\pi\epsilon_0} \frac{1}{r^2} \mathbf{e}_r,$$

where ϵ_0 is the permittivity of free space.

A. Show that

$$\frac{b^2}{rb_0} = 1 + \epsilon \cos \theta, \quad (2.18)$$

where the eccentricity is given by

$$\epsilon \equiv \sqrt{1 + \frac{b^2}{b_0^2}}; \quad b_0 = \frac{|Z_1 Z_2| e^2}{4\pi\epsilon_0 M g^2}. \quad (2.19)$$

B. Show that $\tan \theta_0 = b/b_0$ at the point of closest approach!

C. Show that the Coulomb or Rutherford scattering cross section is given by

$$\sigma = \frac{b_0^2}{4 \sin^4 \left(\frac{\chi}{2} \right)} = \left(\frac{Z_1 Z_2 e^2}{8\pi\epsilon_0 M g \sin^2 \frac{\chi}{2}} \right)^2.$$

Solution

A. The electron experiences the (attractive) force

$$\mathbf{F}(r) = q_1 \mathbf{E}(r) = \frac{e^2 Z_1 Z_2}{4\pi\epsilon_0} \frac{1}{r^2} \mathbf{e}_r = -\frac{e^2 |Z_1 Z_2|}{4\pi\epsilon_0} \frac{1}{r^2} \mathbf{e}_r \quad (2.20)$$

in the Coulomb field of the ion (see Fig. 2.1). Note that Z_1 and Z_2 have opposite signs (attractive force) and therefore the minus sign appears by taking the absolute value of Z_1 and Z_2 . The potential energy at a particular distance r is then given by

$$V(r) = \int_r^\infty \mathbf{F}(r') dr' = -\frac{e^2 |Z_1 Z_2|}{4\pi\epsilon_0} \int_r^\infty \frac{1}{r'^2} dr' = -\frac{e^2 |Z_1 Z_2|}{4\pi\epsilon_0} \frac{1}{r}, \quad (2.21)$$

where we used the fact that the potential energy for $r = \infty$ vanishes, i.e., $V(r = \infty) = 0$. Since the parameter b_0 is per definition a positive number, see Eq. (2.19), we find

$$\frac{V(r)}{M g^2} = -\frac{b_0}{r}. \quad (2.22)$$

By substituting Eq. (2.22) into Eq. (2.17) from the previous problem we obtain

$$\frac{d\theta}{dr} = \pm \frac{b}{r^2} \left[1 + 2 \frac{b_0}{r} - \frac{b^2}{r^2} \right]^{-1/2}. \quad (2.23)$$

Note that Eq. (2.17) was derived under a certain configuration in which the negative root denotes an incoming particle and the positive root denotes an outgoing particle. Now, according to Fig. 2.1 the coordinate system has been rotated so that the configuration is symmetric and each direction can refer to both an incoming or outgoing particle. We have to distinguish between the following two cases:

- a positive root refers to a particle moving into negative y direction
- a negative root refers to a particle moving into positive y direction.

This can easily be understood from Fig. 2.1. For a particle that moves in a negative y direction the angle $\theta(r)$ is always increasing (positive sign), while for a particle moving in a positive y direction the angle $\theta(r)$ is always decreasing (negative sign). The point of closest approach, r_0 , is given by $d\theta/dr = 0$, which means that the square root of Eq. (2.23) has to be zero,

$$r_0^2 + 2b_0r_0 - b^2 = 0. \quad (2.24)$$

The quadratic equation can easily be solved by

$$r_0 = -b_0 \pm \sqrt{b_0^2 + b^2}. \quad (2.25)$$

Since, per definition, the radial distance cannot be negative we have to choose the positive root, $r_0 = -b_0 + \sqrt{b_0^2 + b^2}$! By rearranging the root we can express the minimal distance in terms of the eccentricity (2.19),

$$r_0 = b_0(\epsilon - 1). \quad (2.26)$$

According to Fig. 2.1 we find that the angle under which the closest approach occurs is given by $\theta(r_0) = \pi$. (Note that the angle θ is taken anticlockwise from the positive x axis, the mathematical correct direction.) By considering a particle moving into negative y direction (starting from r_0) we can derive from Eq. (2.23) by choosing the positive root

$$\int_{\pi}^{\pi+\theta(r)} d\theta = \int_{r_0}^r \frac{b}{r'^2} \left[1 + 2\frac{b_0}{r'} - \frac{b^2}{r'^2} \right]^{-1/2} dr'.$$

With the substitution $u = b/r'$ we find (according to [2], Eq. (2.261))

$$\theta(r) = - \int_{b/r_0}^{b/r} du \frac{1}{\sqrt{1 + 2\frac{b_0}{b}u - u^2}} = - \arcsin \left(\frac{u - \frac{b_0}{b}}{\sqrt{1 + \frac{b_0^2}{b^2}}} \right) \Bigg|_{b/r_0}^{b/r}.$$

The denominator in the argument of the arcsin function can be expressed through the eccentricity,

$$\sqrt{1 + \frac{b_0^2}{b^2}} = \frac{b_0}{b} \sqrt{1 + \frac{b^2}{b_0^2}} = \frac{b_0}{b} \epsilon.$$

By using the identity $\arcsin y = \pi/2 - \arccos(y)$ we find

$$\theta(r) = -\arcsin\left(\frac{\frac{b}{b_0}u - 1}{\epsilon}\right)\Big|_{b/r_0}^{b/r} = \left[\arccos\left(\frac{\frac{b}{b_0}u - 1}{\epsilon}\right) - \frac{\pi}{2}\right]_{b/r_0}^{b/r}.$$

The angle θ is then given by

$$\theta(r) = \arccos\left(\frac{\frac{b^2}{b_0 r} - 1}{\epsilon}\right) - \arccos\left(\frac{\frac{b^2}{b_0 r_0} - 1}{\epsilon}\right).$$

Let us consider now the last term on the right side. More specifically, we want to calculate the argument of the arccos function. Therefore, we substitute r_0 (see Eq. 2.26) and we find

$$\frac{b^2}{b_0 r_0 \epsilon} - \frac{1}{\epsilon} = \frac{b^2}{b_0^2 \epsilon (\epsilon - 1)} - \frac{1}{\epsilon} = \frac{\epsilon^2 - 1}{\epsilon (\epsilon - 1)} - \frac{1}{\epsilon} = \frac{\epsilon^2 - 1}{(\epsilon^2 - \epsilon)} - \frac{(\epsilon - 1)}{(\epsilon^2 - \epsilon)} = 1,$$

where we used $b^2/b_0^2 = \epsilon^2 - 1$ in the second step, see also Eq. (2.19). Since $\arccos(1) = 0$ we find immediately

$$\theta(r) = \arccos\left[\frac{1}{\epsilon} \left(\frac{b^2}{r b_0} - 1\right)\right],$$

which leads to

$$\frac{b^2}{r b_0} = 1 + \epsilon \cos \theta. \quad (2.27)$$

- B. From Fig. 2.1 we can deduce that for $r \rightarrow \infty$ the angle $\theta \rightarrow \pi \pm \theta_0$. Note that this is valid for both, a particle moving in positive *and* negative y-direction. Letting $r \rightarrow \infty$ in Eq. (2.27) we find that

$$0 = 1 + \epsilon \cos(\pi \pm \theta_0). \quad \implies \quad \cos(\theta_0) = \frac{1}{\epsilon},$$

since $\cos(\pi \pm x) = -\cos(x)$. It is an easy matter to show then that

$$\tan(\theta_0) = \frac{\sqrt{1 - \cos^2 \theta_0}}{\cos \theta_0} = \sqrt{\epsilon^2 - 1} = \frac{b}{b_0}.$$

C. The cross section is defined by

$$\sigma = \frac{b}{\sin \chi} \left| \frac{db}{d\chi} \right|, \quad (2.28)$$

where χ is the scattering angle (i.e., the angle between the two asymptotes, which describe an incoming/outgoing particle). We know that the point of closest approach is defined by the angle θ_0 with (compare with Fig. 2.1)

$$\tan \theta_0 = \frac{b}{b_0} \quad \text{and} \quad \theta_0 = \frac{\pi}{2} - \frac{\chi}{2}.$$

By combining both equations we obtain

$$\begin{aligned} b &= b_0 \tan \left(\frac{\pi}{2} - \frac{\chi}{2} \right) \\ \left| \frac{db}{d\chi} \right| &= \left| -\frac{b_0}{2} \frac{1}{\cos^2 \left(\frac{\pi}{2} - \frac{\chi}{2} \right)} \right| = \frac{b_0}{2} \frac{1}{\cos^2 \left(\frac{\pi}{2} - \frac{\chi}{2} \right)}. \end{aligned} \quad (2.29)$$

Inserting the last two equations in Eq. (2.28) we obtain for the cross section

$$\sigma = \frac{b_0^2}{2} \frac{\tan \left(\frac{\pi}{2} - \frac{\chi}{2} \right)}{\sin \chi} \frac{1}{\cos^2 \left(\frac{\pi}{2} - \frac{\chi}{2} \right)}. \quad (2.30)$$

By using $\tan x = \sin x / \cos x$ and

$$\begin{aligned} \cos \left(\frac{\pi}{2} - \frac{\chi}{2} \right) &= \sin \left(\frac{\chi}{2} \right) \\ \sin \left(\frac{\pi}{2} - \frac{\chi}{2} \right) &= \cos \left(\frac{\chi}{2} \right) \\ \sin \chi &= 2 \sin \left(\frac{\chi}{2} \right) \cos \left(\frac{\chi}{2} \right) \end{aligned} \quad (2.31)$$

we find the Rutherford cross section

$$\sigma = \frac{b_0^2}{4} \frac{1}{\sin^4 \left(\frac{\chi}{2} \right)}. \quad (2.32)$$

2.3 The Boltzmann Equation and the Fluid Equations

The equations describing the conservation of mass, momentum and energy are

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = \frac{\partial n}{\partial t} + \sum_i \frac{\partial}{\partial x_i}(nu_i) = 0 \quad (2.33)$$

$$mn \left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} \right) = - \sum_j \frac{\partial p_{ij}}{\partial x_j} \quad (2.34)$$

$$\frac{\partial}{\partial t} \left[mn \left(e + \frac{u^2}{2} \right) \right] + \sum_i \frac{\partial}{\partial x_i} \left[mnu_i \left(e + \frac{u^2}{2} \right) + \sum_j u_j p_{ij} + q_i \right] = 0. \quad (2.35)$$

The equations for conservation of mass and energy are scalars; the equation for conservation of momentum describes the i -th component of that vector.

The Maxwell-Boltzmann distribution is given by

$$f(\mathbf{x}, \mathbf{v}, t) = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left[-m \frac{(\mathbf{v} - \mathbf{u})^2}{2k_B T} \right], \quad (2.36)$$

where k_B is Boltzmann's constant, $\mathbf{u}$, n , and T are the bulk velocity, number density, and temperature of the gas, respectively.

Problem 2.7 By using the conservation equations for mass, momentum and energy, derive the evolution equation for p_{ij} assuming the flow is smooth, i.e., the flow has no discontinuities like shock waves or contact discontinuities.

Solution The idea is as follows: To derive an evolution equation for the pressure tensor we start from the equation for conservation of energy (2.35) and transform that equation in such a way that the conservation of mass and momentum can be used for simplification. By expanding Eq. (2.35) we obtain

$$\begin{aligned} m \frac{\partial ne}{\partial t} + \frac{m}{2} \frac{\partial nu^2}{\partial t} + m \sum_i \frac{\partial (nu_i e)}{\partial x_i} \\ + \frac{m}{2} \sum_i \frac{\partial (nu_i u^2)}{\partial x_i} + \sum_i \sum_j \frac{\partial u_j p_{ij}}{\partial x_i} + \sum_i \frac{\partial q_i}{\partial x_i} = 0. \end{aligned} \quad (2.37)$$

Here we used the fact that the mass m is constant, i.e., independent of time and space coordinates. Consider now the second and fourth term in that equation. By expanding we obtain

$$\begin{aligned}
 & \frac{m}{2} \left(\frac{\partial n u^2}{\partial t} + \sum_i \frac{\partial (n u_i u^2)}{\partial x_i} \right) \\
 &= \frac{m}{2} \left(u^2 \frac{\partial n}{\partial t} + n \frac{\partial u^2}{\partial t} + u^2 \sum_i \frac{\partial (n u_i)}{\partial x_i} + \sum_i n u_i \frac{\partial u^2}{\partial x_i} \right) \\
 &= \frac{m}{2} \left(n \frac{\partial u^2}{\partial t} + \sum_i n u_i \frac{\partial u^2}{\partial x_i} \right), \tag{2.38}
 \end{aligned}$$

where we used the continuity equation (2.33) for the summation of the first and third term in the second line. By using $u^2 = u_x^2 + u_y^2 + u_z^2 = \sum_j u_j^2$ for the partial derivatives with respect to t and x_i and by pulling out the number density n we can simplify Eq. (2.38) to obtain

$$\frac{m}{2} \left(\frac{\partial n u^2}{\partial t} + \sum_i \frac{\partial (n u_i u^2)}{\partial x_i} \right) = n m \left(\sum_j u_j \frac{\partial u_j}{\partial t} + \sum_i \sum_j u_i u_j \frac{\partial u_j}{\partial x_i} \right).$$

We change now the indices, i.e., $i \leftrightarrow j$. The change of indices has no influence on the summation. We then pull out the summation over index i multiplied by u_i and obtain

$$\begin{aligned}
 \frac{m}{2} \left(\frac{\partial n u^2}{\partial t} + \sum_i \frac{\partial (n u_i u^2)}{\partial x_i} \right) &= n m \sum_i u_i \left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} \right) \\
 &= - \sum_i \sum_j u_i \frac{\partial p_{ij}}{\partial x_j},
 \end{aligned}$$

where we used the equation of momentum conservation (2.34). We substitute this result back into Eq. (2.37) and obtain

$$m \frac{\partial n e}{\partial t} + m \sum_i \frac{\partial (n u_i e)}{\partial x_i} + \sum_i \sum_j \frac{\partial u_j p_{ij}}{\partial x_i} - \sum_i \sum_j u_i \frac{\partial p_{ij}}{\partial x_j} + \sum_i \frac{\partial q_i}{\partial x_i} = 0. \tag{2.39}$$

Consider now the third term

$$\begin{aligned} \sum_i \sum_j \frac{\partial u_j p_{ij}}{\partial x_i} &= \sum_i \sum_j u_j \frac{\partial p_{ij}}{\partial x_i} + \sum_i \sum_j p_{ij} \frac{\partial u_j}{\partial x_i} \\ &= \sum_i \sum_j u_i \frac{\partial p_{ij}}{\partial x_j} + \sum_i \sum_j p_{ij} \frac{\partial u_j}{\partial x_i}. \end{aligned} \quad (2.40)$$

Note that we changed the summation index in the first term of the last line; this has no influence on the result. We also used the fact that the pressure tensor is symmetric, i.e., $p_{ij} = p_{ji}$, which can be seen from the definition of the pressure tensor (we refer to [5] for further details). With this result Eq.(2.39) can be simplified and we obtain

$$m \frac{\partial ne}{\partial t} + m \sum_i \frac{\partial (nu_i e)}{\partial x_i} + \sum_i \sum_j p_{ij} \frac{\partial u_j}{\partial x_i} + \sum_i \frac{\partial q_i}{\partial x_i} = 0. \quad (2.41)$$

The second term can be described by

$$m \sum_i \frac{\partial (nu_i e)}{\partial x_i} = m \sum_i u_i \frac{\partial ne}{\partial x_i} + m \sum_i ne \frac{\partial u_i}{\partial x_i} \quad (2.42)$$

giving

$$m \frac{\partial ne}{\partial t} + m \sum_i u_i \frac{\partial ne}{\partial x_i} + m \sum_i ne \frac{\partial u_i}{\partial x_i} + \sum_i \sum_j p_{ij} \frac{\partial u_j}{\partial x_i} + \sum_i \frac{\partial q_i}{\partial x_i} = 0.$$

By multiplying the last equation by 2 and substitute $2mne = \sum_j p_{jj}$ we obtain

$$\begin{aligned} \sum_j \frac{\partial p_{jj}}{\partial t} + \sum_i \sum_j u_i \frac{\partial p_{ij}}{\partial x_i} + \sum_i \sum_j p_{ij} \frac{\partial u_i}{\partial x_i} \\ + 2 \sum_i \sum_j p_{ij} \frac{\partial u_j}{\partial x_i} + 2 \sum_i \frac{\partial q_i}{\partial x_i} = 0. \end{aligned} \quad (2.43)$$

This equation can be interpreted as the evolution equation for the pressure tensor.

Problem 2.8 Use the Maxwell-Boltzmann distribution (2.36) to show that the definitions for (A) the number density n , (B) the bulk velocity $\mathbf{u}$, (C) the temperature T , and (D) the pressure tensor p_{ij} do indeed yield these quantities, and that the pressure tensor can be expressed as $p_{ij} = p(\mathbf{x}, t)\delta_{ij}$. Show too that (E) the heat flux $\mathbf{q}$ vanishes.

Solution The Maxwell-Boltzmann distribution is given by Eq. (2.36). In all following calculations we will make use of

$$\int_{-\infty}^{\infty} x^\eta e^{-\beta x^2} dx = 0 \quad \text{for } x = 1, 3, 5, 7, \dots, \quad (2.44)$$

since the Maxwell-Boltzmann distribution satisfies this relation for the expected value $E[X^\eta]$. For more details see also Problem 2.12 later in this chapter, where the expected values are calculated from the moment-generating function.

A. The number density is defined as

$$n = \int f(\mathbf{x}, \mathbf{v}, t) d^3v.$$

With the substitution $\mathbf{c} = \mathbf{v} - \mathbf{u}$ and $c^2 = c_x^2 + c_y^2 + c_z^2$ we find

$$\begin{aligned} \int f(\mathbf{x}, \mathbf{v}, t) d^3v &= n \left(\frac{m}{2\pi kT} \right)^{3/2} \int \exp \left[-m \frac{(\mathbf{v} - \mathbf{u})^2}{2kT} \right] d^3v \\ &= n \left(\frac{m}{2\pi kT} \right)^{3/2} \int \exp \left[-\frac{m\mathbf{c}^2}{2kT} \right] d^3c = n. \end{aligned}$$

B. The bulk velocity is defined by

$$n\mathbf{u} = \int \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d^3v.$$

Consider the i -th component (i.e., nu_i) for which we have

$$\int v_i f(\mathbf{x}, \mathbf{v}, t) d^3v = n \left(\frac{m}{2\pi kT} \right)^{3/2} \int v_i \exp \left[-m \frac{(\mathbf{v} - \mathbf{u})^2}{2kT} \right] d^3v.$$

By using again the substitution $c_i = v_i - u_i$ we find

$$\begin{aligned} \int v_i f(\mathbf{x}, \mathbf{v}, t) d^3v &= n \left(\frac{m}{2\pi kT} \right)^{3/2} \int (c_i + u_i) \exp \left[-\frac{m\mathbf{c}^2}{2kT} \right] d^3c \\ &= n \left(\frac{m}{2\pi kT} \right)^{3/2} \left[\int c_i \exp \left[-\frac{m\mathbf{c}^2}{2kT} \right] d^3c + \int u_i \exp \left[-\frac{m\mathbf{c}^2}{2kT} \right] d^3c \right]. \end{aligned}$$

The first integral yields zero (see Eq. (2.44)) and the second integral can be solved with the help of part A. We find

$$\int v_i f(\mathbf{x}, \mathbf{v}, t) d^3v = nu_i \left(\frac{m}{2\pi kT} \right)^{3/2} \int \exp \left[-\frac{m\mathbf{c}^2}{2kT} \right] d^3c = nu_i.$$

C. The temperature T is defined by

$$\frac{3}{2}nkT = \frac{m}{2} \int (\mathbf{v} - \mathbf{u})^2 f(\mathbf{x}, \mathbf{v}, t) d^3v.$$

Again, by using $\mathbf{c} = \mathbf{v} - \mathbf{u}$ we find

$$\begin{aligned} \frac{m}{2} \int (\mathbf{v} - \mathbf{u})^2 f(\mathbf{x}, \mathbf{v}, t) d^3v &= nm \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{1}{2} \int \mathbf{c}^2 \exp \left[-\frac{m\mathbf{c}^2}{2kT} \right] d^3c \\ &= \frac{1}{2} nm \left(\frac{m}{2\pi kT} \right)^{3/2} 3 \frac{\pi^{3/2}}{2} \left(\frac{2kT}{m} \right)^{5/2} \\ &= \frac{3}{2} nkT, \end{aligned}$$

where the integration yields $(3/2)\pi^{3/2} (2kT/m)^{5/2}$. The factor 3 originates from the summation of $\mathbf{c}^2 = c_x^2 + c_y^2 + c_z^2$.

D. The pressure tensor is given by

$$p_{ij} = \int m(v_i - u_i)(v_j - u_j) f d^3v$$

With the substitution $c_i = v_i - u_i$ we obtain

$$p_{ij} = nm \left(\frac{m}{2\pi kT} \right)^{3/2} \int c_i c_j \exp \left[-\frac{m\mathbf{c}^2}{2kT} \right] d^3c.$$

With $d^3c = dc_i dc_j dc_k$ and $\mathbf{c}^2 = c_i^2 + c_j^2 + c_k^2$ we find that for $i \neq j$ the pressure tensor is zero (see part B). The integral only contributes for $i = j$ and we have

$$\begin{aligned} p_{ii} &= nm \left(\frac{m}{2\pi kT} \right)^{3/2} \int c_i^2 \exp \left[-\frac{m\mathbf{c}^2}{2kT} \right] d^3c \\ &= nm \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{\pi^{3/2}}{2} \left(\frac{2kT}{m} \right)^{5/2} \\ &= nkT. \end{aligned}$$

The pressure tensor can therefore be written as $p_{ij} = p\delta_{ij}$, where $p = nkT$ for ideal gases.

E. The i -th component of the heat flux vector is defined by

$$q_i = \frac{m}{2} \int (v_i - u_i)(\mathbf{v} - \mathbf{u})^2 f d^3v.$$

Substitute again $\mathbf{c} = \mathbf{v} - \mathbf{u}$ and we obtain

$$q_i = n \left(\frac{m}{2\pi kT} \right)^{3/2} \frac{m}{2} \int c_i c^2 \exp \left[-\frac{mc^2}{2kT} \right] d^3 c = 0.$$

The integral yields zero since we have integrals of the form (2.44) in each term.

Problem 2.9 Using the above results, derive the Euler equations.

Solution The *Euler equations* result from assuming that

$$p_{ij} = p(\mathbf{x}, t) \delta_{ij} \quad \text{and} \quad q_i = 0,$$

where $p(\mathbf{x}, t)$ is the scalar pressure. For the conservation of momentum we find from Eq. (2.34) for the i -th component

$$mn \left(\frac{\partial u_i}{\partial t} + \sum_j u_j \frac{\partial u_i}{\partial x_j} \right) = - \sum_j \frac{\partial p}{\partial x_j} \delta_{ij} = - \frac{\partial p}{\partial x_i}$$

or written as a vector

$$mn \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p.$$

For the energy conservation we use Eq. (2.43) and $\sum_j p_{jj} = 3p$ and find

$$\begin{aligned} 3 \frac{\partial p}{\partial t} + 3 \sum_i u_i \frac{\partial p}{\partial x_i} + 3p \sum_i \frac{\partial u_i}{\partial x_i} + 2 \sum_i \sum_j p \delta_{ij} \frac{\partial u_j}{\partial x_i} &= 0 \\ 3 \frac{\partial p}{\partial t} + 3 \sum_i u_i \frac{\partial p}{\partial x_i} + 3p \sum_i \frac{\partial u_i}{\partial x_i} + 2p \sum_i \frac{\partial u_i}{\partial x_i} &= 0 \\ 3 \frac{\partial p}{\partial t} + 3 \sum_i u_i \frac{\partial p}{\partial x_i} + 5p \sum_i \frac{\partial u_i}{\partial x_i} &= 0. \end{aligned}$$

If we divide by 3 and write the summations with the help of vectors we eventually find

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \frac{5}{3} p \nabla \cdot \mathbf{u} = 0.$$

Together with the continuity equation (2.33) we find the Euler equations,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0 \tag{2.45a}$$

$$mn \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p \quad (2.45b)$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p + \frac{5}{3} p \nabla \cdot \mathbf{u} = 0. \quad (2.45c)$$

Problem 2.10 Linearize the 1D Euler equations about the constant state $\Psi_0 = (n_0, u_0, p_0)$, i.e., consider perturbations $\delta\Psi$ such that $\Psi = \Psi_0 + \delta\Psi$. Derive a linear wave equation in terms of a single variable, say δn . Seek solutions to the linear wave equation in the form $\exp[i(\omega t - kx)]$, and show that the Euler equations admit a non-propagating zero-frequency wave and forward and backward propagating acoustic modes satisfying the dispersion relation $\omega' = \omega - u_0 k = \pm C_s k$, where C_s is a suitably defined sound speed.

Solution The 1D Euler equations are given by

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x} (n u_x) = 0 \quad (2.46a)$$

$$nm \left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} \right) = -\frac{\partial p}{\partial x} \quad (2.46b)$$

$$\frac{\partial p}{\partial t} + u_x \frac{\partial p}{\partial x} + \frac{5}{3} p \frac{\partial u_x}{\partial x} = 0. \quad (2.46c)$$

We linearize the 1D Euler equations about the constant state $\Psi_0 = (n_0, u_0, p_0)$, i.e., we consider a small perturbation so that $\Psi = \Psi_0 + \delta\Psi$. The number density n , the velocity u_x , and the pressure p are then given by

$$n = n_0 + \delta n \quad u_x = u_0 + \delta u \quad p = p_0 + \delta p.$$

We substitute these equations into the Euler equations and obtain

$$\frac{\partial \delta n}{\partial t} + u_0 \frac{\partial \delta n}{\partial x} + n_0 \frac{\partial \delta u}{\partial x} = 0 \quad (2.47a)$$

$$mn_0 \frac{\partial \delta u}{\partial t} + mn_0 u_0 \frac{\partial \delta u}{\partial x} = -\frac{\partial \delta p}{\partial x} \quad (2.47b)$$

$$\frac{\partial \delta p}{\partial t} + u_0 \frac{\partial \delta p}{\partial x} + \frac{5}{3} p_0 \frac{\partial \delta u}{\partial x} = 0. \quad (2.47c)$$

Note that we neglected terms of order $\delta\Psi^2$ and $\delta\Psi \partial \delta\Psi / \partial t$ or $\delta\Psi \partial \delta\Psi / \partial x$ and that the derivatives of a constant are zero.

Alternative 1 We assume now that the solutions of the differential equations have the form $\delta\Psi = \delta\Psi_0 \exp[i(\omega t - kx)]$, where $\delta\Psi = \delta n, \delta u, \delta p$, i.e.,

$$\delta n = \delta n_0 e^{i(\omega t - kx)} \quad \delta u = \delta u_0 e^{i(\omega t - kx)} \quad \delta p = \delta p_0 e^{i(\omega t - kx)}.$$

For the partial derivatives with respect to time t and space coordinate x we find in general

$$\frac{\partial \delta \Psi}{\partial t} = i\omega \delta \Psi \qquad \frac{\partial \delta \Psi}{\partial x} = -ik \delta \Psi.$$

The linearized Euler equations can then be written as

$$\begin{aligned} i\omega \delta n - iku_0 \delta n - ikn_0 \delta u &= 0 & (\omega - u_0 k) \delta n - kn_0 \delta u &= 0 \\ mn_0 i\omega \delta u - ikmn_0 u_0 \delta u &= ik\delta p \quad \Rightarrow & (\omega - u_0 k) \delta u - \frac{k}{mn_0} \delta p &= 0 \\ i\omega \delta p - iku_0 \delta p - ik\frac{5}{3}p_0 \delta u &= 0 & (\omega - u_0 k) \delta p - k\frac{5}{3}p_0 \delta u &= 0. \end{aligned}$$

By using $\omega' = \omega - u_0 k$ we introduce the matrix $\underline{\mathbf{A}}$ and the vector $\delta \Psi$,

$$\underline{\mathbf{A}} = \begin{pmatrix} \omega' & -n_0 k & 0 \\ 0 & \omega' & -\frac{k}{mn_0} \\ 0 & -\frac{5}{3}p_0 k & \omega' \end{pmatrix} \qquad \delta \Psi = \begin{pmatrix} \delta n \\ \delta u \\ \delta p \end{pmatrix},$$

so that the set of equations can be written as

$$\underline{\mathbf{A}} \cdot \delta \Psi = 0.$$

The trivial solution is, of course, $\delta \Psi = 0$. Non-trivial solutions are given by the eigenvalues of the matrix $\underline{\mathbf{A}}$. Therefore, we determine first the *characteristic equation* (also called the characteristic polynomial), $\det \underline{\mathbf{A}} = 0$,

$$\det \underline{\mathbf{A}} = \omega' \cdot \left[\omega'^2 - \frac{5}{3} \frac{k^2 p_0}{mn_0} \right] = 0.$$

The eigenvalues are given by the zeros of the characteristic equation, hence,

$$\omega'_1 = 0 \qquad \omega'_{2,3} = \pm \sqrt{\frac{5}{3} \frac{k_B T}{m}} k,$$

where we used $p_0 = n_0 k_B T$ (ideal gas law). The sound speed is defined as $C_s = \sqrt{\gamma k_B T / m}$ with the adiabatic index γ (which is $\gamma = 5/3$ for ideal gases). We find

$$\omega'_1 = 0 \qquad \omega'_{2,3} = \pm C_s k.$$

The Euler equations, indeed, admit a zero frequency (non-propagating) wave, and forward and backward propagating acoustic modes with sound speed C_s .

Alternative 2 Our starting point are again the linearized 1D Euler equations (2.47a)–(2.47c). By introducing the convective derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_0 \frac{\partial}{\partial x}, \quad (2.48)$$

we may write the linearized 1D Euler equations as

$$\begin{aligned} \frac{D\delta n}{Dt} + n_0 \frac{\partial \delta u}{\partial x} &= 0 \\ \frac{D\delta u}{Dt} + \frac{1}{n_0 m} \frac{\partial \delta p}{\partial x} &= 0 \\ \frac{D\delta p}{Dt} + \frac{5}{3} p_0 \frac{\partial \delta u}{\partial x} &= 0. \end{aligned}$$

This is a set of three differential equations with three *unknowns* (δn , δu , and δp). We want to solve that set of differential equation for δp . Therefore, we take the convective derivative of the first and third equation, and the partial derivative with respect to x of the second equation resulting in

$$\begin{aligned} \frac{D^2 \delta n}{Dt^2} + n_0 \frac{\partial}{\partial x} \frac{D\delta u}{Dt} &= 0 \\ \frac{\partial}{\partial x} \frac{D\delta u}{Dt} + \frac{1}{n_0 m} \frac{\partial^2 \delta p}{\partial x^2} &= 0 \\ \frac{D^2 \delta p}{Dt^2} + \frac{5}{3} p_0 \frac{\partial}{\partial x} \frac{D\delta u}{Dt} &= 0. \end{aligned}$$

We substitute now the term with δu in the second and third equation by using the first equation (continuity equation). We obtain a set of two differential equations,

$$\begin{aligned} -\frac{D^2 \delta n}{Dt^2} + \frac{1}{m} \frac{\partial^2 \delta p}{\partial x^2} &= 0 \\ \frac{D^2 \delta p}{Dt^2} - \frac{5}{3} \frac{p_0}{n_0} \frac{D^2 \delta n}{Dt^2} &= 0. \end{aligned}$$

Consider now an ideal gas with $p_0 = n_0 k_B T$. By substituting the term with δn in the second equation we obtain

$$\frac{D^2 \delta p}{Dt^2} - \frac{5}{3} \frac{k_B T}{m} \frac{\partial^2 \delta p}{\partial x^2} = \frac{D^2 \delta p}{Dt^2} - C_s^2 \frac{\partial^2 \delta p}{\partial x^2} = 0,$$

where we defined the sound speed C_s as above. This equation is a wave equation. By using the above definition of the convective derivative we obtain

$$\frac{D^2}{Dt^2} = \frac{\partial^2}{\partial t^2} + 2u_0 \frac{\partial^2}{\partial t \partial x} + u_0^2 \frac{\partial^2}{\partial x^2},$$

so that

$$\frac{\partial^2 \delta p}{\partial t^2} + 2u_0 \frac{\partial^2 \delta p}{\partial t \partial x} + u_0^2 \frac{\partial^2 \delta p}{\partial x^2} - C_s^2 \frac{\partial^2 \delta p}{\partial x^2} = 0. \quad (2.49)$$

We seek now solutions of the form $\delta p = \delta \bar{p} \exp [i(\omega t - kx)]$. We obviously find

$$\frac{\partial^2 \delta p}{\partial t^2} = -\omega^2 \delta p \quad \frac{\partial^2 \delta p}{\partial x^2} = -k^2 \delta p \quad \frac{\partial^2 \delta p}{\partial t \partial x} = \omega k \delta p,$$

so that Eq. (2.49) becomes

$$(-\omega^2 + 2u_0 \omega k - u_0^2 k^2) + C_s^2 k^2 = 0.$$

The term in brackets can be written as $(-\omega^2 + 2u_0 \omega k - u_0^2 k^2) = -(\omega - u_0 k)^2$ so that by using ω' as defined above

$$\omega'^2 \equiv (\omega - u_0 k)^2 = C_s^2 k^2 \quad \implies \quad \omega' \equiv \omega - u_0 k = \pm C_s k.$$

Note that ω' is the frequency of the wave seen by an observer who is co-moving with the background flow at speed u_0 , and ω is the frequency seen by an observer outside (not co-moving) with the background flow.

A wave (seen from an observer outside the co-moving frame) that travels with the same velocity as the background medium u_0 and with dispersion $\omega = u_0 k$ has a zero frequency in the co-moving frame

$$\omega' = u_0 k - u_0 k = 0,$$

and is therefore, non-propagating in that co-moving frame.

2.4 The Chapman-Enskog Expansion

We consider an expansion of the distribution function f about the equilibrium or Maxwellian distribution f_0 in the form

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots,$$

(continued)

where $f_1, f_2, \dots$ are successive corrections to f_0 . We consider also the force-free Boltzmann equation,

$$\frac{\partial f}{\partial t} + v_k \frac{\partial f}{\partial x_k} = -\nu(f - f_0),$$

where the collision operator on the right side is approximated by a scattering frequency ν . By using the above expansion for the distribution function we find

$$\frac{\partial f_0}{\partial t} + v_k \frac{\partial f_0}{\partial x_k} = -\nu f_1. \quad (2.50)$$

Since f_0 is the Maxwell-Boltzmann distribution we can evaluate the left side and find an expression for the first correction f_1 .

Problem 2.11 Complete the details for the derivation of the expressions above for $\partial f_0 / \partial t$ and $\partial f_0 / \partial x_k$. Use these results to complete the derivation of the expression for f_1 .

Solution The first correction f_1 to the Maxwell-Boltzmann distribution f_0 can be calculated by using Eq. (2.50), where the Maxwell-Boltzmann distribution (2.36) is given by

$$f_0 = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \exp \left[-\frac{m(\mathbf{v} - \mathbf{u})^2}{2k_B T} \right]. \quad (2.51)$$

Note that the density n , velocity $\mathbf{u}$, and temperature T are functions of time and space, so that $n = n(\mathbf{x}, t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$, and $T = T(\mathbf{x}, t)$. By introducing these new variables we have to transform the derivatives according to the new time and spatial dependencies of the new variables (see also Problems 2.1 and 2.2).

The idea is as follows: First, we derive the transformations for the time derivative and the spatial derivative, respectively. Finally, these expressions will be substituted back into Eq. (2.50), which will lead to an expression for the first correction in terms of f_0 .

A. The time derivative transforms as follows

$$\frac{\partial f_0}{\partial t} = \frac{\partial f_0}{\partial n} \frac{\partial n}{\partial t} + \frac{\partial f_0}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial t} + \frac{\partial f_0}{\partial T} \frac{\partial T}{\partial t}, \quad (2.52)$$

where (using the definition of the Maxwell-Boltzmann distribution)

$$\frac{\partial f_0}{\partial n} = \frac{f_0}{n}, \quad (2.53a)$$

$$\frac{\partial f_0}{\partial \mathbf{u}} = f_0 \frac{m(\mathbf{v} - \mathbf{u})}{k_B T}, \quad (2.53b)$$

$$\frac{\partial f_0}{\partial T} = -\frac{3}{2} \frac{1}{T} f_0 + \frac{m(\mathbf{v} - \mathbf{u})^2}{2k_B T} \frac{1}{T} f_0. \quad (2.53c)$$

All derivatives can be expressed in terms of the Maxwell-Boltzmann distribution f_0 . By substituting these results into Eq. (2.52) we obtain

$$\frac{\partial f_0}{\partial t} = f_0 \left[\frac{1}{n} \frac{\partial n}{\partial t} + \frac{m(\mathbf{v} - \mathbf{u})}{k_B T} \frac{\partial \mathbf{u}}{\partial t} - \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial t} + \frac{m(\mathbf{v} - \mathbf{u})^2}{2k_B T} \frac{1}{T} \frac{\partial T}{\partial t} \right].$$

By substituting $\mathbf{c} = \mathbf{v} - \mathbf{u}$ and by using a component description we find for the i -th component

$$\frac{\partial f_0}{\partial t} = f_0 \left[\frac{1}{n} \frac{\partial n}{\partial t} + \frac{mc_i}{k_B T} \frac{\partial u_i}{\partial t} - \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial t} + \frac{mc^2}{2k_B T} \frac{1}{T} \frac{\partial T}{\partial t} \right]. \quad (2.54)$$

Equation (2.54) includes time derivatives of the density n , velocity u_i , and temperature T . To replace these time derivatives we use the Euler equations (2.45a)–(2.45c) (with Einstein's summation convention),

$$\begin{aligned} \frac{\partial n}{\partial t} + \frac{\partial}{\partial x_k} (nu_k) &= 0 \\ mn \left(\frac{\partial u_i}{\partial t} + u_k \frac{\partial u_i}{\partial x_k} \right) &= -\frac{\partial p}{\partial x_k} \\ \frac{\partial p}{\partial t} + u_k \frac{\partial p}{\partial x_k} + \frac{5}{3} p \frac{\partial u_k}{\partial x_k} &= 0. \end{aligned}$$

With $p = nk_B T$ we obtain

$$\begin{aligned} \frac{1}{n} \frac{\partial n}{\partial t} &= -\frac{\partial u_k}{\partial x_k} - \frac{u_k}{n} \frac{\partial n}{\partial x_k} \\ \frac{\partial u_i}{\partial t} &= -u_k \frac{\partial u_i}{\partial x_k} - \frac{k_B T}{mn} \frac{\partial n}{\partial x_i} - \frac{k_B}{m} \frac{\partial T}{\partial x_i} \\ \frac{1}{T} \frac{\partial T}{\partial t} &= -\frac{2}{3} \frac{\partial u_k}{\partial x_k} - \frac{u_k}{T} \frac{\partial T}{\partial x_k}. \end{aligned}$$

For the last equation we used the continuity equation to eliminate the terms proportional to $\partial n/\partial t$ and $\partial n/\partial x_k$. By substituting these results back into Eq. (2.54) we obtain

$$\begin{aligned} \frac{1}{f_0} \frac{\partial f_0}{\partial t} = & -\frac{u_k}{n} \frac{\partial n}{\partial x_k} - u_k \frac{mc_i}{k_B T} \frac{\partial u_i}{\partial x_k} - \frac{c_i}{n} \frac{\partial n}{\partial x_i} - \frac{c_i}{T} \frac{\partial T}{\partial x_i} \\ & + \frac{3}{2} \frac{u_k}{T} \frac{\partial T}{\partial x_k} - \frac{2}{3} \frac{mc^2}{2k_B T} \frac{\partial u_k}{\partial x_k} - \frac{mc^2}{2k_B T} \frac{u_k}{T} \frac{\partial T}{\partial x_k}. \end{aligned} \quad (2.55)$$

Equation (2.55) is just the transformation of the time derivative in Eq. (2.50).

- B. We also need to calculate the spatial derivatives in Eq. (2.50), which is done similarly to the time derivative (2.52) and we obtain

$$\begin{aligned} \frac{\partial f_0}{\partial x_k} = & \frac{\partial f_0}{\partial n} \frac{\partial n}{\partial x_k} + \frac{\partial f_0}{\partial \mathbf{u}} \frac{\partial \mathbf{u}}{\partial x_k} + \frac{\partial f_0}{\partial T} \frac{\partial T}{\partial x_k} \\ = & f_0 \left[\frac{1}{n} \frac{\partial n}{\partial x_k} + \frac{m(\mathbf{v} - \mathbf{u})}{k_B T} \frac{\partial \mathbf{u}}{\partial x_k} - \frac{3}{2} \frac{1}{T} \frac{\partial T}{\partial x_k} + \frac{m(\mathbf{v} - \mathbf{u})^2}{2k_B T} \frac{1}{T} \frac{\partial T}{\partial x_k} \right]. \end{aligned}$$

where we used Eqs. (2.53a)–(2.53c) to replace the derivatives of the Maxwell-Boltzmann distribution, as before. By multiplying this equation with v_k and dividing by f_0 we obtain

$$\frac{v_k}{f_0} \frac{\partial f_0}{\partial x_k} = \frac{v_k}{n} \frac{\partial n}{\partial x_k} + \frac{m}{k_B T} c_i v_k \frac{\partial u_i}{\partial x_k} - \frac{3}{2} \frac{1}{T} v_k \frac{\partial T}{\partial x_k} + \frac{mc^2}{2k_B T} \frac{1}{T} v_k \frac{\partial T}{\partial x_k}. \quad (2.56)$$

We have also used $\mathbf{c} = \mathbf{v} - \mathbf{u}$ and substituted $v_k = c_k + u_k$.

Now we substitute the results (2.55) and (2.56) back into Eq. (2.50) and obtain the somewhat lengthy expression

$$\begin{aligned} -v \frac{f_1}{f_0} = & -\frac{u_k}{n} \frac{\partial n}{\partial x_k} - u_k \frac{mc_i}{k_B T} \frac{\partial u_i}{\partial x_k} - \frac{c_i}{n} \frac{\partial n}{\partial x_i} - \frac{c_i}{T} \frac{\partial T}{\partial x_i} + \frac{3}{2} \frac{u_k}{T} \frac{\partial T}{\partial x_k} \\ & - \frac{2}{3} \frac{mc^2}{2k_B T} \frac{\partial u_k}{\partial x_k} - \frac{mc^2}{2k_B T} \frac{u_k}{T} \frac{\partial T}{\partial x_k} + \frac{c_k + u_k}{n} \frac{\partial n}{\partial x_k} \\ & + \frac{m}{k_B T} c_i (c_k + u_k) \frac{\partial u_i}{\partial x_k} - \frac{3}{2} \frac{1}{T} (c_k + u_k) \frac{\partial T}{\partial x_k} + \frac{mc^2}{2k_B T} \frac{1}{T} (c_k + u_k) \frac{\partial T}{\partial x_k}. \end{aligned}$$

Note that $c_i \partial/\partial x_i = c_k \partial/\partial x_k$, since we use Einstein's summation convention. After some simplifications we obtain

$$-v \frac{f_1}{f_0} = -\frac{2}{3} \frac{mc^2}{2k_B T} \frac{\partial u_k}{\partial x_k} + \frac{m}{k_B T} c_i c_k \frac{\partial u_i}{\partial x_k} - \frac{5}{2} \frac{1}{T} c_k \frac{\partial T}{\partial x_k} + \frac{mc^2}{2k_B T} \frac{1}{T} c_k \frac{\partial T}{\partial x_k}.$$

From this equation it follows that

$$-vf_1 = f_0 \left[\frac{m}{k_B T} \left(c_i c_k - \frac{1}{3} c^2 \delta_{ik} \right) \frac{\partial u_i}{\partial x_k} + c_k \left(\frac{mc^2}{2k_B T} - \frac{5}{2} \right) \frac{1}{T} \frac{\partial T}{\partial x_k} \right]. \quad (2.57)$$

The first correction can be described through the Maxwell-Boltzmann distribution function.

Problem 2.12 Consider the 1D pdf

$$f(x) = \sqrt{\frac{\beta}{\pi}} e^{-\beta x^2} \quad \text{for } -\infty < x < \infty. \quad (2.58)$$

(A) Show that the moment-generating function is given by $M(t) = \exp(t^2/4\beta)$. (B) Derive the expectations $E(X)$, $E(X^2)$, $E(X^3)$, $E(X^4)$, $E(X^5)$ and $E(X^6)$. (C) Hence show that one obtains the integrals

$$\frac{\sqrt{\pi}}{2\beta^{3/2}} = \int_{-\infty}^{\infty} x^2 e^{-\beta x^2} dx \quad (2.59a)$$

$$\frac{3\sqrt{\pi}}{4\beta^{5/2}} = \int_{-\infty}^{\infty} x^4 e^{-\beta x^2} dx \quad (2.59b)$$

$$\frac{15\sqrt{\pi}}{8\beta^{7/2}} = \int_{-\infty}^{\infty} x^6 e^{-\beta x^2} dx. \quad (2.59c)$$

Solution

A. The moment-generating function is defined by Eq. (1.15), so that

$$M(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx = \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} e^{-\beta x^2 + tx} dx.$$

Consider the exponent. With the transformation

$$-\beta x^2 + tx = -\beta \left(x - \frac{t}{2\beta} \right)^2 + \frac{t^2}{4\beta} = -\beta y^2 + \frac{t^2}{4\beta},$$

where we substituted $y = x - t/2\beta$, we obtain

$$M(t) = \sqrt{\frac{\beta}{\pi}} \exp\left(\frac{t^2}{4\beta}\right) \int_{-\infty}^{\infty} e^{-\beta y^2} dy = \exp\left(\frac{t^2}{4\beta}\right),$$

since the integration yields $\sqrt{\pi/\beta}$.

B. The n -th moment (or the n -th derivative) of the moment-generating function is given by

$$M^n(t) = \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} x^n e^{-\beta x^2 + tx} dx = \frac{d^n}{dt^n} \left[\exp \left(\frac{t^2}{4\beta} \right) \right].$$

Since $E(X^n) = M^n(t=0)$ we find with $M(0) = 1$ the following results

$$\begin{aligned} M'(t) &= \frac{t}{2\beta} M(t) & E(X) &= 0 \\ M''(t) &= \left(\frac{1}{4} \frac{t^2}{\beta^2} + \frac{1}{2\beta} \right) M(t) & E(X^2) &= \frac{1}{2\beta} \\ M'''(t) &= \left(\frac{3}{4} \frac{t}{\beta^2} + \frac{1}{8} \frac{t^3}{\beta^3} \right) M(t) & E(X^3) &= 0 \\ M^4(t) &= \left(\frac{3}{4} \frac{1}{\beta^2} + \frac{3}{4} \frac{t^2}{\beta^3} + \frac{1}{16} \frac{t^4}{\beta^4} \right) M(t) & E(X^4) &= \frac{3}{4} \frac{1}{\beta^2} \\ M^5(t) &= \left(\frac{15}{8} \frac{t}{\beta^3} + \frac{5}{8} \frac{t^3}{\beta^4} + \frac{1}{32} \frac{t^5}{\beta^5} \right) M(t) & E(X^5) &= 0 \\ M^6(t) &= \left(\frac{15}{8} \frac{1}{\beta^3} + \frac{45}{16} \frac{t^2}{\beta^4} + \frac{15}{32} \frac{t^4}{\beta^5} + \frac{1}{64} \frac{t^6}{\beta^6} \right) M(t) & E(X^6) &= \frac{15}{8} \frac{1}{\beta^3}. \end{aligned}$$

C. With $E(X^n) = M^n(t=0)$ it follows immediately that

$$\begin{aligned} E(X^n) &= \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{\infty} x^n e^{-\beta x^2} dx \\ \Rightarrow \int_{-\infty}^{\infty} x^n e^{-\beta x^2} dx &= \sqrt{\frac{\pi}{\beta}} E(X^n). \end{aligned} \quad (2.60)$$

By combining the expectations derived in part (B) with Eq. (2.60) we find the equations given by (2.59a)–(2.59c).

Problem 2.13 Show that the Chapman-Enskog expression for f_1 satisfies the constraints

$$\int f_1 d^3v = 0 \quad \int c f_1 d^3v = 0 \quad \int c^2 f_1 d^3v = 0 \quad (2.61)$$

Solution The first order correction term of the Chapman-Enskog expansion is given by

$$f_1 = -\frac{f_0}{v} \left[\frac{m}{k_B T} \left(c_i c_k - \frac{1}{3} c^2 \delta_{ik} \right) \frac{\partial u_i}{\partial x_k} + c_k \left(\frac{mc^2}{2k_B T} - \frac{5}{2} \right) \frac{1}{T} \frac{\partial T}{\partial x_k} \right], \quad (2.62)$$

with the scattering frequency ν and the Maxwell-Boltzmann distribution f_0 (compare with Eq. (2.57)). For the sake of brevity we write the Maxwell-Boltzmann distribution function as

$$f_0 = n \left(\frac{\beta}{\pi} \right)^{3/2} e^{-\beta c^2}, \quad (2.63)$$

where $\beta = m/(2kT)$ and $\mathbf{c} = \mathbf{v} - \mathbf{u}$. This has the advantage that we can use the results of the previous Problem 2.12; compare with the probability density function given by Eq. (2.51). With $d^3v = d^3c$ the constraints can also be written as

$$\begin{aligned} 0 &= \int \mathbf{c}^\alpha f_1 d^3c \\ &= \frac{1}{\nu} \int \mathbf{c}^\alpha f_0 \left[\frac{m}{k_B T} \frac{\partial u_i}{\partial x_k} \left(c_i c_k - \frac{1}{3} c^2 \delta_{ik} \right) + \frac{1}{T} \frac{\partial T}{\partial x_k} c_k \left(\frac{mc^2}{2kT} - \frac{5}{2} \right) \right] d^3c. \end{aligned} \quad (2.64)$$

where $\alpha = 0, 1, 2$ for the zeroth, first, and second constraint respectively. In Eq. (2.64) the constants k_B, T, m and the partial derivatives with respect to x_k are independent of d^3c and, thus, the integral operates only on the terms in rounded brackets (which include the velocity c). Each of the integrals on the right hand side has to vanish independently, therefore, we consider both integrals separately. For simplicity we substitute f_0 by Eq. (2.63) and neglect all constant factors and obtain the following two conditions

$$A : \quad \int \mathbf{c}^\alpha e^{-\beta c^2} \left(c_i c_k - \frac{1}{3} c^2 \delta_{ik} \right) d^3c = 0 \quad (2.65a)$$

$$B : \quad \int \mathbf{c}^\alpha e^{-\beta c^2} c_k \left(\beta c^2 - \frac{5}{2} \right) d^3c = 0. \quad (2.65b)$$

Basically, the constraints (2.61) reduce to the two conditions A and B.

First Constraint: $\alpha = 0$

A. In this case the expression given by Eq. (2.65a) becomes

$$\int_{-\infty}^{\infty} e^{-\beta c^2} \left(c_i c_k - \frac{1}{3} c^2 \delta_{ik} \right) d^3c. \quad (2.66)$$

Here we have to distinguish between the two cases $i = k$ and $i \neq k$.

Case $i = k$: We find for Eq. (2.66)

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_j^2 + c_k^2)} \left[c_i^2 - \frac{1}{3}(c_i^2 + c_j^2 + c_k^2) \right] dc_i dc_j dc_k.$$

Note that the integral of each component within the squared brackets yields the same result, i.e., the integrals with c_j^2 and c_k^2 yield the same result as c_i^2 , hence we set $(c_i^2 + c_j^2 + c_k^2) = 3c_i^2$ and obtain

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_j^2 + c_k^2)} [c_i^2 - c_i^2] dc_i dc_j dc_k = 0.$$

Case $i \neq k$: In this case Eq. (2.66) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_j^2 + c_k^2)} c_i c_k dc_i dc_j dc_k = 0.$$

The result is zero due to odd orders of c under the integral, compare with Eq. (2.44).

B. Let us now consider the second integral given by Eq. (2.65b). For $\alpha = 0$ we obtain

$$\int_{-\infty}^{\infty} e^{-\beta c^2} c_k \left[\beta c^2 - \frac{5}{2} \right] d^3 c. \quad (2.67)$$

By expanding $c^2 = c_i^2 + c_j^2 + c_k^2$ we find

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_j^2 + c_k^2)} \left[\beta(c_k c_i^2 + c_k c_j^2 + c_k^3) - \frac{5}{2} c_k \right] dc_i dc_j dc_k = 0.$$

This integral is zero, where we used again Eq. (2.44).

Obviously, the first constraint is fulfilled.

Second Constraint: $\alpha = 1$ The result of this constraint is a vector, but for convenience we consider only the j -th component of the vector $\mathbf{c}$.

A. For the first integral we find

$$\int e^{-\beta c^2} \left(c_i c_j c_k - \frac{1}{3} c_j c^2 \delta_{ik} \right) d^3 c. \quad (2.68)$$

Here we have to distinguish the following cases:

Case $i = k = j$: In this case Eq. (2.68) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_l^2 + c_m^2)} \left[c_i^3 - \frac{1}{3} c_i (c_i^2 + c_l^2 + c_m^2) \right] dc_i dc_l dc_m = 0,$$

where we used the indices i, l, m to avoid confusion with the indices j, k . One can see immediately that the integral vanishes according to Eq. (2.44), since we have always an odd order of c_i under the integral.

Case $i = k \neq j$: In this case Eq. (2.68) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_j^2 + c_l^2)} \left[c_i^2 c_j - \frac{1}{3} c_j (c_i^2 + c_j^2 + c_l^2) \right] dc_i dc_j dc_l = 0,$$

which vanishes according to Eq. (2.44), since we have always odd orders of c_j under the integral.

Case $i \neq k = j$: In this case Eq. (2.68) becomes

$$\int e^{-\beta(c_i^2 + c_j^2 + c_l^2)} c_i c_j^2 dc_i dc_j dc_l = 0,$$

which vanishes also according to Eq. (2.44).

Case $i \neq k \neq j$ and $i \neq j$: In this last case, where all indices are mutually distinct, Eq. (2.68) becomes

$$\int e^{-\beta(c_i^2 + c_j^2 + c_k^2)} c_i c_j c_k dc_i dc_j dc_k = 0,$$

which vanishes also, because we have an odd order of c for all indices i, j , and k .

B. For the second integral we obtain (by using the j -th component of the vector)

$$\int e^{-\beta c^2} \left(\beta c_j c_k c^2 - \frac{5}{2} c_j c_k \right) d^3 c, \quad (2.69)$$

where we pulled c_j and c_k into the brackets. We have to distinguish between the cases $j = k$ and $j \neq k$.

Case $j = k$: In this case Eq. (2.69) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_j^2 + c_l^2 + c_m^2)} \left[\beta(c_j^2 + c_l^2 + c_m^2) c_j^2 - \frac{5}{2} c_j^2 \right] dc_j dc_l dc_m.$$

The first part of that integral becomes then

$$\beta \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_j^2 + c_l^2 + c_m^2)} (c_j^4 + c_j^2 c_l^2 + c_j^2 c_m^2) dc_j dc_l dc_m = \frac{5}{4} \frac{\pi^{3/2}}{\beta^{5/2}}.$$

Note that while evaluating the integral we can set $c_j^2 c_l^2 + c_j^2 c_m^2 = 2c_j^2 c_l^2$, since both integrals provide the same result. The second part of the integral yields

$$-\frac{5}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_j^2 e^{-\beta(c_j^2 + c_l^2 + c_m^2)} dc_j dc_l dc_m = -\frac{5}{4} \frac{\pi^{3/2}}{\beta^{5/2}}.$$

By adding up both parts we obtain

$$\int e^{-\beta c^2} \left(\beta c_j c_k c^2 - \frac{5}{2} c_j c_k \right) d^3 c = 0.$$

Case $j \neq k$: In this case Eq. (2.69) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_j^2 + c_k^2 + c_l^2)} \left[\beta(c_j^2 + c_k^2 + c_l^2) c_j c_k - \frac{5}{2} c_j c_k \right] dc_j dc_k dc_l.$$

Obviously, this integral yields zero according to Eq. (2.44), since we have odd orders of c_j and c_k under the integral.

Obviously, the second constraint is also fulfilled.

Third Constraint: $\alpha = 2$

A. In this case Eq. (2.65a) becomes

$$\int e^{-\beta c^2} \left(c^2 c_i c_k - \frac{1}{3} c^4 \delta_{ik} \right) d^3 c, \quad (2.70)$$

where we have pulled the c^2 into the brackets. Here we have to distinguish between the cases $i = k$ and $i \neq k$.

Case $i = k$: In this case Eq. (2.70) becomes

$$\int e^{-\beta c^2} \left(c^2 c_i^2 - \frac{1}{3} c^4 \right) d^3 c = \Lambda_1 + \Lambda_2.$$

Let us consider both terms separately and we obtain for the first part

$$\Lambda_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_l^2 + c_m^2)} (c_i^4 + c_i^2 c_l^2 + c_i^2 c_m^2) dc_i dc_l dc_m.$$

Note that the integration of the terms containing $c_i^2 c_l^2$ and $c_i^2 c_m^2$ yield the same result. Therefore we simplify the equation and obtain for the first term

$$\Lambda_1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_l^2 + c_m^2)} (c_i^4 + 2c_i^2 c_l^2) dc_i dc_l dc_m.$$

For the second part of the integral we consider first the expression c^4 which can be written as

$$\begin{aligned}(c_i^2 + c_l^2 + c_m^2)^2 &= c_i^4 + c_l^4 + c_m^4 + 2c_i^2 c_l^2 + 2c_i^2 c_m^2 + 2c_l^2 c_m^2 \\ &= 3c_i^4 + 6c_i^2 c_l^2,\end{aligned}$$

where we used the fact, that the integrations over c_i^4 , c_l^4 , and c_m^4 yield the same result as well as the integrations over $c_i^2 c_l^2$, $c_i^2 c_m^2$, and $c_l^2 c_m^2$. The second part of the integral can then be written as

$$\Lambda_2 = - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_l^2 + c_m^2)} \frac{1}{3} (3c_i^4 + 6c_i^2 c_l^2) dc_i dc_l dc_m.$$

Since $\Lambda_1 = -\Lambda_2$ we find that $\Lambda_1 + \Lambda_2 = 0$ and therefore the integral vanishes for $i = k$.

Case $i \neq k$: In this case Eq. (2.70) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\beta(c_i^2 + c_k^2 + c_l^2)} (c_i^2 + c_k^2 + c_l^2) c_i c_k dc_i dc_k dc_l = 0.$$

This integral vanishes also, since we have odd orders of c_i and c_k under the integral.

With both results we find

$$\int c^2 e^{-\beta c^2} \left(c_i c_k - \frac{1}{3} c^2 \delta_{ik} \right) d^3 c = 0.$$

B. For the second integral we find

$$\int e^{-\beta c^2} c_k \left(\beta c^4 - \frac{5}{2} c^2 \right) d^3 c = 0,$$

where we pulled c^2 into the brackets. One can see immediately that this integral vanishes, since we have always an odd order of c_k under the integral.

By adding both results up we find that the third constraint is also fulfilled. All results show that the first correction f_1 to the Maxwell-Boltzmann distribution f_0 indeed satisfies the above mentioned conditions (2.61).

Problem 2.14 Show that the terms $\propto (1/T) \partial T / \partial x_k$ in the pressure term of Eq. (2.57) vanish identically.

Solution The first correction to the pressure term is given by

$$p_{ij}^1 = m \int c_i c_j f_1 d^3 c.$$

According to Eq. (2.57) the term of f_1 that is proportional to $\propto (1/T)\partial T/\partial x_k$ is given by

$$\tilde{f}_1 = \frac{f_0}{v} c_k \left(\beta c^2 - \frac{5}{2} \right) \frac{1}{T} \frac{\partial T}{\partial x_k}.$$

Let's consider both terms separately. Since the temperature and the scattering frequency are independent of c we have

$$A : \quad \int_{-\infty}^{\infty} c_i c_j c_k c^2 e^{-\beta c^2} d^3 c = 0 \quad (2.71a)$$

$$B : \quad \int_{-\infty}^{\infty} c_i c_j c_k e^{-\beta c^2} d^3 c = 0, \quad (2.71b)$$

where we used the Maxwell-Boltzmann distribution function f_0 from Eq. (2.63). We consider first the integral of Eq. (2.71a).

A. Here we consider Eq. (2.71a). We have to distinguish between the following cases: (a) $i = j = k$, (b) $i = j \neq k$ (it is an easy matter to show that $i \neq j = k$ yields the same results) and (c) $i \neq j \neq k$ and $i \neq k$.

Case $i = j = k$: In this case Eq. (2.71a) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_i^3 (c_i^2 + c_l^2 + c_m^2) e^{-\beta(c_i^2 + c_l^2 + c_m^2)} dc_i dc_l dc_m = 0.$$

This integral vanishes, since we have an odd order of c_i under the integral in each term, see Eq. (2.44).

Case $i \neq j = k$: In this case Eq. (2.71a) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_i c_j^2 (c_i^2 + c_j^2 + c_l^2) e^{-\beta(c_i^2 + c_j^2 + c_l^2)} dc_i dc_j dc_l.$$

As before we have an odd order of c_i under the integral in each term.

Therefore, the integral vanishes.

Case $i \neq j \neq k$ and $i \neq k$: In this case Eq. (2.71a) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_i c_j c_k (c_i^2 + c_j^2 + c_k^2) e^{-\beta(c_i^2 + c_j^2 + c_k^2)} dc_i dc_j dc_k.$$

This integral vanishes also, since we have odd orders of c_i , c_j , and c_k under the integral in each term.

B. For the second integral of Eq. (2.71b) we have to distinguish the same three cases:

Case $i = j = k$: In this case Eq. (2.71b) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_i^3 e^{-\beta(c_i^2 + c_l^2 + c_m^2)} dc_i dc_l dc_m = 0,$$

since we have an odd order of c_i under the integral.

Case $i \neq j = k$: In this case Eq. (2.71b) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_i c_j^2 e^{-\beta(c_i^2 + c_j^2 + c_l^2)} dc_i dc_j dc_l,$$

which will also vanish due to odd orders of c_i under the integral.

Case $i \neq j \neq k$ and $i \neq k$: In this case Eq. (2.71a) becomes

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} c_i c_j c_k e^{-\beta(c_i^2 + c_j^2 + c_k^2)} dc_i dc_j dc_k,$$

which also vanishes due to odd orders of c under the integral.

By combining the results from part A and B we find indeed that the terms proportional to $(1/T)\partial T/\partial x_k$ in the pressure term vanish identically.

Problem 2.15 Show that the heat flux vector is given by

$$q_i = -\lambda \frac{\partial T}{\partial x_i} \quad \text{with} \quad \lambda = \frac{5}{2} \frac{nk^2 T}{mv}.$$

Solution The heat flux vector can be calculated by

$$q_i = \frac{m}{2} \int c_i c^2 f_1 d^3 c,$$

with the first order correction f_1 given by Eq. (2.62), and the Maxwell-Boltzmann distribution function f_0 given by Eq. (2.63), with $\beta = m/(2kT)$ and $\mathbf{c} = \mathbf{v} - \mathbf{u}$. Again, the integral operates only on the terms in rounded brackets

$$\begin{aligned} q_i = & -\frac{m}{2v} \left\{ \frac{m}{kT} \int f_0 c_i c^2 \left(c_j c_k - \frac{1}{3} c^2 \delta_{jk} \right) d^3 c \frac{\partial u_i}{\partial x_k} \right. \\ & \left. + \int c_i c^2 f_0 c_k \left(\beta c^2 - \frac{5}{2} \right) d^3 c \frac{1}{T} \frac{\partial T}{\partial x_k} \right\}. \end{aligned}$$

It can easily be seen that the first integral is

$$\int f_0 c_i c^2 \left(c_j c_k - \frac{1}{3} c^2 \delta_{jk} \right) d^3 c = 0,$$

since we find in each term odd orders of c , see Eq. (2.44). The heat flux is therefore, given by

$$q_i = -\frac{m}{2\nu} \frac{1}{T} \frac{\partial T}{\partial x_k} \int c_i c_k c^2 f_0 \left(\beta c^2 - \frac{5}{2} \right) d^3 c.$$

As before, for $i \neq k$ this integral is zero, therefore we consider only the case $i = k$ and obtain

$$q_i = -n \left(\frac{\beta}{\pi} \right)^{3/2} \frac{m}{2\nu} \frac{1}{T} \frac{\partial T}{\partial x_i} \int c_i^2 c^2 e^{-\beta c^2} \left(\beta c^2 - \frac{5}{2} \right) d^3 c,$$

where we substituted f_0 by Eq. (2.63). Let us consider the first term of the sum. We obtain

$$n \left(\frac{\beta}{\pi} \right)^{3/2} \beta \frac{m}{2\nu} \frac{1}{T} \frac{\partial T}{\partial x_i} \int c_i^2 c^4 e^{-\beta c^2} d^3 c = n \frac{35}{16} \frac{1}{\beta^2} \frac{m}{\nu T} \frac{\partial T}{\partial x_i}. \quad (2.72)$$

For the second term we find

$$n \frac{5}{2} \left(\frac{\beta}{\pi} \right)^{3/2} \frac{m}{2\nu} \frac{1}{T} \frac{\partial T}{\partial x_i} \int c_i^2 c^2 e^{-\beta c^2} d^3 c = n \frac{25}{16} \frac{1}{\beta^2} \frac{m}{\nu T} \frac{\partial T}{\partial x_i}. \quad (2.73)$$

By subtracting Eq. (2.73) from Eq. (2.72) we find for the heat flux

$$q_i = -n \frac{35}{16} \frac{1}{\beta^2} \frac{m}{\nu T} \frac{\partial T}{\partial x_i} - n \frac{25}{16} \frac{1}{\beta^2} \frac{m}{\nu T} \frac{\partial T}{\partial x_i} = -n \frac{5}{8} \frac{1}{\beta^2} \frac{m}{\nu T} \frac{\partial T}{\partial x_i} = -\frac{5}{2} \frac{nk^2 T}{m\nu} \frac{\partial T}{\partial x_i},$$

where we used $\beta = m/2kT$.

2.5 Application 1: Structure of Weak Shock Waves

The one-dimensional *Rankine-Hugoniot* conditions are given by

$$s[\rho] = [\rho u] \equiv [m]$$

$$s[\rho u] = [\rho u^2 + p]$$

$$s[e] = [(e + p)u],$$

(continued)

where $s = dx/dt$ is the speed of the discontinuity and e is the total energy with

$$e \equiv \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1} = \frac{1}{2}\rho u^2 + \rho\epsilon. \quad (2.74)$$

Here, $\epsilon = p/\rho(\gamma - 1)$ is the expression for the internal energy. Since the Euler equations are Galilean invariant, we may transform the Rankine-Hugoniot conditions into a coordinate system moving with a uniform velocity such that the speed of the discontinuity is zero, $s = 0$. The steady-state Rankine-Hugoniot conditions can then be written as

$$\rho_0 u_0 = \rho_1 u_1 \quad (2.75a)$$

$$\rho_0 u_0^2 + p_0 = \rho_1 u_1^2 + p_1 \quad (2.75b)$$

$$(e_0 + p_0)u_0 = (e_1 + p_1)u_1. \quad (2.75c)$$

If we let $m = \rho_0 u_0 = \rho_1 u_1$, we can distinguish between two classes of discontinuities. If $m = 0$, the discontinuity is called a *contact discontinuity* or *slip line*. Since $u_0 = u_1 = 0$, these discontinuities convect with the fluid. From (2.75b) we observe that $p_0 = p_1$ across a contact discontinuity but in general $\rho_0 \neq \rho_1$. By contrast, if $m \neq 0$, then the discontinuity is called a *shock wave*. Since $u_0 \neq 0$ and $u_1 \neq 0$, the gas crosses the shock, or equivalently, the shock propagates through the fluid. The side of the shock that comprises gas that has *not* been shocked is the *front* or *upstream* of the shock, while the shocked gas is the *back* or *downstream* of the shock.

Problem 2.16 Explicitly, derive the $O(\varepsilon)$ and $O(\varepsilon^2)$ expansions of the Euler equations (e.g., Eqs. (2.46a)–(2.46c)).

Solution The 1D Euler equations can be rewritten as

$$\begin{aligned} \frac{\partial \bar{\rho}}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{\rho} \bar{u}) &= 0 \\ \bar{\rho} \left(\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} \right) &= -\frac{a_{c0}^2}{\gamma V_p^2} \frac{\partial \bar{p}}{\partial \bar{x}} \\ \frac{\partial \bar{p}}{\partial \bar{t}} + \bar{u} \frac{\partial \bar{p}}{\partial \bar{x}} + \gamma \bar{p} \frac{\partial \bar{u}}{\partial \bar{x}} &= 0, \end{aligned}$$

where we introduced the dimensionless variables $\bar{t} = t/T$, $\bar{x} = x/L$, $\bar{\rho} = \rho/\rho_0$, $\bar{p} = p/p_0$, and $\bar{u} = u/V_p$. Here, T and L are a characteristic time and length scale respectively, and V_p is a characteristic phase velocity. Also ρ_0 and p_0 are equilibrium values for the density and pressure far upstream of any shock transition.

By introducing fast and slow variables $\xi = \bar{x} - \bar{t}$ and $\tau = \epsilon \bar{t}$ we find

$$\frac{\partial}{\partial \bar{x}} = \frac{\partial \xi}{\partial \bar{x}} \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi} \quad \frac{\partial}{\partial \bar{t}} = \frac{\partial \tau}{\partial \bar{t}} \frac{\partial}{\partial \tau} + \frac{\partial \xi}{\partial \bar{t}} \frac{\partial}{\partial \xi} = \epsilon \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi}. \quad (2.76)$$

If we expand the flow variables about a uniform far-upstream background, we obtain

$$\bar{\rho} = 1 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \dots$$

$$\bar{u} = \epsilon u_1 + \epsilon^2 u_2 + \dots$$

$$\bar{p} = 1 + \epsilon p_1 + \epsilon^2 p_2 + \dots$$

A. Continuity Equation

For the continuity equation we find the general expression

$$\begin{aligned} & \left(\epsilon \frac{\partial}{\partial \tau} - \frac{\partial}{\partial \xi} \right) (1 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \dots) \\ & + \frac{\partial}{\partial \xi} [(1 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \dots) (\epsilon u_1 + \epsilon^2 u_2 + \dots)] = 0. \end{aligned}$$

- Terms of order $O(\epsilon)$:

$$-\frac{\partial \rho_1}{\partial \xi} + \frac{\partial u_1}{\partial \xi} = 0 \quad \Rightarrow \quad \rho_1 = u_1. \quad (2.77)$$

- Terms of order $O(\epsilon^2)$:

$$\frac{\partial \rho_1}{\partial \tau} - \frac{\partial \rho_2}{\partial \xi} + \frac{\partial u_2}{\partial \xi} + \frac{\partial \rho_1 u_1}{\partial \xi} = 0.$$

B. Momentum Equation

The general form of the momentum equation is given by

$$\begin{aligned} & \epsilon (1 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \dots) \frac{\partial}{\partial \tau} (\epsilon u_1 + \epsilon^2 u_2 + \dots) \\ & - (1 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \dots) \frac{\partial}{\partial \xi} (\epsilon u_1 + \epsilon^2 u_2 + \dots) \\ & + (1 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \dots) (\epsilon u_1 + \epsilon^2 u_2 + \dots) \frac{\partial}{\partial \xi} (\epsilon u_1 + \epsilon^2 u_2 + \dots) \\ & = -\frac{\bar{a}_{c0}^2}{\gamma} \frac{\partial}{\partial \xi} (1 + \epsilon p_1 + \epsilon^2 p_2 + \dots), \end{aligned}$$

where we used $\bar{a}_{c0}^2 = a_{c0}^2 / V_p^2$.

- Terms of order $O(\varepsilon)$:

$$\frac{\partial u_1}{\partial \xi} = \frac{\bar{a}_{c0}^2}{\gamma} \frac{\partial p_1}{\partial \xi} \quad \Rightarrow \quad u_1 = \frac{\bar{a}_{c0}^2}{\gamma} p_1. \quad (2.78)$$

- Terms of order $O(\varepsilon^2)$:

$$\frac{\partial u_1}{\partial \tau} - \frac{\partial u_2}{\partial \xi} - \rho_1 \frac{\partial u_1}{\partial \xi} + u_1 \frac{\partial u_1}{\partial \xi} = -\frac{\bar{a}_{c0}^2}{\gamma} \frac{\partial p_2}{\partial \xi}.$$

C. Energy Equation

The general form of the energy equation is given by

$$\begin{aligned} & \varepsilon \frac{\partial}{\partial \tau} (1 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots) - \frac{\partial}{\partial \xi} (1 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots) \\ & + (\varepsilon u_1 + \varepsilon^2 u_2 + \dots) \frac{\partial}{\partial \xi} (1 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots) \\ & + \gamma (1 + \varepsilon p_1 + \varepsilon^2 p_2 + \dots) \frac{\partial}{\partial \xi} (\varepsilon u_1 + \varepsilon^2 u_2 + \dots) = 0. \end{aligned}$$

- Terms of order $O(\varepsilon)$:

$$-\frac{\partial p_1}{\partial \xi} + \gamma \frac{\partial u_1}{\partial \xi} = 0 \quad \Rightarrow \quad p_1 = \gamma u_1. \quad (2.79)$$

- Terms of order $O(\varepsilon^2)$:

$$\frac{\partial p_1}{\partial \tau} - \frac{\partial p_2}{\partial \xi} + u_1 \frac{\partial p_1}{\partial \xi} + \gamma \frac{\partial u_2}{\partial \xi} + \gamma p_1 \frac{\partial u_1}{\partial \xi} = 0.$$

Problem 2.17 Derive the nonlinear wave equation

$$\frac{\partial u_1}{\partial \tau} + \frac{\gamma + 1}{2} u_1 \frac{\partial u_1}{\partial \xi} = 0, \quad (2.80)$$

which is called the *inviscid* form of Burgers' equation.

Solution From the first order expansions of the Euler equations (see previous Problem 2.16), we find the relations

$$\rho_1 = u_1 \quad u_1 = \frac{\bar{a}_{c0}^2}{\gamma} p_1 \quad p_1 = \gamma u_1. \quad (2.81)$$

From the last two relations it follows that $\bar{a}_{c0}^2 = 1$. For the second order equations, $O(\varepsilon^2)$, we find

$$-\frac{\partial \rho_2}{\partial \xi} + \frac{\partial u_2}{\partial \xi} = -\frac{\partial}{\partial \xi}(\rho_1 u_1) - \frac{\partial \rho_1}{\partial \tau} \quad (2.82a)$$

$$-\frac{\partial u_2}{\partial \xi} + \frac{\bar{a}_{c0}^2}{\gamma} \frac{\partial p_2}{\partial \xi} = \rho_1 \frac{\partial u_1}{\partial \xi} - \frac{\partial u_1}{\partial \tau} - u_1 \frac{\partial u_1}{\partial \xi} \quad (2.82b)$$

$$-\frac{\partial p_2}{\partial \xi} + \gamma \frac{\partial u_2}{\partial \xi} = -\frac{\partial p_1}{\partial \tau} - \gamma p_1 \frac{\partial u_1}{\partial \xi} - u_1 \frac{\partial p_1}{\partial \xi}. \quad (2.82c)$$

Equation (2.82c) can be rewritten as

$$\frac{\partial p_2}{\partial \xi} = \frac{\partial p_1}{\partial \tau} + \gamma p_1 \frac{\partial u_1}{\partial \xi} + u_1 \frac{\partial p_1}{\partial \xi} + \gamma \frac{\partial u_2}{\partial \xi}.$$

Substituting this result into Eq. (2.82b) and setting $\bar{a}_{c0}^2 = 1$ we obtain

$$-\frac{\partial u_2}{\partial \xi} + \frac{1}{\gamma} \left(\frac{\partial p_1}{\partial \tau} + \gamma p_1 \frac{\partial u_1}{\partial \xi} + u_1 \frac{\partial p_1}{\partial \xi} + \gamma \frac{\partial u_2}{\partial \xi} \right) = \rho_1 \frac{\partial u_1}{\partial \xi} - \frac{\partial u_1}{\partial \tau} - u_1 \frac{\partial u_1}{\partial \xi},$$

which can be simplified to

$$\frac{1}{\gamma} \frac{\partial p_1}{\partial \tau} + p_1 \frac{\partial u_1}{\partial \xi} + \frac{1}{\gamma} u_1 \frac{\partial p_1}{\partial \xi} = \rho_1 \frac{\partial u_1}{\partial \xi} - \frac{\partial u_1}{\partial \tau} - u_1 \frac{\partial u_1}{\partial \xi}.$$

Now we substitute $p_1 = \gamma u_1$ and $\rho_1 = u_1$ from Eq. (2.81) and obtain

$$\frac{\partial u_1}{\partial \tau} + \gamma u_1 \frac{\partial u_1}{\partial \xi} + u_1 \frac{\partial u_1}{\partial \xi} = u_1 \frac{\partial u_1}{\partial \xi} - \frac{\partial u_1}{\partial \tau} - u_1 \frac{\partial u_1}{\partial \xi}.$$

Finally, this can be simplified to

$$\frac{\partial u_1}{\partial \tau} + \frac{\gamma + 1}{2} u_1 \frac{\partial u_1}{\partial \xi} = 0,$$

which is called the *inviscid* form of Burgers' equation.

Problem 2.18 Solve the linear wave equation

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0, \quad (2.83)$$

where c is a constant and the initial condition $u(x, t = 0) = f(x)$. Write down the solution if $f(x) = \sin(kx)$.

Solution The initial curve at time $t = 0$ can be parameterized through

$$t = 0 \quad x = x_0 \quad u(x, t = 0) = f(x_0). \quad (2.84)$$

The set of characteristic equations (see also Problem 2.3) and their solutions are given by

$$\begin{array}{lll} \frac{dt}{d\tau} = 1 & t = \tau + \text{const}_1 & t = \tau \\ \frac{dx}{d\tau} = c & x = c\tau + \text{const}_2 & x = c\tau + x_0 \\ \frac{du}{d\tau} = 0 & u = \text{const}_3 & u = f(x_0). \end{array}$$

From the last equations it follows that u is constant along the characteristic curve $x_0 = x - ct$. In particular, $x_0 = x_0(x, t)$. The characteristics can be inverted and we find $u = f(x_0) = f(x - ct)$. Together with $f(x) = \sin(kx)$ the solution is given by

$$u = \sin(kx - ckt) = \sin(kx - \omega t), \quad (2.85)$$

where we used the dispersion $\omega = ck$.

Problem 2.19 Consider the initial data

$$U(x, 0) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases} \quad (2.86)$$

for the partial differential equation written in conservative form

$$\frac{\partial U}{\partial t} + \frac{1}{2} \frac{\partial U^2}{\partial x} = 0.$$

Sketch the characteristics. What is the shock propagation speed necessary to prevent the characteristics from crossing?

Solution First, we rewrite Burgers' equation in the form

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = 0.$$

The set of characteristic equations is then given by

$$\frac{dt}{ds} = 1 \quad \frac{dx}{ds} = U \quad \frac{dU}{ds} = 0.$$

From the last equation, $dU/ds = 0$, we find that $U = \text{const.}$ along the characteristics. In fact we have $U = U_0(x_0)$, where $U_0(x_0) = U(x, 0)$ is given by the initial data, Eq. (2.86). Since U is constant it follows that

$$\frac{dx}{ds} = U = U_0(x_0) \quad \implies \quad x(s) = U_0(x_0)s + x_0.$$

And since $dt/ds = 1$, so that $s = t$, we find for the characteristic curve

$$x(t) = U_0(x_0)t + x_0 \quad \implies \quad x(t) = \begin{cases} x_0 & \text{for } x_0 \geq 0 \\ x_0 + t & \text{for } x_0 < 0 \end{cases},$$

where we used the initial conditions given in Eq. (2.86). The curves are shown in Fig. 2.2. To prevent the characteristics from crossing, we introduce a shock with propagating speed (valid for the inviscid Burgers' equation)

$$s = \frac{[\frac{1}{2}U^2]}{[U]} = \frac{U_0 + U_1}{2} = \frac{1}{2},$$

with $U_0 = 0$ and $U_1 = 1$, and where $s = (U_0 + U_1)/2$ is the shock jump relation for the inviscid Burgers' equation, connecting the speed of propagation s of the discontinuity with the amounts by which the velocity U jumps. Figure 2.3 shows this case.

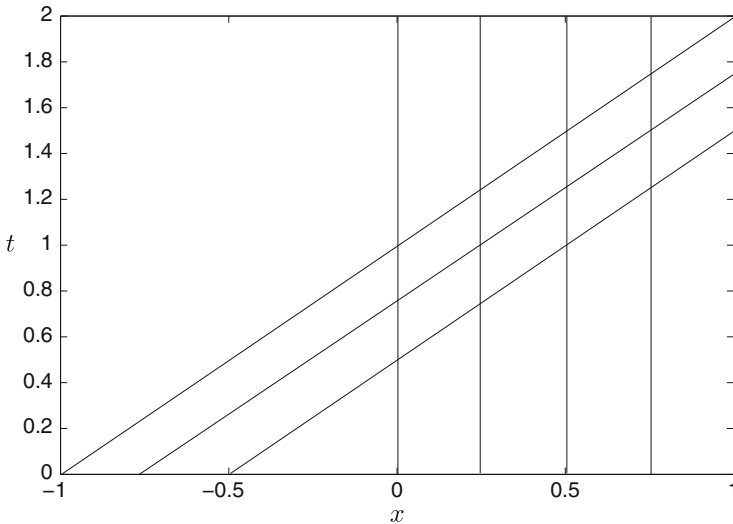


Fig. 2.2 Shown are the characteristic curves. Apparently, the curves intersect

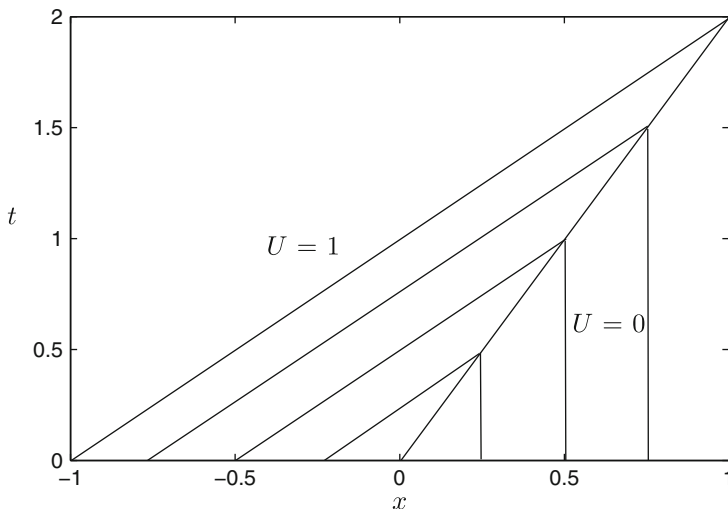


Fig. 2.3 Shown are the characteristic curves with the shock speed $s = 0.5$

Problem 2.20 Starting from the stationary Rankine-Hugoniot conditions (2.75a)–(2.75c), show that

$$m^2 = -\frac{p_0 - p_1}{\tau_0 - \tau_1},$$

where $\tau \equiv 1/\rho$. Show also that $e_0\tau_0 - e_1\tau_1 = p_1\tau_1 - p_0\tau_0$ and hence that

$$\epsilon_1 - \epsilon_0 + \frac{p_0 + p_1}{2}(\tau_1 - \tau_0) = 0,$$

(the Hugoniot equation for the shock) where $\epsilon \equiv p\tau/(\gamma - 1)$.

Solution

A. We begin with Eq. (2.75b) and rewrite this equation as

$$\rho_1 u_1^2 - \rho_0 u_0^2 = p_0 - p_1 \quad \implies \quad \frac{\rho_1^2 u_1^2}{\rho_1} - \frac{\rho_0^2 u_0^2}{\rho_0} = p_0 - p_1,$$

where we multiplied each term on the left-hand side with $1 = \rho_1/\rho_1 = \rho_0/\rho_0$. Using now Eq. (2.75a) and letting $\rho_0 u_0 = \rho_1 u_1 = m$, we find immediately

$$m^2 \left[\frac{1}{\rho_1} - \frac{1}{\rho_0} \right] = p_0 - p_1 \quad \implies \quad m^2 = -\frac{p_0 - p_1}{\tau_0 - \tau_1},$$

where $\tau = 1/\rho$. Note that, since $\rho u = m$ and thus $\tau = u/m$, we can rewrite the expression as

$$m = -\frac{p_0 - p_1}{u_0 - u_1} \implies (u_0 - u_1) = -\frac{p_0 - p_1}{m}. \quad (2.87)$$

- B. We begin by rewriting the condition (2.75a) in the form $u_1 = u_0 \rho_0 / \rho_1$, so that condition (2.75c) can be written as

$$(e_0 + p_0)u_0 = (e_1 + p_1)u_0 \frac{\rho_0}{\rho_1} \implies \tau_0(e_0 + p_0) = \tau_1(e_1 + p_1),$$

where we divided both sides by u_0 and ρ_0 and replaced $\tau = 1/\rho$. Reordering the equation, we obtain

$$e_0 \tau_0 - e_1 \tau_1 = p_1 \tau_1 - p_0 \tau_0. \quad (2.88)$$

- C. We begin with the left-hand side of Eq. (2.88) and substitute

$$e = \rho \epsilon + \frac{1}{2} \rho u^2 = \left(\epsilon + \frac{1}{2} u^2 \right) \frac{1}{\tau},$$

where $\tau = 1/\rho$, ϵ is the internal energy, and e is the specific total energy (see the introduction of this section). We obtain

$$\begin{aligned} \left(\epsilon_0 + \frac{1}{2} u_0^2 \right) \tau_0 - \left(\epsilon_1 + \frac{1}{2} u_1^2 \right) \tau_1 &= p_1 \tau_1 - p_0 \tau_0 \\ \epsilon_0 - \epsilon_1 + \frac{1}{2} (u_0^2 - u_1^2) &= p_1 \tau_1 - p_0 \tau_0 \\ \epsilon_0 - \epsilon_1 + \frac{1}{2} (u_0 - u_1) (u_0 + u_1) &= p_1 \tau_1 - p_0 \tau_0. \end{aligned}$$

We substitute now $(u_0 - u_1)$ by Eq. (2.87) and use $u = m\tau$ for the term $(u_0 + u_1)$, and obtain

$$\begin{aligned} \epsilon_0 - \epsilon_1 - \frac{1}{2} \frac{p_0 - p_1}{m} (m\tau_0 + m\tau_1) &= p_1 \tau_1 - p_0 \tau_0 \\ \epsilon_0 - \epsilon_1 - \frac{p_0 - p_1}{2} (\tau_0 + \tau_1) &= p_1 \tau_1 - p_0 \tau_0 \\ \epsilon_0 - \epsilon_1 - \frac{1}{2} p_0 \tau_0 - \frac{1}{2} p_0 \tau_1 + \frac{1}{2} p_1 \tau_0 + \frac{1}{2} p_1 \tau_1 &= p_1 \tau_1 - p_0 \tau_0 \end{aligned}$$

$$\begin{aligned}\epsilon_0 - \epsilon_1 + \frac{1}{2}p_0\tau_0 - \frac{1}{2}p_0\tau_1 + \frac{1}{2}p_1\tau_0 - \frac{1}{2}p_1\tau_1 &= 0 \\ \epsilon_0 - \epsilon_1 + \frac{1}{2}(p_0 + p_1)(\tau_0 - \tau_1) &= 0,\end{aligned}$$

which is the Hugoniot equation for a shock.

Problem 2.21 Show that the Cole-Hopf transformation

$$u = -2\kappa \frac{\phi_x}{\phi} \quad (2.89)$$

removes the nonlinear term in the Burgers' equation

$$u_t + uu_x = \kappa u_{xx},$$

and yields the heat equation as the transformed equation. For the initial problem $u(x, t = 0) = F(x)$, show that this transforms to the initial problem

$$\Phi = \Phi(x) = \exp \left[-\frac{1}{2\kappa} \int_0^x F(\eta) d\eta \right], \quad t = 0,$$

for the heat equation. Show that the solution for u is

$$u(x, t) = \frac{\int_{-\infty}^{\infty} \frac{x-\eta}{t} e^{-G/2\kappa} d\eta}{\int_{-\infty}^{\infty} e^{-G/2\kappa} d\eta},$$

where

$$G(\eta; x, t) = \int_0^\eta F(\eta') d\eta' + \frac{(x - \eta)^2}{2t}.$$

Solution Note that the velocity u is a one-dimensional function of time and space, thus $u = u(x, t)$. The Cole-Hopf transformation (2.89) introduces a function ϕ which is also a function of time and space, $\phi = \phi(x, t)$. To avoid any ambiguities we define the initial condition at time $t = 0$ as $\Phi = \Phi(x) = \phi(x, 0)$.

A. With the Cole-Hopf transformation the derivatives of u with respect to t and x are given by

$$\begin{aligned}u_x &= -2\kappa \frac{\phi \phi_{xx} - \phi_x^2}{\phi^2}, & u_t &= -2\kappa \frac{\phi \phi_{xt} - \phi_x \phi_t}{\phi^2}, \\ u_{xx} &= -2\kappa \frac{\phi_{xxx}}{\phi} + 6\kappa \frac{\phi_x \phi_{xx}}{\phi^2} - 4\kappa \frac{\phi_x^3}{\phi^3}.\end{aligned}$$

Burgers' equation becomes then

$$\begin{aligned} u_t + uu_x - \kappa u_{xx} &= -\frac{\phi_{xt}}{\phi} + \frac{\phi_x \phi_t}{\phi^2} + \kappa \frac{\phi_{xxx}}{\phi} - \kappa \frac{\phi_x \phi_{xx}}{\phi^2} \\ &= \phi (\kappa \phi_{xx} - \phi_t)_x - \phi_x (\kappa \phi_{xx} - \phi_t) = 0. \end{aligned}$$

Note that the expressions in both brackets are identical. For any non-trivial solution of ϕ , i.e., $\phi \neq 0$, this equation is fulfilled if the term in brackets equals zero, and thus

$$\kappa \phi_{xx} = \phi_t. \quad (2.90)$$

Equation (2.90) is called the *heat equation* and we note that the Cole-Hopf transformation reduces Burgers' equation to the problem of solving the heat equation.

- B. By using Eq. (2.89) we find for the initial problem (setting $t = 0$, so that $\phi(x, 0) = \Phi(x)$) the following ordinary differential equation

$$F(x) = -2\kappa \frac{\phi_x(x, 0)}{\phi(x, 0)} \implies \frac{d\Phi(x)}{\Phi(x)} = -\frac{F(x)}{2\kappa} dx,$$

where $u(x, t = 0) = F(x)$. The general solution is given by

$$\begin{aligned} \ln \Phi(x) - \ln \Phi(0) &= -\frac{1}{2\kappa} \int_0^x F(\eta) d\eta \\ \Phi(x) &= C \exp \left[-\frac{1}{2\kappa} \int_0^x F(\eta) d\eta \right], \end{aligned} \quad (2.91)$$

where we used $C = \Phi(0)$.

- C. In order to solve Burgers' equation we essentially need to solve the heat equation (2.90) and transform the solution back according to the Cole-Hopf transformation (2.89). Let us begin by rewriting the heat equation as

$$\begin{aligned} \frac{\partial \phi(x, t)}{\partial t} &= \kappa \frac{\partial^2 \phi(x, t)}{\partial x^2} & -\infty < x < \infty, \quad 0 < t \\ \phi(x, t = 0) &= \Phi(x) & -\infty < x < \infty. \end{aligned} \quad (2.92)$$

with the initial profile $\Phi(x)$ at time $t = 0$, given by Eq. (2.91). A very common approach is to transform the function $\phi(x, t)$ into Fourier space

$$\phi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{\phi}(k, t) e^{ikx}, \quad \hat{\phi}(k, t) = \int_{-\infty}^{\infty} dx \phi(x, t) e^{-ikx} \quad (2.93a)$$

$$\Phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{\Phi}(k) e^{ikx}, \quad \hat{\Phi}(k) = \int_{-\infty}^{\infty} dx \Phi(x) e^{-ikx}. \quad (2.93b)$$

where, $\hat{\phi}(k, t)$ and $\hat{\Phi}(k)$ are the Fourier transforms of $\phi(x, t)$ and $\Phi(x)$. By substituting the Fourier transform of $\phi(x, t)$ into the heat equation (2.92) we obtain

$$\begin{aligned} \frac{\partial \hat{\phi}(k, t)}{\partial t} &= -\kappa k^2 \hat{\phi}(k, t) & k \in \mathbb{R}, \quad 0 \leq t \\ \hat{\phi}(k, 0) &= \hat{\Phi}(k) & k \in \mathbb{R}. \end{aligned} \quad (2.94)$$

For each constant wave mode k , the function $\hat{\phi}(k, t)$ fullfils the initial problem with the initial condition $\hat{\phi}(k, 0) = \hat{\Phi}(k)$. The ordinary differential equation (2.94) can easily be solved,

$$\frac{\partial \hat{\phi}(k, t)}{\hat{\phi}(k, t)} = -\kappa k^2 \partial t \quad \Rightarrow \quad \ln \hat{\phi}(k, t) - \ln \hat{\phi}(k, 0) = -\kappa k^2 t,$$

where we integrated from $t = 0$ to t . Since $\hat{\phi}(k, 0) = \hat{\Phi}(k)$ we find our solution

$$\hat{\phi}(k, t) = \hat{\Phi}(k) e^{-\kappa k^2 t}. \quad (2.95)$$

Now we transform this solution back into real space, using Eq. (2.93a),

$$\phi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \hat{\Phi}(k) e^{-\kappa k^2 t} e^{ikx}.$$

Substituting now the initial condition $\hat{\Phi}(k)$ by Eq. (2.93b) we find

$$\phi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \Phi(x') \int_{-\infty}^{\infty} dk e^{-ikx'} e^{-\kappa k^2 t} e^{ikx}.$$

Note that we use the x' -coordinate for the back transformation of the initial condition. The integral with respect to k is readily solved by (see Problem 2.27 for a detailed analysis of the integration)

$$K(x - x') = \int_{-\infty}^{\infty} e^{-ikx'} e^{-\kappa k^2 t} e^{ikx} dk = \sqrt{\frac{\pi}{\kappa t}} e^{-\frac{(x-x')^2}{4\kappa t}}, \quad (2.96)$$

so that

$$\phi(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx' \Phi(x') K(x - x'), \quad (2.97)$$

where $K(x - x')$ is the so-called *heat kernel*. Note that this integration is only convergent for $\kappa > 0$ and $t > 0$. Thus, we find the *general solution*

$$\phi(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} \Phi(x') e^{-\frac{(x-x')^2}{4\kappa t}} dx'.$$

This solution is a convolution of the *fundamental solution*

$$\psi(x, t) = \frac{1}{\sqrt{4\pi\kappa t}} \exp\left[-\frac{x^2}{4\kappa t}\right]$$

and the function $\Phi(x')$. Substituting $\Phi(x')$ by Eq. (2.91) we obtain

$$\phi(x, t) = \frac{C}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\kappa} \int_0^{x'} F(\eta) d\eta - \frac{(x-x')^2}{4\kappa t}\right] dx'.$$

The spatial derivative is then

$$\phi(x, t)_x = -\frac{C}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} \frac{x-x'}{2\kappa t} \exp\left[-\frac{1}{2\kappa} \int_0^{x'} F(\eta) d\eta - \frac{(x-x')^2}{4\kappa t}\right] dx'.$$

According to Eq. (2.89) the solution of Burgers' equation is then

$$u(x, t) = \frac{\int_{-\infty}^{\infty} \frac{x-x'}{t} \exp\left[-\frac{1}{2\kappa} \int_0^{x'} F(\eta) d\eta - \frac{(x-x')^2}{4\kappa t}\right] dx'}{\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2\kappa} \int_0^{x'} F(\eta) d\eta - \frac{(x-x')^2}{4\kappa t}\right] dx'}. \quad (2.98)$$

Problem 2.22 Show that the exponential solution of the characteristic form of the steady Burgers' equation admits a solution that can be expressed as a hyperbolic tan (tanh) profile, given $u(-\infty) = u_0$ and $u(\infty) = u_1$.

Solution Burgers' equation is given by

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \kappa \frac{\partial^2 u}{\partial x^2}$$

The steady state Burger's equation can also be written as

$$\frac{d}{dx} \left[\frac{u^2}{2} - \frac{d}{dx} (\kappa u) \right] = 0 \quad \implies \quad \frac{d}{dx} \left[u^2 - 2\kappa \frac{du}{dx} \right] = 0.$$

where we multiplied by 2 and assumed that κ is independent of x . Obviously, the term in brackets has to be constant, and the dimensions of that constant are of the

order velocity squared. Thus, we introduce the constant u_c and have

$$u^2 - 2\kappa \frac{du}{dx} = u_c^2,$$

where u_c is a constant velocity that has to be determined by the boundary conditions. We can rewrite that equation and obtain

$$\int dx = \int du \frac{2\kappa}{u^2 - u_c^2} = -\frac{2\kappa}{u_c^2} \int du \frac{1}{1 - \frac{u^2}{u_c^2}} = -\frac{2\kappa}{u_c} \int dy \frac{1}{1 - y^2},$$

where we used the substitution $y = u/u_c$. According to Gradshteyn and Ryzhik [2], Eq. (2.01–16), the integral can be solved by

$$x = -\frac{2\kappa}{u_c} \operatorname{arctanh}\left(\frac{u(x)}{u_c}\right) + x_0,$$

where x_0 is an integration constant and $u(x)$ obeys the condition $-u_c < u(x) < u_c$. Solving this equation for $u(x)$ gives

$$u(x) = -u_c \tanh\left[(x - x_0) \frac{u_c}{2\kappa}\right],$$

where we used $\tanh(-a) = -\tanh(a)$. Now it is clear that the integration constant x_0 shifts the $\tanh(x)$ by a distance x_0 in the positive or negative x direction, depending on the sign of x_0 . The constant $2\kappa/u_c$ has the dimension of a length. For convenience, we introduce a characteristic length scale $l = 2\kappa/u_c$ so that our solution is

$$u(x) = -u_c \tanh\left(\frac{x - x_0}{l}\right). \quad (2.99)$$

Example: Shock. Here we consider the example of a shock located at x_0 and a background flow with the upstream velocity $u_0 = u(x = -\infty)$ and the downstream velocity $u_1 = u(x = +\infty)$.

- A. By using $\tanh(\pm\infty) = \pm 1$ and Eq. (2.99) we find the relations $u_0 = u_c$ and $u_1 = -u_c$, and therefore $u_0 = -u_1$. This also implies $u(x = x_0) = 0$, because $\tanh(0) = 0$, which means, that the shock speed at the position of the shock x_0 is zero, thus $u(x_0) = 0$. However, in some cases one is rather interested in $u_0 \neq -u_1$. It is obvious that, in this case, we have to add a constant $C \neq 0$ to Eq. (2.99),

$$u(x) = -u_c \tanh\left(\frac{x - x_0}{l}\right) + C, \quad (2.100)$$

to guarantee that $u(x_0) \neq 0$. We find immediately from Eq. (2.100) that

$$u(x = -\infty) = u_c + C \quad (2.101a)$$

$$u(x = x_0) = C \quad (2.101b)$$

$$u(x = +\infty) = -u_c + C. \quad (2.101c)$$

To specify the constant C we impose another boundary condition,

$$u(x = x_0) = \frac{u_0 + u_1}{2} \equiv C, \quad (2.102)$$

which means, that at the shock position x_0 the shock speed has decreased to the constant arithmetic mean $C = (u_0 + u_1)/2$, which is zero only for $u_0 = -u_1$. The constant u_c is determined by Eqs. (2.101a) and (2.101c). Subtracting the second equation from the first equation and dividing by 2 gives

$$u_c = \frac{u_0 - u_1}{2}.$$

Substituting C and u_c back into Eq. (2.100) gives

$$u(x) = \frac{u_0 + u_1}{2} - \frac{u_0 - u_1}{2} \tanh\left(\frac{x - x_0}{l}\right), \quad (2.103)$$

which provides the correct results for the boundary conditions. In the case that $C = 0$ and, thus, $u_0 = -u_1 \equiv u_e$ we find the result as given by Eq. (2.99). As an example, Fig. 2.4 shows the velocity profile (2.103) across a shock for $u_0 = 1$ and $u_1 = 0.25$ and the shock position $x_0 = 5$.

B. The exponential solution of the characteristic form of Burgers' equation is given by

$$u = u_0 + \frac{u_1 - u_0}{e^{2z} + 1}, \quad \text{where} \quad z = \frac{u_1 - u_0}{4\kappa} \left(x - \frac{u_0 + u_1}{2} t \right). \quad (2.104)$$

Note that the hyperbolic tangent can be expressed by

$$\tanh z = 1 - \frac{2}{e^{2z} + 1}.$$

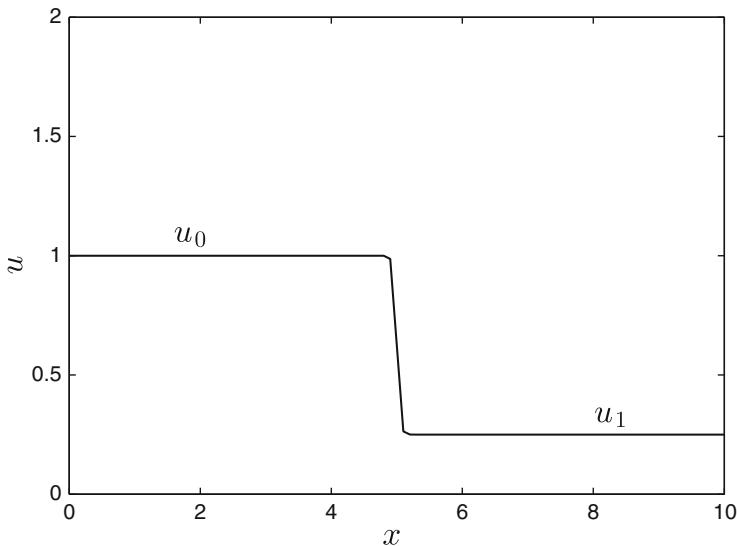


Fig. 2.4 Shown is the velocity profile for $x_0 = 5$, $u_0 = 1$ and $u_1 = 0.25$

By expressing $u_0 = u_0/2 + u_0/2$ in Eq.(2.104) and adding a ‘zero’ in the form of $0 = u_1/2 - u_1/2$, we obtain

$$\begin{aligned} u &= \frac{u_0}{2} + \frac{u_1}{2} + \frac{u_0}{2} - \frac{u_1}{2} + \frac{u_1 - u_1}{e^{2z} + 1} = \frac{u_0 + u_1}{2} + \frac{u_0 - u_1}{2} - \frac{u_0 - u_1}{e^{2z} + 1} \\ &= \frac{u_0 + u_1}{2} + \frac{u_0 - u_1}{2} \left[1 - \frac{2}{e^{2z} + 1} \right] = \frac{u_0 + u_1}{2} + \frac{u_0 - u_1}{2} \tanh(z). \end{aligned}$$

This result is identical to Eq. (2.103).

2.6 Application 2: The Diffusion and Telegrapher Equations

Legendre’s differential equation is an ordinary differential equation, given by

$$0 = (1 - \mu^2) \frac{dP_n^2(\mu)}{d\mu^2} - 2\mu \frac{dP_n(\mu)}{d\mu} + n(n+1)P_n(\mu) \quad (2.105a)$$

$$= \frac{d}{d\mu} \left[(1 - \mu^2) \frac{dP_n}{d\mu} \right] + n(n+1)P_n, \quad (2.105b)$$

(continued)

with $n = 0, 1, 2, \dots$. Solutions to this differential equation are called Legendre polynomials, where each Legendre polynomial is an n -th degree polynomial and can be determined by

$$P_n(\mu) = F\left(n+1, -n, 1; \frac{1-\mu}{2}\right) = \frac{1}{2^n n!} \frac{d^n (\mu^2 - 1)^n}{d\mu^n}, \quad (2.106)$$

where $F(\dots)$ is the hypergeometric function. The last expression on the right side is also called *Rodrigues' formula*. The generating function is

$$L(\mu, t) = \frac{1}{\sqrt{1-2\mu t+t^2}} = \sum_{n=0}^{\infty} P_n(\mu) t^n, \quad |t| < 1. \quad (2.107)$$

An important relation is given by

$$\int_{-1}^1 P_n(\mu) P_m(\mu) d\mu = \frac{2}{2m+1} \delta_{nm}. \quad (2.108)$$

Problem 2.23 Legendre polynomials $P_n(\mu)$ and $P_m(\mu)$ satisfy Legendre's differential equation (2.105a). Show that for $n \neq m$, the following orthogonality condition holds,

$$\int_{-1}^1 P_n(\mu) P_m(\mu) d\mu = 0 \quad \text{for } n \neq m. \quad (2.109)$$

Solution Using Eq. (2.105a), Legendre's differential equation (for two Legendre polynomials P_n and P_m) is described by

$$\begin{aligned} \frac{d}{d\mu} \left[(1-\mu^2) \frac{dP_n}{d\mu} \right] + n(n+1)P_n &= 0 \\ \frac{d}{d\mu} \left[(1-\mu^2) \frac{dP_m}{d\mu} \right] + m(m+1)P_m &= 0, \end{aligned}$$

where $n \neq m$. Multiply now the first equation with P_m and the second with P_n , then subtract the second from the first. We obtain

$$\begin{aligned} &[m(m+1) - n(n+1)] P_m P_n \\ &= P_m \frac{d}{d\mu} \left[(1-\mu^2) \frac{dP_n}{d\mu} \right] - P_n \frac{d}{d\mu} \left[(1-\mu^2) \frac{dP_m}{d\mu} \right] \\ &= \frac{d}{d\mu} \left[(1-\mu^2) P_m \frac{dP_n}{d\mu} - (1-\mu^2) P_n \frac{dP_m}{d\mu} \right]. \end{aligned}$$

It is easy to show that the last line equals the second. Now we integrate both sides with respect to μ and obtain

$$\begin{aligned} & [m(m+1) - n(n+1)] \int_{-1}^1 P_m P_n d\mu \\ &= \int_{-1}^{+1} \frac{d}{d\mu} \left[(1 - \mu^2) \left(P_m \frac{dP_n}{d\mu} - P_n \frac{dP_m}{d\mu} \right) \right] d\mu \\ &= \left[(1 - \mu^2) \left(P_m \frac{dP_n}{d\mu} - P_n \frac{dP_m}{d\mu} \right) \right]_{-1}^{+1} = 0, \end{aligned}$$

because the term $(1 - \mu^2)$ vanishes for $\mu = \pm 1$. For $n \neq m$ we have the orthogonality condition given by Eq. (2.109).

Problem 2.24 The generating function for the Legendre polynomials is given by Eq. (2.107). By differentiating the generating function with respect to t and equating coefficients, derive the recursion relation

$$(n+1)P_{n+1} + nP_{n-1} = (2n+1)\mu P_n, \quad n = 1, 2, 3, \dots$$

Solution We consider the partial derivative of Eq. (2.107) with respect to t and obtain

$$\frac{\partial L}{\partial t} = \frac{\mu - t}{(1 - 2\mu t + t^2)^{3/2}} = \sum_{n=0}^{\infty} n P_n t^{n-1}.$$

By multiplying both sides with $(1 - 2\mu t + t^2)$ we find

$$\frac{\mu - t}{\sqrt{1 - 2\mu t + t^2}} \equiv (\mu - t) \sum_{n=0}^{\infty} P_n t^n = (1 - 2\mu t + t^2) \sum_{n=0}^{\infty} n P_n t^{n-1},$$

where we used the definition of the generating function (2.107) for the first equality. By expanding the parenthesis in front of both sums we find

$$\sum_{n=0}^{\infty} \mu P_n t^n - \sum_{n=0}^{\infty} P_n t^{n+1} = \sum_{n=0}^{\infty} n P_n t^{n-1} - \sum_{n=0}^{\infty} 2\mu n P_n t^n + \sum_{n=0}^{\infty} n P_n t^{n+1}.$$

Compare now each coefficient with the same power of t . We find

$$\mu P_n - P_{n-1} = (n+1)P_{n+1} - 2\mu n P_n + (n-1)P_{n-1}.$$

Rearranging leads to the recursion relation

$$(n+1)P_{n+1} + nP_{n-1} = (2n+1)\mu P_n.$$

Problem 2.25 By using the generating function and Problem 2.23 above, show that

$$\int_{-1}^1 P_n^2(\mu) d\mu = \frac{2}{2n+1}. \quad (2.110)$$

Solution By multiplying the generating function (2.107) with itself we obtain

$$\frac{1}{1-2\mu t+t^2} = \left(\sum_{n=0}^{\infty} P_n t^n \right) \left(\sum_{m=0}^{\infty} P_m t^m \right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_n P_m t^{n+m},$$

where $|t| < 1$. By integrating with respect to μ we find

$$\int_{-1}^1 \frac{1}{1-2\mu t+t^2} d\mu = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_{-1}^1 P_n P_m d\mu t^{n+m} = \sum_{n=0}^{\infty} \int_{-1}^1 P_n^2 d\mu t^{2n},$$

since we know from Problem 2.23 that $\int_{-1}^1 P_n P_m d\mu = 0$ for $m \neq n$. Let us consider now the left-hand side of that equation. By using $\alpha = 1+t^2$ and $x = -2\mu t$ we have

$$\begin{aligned} \int_{-1}^1 \frac{1}{1-2\mu t+t^2} d\mu &= \frac{1}{2t} \int_{-2t}^{2t} \frac{1}{\alpha+x} dx = \frac{1}{2t} [\ln(\alpha+x)]_{-2t}^{2t} \\ &= \frac{1}{2t} \ln \left(\frac{\alpha+2t}{\alpha-2t} \right) = \frac{1}{2t} \ln \left(\frac{1+2t+t^2}{1-2t+t^2} \right) \\ &= \frac{1}{2t} \ln \left(\left(\frac{1+t}{1-t} \right)^2 \right) = \frac{1}{t} \ln \left(\frac{1+t}{1-t} \right). \end{aligned}$$

According to Gradshteyn and Rhyzhik [2], Eq. (1.513–1) the logarithm can be written as an infinite sum

$$\ln \left(\frac{1+t}{1-t} \right) = 2 \sum_{n=0}^{\infty} \frac{1}{2n+1} t^{2n+1} \quad \text{with} \quad |t| \leq 1.$$

Together with the factor $1/t$ we find for the left-hand side

$$\int_{-1}^1 \frac{1}{1-2\mu t+t^2} d\mu = \sum_{n=0}^{\infty} \frac{2}{2n+1} t^{2n} = \sum_{n=0}^{\infty} \int_{-1}^1 P_n^2 d\mu t^{2n}.$$

By comparing the coefficients for each n we find

$$\int_{-1}^1 P_n^2 d\mu = \frac{2}{2n+1}.$$

A basic problem in space physics and astrophysics is the transport of charged particles in the presence of a magnetic field that is ordered on some large scale and highly random and temporal on the other scales. We discuss a simplified form of the Fokker-Planck transport equation that describes particle transport via particle scattering in pitch angle in a magnetically turbulent medium since it resembles closely the basic Boltzmann equation. In the absence of both focusing and adiabatic energy changes, the BGK form of the Boltzmann equation reduces to the simplest possible integro-differential equation

$$\frac{\partial f}{\partial t} + \mu v \frac{\partial f}{\partial r} = \frac{\langle f \rangle - f}{\tau}, \quad (2.111)$$

where $f(r, t, v, \mu)$ is a gyrophase averaged velocity distribution function at position r and time t for particles of speed v and pitch-angle cosine $\mu = \cos \theta$ with $\mu \in [-1, 1]$ and where

$$\langle f \rangle = \frac{1}{2} \int_{-1}^1 f d\mu$$

is the mean or isotropic distribution function averaged over μ .

Problem 2.26 Starting from Eq. (2.111) derive the infinite set of partial differential equations

$$(2n+1) \frac{\partial f_n}{\partial t} + (n+1)v \frac{\partial f_{n+1}}{\partial r} + nv \frac{\partial f_{n-1}}{\partial r} + (2n+1) \frac{f_n}{\tau} = \frac{f_0}{\tau} \delta_{n0} \quad (2.112)$$

with $n = 0, 1, 2, \dots$

Solution By expanding $f = f(r, t, v, \mu)$ in an infinite series of Legendre polynomials $P_n(\mu)$,

$$f = \frac{1}{4\pi} \sum_{n=0}^{\infty} (2n+1) P_n(\mu) f_n(r, t, v),$$

we can rewrite Eq. (2.111) as (neglecting the factor $1/4\pi$)

$$\begin{aligned} & \sum_{n=0}^{\infty} (2n+1) P_n \frac{\partial f_n}{\partial t} + \mu v \sum_{n=0}^{\infty} (2n+1) P_n \frac{\partial f_n}{\partial r} \\ & + \frac{1}{\tau} \sum_{n=0}^{\infty} (2n+1) P_n f_n = \frac{1}{\tau} \frac{1}{2} \int_{-1}^1 \sum_{n=0}^{\infty} (2n+1) P_n f_n d\mu. \end{aligned}$$

The right-hand side of that equation can be written as

$$\frac{1}{\tau} \frac{1}{2} \int_{-1}^1 \sum_{n=0}^{\infty} (2n+1) P_n f_n d\mu = \frac{1}{\tau} \frac{1}{2} \sum_{n=0}^{\infty} (2n+1) f_n \int_{-1}^1 P_n P_0 d\mu,$$

where we included $P_0(\mu) = 1$ in the integral. With the orthogonality relation (2.108),

$$\int_{-1}^1 P_n P_0 d\mu = \delta_{n0} \frac{2}{2n+1},$$

we find

$$\frac{1}{\tau} \frac{1}{2} \int_{-1}^1 \sum_{n=0}^{\infty} (2n+1) P_n f_n d\mu = \frac{1}{\tau} \sum_{n=0}^{\infty} f_n(r, t, v) \delta_{n0} = \frac{f_0}{\tau},$$

since the delta function contributes only for $n = 0$. The differential equation reduces to

$$\sum_{n=0}^{\infty} (2n+1) P_n \frac{\partial f_n}{\partial t} + \mu v \sum_{n=0}^{\infty} (2n+1) P_n \frac{\partial f_n}{\partial r} + \frac{1}{\tau} \sum_{n=0}^{\infty} (2n+1) P_n f_n = \frac{f_0}{\tau}.$$

With the recurrence relation $(2n+1)\mu P_n = (n+1)P_{n+1} + nP_{n-1}$ we can rewrite the second term and find

$$\begin{aligned} & \sum_{n=0}^{\infty} (2n+1) P_n \frac{\partial f_n}{\partial t} + v \sum_{n=0}^{\infty} (n+1) P_{n+1} \frac{\partial f_n}{\partial r} \\ & + v \sum_{n=0}^{\infty} n P_{n-1} \frac{\partial f_n}{\partial r} + \frac{1}{\tau} \sum_{n=0}^{\infty} (2n+1) P_n f_n = \frac{f_0}{\tau}. \end{aligned}$$

Now we multiply the equation by P_m and integrate with respect to μ ,

$$\begin{aligned} & \sum_{n=0}^{\infty} (2n+1) \int_{-1}^1 P_m P_n d\mu \frac{\partial f_n}{\partial t} + v \sum_{n=0}^{\infty} (n+1) \int_{-1}^1 P_m P_{n+1} d\mu \frac{\partial f_n}{\partial r} \\ & + v \sum_{n=0}^{\infty} n \int_{-1}^1 P_m P_{n-1} d\mu \frac{\partial f_n}{\partial r} + \frac{1}{\tau} \sum_{n=0}^{\infty} (2n+1) \int_{-1}^1 P_m P_n d\mu f_n \\ & = \frac{f_0}{\tau} \int_{-1}^1 P_m d\mu = \frac{f_0}{\tau} \int_{-1}^1 P_0 P_m d\mu. \end{aligned}$$

With the orthogonality relation Eq. (2.108) we find

$$\begin{aligned}
 & \sum_{n=0}^{\infty} (2n+1) \frac{2}{2m+1} \delta_{mn} \frac{\partial f_n}{\partial t} + v \sum_{n=0}^{\infty} (n+1) \frac{2}{2m+1} \delta_{mn+1} \frac{\partial f_n}{\partial r} \\
 & + v \sum_{n=0}^{\infty} n \frac{2}{2m+1} \delta_{mn-1} \frac{\partial f_n}{\partial r} + \frac{1}{\tau} \sum_{n=0}^{\infty} (2n+1) \frac{2}{2m+1} \delta_{mn} f_n \\
 & = \frac{f_0}{\tau} \frac{2}{2m+1} \delta_{m0}.
 \end{aligned}$$

Note that the second term contributes only for $n = m - 1$, while the third term contributes only for $n = m + 1$. All other terms on the left side contribute only for $n = m$, so that

$$\frac{\partial f_m}{\partial t} + v \frac{2m}{2m+1} \frac{\partial f_{m-1}}{\partial r} + v \frac{2(m+1)}{2m+1} \frac{\partial f_{m+1}}{\partial r} + \frac{1}{\tau} f_m = \frac{f_0}{\tau} \frac{2}{2m+1} \delta_{m0}.$$

By multiplying this equation with $(2m+1)/2$ and swapping the indices $m \leftrightarrow n$ we obtain Eq. (2.112).

Problem 2.27 Show that the integral

$$\int_{-\infty}^{\infty} \exp(-\beta\omega^2 t - i\alpha\omega t - i\omega r) d\omega = \sqrt{\frac{\pi}{\beta t}} \exp\left(-\frac{(r + \alpha t)^2}{4\beta t}\right). \quad (2.113)$$

Solution The exponential function can be rewritten as

$$\begin{aligned}
 \exp(-\beta\omega^2 t - i\alpha\omega t - i\omega r) &= \exp\left[-\beta t \left(\omega^2 + i\omega \frac{\alpha t + r}{\beta t}\right)\right] \\
 &= \exp\left[-\beta t \left(\omega + i \frac{\alpha t + r}{2\beta t}\right)^2 - \frac{(\alpha t + r)^2}{4\beta t}\right].
 \end{aligned}$$

By substituting $x = \omega + i(\alpha t + r)/2\beta t$ we obtain

$$\begin{aligned}
 \int_{-\infty}^{\infty} \exp(-\beta\omega^2 t - i\alpha\omega t - i\omega r) d\omega &= \exp\left[-\frac{(\alpha t + r)^2}{4\beta t}\right] \int_{-\infty}^{\infty} e^{-\beta t x^2} dx \\
 &= \sqrt{\frac{\pi}{\beta t}} \exp\left[-\frac{(\alpha t + r)^2}{4\beta t}\right].
 \end{aligned}$$

Transport Processes in Space Physics and Astrophysics
Problems and Solutions

Dosch, A.; Zank, G.P.

2016, VIII, 257 p. 5 illus. in color., Softcover

ISBN: 978-3-319-24878-3