

Chapter 2

Stochastic Resonance

Noise is often thought of as interfering with signal detection and information transmission. Static on a radio-station, ancillary conversations in a crowded room and flashing neon light along a busy thoroughfare all tend to obscure or distract from the desired information. Now, it has been realized that certain noise-induced phenomena are of great use in various contexts. One such noise-induced phenomenon which has a wide range of applications in various branches of science and engineering is stochastic resonance. Consider a double-well potential system (bistable system) driven by a weak periodic force of frequency ω . The amplitude of the external driving force is so small that there is no cross-well motion. When noise is added to the system, then at an appropriate noise intensity, a periodic switching between the two wells takes place. At this optimum value of noise intensity the signal-to-noise ratio measured at the frequency ω becomes a maximum. This noise-induced phenomenon is called *stochastic resonance* [1, 2].

The stochastic resonance phenomenon was first introduced by the Italian physicists Benzi et al. [1] to explain the periodicity of earth's ice ages. The eccentricity of the orbit of the earth varies with a periodicity of about 10^5 years, but according to current theories the variation is not strong enough to cause a dramatic climate change. Suppose the ice-covered and water-covered earth correspond to the two local minima. The variation of eccentricity with the period 10^5 years is too weak to induce the transition from ice-covered to water-covered earth and vice-versa. By introducing a bistable potential Benzi et al. [1] suggested that a cooperative phenomenon between the weak periodic variation in the eccentricity (the signal) and the other random fluctuations might account for the strong periodicity observed. Motivated by their work, the effect of noise has been investigated in nonlinear systems. Many fascinating phenomena like stochastic resonance [3, 4], noise-enhanced stability [5, 6], noise-delayed extinction [7], noise-induced intermittency [8, 9], synchronization [10–12], phase transition [13], pattern formation [14], linearization [15, 16], stochastic transport in ratchets [17], mixed mode oscillations [18, 19] and collective firing in excitable media [20, 21] have been reported. A phenomenon

similar to stochastic resonance was found in an experiment with acoustically excited turbulent, submerged jets [22] where the role of noise is played by turbulence. Being noisy, nature takes advantages of such noise-induced processes to employ random fluctuations as an agent of self-organization and improved performance. This is the prime reason why living systems work so reliably in spite of the presence of various sources of noise.

Stochastic resonance was found in numerous systems ranging from astronomic macrocosm to quantum micro-cosmology. For example, it was found in a ring laser [23], Schmidt trigger [24, 25], tunnel diode [26], electron-paramagnetic-resonance systems [27], monostable systems [28], nanomechanical oscillators [29], carbon nanotube transistors [30, 31], small world networks [32], delayed-feedback systems [33–35], chemical systems [36], financial models [37] and social systems [38]. Stochastic resonance has been investigated in threshold crossing (TC) systems also called *excitable systems* because their output signal consist of pulses which can be emitted when the noisy input crosses some threshold [39, 40].

The present chapter, briefly summarizes first the characterization of stochastic resonance in terms of signal-to-noise ratio, mean residence time and probability distribution of residence times. This is followed by a numerical illustration of stochastic resonance in a double-well Duffing oscillator. Next, a theory of stochastic resonance for a overdamped bistable system is given. The occurrence of stochastic resonance in an optomechanically coupled oscillator, a magnetic system and in a monostable system is also discussed. Realization of stochastic resonance in quantum systems is brought out. Finally, some of its applications are enumerated.

2.1 Characterization of Stochastic Resonance

Consider a bistable system driven by a weak periodic force and an additive noise exhibiting stochastic resonance. Quantities such as signal-to-noise ratio (*SNR*), input-output gains, cross-correlation, mutual information, channel capacity, detection probability and propagation distance are useful to analyze the performance of the system in the presence of noise.

What are the signatures of stochastic resonance? The most common measure used to characterize stochastic resonance is *SNR*. Suppose the input signal is the sine-wave $S(t) = f \sin \omega t$. The power spectrum of the noisy signal is a superposition of a background power spectral density and delta-spikes centered at $\Omega = (2n + 1)\omega$ with $n = 0, \pm 1, \dots$. *SNR* is the ratio of the Fourier coefficient and the value of the noise at the frequency ω . That is, it measures how much the system output, say $x(t)$, contains the input signal frequency ω . Often it is defined as

$$SNR = 10 \log_{10} \left(\frac{S(\omega)}{N(\omega)} \right) \text{ dB} . \quad (2.1)$$

The signal power $S = |X(\omega)|^2$ is the magnitude of the output power spectrum $X(\Omega)$ at the frequency ω . S can be used as a measure of the response of the system to the external driving force. The background noise spectrum $N(\omega)$ at the input frequency ω is some average of $|X(\Omega)|^2$ at nearby frequencies. In a typical stochastic resonance phenomenon, SNR increases with the noise intensity D , peaks at an optimum value of D and then decreases smoothly for higher values of D .

Other statistical tools useful for the characterization of stochastic resonance are the mean residence time (T_{MR}) and probability distribution of residence time ($P(T_R)$). A residence time in a well is defined as a time duration spent by the system in it before switching to another well. Mean residence time is the average over a large number of residence times. In the absence of noise, the motion of the system is confined to a well or an equilibrium state. That is, before the noise-induced dynamics, the residence time of the system in each well is infinite. When the noise intensity is increased at a certain value of the noise intensity, say D_c , the system begins to visit the other well also. For D values, just above D_c , T_{MR} of the system in a well is very large, say, $T_{MR} \gg T/2$ where $T = 2\pi/\omega$ is the period of the input periodic signal. In a symmetric bistable system T_{MR} in the two wells are equal and moreover at resonance (at which SNR is maximum) $T_{MR} = T/2$. In asymmetric systems at resonance periodic switching between the wells occur but with different switching rates for the left- and right-wells. For large values of the noise intensity, erratic switching between the wells occur.

The $P(T_R)$ demonstrates coherence of the stochastic response with the modulation. This quantity shows a sequence of strong Gaussian-like peaks centered at odd integer multiples of the modulation half period $T/2$ and exponentially decaying maximum amplitudes.

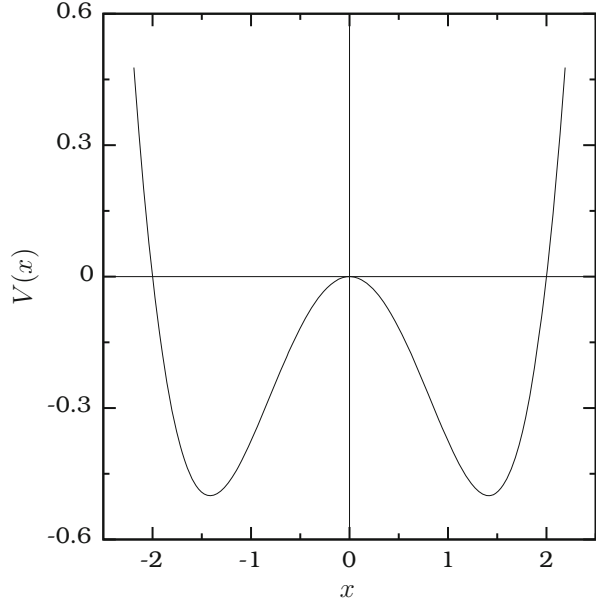
2.2 Stochastic Resonance in Duffing Oscillator

Let us illustrate the stochastic resonance phenomenon in the Duffing oscillator

$$\ddot{x} + d\dot{x} + \omega_0^2 x + \beta x^3 = f \sin \omega t + \xi(t) . \quad (2.2)$$

The term $f \sin \omega t$ is normally interpreted as representing the signal which is to be amplified. The noise term is often an additive Gaussian white noise with the moments $\langle \xi(t) \rangle = 0$, $\langle \xi(t) \xi(s) \rangle = D \delta(t - s)$ and D is the noise intensity (variance). Fix the parameters as $\omega_0^2 = -1$, $\beta = 0.5$, $d = 0.5$, $\omega = 0.05$. In the absence of the noise term $\xi(t)$, for small values of the amplitude f of the driving signal, two period- T ($= 2\pi/\omega$) orbits coexist. Each of the two wells of the potential (Fig. 2.1) has one periodic orbit. As the value of f increases, the interval of the state variable x covered by the periodic orbits increase. For $0 < f < 1$ period- T orbits alone exist. However, at a critical value of f , the trajectory makes a periodic transition from one well to another well forming a cross-well periodic orbit. The critical value of f , f_c , at which cross-well orbit first occurs is 0.56.

Fig. 2.1 Double-well potential $V(x) = \frac{1}{2}\omega_0^2 x^2 + \frac{1}{4}\beta x^4$ of the Duffing oscillator with $\omega_0^2 = -1$ and $\beta = 0.5$



Now, fix f at 0.38, a value $< f_c$, so that the additional force is weak. Here, the term weak means that in the absence of noise the periodic force alone is unable to move a particle from one well to the other one. Let us investigate the effect of added noise by varying the noise intensity D . In the numerical simulation Eq. (2.2) is integrated with the time step $\Delta t = (2\pi/\omega)/2000$ from t to $t + \Delta t$ without the noise term. Then, the noise is added to the state variable x as $x(t + \Delta t) \rightarrow x(t + \Delta t) + \sqrt{D\Delta t} \xi(t)$ where $\xi(t)$ represents Gaussian random numbers with zero mean and variance D . This procedure is repeated.

2.2.1 Time-Series Plot

Figure 2.2 shows a time series plot for few values of D . For small values of D the motion is confined to one well alone as in Fig. 2.2a. The system exhibits the behaviour similar to that of the noise free case but slightly perturbed by the noise. This type of behaviour occurs for $D < D_c = 0.011$. At this critical value of D cross-well behaviour occurs. The trajectory jumps randomly from one well to another. In Fig. 2.2b for $D = 0.05$ just above D_c , the state variable x switches irregularly and rarely between positive and negative values, that is, between the two wells. In the presence of forcing, the system initially in the well, say, V_+ ($x > 0$) is forced by the noise to leave the well. Then, the system enters the well V_- ($x < 0$) and wanders irregularly there for some time and jumps back to the well V_+ and so on. The switching is not periodic. As the value of D increases further, the switching

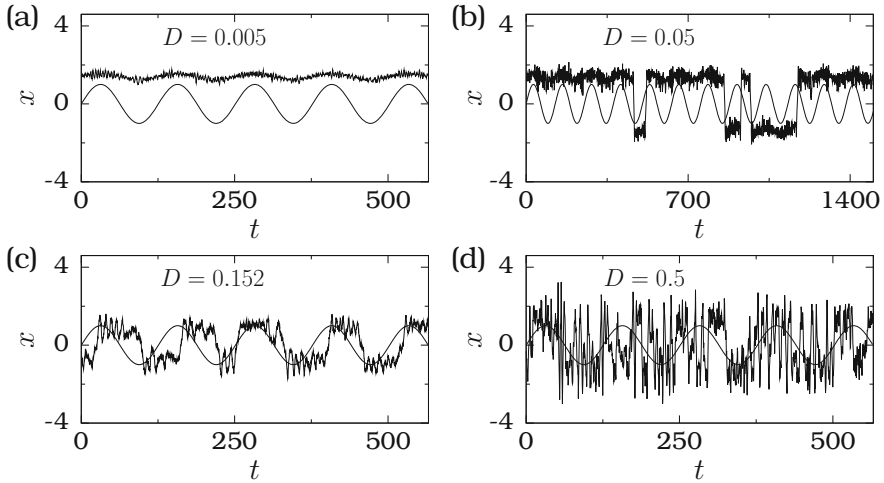


Fig. 2.2 Time series plot of Eq. (2.2) in the presence of periodic external force for four values of noise intensity D . The parameters of the system are $\omega_0^2 = -1$, $\beta = 0.5$, $d = 0.5$, $f = 0.38$ and $\omega = 0.05$. The external force $f \sin \omega t$ is also shown

between the wells increases. At $D = 0.152$ (Fig. 2.2c) almost periodic switching is observed. The periodicity of the noise-induced oscillation in the time behaviour of x is the same of the driving periodic force. This is the signature of the stochastic resonance. Further increase of the noise intensity produces a loss of coherence. For sufficiently large values of D , the motion is strongly dominated by the noise. In this case intermittent dynamics disappears and the trajectory jumps erratically between the wells. This is shown in Fig. 2.2d for $D = 0.5$. From the above, it is clear that the phenomenon of stochastic resonance is simply an event of potential barrier crossing, where the transition from one local equilibrium situation to the other happens in the presence of noise. An indication of stochastic resonance is that the flow of information through a system is maximized when the input noise intensity is set to a certain value.

Let us point out the mechanism of stochastic resonance. When a weak periodic signal is applied to a bistable system, it serves to periodically modulate the potential by raising and lowering the wells as shown in Fig. 2.3. Essentially, the additive forcing changes alternately the relative depth of the potential wells, increasing the probability of jumps between wells twice per modulation period. At a critical value of the noise intensity, the particle in a well arrives at the neighbourhood of the local maximum (barrier), almost when the barrier height seen by it is minimum so that the noise is able to push it to the other well. Thus, there is a synchronization between the periodic force and the output signal resulting in a periodic switching of the particle from one well to another well. The essential ingredients for stochastic resonance consists of a nonlinear system, a weak signal and a source of noise. In bistable systems, the underlying mechanism of stochastic resonance is easily appreciated,

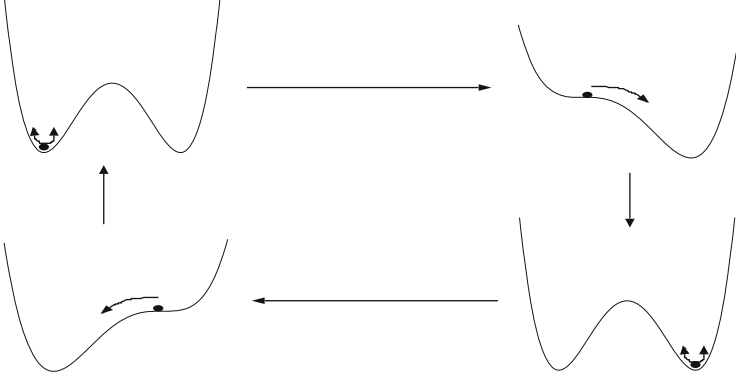


Fig. 2.3 Periodic modulation of the potential wells by the driving periodic force and the switching motion of the particle from one well to another induced by the additive noise

and in fact has been known since 1929 with the work of Peter Debye (1884–1966), a Dutch-American physicist and physical chemist, and Nobel laureate in Chemistry, on reorienting polar molecules [41].

2.2.2 Mean Residence Time

In order to get a deep understanding of the observed dynamics and the influence of noise one can compute the mean residence time, switching rate and *SNR*. For a fixed noise intensity D , residence times T_R on each well are computed for a set of 10^5 transitions. Then mean residence time T_{MR} is calculated. One can define the switching rate T' as the number of times the particle jumped from one well to another per driving period of the external periodic force. Figure 2.4 shows the numerically computed T_{MR} and T' as a function of the noise intensity D . For $D < D_c$, there is no switching and so T_{MR} is infinity and T' is zero. T_{MR} decreases with increase in D . For values of D just above D_c , the mean residence time is much larger than the period of the driving force indicating that the chance for the system to switch from one potential well to another is very small.

For $D = 0.152$ $T_{MR} \approx \pi/\omega \approx 62.83185 \dots$. In this case nearly periodic switching between the two wells V_+ and V_- occur. This is clearly evident in Fig. 2.4. At this critical value of D the switching rate is 2. Later, it will be shown that the *SNR* is maximum at this critical value of D . Denote the value of D at which $T_{MR} \approx \pi/\omega$ and $T' = 2$ as D_{MAX} . When $T' = 2$ there is a co-operation between the periodic driving force and the noise. The trajectory switches between the positive and negative values with the period approximately half of the period of the applied external periodic force. Nearly at the end of one half of a drive cycle the trajectory in one well is likely to jump to the other well and after the next half cycle it is likely

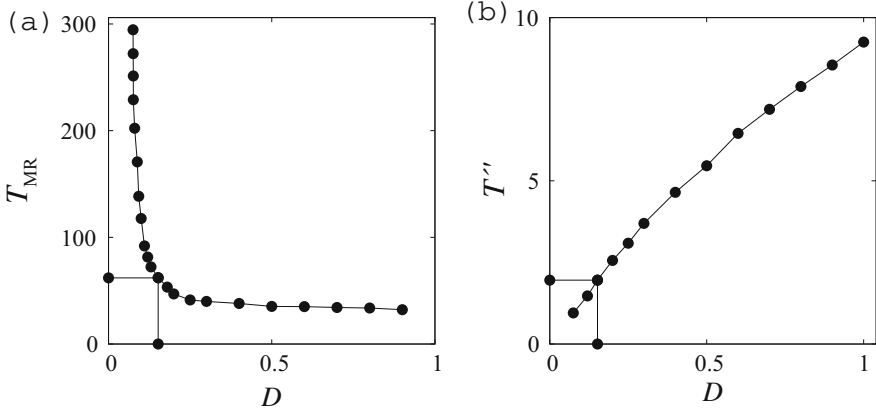


Fig. 2.4 Mean residence time T_{MR} and switching rate T' as a function of noise intensity D for the Duffing oscillator. The *solid circle* on the D -axis denotes the value of D at which resonance occurs and the corresponding *solid circles* on the y -axis in the subplots (a) and (b) denote the corresponding values of $T_{MR}(= T/2)$ and $T'(= 2)$, respectively

to return back. This is the signature of stochastic resonance. Note that at $D = D_{MAX}$ not only $T_{MR} \approx \pi/\omega$ but residence times T_R are all $\approx \pi/\omega$. For other values of D with $D > D_c$, T_R are randomly distributed over a range. Thus the relevant condition for resonance to occur is to tune the noise intensity to an optimum value so that synchronization between the drive and the output signal occurs. In symmetric bistable systems this corresponds to setting T_{MR} as $T/2$. In asymmetric bistable systems T_{MR} at resonance is different [42]. For $D > D_{MAX}$ loss of coherence is produced and the mean residence time of the system is much smaller than the value $T/2$. This means that the system will not wait for the relevant potential barrier to assume a minimum value.

2.2.3 Power Spectrum and SNR

Now, characterize stochastic resonance by using SNR , which can be calculated from the power spectrum. To obtain the power spectrum of the variable x , a set of 2^{10} data collected at a time interval of $(2\pi/\omega)/10$ is used. The output of the fast Fourier transform (FFT) routine is the spectral density of the output signal. More accurate spectral densities are obtained by averaging over 25 different realizations of Gaussian random numbers. To calculate SNR , the peak height of the signal and the broad-band noise level at the signal frequency ω are measured. The peak height is directly read from the FFT data. To calculate the background of the power spectrum about ω , consider the power spectrum in the interval $[\omega - \Delta\omega, \omega + \Delta\omega]$ after

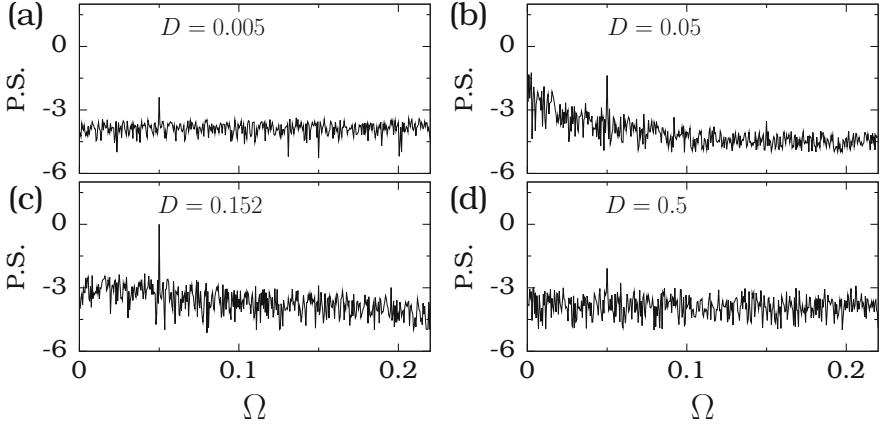


Fig. 2.5 Power spectral density of x -component of the state variable of the Duffing oscillator for different values of noise intensity D . The main peak in the spectrum is at the frequency ($\omega = 0.05$) of the periodic driving force

subtracting the spike at the frequency ω . The average value of the power spectrum in the above interval is taken as the background noise level at ω .

Figure 2.5 depicts the power spectra for four different values of D . In all the subplots, the power spectrum has a peak at the frequency $\omega = 0.05$ of the system riding on a broad noise background. The interplay between noise and periodic driving force results in a sharp increase of the signal power spectrum about the forcing frequency ω . When the noise intensity D is increased from D_c , the height of the peak increases for a while and then decreases. This is a signature of stochastic resonance. The amplitude of the noise background is obtained by averaging the spectral density for Ω in the interval 0.04–0.06 after eliminating the peak at the periodic driving frequency $\omega = 0.05$. Equation (2.1) gives the value of SNR in units of decibel. It is calculated for a range of noise intensity above $D_c = 0.011$.

Signal and noise power densities used to calculate SNR are plotted against D in Fig. 2.6a. Figure 2.6b shows the plot of SNR as a function of D . In Fig. 2.6a as D increases, the noise and signal levels increase. The noise level attains a maximum at $D = 0.09$. When D is increased further, the noise level is almost flat. On the other hand, the signal level increases for D values above D_c and attains a maximum level at $D = 0.152$ and then begins to decrease. Though for D in the interval $[0, 4]$ both the signal and noise levels increase, the former increases relatively at a higher rate. As a result the SNR increases with noise intensity D and peaks at $D_{MAX} = 0.152$. For $D > D_{MAX}$ the noise level is almost constant whereas the signal level decreases. Consequently, the SNR decreases with D for $D > D_{MAX}$. For D values just above D_c , the time series plot shows rare switching between the wells. That is, for low intensity the combination of noise and external periodic force occasionally gives the system a kick sufficiently large to cross the barrier between the two wells. As D increases, at 0.152 a transition between the two wells is induced for almost over every half

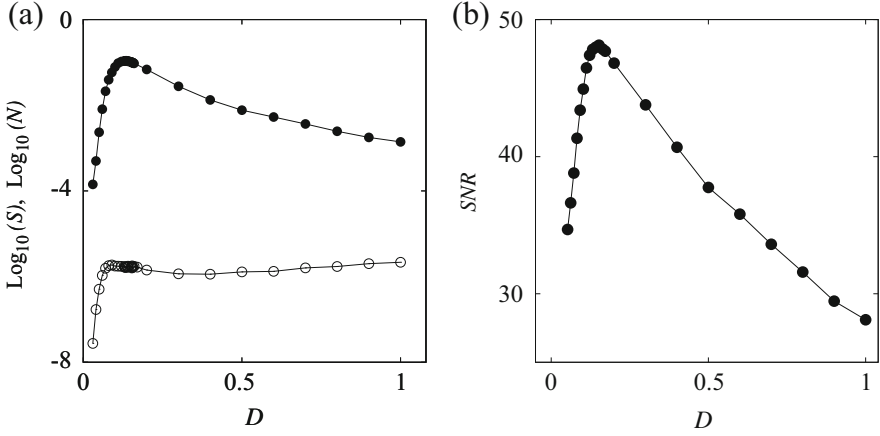


Fig. 2.6 (a) Output signal density (marked by *filled circle*) and noise power density (marked by *open circle*) as a function of input noise intensity D for the Duffing oscillator. (b) Signal-to-noise ratio (SNR) as a function of input noise intensity for $\omega_0^2 = -1$, $\beta = 0.5$, $d = 0.5$, $f = 0.38$ and $\omega = 0.05$

period of the driving force which resulted in the maximum of SNR . As D is further increased, switching become frequent and irregular so that the power in the Fourier spectra is distributed widely over a wide range of frequencies thereby leading to a decrease in SNR . Stochastic resonance indicates that the flow of information through the system is maximized when the input noise intensity is set to an optimum value (D_{MAX}). Note that SNR vanishes for both $D \rightarrow 0$ and $D \rightarrow \infty$ and it peaks at a critical value. The point is that the performance of the system with an optimum noise is better than its performance without noise.

2.2.4 Probability Distribution of Residence Times

Although the power spectrum is the most widely used coherence measure, it is not the only possibility. An alternative quantity, which also clearly demonstrates the stochastic resonance phenomenon is the probability distribution of normalized residence times. This is obtained as follows. For a fixed noise intensity D , 10^5 residence times T_R on a well are computed. Then, normalized residence times are obtained by dividing T_R by T , where $T = 2\pi/\omega$ is the period of the weak periodic force $f \sin \omega t$. The distribution of normalized residence times is shown in Fig. 2.7 for four different values of the noise intensity D with $\omega = 0.05$. The distribution shows a sequence of strong Gaussian-like peaks centered near the discrete set $T_R/T = n + \frac{1}{2}$, $n = 0, 1, 2, \dots$. That is, $P(T_R)$ has peaks at odd integral multiples of half of the forcing period, $T/2$. The height of the peaks decrease with their order n . These peaks correspond to the appropriate times for the system to make transitions

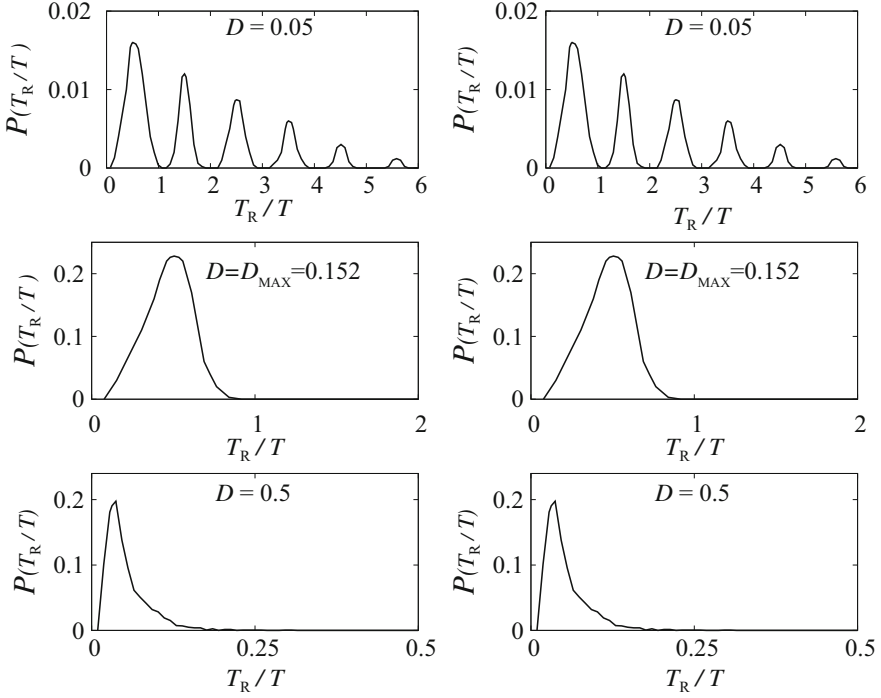
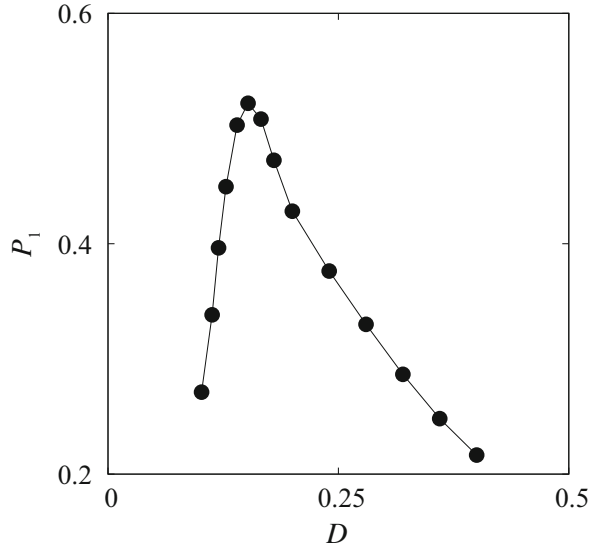


Fig. 2.7 Normalized residence time distribution for different values of noise intensity D . *Left-column and right-column* subplots are for the left-well and the right-well, respectively

between the potential wells. At these times the relevant potential barrier becomes a minimum. This happens when the potential $V(x, t) = V(x) - fx \sin \omega t$ is tilted most extremely to the left or right. If the system moves at this time into the other well, then it will reside in that well for almost $T/2$ time duration until the new relevant barrier height becomes a minimum. Thus, $T/2$ is a more suitable residence time interval. If the system is unable to switch to the other well in a $T/2$ time interval, then it has to wait another one complete period of the drive until the relevant potential barrier takes a minimum value. In this case, the residence time becomes $3T/2$ and is the location of the second peak in Fig. 2.7.

Let us denote the maximum of peaks in the successive Gaussian type distribution as P_n . A meaningful criterion of stochastic resonance based on normalized residence times distribution is the variation of the height of the peak (P_1) at the half driving period [4]. Figure 2.8 shows the numerically computed P_1 as a function of D . For values of D nearly above D_c , T_R/T is distributed relatively over a wide interval of time. As D increases, the range of T_R decreases and hence $P(T_R)$ of smaller T_R increases. This happens up to $D = D_{\text{MAX}} = 0.152$. Consequently, P_1 increases from a small value and reaches a maximum at $D = D_{\text{MAX}}$. As D is further increased P_1 decreases.

Fig. 2.8 Numerically computed P_1 as a function of D for the Duffing oscillator



Stochastic resonance in the electronic circuit simulation of the Duffing oscillator, Eq. (2.2), has been reported in [43]. Badzey and Mohanty [44] fabricated two double-clamped damped nanomechanical beams from single-crystal of silicon making use of an e-beam lithography and dry etching and observed stochastic resonance.

2.3 Theory of Stochastic Resonance

The overdamped bistable system

$$\dot{x} = -\frac{dV}{dx} + \xi(t), \quad V(x, t) = -\frac{1}{2}\omega_0^2 x^2 + \frac{1}{4}\beta x^4 - fx \cos \omega t \quad (2.3)$$

with ω_0^2 and $\beta > 0$ is used as a prototype model equation for developing theoretical approaches and investigating various features of stochastic resonance. In the above equation ξ is a Gaussian white noise with variance D . In the absence of the periodic driving the double-well potential has two minima $x_{\pm} = \pm c = \pm \sqrt{\omega_0^2/\beta}$ and a maximum x' (the location of the maximum of the potential separating the two wells) at the origin. The barrier heights of the two wells are equal and are $h_{\pm} = \omega_0^4/(4\beta)$. It is possible to obtain expressions for signal and noise output from bistable systems which agree with numerical and experimental analysis thereby providing much insight into the physical source of the stochastic resonance phenomenon. McNamara

and Wiesenfeld [45] developed a theory for stochastic resonance based on a rate equation approach. In the following, their theory is briefly presented.

2.3.1 Analytical Expression for Power Spectrum

Let us treat the dynamical variable as discrete. It can be either x_+ or x_- with probabilities $n_{\pm} = P(x = x_{\pm})$. For a continuous variable, as with the double-well system, write

$$n_- = 1 - n_+ = \int_{-\infty}^{x'} P(x) dx , \quad (2.4)$$

where the probability density is

$$P(x, t) = n_+(t)\delta(x - x_+) + n_-(t)\delta(x - x_-) . \quad (2.5)$$

The values of $x_{\pm}$ are chosen suitably so that the error in the variance of x is minimum. For simplicity, assume that the system is symmetrical about $x = 0$ and hence $x_{\pm} = \pm c$. The variance of an unmodulated two-state system in its steady state ($n_{\pm} = 1/2$) is

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x) dx = x_+^2 n_+ + x_-^2 n_- = c^2 . \quad (2.6)$$

The master equation [46] governing the evolution of n_+ is

$$\begin{aligned} \frac{dn_+}{dt} &= -\frac{dn_-}{dt} = W_-(t)n_- - W_+(t)n_+ \\ &= W_- - [W_- + W_+]n_+ , \end{aligned} \quad (2.7)$$

where $W_{\pm}(t)$ is the transition rate out of the state $x_{\pm}$. The solution of Eq. (2.7) is

$$n_+(t) = \frac{1}{g(t)} \left[n_+(t_0)g(t_0) + \int_{t_0}^t W_-(t')g(t') dt' \right] , \quad (2.8a)$$

where

$$g(t) = e^{\int^t W_+(t') + W_-(t') dt'} . \quad (2.8b)$$

If the system is subjected to a periodic signal $f_0 \cos \omega t$ then

$$W_{\pm}(t) = F(\mu \pm f_0 \cos \omega t) . \quad (2.9)$$

The time dependence of $W_{\pm}(t)$ is due to the external driving force. $W_{\pm}$ can be expanded in the small parameter $\eta = f_0 \cos \omega t$ as

$$W_{\pm} = \frac{1}{2} (\alpha_0 \mp \alpha_1 f_0 \cos \omega t + \alpha_2 f_0^2 \cos^2 \omega t + \dots) , \quad (2.10a)$$

$$W_+ + W_- = \alpha_0 + \alpha_2 f_0^2 \cos^2 \omega t + \dots , \quad (2.10b)$$

where $\alpha_0 = 2F(\mu)$ and $\alpha_n = 2(-1)^n d^n F / d\eta^n$. Now, integration of Eq. (2.7) gives, to first-order in η ,

$$\begin{aligned} n_+(t|x_0, t_0) = \frac{1}{2} \left\{ e^{-\alpha_0(t-t_0)} \left[2\delta_{x_0c} - 1 - \frac{\alpha_1 f_0 \cos(\omega t_0 - \phi)}{(\alpha_0^2 + \omega^2)^{1/2}} \right] \right. \\ \left. + 1 + \frac{\alpha_1 f_0 \cos(\omega t - \phi)}{(\alpha_0^2 + \omega^2)^{1/2}} \right\} , \end{aligned} \quad (2.11)$$

where $\phi = \tan^{-1}(\omega/\alpha_0)$. In Eq. (2.11) the Kronecker delta function is 1 if the system is initially in the ‘+’ state and 0 if it is in the ‘-’ state. $n_+(t|x_0, t_0)$ is the conditional probability that $x(t)$ was in the + state at time t given that the state at time t_0 was x_0 .

Equation (2.11) provides a useful statistical information. For example, the auto-correlation function is given by McNamara and Wiesenfeld [45]

$$\begin{aligned} \langle x(t)x(t+\tau)|x_0, t_0 \rangle &= c^2 n_+(t+\tau|+c, t) n_+(t|x_0, t_0) \\ &\quad - c^2 n_+(t+\tau|-c, t) n_-(t|x_0, t_0) \\ &\quad - c^2 n_-(t+\tau|+c, t) n_+(t|x_0, t_0) \\ &\quad + c^2 n_-(t+\tau|-c, t) n_-(t|x_0, t_0) \\ &= c^2 \{ [2n_+(t+\tau|+c, t) - 1 + 2n_+(t+\tau|-c, t) - 1] \\ &\quad \times n_+(t|x_0, t_0) - [2n_+(t+\tau|-c, t) - 1] \} . \end{aligned} \quad (2.12)$$

In the limit $t_0 \rightarrow -\infty$

$$\begin{aligned} \langle x(t)x(t+\tau) \rangle &= c^2 e^{-\alpha_0|\tau|} \left[1 - \frac{\alpha_1^2 f_0^2 \cos^2(\omega t - \phi)}{\alpha_0^2 + \omega^2} \right] \\ &\quad + \frac{c^2 \alpha_1^2 f_0^2 \{ \cos \omega \tau + \cos [\omega(2t + \tau) + 2\phi] \}}{2(\alpha_0^2 + \omega^2)} . \end{aligned} \quad (2.13)$$

The power spectrum, which is the Fourier transform of the auto-correlation function, is a function of t and Ω . In an experiment, typically, one may take an

ensemble of many time series $t_1, t_2, \dots$, compute the power spectrum for each one and then take the average of them. The result is

$$\begin{aligned}
 \langle S(\Omega) \rangle_t &= \frac{\omega}{2\pi} \int_0^{2\pi/\omega} S(\Omega, t) dt \\
 &= \frac{\omega}{2\pi} \int_{-\infty}^{\infty} \int_0^{2\pi/\omega} \langle x(t)x(t+\tau) \rangle e^{-i\Omega\tau} d\tau dt \\
 &= \int_{-\infty}^{\infty} \left\{ c^2 e^{-\alpha_0|\tau|} \left(1 - \frac{\alpha_1^2 f_0^2}{2(\alpha_0^2 + \omega^2)} \right) + \frac{c^2 \alpha_1^2 f_0^2 \cos \omega\tau}{2(\alpha_0^2 + \omega^2)} \right\} e^{-i\Omega\tau} d\tau \\
 &= \frac{2c^2 \alpha_0}{\alpha_0^2 + \Omega^2} \left(1 - \frac{\alpha_1^2 f_0^2}{2(\alpha_0^2 + \omega^2)} \right) \\
 &\quad + \frac{\pi c^2 \alpha_1^2 f_0^2}{2(\alpha_0^2 + \omega^2)} [\delta(\Omega - \omega) + \delta(\Omega + \omega)]. \tag{2.14}
 \end{aligned}$$

Then

$$\begin{aligned}
 \langle S(\Omega) \rangle_t &= \langle S(\Omega) \rangle_t + \langle S(-\Omega) \rangle_t \\
 &= \frac{4c^2 \alpha_0}{\alpha_0^2 + \Omega^2} \left(1 - \frac{\alpha_1^2 f_0^2}{2(\alpha_0^2 + \omega^2)} \right) + \frac{\pi c^2 \alpha_1^2 f_0^2}{\alpha_0^2 + \omega^2} \delta(\Omega - \omega). \tag{2.15}
 \end{aligned}$$

The power spectrum given by Eq. (2.15) consists of two parts:

- The signal output (the second term in the equation): It is a delta-function at the input signal frequency ω .
- The broad-band noise output: It is a Lorentzian hump centered at $\Omega = 0$. The noise spectrum is the product of the Lorentzian obtained with $f = 0$ and a correction factor representing the effect of the signal on the noise. For sufficiently small signal amplitude, this factor is nearly the unity.

The effect of the correction factor is the reduction of the noise power. This reduction is most pronounced for low-frequency ($\omega \ll \alpha_0$) and large amplitude of the input signal. The signal has the effect of transferring power from the broad-band into the delta-function spike. Assume that the total output power (signal+noise) is independent of the parameter f_0 and ω . This is a consequence of Parseval's relation, namely, the time integral of the square of the signal is equal to the integral of the power spectrum over all frequencies and moreover the system takes on discrete values $\pm c$ at all times.

2.3.2 Determination of Signal-to-Noise Ratio

The above two-state theory is valid when the drive frequency and the inter-well transition rates are much slower than the intra-well relaxation rate. The correctness of Eq. (2.15) has been verified by numerically integrating Eq. (2.3) and from the obtained power spectrum [45]. The potential $V(x, t)$ given in Eq. (2.3) can be rewritten as $V(x, t) = h_+[-2(x/c)^2 + (x/c)^4] - V_1(x/c) \cos \omega t$ with $V_1 = fc$. When $f = 0$, in the system (2.3), the mean first-passage time is given by the Kramers time (the average time taken by a particle to cross the region $x < 0$ ($x > 0$) for a set of initial conditions taken around $x = x_-$ (x_+))

$$\tau_{\text{MFP}} = \frac{1}{W} = \frac{2\pi e^{2h_+/D}}{(|V''(0)|V''(c))^{1/2}} = \frac{\sqrt{2}\pi e^{2h_+/D}}{\omega_0^2}. \quad (2.16)$$

The Kramers rate formula is derived under the assumption that the probability density within a well is roughly at equilibrium and is a Gaussian distribution centered about the minimum. For $f \neq 0$ there is an interaction between the drive and the noise resulting in a resonance in the *SNR*. Assume that

$$W_{\pm}(t) = \frac{\omega_0^2}{\sqrt{2}\pi} e^{-2(h_+ \pm V_1 \cos \omega t)/D}. \quad (2.17)$$

In Eq. (2.17) ω must be much smaller than the characteristic rate for probability to equilibrate within a well. Since the rate is $V''(\pm c)$ the value of ω must be $\ll V''(\pm c) = 2\omega_0^2$.

A comparison of Eqs. (2.9) and (2.17) gives $\mu = h_+/D$, $f_0 = V_1/D = fc/D$ and

$$F(\mu + f_0 \cos \omega t) = \frac{\omega_0^2}{\sqrt{2}\pi} e^{-2(\mu + f_0 \cos \omega t)}, \quad (2.18a)$$

$$\alpha_0 = 2F(\eta = 0) = \frac{\sqrt{2}\omega_0^2}{\pi} e^{-2h_+/D}, \quad (2.18b)$$

$$\alpha_1 = -2 \left. \frac{dF}{d\eta} \right|_{\eta=0} = \frac{2\sqrt{2}\omega_0^2}{\pi} e^{-2h_+/D} = 2\alpha_0. \quad (2.18c)$$

A substitution of the above in the expression for S gives $S = S_s(\omega)\delta(\Omega - \omega) + S_n(\Omega, \omega)$ where

$$S_s = \left[\frac{\frac{8\omega_0^4 f^2 c^4}{\pi D^2} e^{-4h_+/D}}{\frac{2\omega_0^4}{\pi^2} e^{-4h_+/D} + \omega^2} \right], \quad (2.19)$$

$$S_n = \left[1 - \frac{\frac{4\omega_0^4 f^2 c^2}{\pi^2 D^2} e^{-4h_+/D}}{\frac{2\omega_0^4}{\pi^2} e^{-4h_+/D} + \omega^2} \right] \left[\frac{\frac{4\sqrt{2}\omega_0^2 c^2}{\pi} e^{-2h_+/D}}{\frac{2\omega_0^4}{\pi^2} e^{-4h_+/D} + \Omega^2} \right]. \quad (2.20)$$

Then SNR is worked out as

$$\begin{aligned} SNR &= \frac{S_s(\omega)}{S_n(\Omega = \omega, \omega)} \\ &= \frac{\sqrt{2}\omega_0^2 f^2 c^2}{D^2} e^{-2h_+/D} \left[1 - \frac{\frac{4\omega_0^4 f^2 c^2}{\pi^2 D^2} e^{-4h_+/D}}{\frac{2\omega_0^4}{\pi^2} e^{-4h_+/D} + \omega^2} \right]^{-1} \\ &\approx \frac{\sqrt{2}\omega_0^2 f^2 c^2}{D^2} e^{-2h_+/D}. \end{aligned} \quad (2.21)$$

Notice that for $D \ll h_+$ the exponential term decays to 0 very rapidly than the term D^2 and so $SNR \rightarrow 0$. On the other hand, for very large values of D , $e^{-2h_+/D} \approx 1$ while D^2 diverges leading to $SNR \rightarrow 0$. At a moderate noise level, SNR will become a maximum and the corresponding noise intensity, denoted as D_{MAX} , becomes $\approx h_+$. From the expression of SNR , one can observe that the signal output increases with increase in the input signal amplitude and the output noise decreases very slightly. Figure 2.9 shows SNR versus D for two values of ω_0^2 .

The above two-state theory was extended to an asymmetric double-well potential system [47]. Landa et al. [48] proposed a theory of stochastic resonance for weakly damped bistable systems. In their approach, the response to a harmonic signal of a nonlinear stochastic system was represented by the response to the same signal of

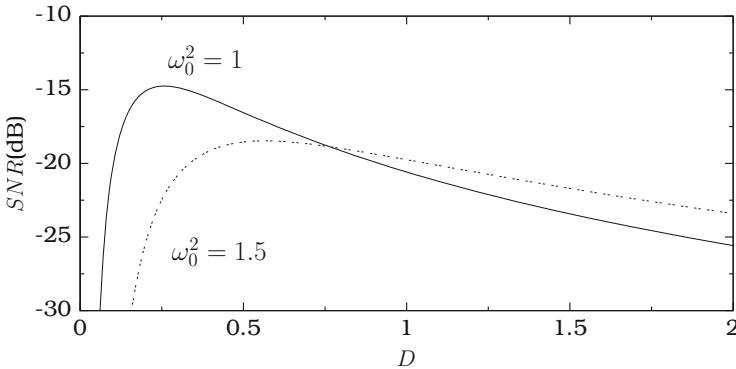


Fig. 2.9 SNR versus D with $f = 0.1$, $\beta = 1.0$, $\omega = 0.1$ and for two values of ω_0^2 for the overdamped bistable system

an effective linear noise free system. The latter is defined by an effective stiffness and an effective damping which depend on the nonlinearity of the system, the parameters of the signal and the intensity of the noise. An analytical two-state theory for stochastic resonance for a bistable system driven by noise and a rectangular periodic signal was also developed [49]. Stochastic resonance in multistable systems was investigated in [50, 51].

2.4 Stochastic Resonance in a Coupled Oscillator

Mueller et al. [52] reported their observation of stochastic resonance in a macroscopic torsion oscillator. Figure 2.10 depicts the experimental system [52]. The torsion balance oscillator shown in Fig. 2.10a is a precision force measurement device, sensitive down to the femto-Newton range. The oscillator is made of a gold coated glass plate of size $50 \text{ mm} \times 10 \text{ mm} \times 0.15 \text{ mm}$ and doubly suspended on a 15 cm long, $25 \mu\text{m}$ diameter tungsten wire. The mass and moment of inertia (I) of the oscillator body are $\sim 0.2 \text{ g}$ and $4.6 \times 10^{-8} \text{ Kg m}^2$, respectively. The natural frequency of the oscillator is $\omega_0 = 0.36 \text{ Hz}$ and the quality factor is ~ 2600 .

In the experiment, a laser beam was reflected from the centre of the oscillator and detected by a quadrant diode detector followed by a lock-in-amplifier (LIA). The angular position voltage was sampled at a rate of 5 kHz. The digitized signal was fed to a computerized digital control loop and was then converted to an analog output signal applied to two electronic electrodes. To generate optomechanical coupling, the gold coated glass plate was served as the moving flat mirror of a

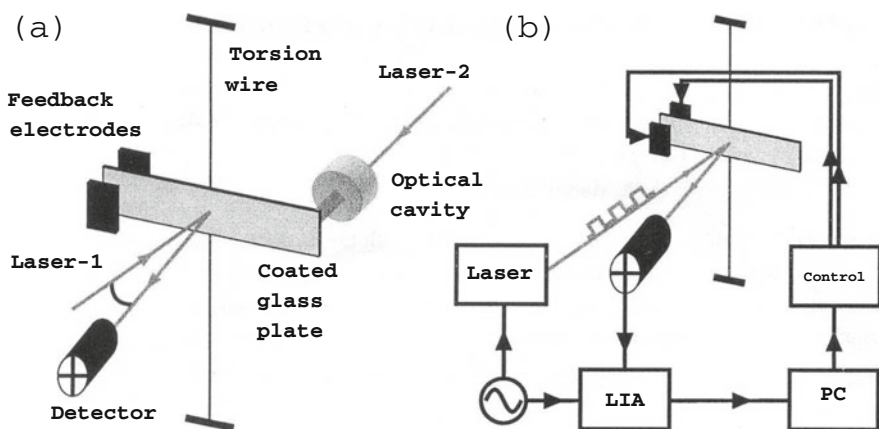


Fig. 2.10 Experimental arrangement of the optomechanically coupled oscillator [52]. (a) The torsion balance oscillator. (b) Block diagram of the set up. (Reprinted with permission from F. Mueller, S. Heugel, L.J. Wang, *Phys. Rev. A* **79**, 031804(R) (2009). Copyright (2009) by the American Physical Society.)

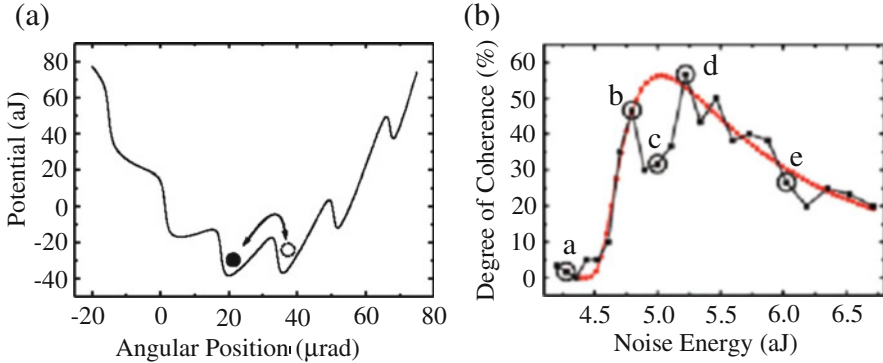


Fig. 2.11 (a) The multistable optomechanical potential generated with an optical cavity input power of 32mW [52]. (b) Degree of coherence versus noise energy. Dots are the experimental result and continuous smooth curve is the theoretical result given by Eq. (2.21) [52]. (Reprinted with permission from F. Mueller, S. Heugel, L.J. Wang, Phys. Rev. A **79**, 031804(R) (2009). Copyright (2009) by the American Physical Society.)

hemispherical optical cavity. Another spherical mirror with a curvature radius of 25 mm was mounted opposite to the glass plate at a distance of 12.5 mm. A second laser with a wavelength of ~ 660 nm was coupled in. This cavity formed Laguerre–Gaussian TEM_{00} and TEM_{20} modes. The finesse of the optical cavity was $F = 11$ and the mean mirror reflectivity was $R = 0.87$. The measurement sensitivity of the oscillator was 100 fN ($15 \mu\text{W}$ of optical power) for the detection of radiation pressure in total reflection. The entire set up was kept in a high vacuum (10^{-7} mbar) and was mounted on top of an active vibration isolation system.

The multistable potential of the system is shown in Fig. 2.11a. The minima of the potential were formed by TEM_{00} cavity modes with an angular spacing of $\sim 16 \mu\text{rad}$. In the experiment, the mechanical torsion constant of the free system was set to $\tau = 9.6 \times 10^{-8}$ Nm/rad, for which the period of the oscillation was $T_0 = 4.3$ s. Using a cavity optical input power $P_{\text{in}} = 32$ mW the torsion balance was optomechanically coupled. An electronic square-wave signal was applied to the feedback electrodes with $\omega = 200 \pi \text{ mHz}$ and $f = 0.79 \text{ pNm}$. Noise energy is varied from 4.4 to 6 aJ. The two centered TEM_{00} mode potential with minima at $\sim 20 \mu\text{rad}$ and $\sim 36 \mu\text{rad}$ with an average potential depth of ~ 20 aJ were considered for signal analysis. The period of the excitation signal, $T = 10$ s, was divided into number of intervals of width 0.15 s. Mueller, Heugel and Wang determined the number of residence times in each bin from the time series. Figure 2.11b shows the plot of degree of coherence, ratio of number of occurrences of residence times in the range $T/2 \pm 0.5$ s and total number of residence times in the range 0 to T , versus the noise energy. The degree of coherence was found to be maximum at the noise energy 5.2 aJ. In the Fig. 2.11b the noise energies corresponding to the data points marked as a, b, c, d and e are 4.4aJ, 4.8aJ, 5aJ, 5.2aJ and 6aJ, respectively.

2.5 Stochastic Resonance in a Magnetic System

An example of bistable magnetic elements is the single-domain magnetic particles with anisotropy in the direction of easy magnetization. Here, two stable states have opposite orientations of the magnetic moment vector along the direction of easy magnetization. Isavnin [53] theoretically studied the occurrence of stochastic resonance dynamics of the magnetic moment vector of a superparamagnetic particle with the additional external constant magnetic field applied along one of the directions of easy magnetization. Temperature (T) is considered in place of the noise intensity D .

Let us consider the uniaxial single-domain ferromagnetic particle and denote the angle between the magnetization vector M and the direction of the easy magnetization as θ , the volume of the particle as V , and K as the anisotropy constant. The energy due to the interaction of the magnetic moment of the superparamagnetic particle with the anisotropy field is $-KV \cos^2 \theta$. The interaction with the external constant magnetic field is $-\mu_0 M H_1 V \cos \theta$. Then the magnetic energy of the particle is

$$E(\theta) = -KV \cos^2 \theta - \mu_0 M H_1 V \cos \theta. \quad (2.22)$$

In the presence of the external field $H \cos \omega t$ applied along the direction of easy magnetization $E(\theta, t)$ is

$$E(\theta, t) = E(\theta) - \mu_0 M H V \cos \theta \cos \omega t. \quad (2.23)$$

For the two wells of $E(\theta, t = 0)$ the two local minima are at $\theta_1 = 0$ and $\theta_2 = \pi$ and a local maximum is at $\theta_3 = \cos^{-1}(-\mu_0 M H_1 / (2K))$. The potential is shown in Fig. 2.12. The barrier heights of the left-well and the right-well are

$$h_- = E(\theta_3) - E(\theta_1) = \frac{(\mu_0 M H_1)^2}{4K} + KV + \mu_0 M H_1 V, \quad (2.24a)$$

$$h_+ = E(\theta_3) - E(\theta_2) = \frac{(\mu_0 M H_1)^2}{4K} + KV - \mu_0 M H_1 V. \quad (2.24b)$$

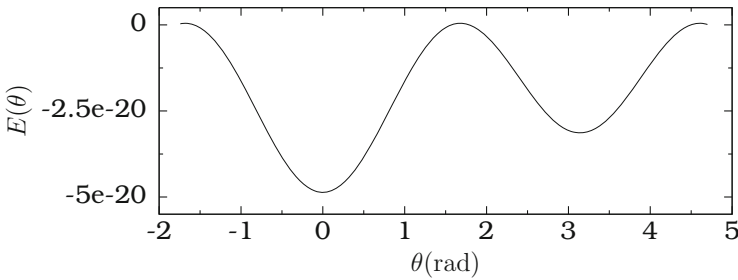


Fig. 2.12 $E(\theta)$ versus θ of a single-domain ferromagnetic particle system. The values of the parameters are $K = 4 \times 10^4 \text{ J/m}^3$, $V = 10^{-24} \text{ m}^3$, $M = 1.72 \times 10^6 \text{ A/m}$, $H_1 = 4 \times 10^3 \text{ A/m}$ and $\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$

The field is assumed to be sufficiently weak so that the orientation of the magnetic moment vector of the particle is unchanged. This is the case for $\mu_0 MHV < h_{\pm}$.

To apply the theory discussed in Sect. 2.3, assume that the magnetic moment vector takes only two states corresponding to the minima of the double-well potential. By introducing $x = M \cos \theta$, one can specify the projection of the magnetization vector on to the direction of easy magnetization having two values $\pm M$ at $\theta = \theta_2 = \pi$ and $\theta = \theta_1 = 0$. Denote $n_{\pm}(t)$ as the probabilities for x to be $x_{\pm} = \pm M$ and $W_{\pm}(t)$ as the rates of particle escape from $x_{\pm}$ states. Then the evolution equations for n_+ and n_- are given by Eq. (2.7). $W_{\pm}(t)$ are given by Isavnin [53]

$$W_{\pm}(t) = \alpha_0 e^{-\frac{h_{\pm}}{k_B T} \pm f \cos \omega t}, \quad f = \frac{\mu_0 MHV}{k_B T}, \quad (2.25)$$

where k_B is the Boltzmann constant, T is the temperature and α_0 is of the order of the frequency of ferromagnetic resonance which is $10^9 - 10^{10} \text{ s}^{-1}$ for single-domain iron particles. One can note that $W_+(t) \neq W_-(t)$. With the probability density function $P(x, t) = n_+ \delta(x - x_+) + n_- \delta(x - x_-)$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} x^2 P(x) dx = x_+^2 n_+ + x_-^2 n_- = M^2. \quad (2.26)$$

The solution $n_+(t)$ is given by Eq. (2.8).

Expanding $W_{\pm}(t)$ in a series in the small parameter $\eta = f \cos \omega t$ one obtains the lowest-order

$$W = W_-(t) + W_+(t) = \alpha_0 e^{-h_-/(k_B T)} + \alpha_0 e^{-h_+/(k_B T)}. \quad (2.27)$$

Then

$$\begin{aligned} n_+(t|t_0) = e^{-W(t-t_0)} & \left[n_+(t_0) - \frac{F}{W} - \frac{fF \cos(\omega t_0 - \phi)}{\sqrt{W^2 + \omega^2}} \right] \\ & + \frac{F}{W} + \frac{fFW \cos(\omega t - \phi)}{\sqrt{W^2 + \omega^2}}, \end{aligned} \quad (2.28a)$$

where

$$F = \alpha_0 e^{-h_-/(k_B T)}, \quad \phi = \tan^{-1}(\omega/W). \quad (2.28b)$$

F is the Kramers rate of escape of the system from the left-well of the asymmetric potential. In the limit of $t_0 \rightarrow -\infty$

$$n_+(t) = \frac{F}{W} + \frac{fF \cos(\omega t - \phi)}{\sqrt{W^2 + \omega^2}}. \quad (2.29)$$

Next,

$$\begin{aligned}
 \langle x(t) \rangle &= \int_{-\infty}^{\infty} xP(x) dx \\
 &= M(2n_+(t) - 1) \\
 &= M \left[\frac{2F}{W} + \frac{2Ff \cos(\omega t - \phi)}{\sqrt{W^2 + \omega^2}} - 1 \right].
 \end{aligned} \tag{2.30}$$

The component of $\langle x(t) \rangle$ which changes with the external driving frequency is

$$\begin{aligned}
 \langle x_\omega(t) \rangle &= \frac{2MFf \cos(\omega t - \phi)}{\sqrt{W^2 + \omega^2}} \\
 &= \frac{2MFf(W \cos \omega t + \omega \sin \omega t)}{W^2 + \omega^2}.
 \end{aligned} \tag{2.31}$$

Writing the modulating signal as $H(t) = He^{i\omega t}$ the magnetization $M(t)$ can be expressed as

$$M(t) = H(\text{Re}\chi \cos \omega t + \text{Im}\chi \sin \omega t), \tag{2.32}$$

where χ is the complex susceptibility. The above two equations give

$$\text{Re}\chi = \frac{2MfFW}{H(W^2 + \omega^2)} = \frac{2\mu_0 VM^2 FW}{k_B T(W^2 + \omega^2)}, \tag{2.33a}$$

$$\text{Im}\chi = \frac{2MfF\omega}{H(W^2 + \omega^2)} = \frac{2\mu_0 VM^2 F\omega}{k_B T(W^2 + \omega^2)}, \tag{2.33b}$$

$$|\chi| = \frac{2\mu_0 VM^2 F}{k_B T \sqrt{W^2 + \omega^2}}. \tag{2.33c}$$

χ exhibits resonance dynamics when the temperature parameter is varied. For the analysis fix the values of the parameters as $K = 4 \times 10^4 \text{ J/m}^3$, $V = 10^{-24} \text{ m}^3$, $M = 1.72 \times 10^6 \text{ A/m}$, $H = 10^3 \text{ A/m}$ and $\mu_0 = 4\pi \times 10^{-7} \text{ H/m}$. Figure 2.13 shows $\text{Re}\chi$, $\text{Im}\chi$ and $|\chi|$ as a function of temperature T for three values of H_1 . The maximum value of susceptibility decreases with increase in H_1 . In Fig. 2.14 $|\chi|$ is plotted for three values of the anisotropy parameter K . The effect of K can be clearly seen in this figure.

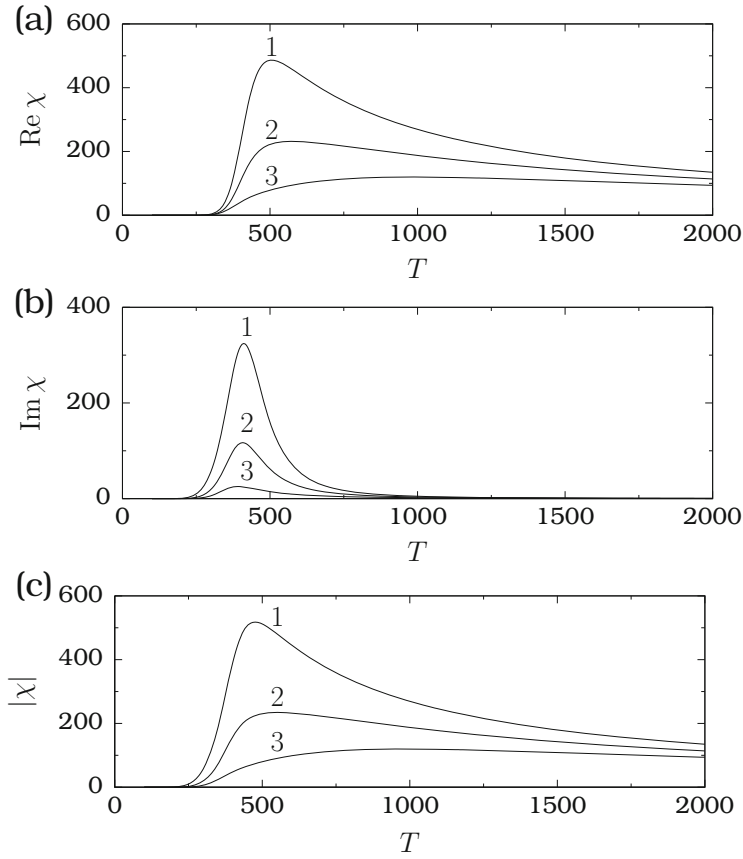


Fig. 2.13 Variation of (a) $\text{Re } \chi$, (b) $\text{Im } \chi$ and (c) $|\chi|$ as a function of temperature T (in kelvin). The curves 1, 2 and 3 are for $H_1 = 0$ A/m, 5×10^3 A/m and 10^4 A/m, respectively

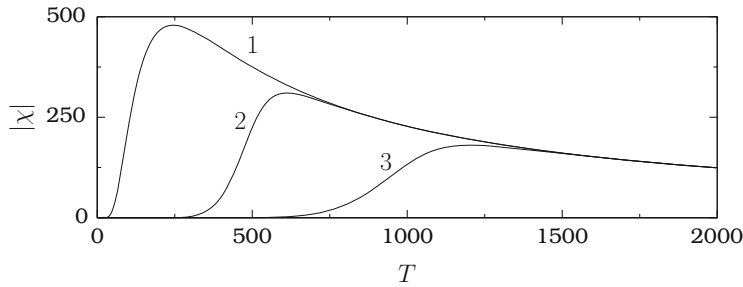


Fig. 2.14 $|\chi|$ versus T (in kelvin). The curves 1, 2 and 3 are for $K = 10^4$ J/m³, 5×10^4 J/m³ and 10^5 J/m³, respectively, with $H_1 = 10^3$ A/m

2.6 Stochastic Resonance in a Monostable System

In a monostable system with additive noise nonmonotonic increase in the signal amplitude and continuous decrease in the noise amplitude take place [54]. However, stochastic resonance does not occur. Consider the overdamped nonlinear system

$$\dot{x} + \beta x^3 - \gamma - x\xi(t) = f \cos \omega t + \eta(t) , \quad (2.34)$$

where $\beta > 0$, γ is the bias term and $\xi(t)$ and $\eta(t)$ are uncorrelated white noise with zero mean and the variance given by $\langle \xi(t)\xi(s) \rangle = 2D_m\delta(t-s)$ and $\langle \eta(t)\eta(s) \rangle = 2D_a\delta(t-s)$, respectively. The intensities of the additive and multiplicative noises are denoted as D_a and D_m , respectively. In the absence of noise and periodic force the system has only one stable state. Guo et al. [55] have obtained an analytic expression for SNR for the system (2.34).

The Fokker–Planck equation for the probability density $P(x, t)$ is

$$\frac{\partial P(x, t)}{\partial t} = -\frac{\partial}{\partial x} F(x, t) P(x, t) + \frac{\partial^2}{\partial x^2} [G(x) P(x, t)] , \quad (2.35a)$$

where

$$F(x, t) = -\beta x^2 + \gamma + D_m x + f \cos \omega t , \quad G(x) = D_m x^2 + D_a . \quad (2.35b)$$

In the adiabatic limit, that is, a small enough value of ω so that the time for the system to reach a local equilibrium is smaller than the period $1/\omega$,

$$P_{\text{st}}(x) = \frac{C_{\text{st}}}{\sqrt{G(x)}} e^{-V_{\text{eff}}(x)/D_m} , \quad (2.36)$$

where C_{st} is the normalization constant and

$$V_{\text{eff}}(x) = \int_{-\infty}^x \frac{D_m}{G(x)} (-V'(x) + \gamma + f \cos \omega t) dx , \quad (2.37a)$$

$$V(x) = -\frac{1}{2} D_m x^2 + \frac{1}{4} \beta x^4 . \quad (2.37b)$$

In the presence of multiplicative noise the effective potential $V_{\text{eff}}(x)$ of the monostable system becomes a bistable form with the stable equilibrium points at $x_{\pm} = \pm \sqrt{D_m/\beta}$ and the unstable point at $x_0 = 0$.

The transition rates for the system to move out of $x_{\pm}$ are obtained as

$$\begin{aligned} W_{\pm}(t) &= \frac{1}{2\pi} \sqrt{|V''(x_0) V''(x_{\pm})|} e^{[V_{\text{eff}}(x_{\pm}) - V_{\text{eff}}(x_0)]/D_m} \\ &= W_{\pm 0} e^{\mp k f \cos \omega t} , \end{aligned} \quad (2.38a)$$

where

$$W_{\pm 0} = \frac{D_m}{\sqrt{2}\pi} e^{\mp k\gamma - \Delta\phi/(2D_m)} \quad (2.38b)$$

with

$$k = \frac{1}{\sqrt{D_m D_a}} \tan^{-1} \left(\frac{D_m}{\sqrt{\beta D_a}} \right), \quad (2.38c)$$

$$\Delta\phi = D_m \left[\left(1 + \frac{\beta}{D_a} \right) \ln \left(\frac{D_m^2 + \beta D_a}{\beta D_a} \right) - 1 \right]. \quad (2.38d)$$

The master equations governing the probabilities $n_{\pm} = P(x = x_{\pm}^s)$ are given by

$$\frac{dn_{\pm}}{dt} = \mp W_{+}(t)n_{+} \pm W_{-}(t)n_{-}. \quad (2.39)$$

Then, following the procedure described in Sect. 2.3 for bistable systems it is easy to obtain [55]

$$S = S_s(\Omega)\delta(\Omega - \omega) + S_n(\Omega, \omega), \quad (2.40a)$$

where

$$S_s(\Omega) = \frac{4\pi D_m v^2}{\beta(u^2 + \omega^2)}, \quad (2.40b)$$

$$S_n(\Omega, \omega) = \frac{4D_m u}{\beta(u^2 + \Omega^2)} \left(\frac{1}{\cosh^2(k\gamma)} - \frac{2v^2}{u^2 + \Omega^2} \right), \quad (2.40c)$$

$$u = \frac{\sqrt{2} D_m \cosh(k\gamma)}{\pi} e^{-\Delta\phi/(2D_m)}, \quad (2.40d)$$

$$v = \frac{D_m f k}{\sqrt{2\pi} \cosh(k\gamma)} e^{-\Delta\phi/(2D_m)}. \quad (2.40e)$$

Then

$$SNR = \frac{D_m f^2 k^2 (u^2 + \omega^2) e^{-\Delta\phi/(2D_m)}}{2\sqrt{2} [u^2 + \omega^2 - 2v^2 \cosh^2(k\gamma)] \cosh(k\gamma)}. \quad (2.41)$$

Figure 2.15 depicts the variation of SNR with D_m for $D_a = 0.1$, $\gamma = 0.7$, $\omega = 0.01$ and $f = 0.01$ and for three values of the coefficient β of the nonlinear term in Eq. (2.34). As β increases, the critical value of D_m at which SNR becomes maximum also increases. The SNR_{MAX} decreases with increase in β . In Fig. 2.16 the dependence of SNR on the bias term γ for three values of the amplitude of the

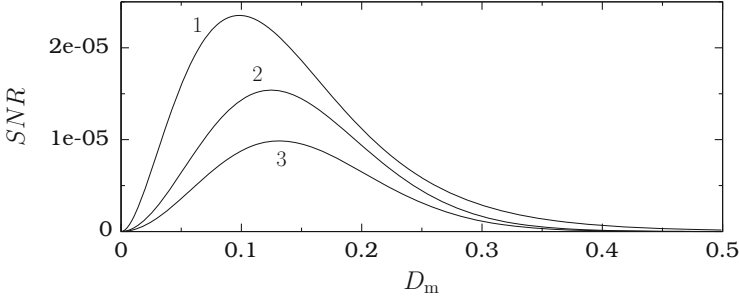


Fig. 2.15 SNR versus D_m for $D_a = 0.1$, $\gamma = 0.7$, $\omega = 0.01$, $f = 0.01$. The curves 1, 2 and 3 are for $\beta = 1, 5$ and 10 , respectively

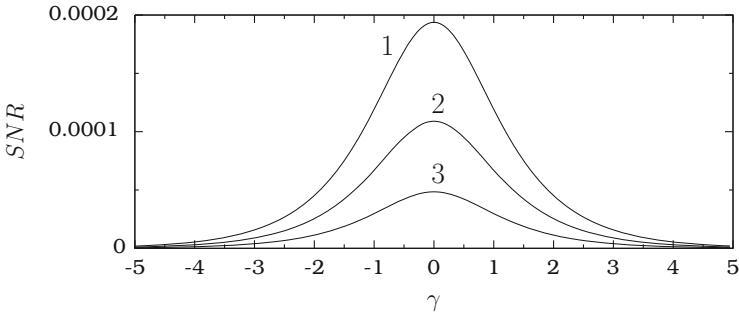


Fig. 2.16 SNR versus the bias term γ for $\beta = 2$, $D_a = 0.5$, $D_m = 0.8$ and $\omega = 0.01$. The curves 1, 2 and 3 are for $f = 0.01, 0.015$ and 0.02 , respectively

driving force is plotted. For a fixed value of f the SNR decreases with increase in γ due to the increase in the barrier height of V_{eff} . Further, SNR is maximum always at $\gamma = 0$. The reason for this is the following. The barrier heights of the two wells are identical for $\gamma = 0$ whereas for $\gamma \neq 0$ the barrier height of one well is higher than the other and also higher than that of the case $\gamma = 0$. Consequently, when γ is varied from a negative value, stochastic resonance occurs at $\gamma = 0$.

Experimentally, the occurrence of stochastic resonance in NbN superconducting strip-line resonators in a monostable zone with one stable state and one unstable state has been reported [56]. The system was driven by an amplitude modulated signal and noise. Maximum signal amplification was observed at an optimum value of mean injected pump power. At the resonance, during one half of the modulation period, nearly regular spikes in reflected power were noticed, while during the second half only few noise-induced spikes occurred. This dynamics was found to give rise to a very strong gain. Stochastic resonance has been studied in an asymmetrical monostable system with two periodic forces and multiplicative and additive noise [57].

2.7 Linear Systems with Additive and Multiplicative Noises

Three ingredients namely nonlinearity, a periodic signal and a random force were thought of as the necessary conditions for the occurrence of stochastic resonance. Let us first identify the effect of additive Gaussian noise in a linear system.

2.7.1 Effect of Additive Noise Only

Consider the motion of a particle in a single-well potential $V(x, t) = \frac{1}{2}\omega_0^2 x^2 - fx \cos \omega t$. The equation of motion of the overdamped version of the system is

$$\dot{x} = -\omega_0^2 x + f \cos \omega t + \xi(t) . \quad (2.42)$$

The autocorrelation function, one-sided power spectrum and SNR are obtained as [45]

$$\langle x(t)x(t + \tau) \rangle = \frac{D}{2\omega_0^2} e^{-\omega_0^2 |\tau|} + \frac{f^2}{2(\omega_0^4 + \omega^2)} \cos \omega \tau , \quad (2.43)$$

$$S(\Omega) = \frac{2D}{\omega_0^4 + \Omega^2} + \frac{\pi f^2}{\omega_0^4 + \omega^2} \delta(\Omega - \omega) , \quad (2.44)$$

$$SNR = \frac{\pi f^2}{2D} . \quad (2.45)$$

The SNR of the linear system is independent of the parameters ω_0^2 and ω and is simply the input SNR . SNR continuously decreases with increase in noise implying degrading of the performance of the system.

The influence of noises other than Gaussian white noise in additive and multiplicative forms has been investigated by few groups on linear systems. For example, stochastic resonance was observed in linear systems for Gaussian coloured noise [58–60], Poissonian noise [61], composite noise [62], signal modulated coloured noise [63], correlated noise [64, 65], multiplicative asymmetric dichotomous noise [55, 66] and multiplicative noise modulated by a bias periodic external force [67].

2.7.2 Effect of Multiplicative Noise Only

The effect of additive noise in a nonlinear system is changing of the internal thermal motion of the system by switching it between the coexisting equilibrium states. In a linear system there is only one potential well or one equilibrium state. Note that the internal structure of the system, for example, the shape of the potential

or the number of equilibrium states cannot be altered by the additive noise. Thus, stochastic resonance cannot be observed in a linear system driven by additive noise only. In Sect. 2.6 a creation of bistability in the monostable nonlinear system by a multiplicative noise is observed. The point is that a multiplicative noise can be able to induce a bistability in a linear system also. Li and Han [66] have shown the occurrence of stochastic resonance in an overdamped linear system with multiplicative asymmetric dichotomous noise where the noise takes two asymmetric values $-E$ and kE , with $E, k > 0$. They obtained an analytical expression for SNR [66]. Berdichevsky and Gitterman [68] considered the following linear system:

$$\dot{x} = -\omega_0^2 x - a_1 x \xi(t) + f \sin \omega t. \quad (2.46)$$

When the noise term $\xi(t)$ is chosen as a Gaussian white noise, the quantity $\langle x(t) \rangle$ is found to be remain bounded and is a monotonic function of the noise intensity D . There is no stochastic resonance in this case. For the exponentially correlated noise with the auto-correlation time τ , that is,

$$\langle \xi(t) \xi(t') \rangle = e^{-|t-t'|/\tau}, \quad (2.47)$$

they obtained $\langle x(t) \rangle$. In this case, also a monotonic variation of $\langle x(t) \rangle$ is noticed.

In the case of dichotomous noise with $a_1 \tau < 1$, $a_1^2 < \omega_0^4 + \omega_0^2/\tau$, the amplitude of the stationary solution depends on the parameters $\omega_0^2, f, \omega, a_1$ and τ . $\langle x(t) \rangle$ is found to be maximum at

$$(a_1)_{\max} = \sqrt{\omega_0^4 + \frac{\omega_0^2}{\tau} - \omega^2}. \quad (2.48)$$

In the presence of dichotomous noise, the system moves along the parabola $U_1 = (\omega_0^2 + a_1)x^2/2$, then jumps to the parabola $U_2 = (\omega_0^2 - a_1)x^2/2$ at the rate $1/\tau$ and so on. For $a_1 > \omega_0^2$ but $a_1 < \sqrt{\omega_0^4 + \frac{\omega_0^2}{\tau}}$ the parabolas U_1 and U_2 have curvatures with opposite sign and act in opposite direction tending to increase (decrease) the x of a particle. Accordingly, the amplitude of the stationary signal has a maximum as a function of the noise intensity.

An overdamped linear system with quadratic multiplicative coloured noise and driven by a periodic signal has been considered in [69]. Applying functional integral techniques, an analytical expression for the mean value of the state variable x , $\langle x \rangle_{\text{st}}$, was obtained. The amplitude of $\langle x \rangle_{\text{st}}$ was found to show a stochastic resonance-like profile with the inverse of the correlation time of the coloured noise.

2.7.3 Effect of Multiplicative and Additive Noises

Consider the system

$$\dot{x} = -(\omega_0^2 + \xi(t))x + f \cos \omega t + \eta(t) , \quad (2.49)$$

where ξ and η are a Gaussian noise with zero mean and correlation functions given by

$$\langle \xi(t)\xi(s) \rangle = \sigma_1 e^{-\lambda|t-s|} , \quad (2.50a)$$

$$\langle \eta(t)\eta(s) \rangle = \sigma_2 e^{-\lambda|t-s|} , \quad (2.50b)$$

$$\langle \xi(t)\eta(s) \rangle = \langle \eta(t)\xi(s) \rangle = 0 . \quad (2.50c)$$

Choosing the noise as asymmetric dichotomous, Ning and Xu [70] derived an analytical expression for *SNR* and have shown the occurrence of stochastic resonance in the above linear system.

Let us assume that $\xi(t)$ assumes the values A_1 and $-B_1$ and $\eta(t)$ takes the values A_2 and $-B_2$ with A_1, A_2, B_1 and $B_2 > 0$. The rate for the transition from A_1 to $-B_1$ is say α_1 while from $-B_1$ to A_1 is α_2 . β_1 and β_2 are the rates for the transitions A_2 to $-B_2$ and $-B_2$ to A_2 , respectively. Writing

$$\sigma_1 = A_1 B_1, \quad \lambda = \alpha_1 + \alpha_2, \quad \Lambda_1 = A_1 - B_1, \quad (2.51a)$$

$$\sigma_2 = A_2 B_2, \quad \lambda = \beta_1 + \beta_2, \quad \Lambda_2 = A_2 - B_2 \quad (2.51b)$$

one can obtain

$$\frac{d\langle x \rangle}{dt} = -\omega_0^2 \langle x \rangle - \langle \xi(t)x \rangle + f \cos \omega t , \quad (2.52a)$$

$$\frac{d\langle x^2 \rangle}{dt} = -2\omega_0^2 \langle x^2 \rangle - 2\langle \xi(t)x^2 \rangle + 2f\langle x \rangle \cos \omega t + 2\langle \eta(t)x \rangle . \quad (2.52b)$$

Multiplying Eq. (2.49) by $\xi(t)$, averaging and using

$$\frac{d}{dt} \langle \xi(t)x \rangle = \left\langle \xi(t) \frac{dx}{dt} \right\rangle - \lambda \langle \xi(t)x \rangle \quad (2.53)$$

one gets

$$\begin{aligned} \frac{d}{dt} \langle \xi(t)x \rangle &= -(\omega_0^2 + \lambda) \langle \xi(t)x \rangle - \langle \xi^2(t)x \rangle \\ &= -(\omega_0^2 + \lambda) \langle \xi(t)x \rangle - A_1 B_1 \langle x \rangle - (A_1 - B_1) \langle \xi(t)x \rangle \\ &= -(\omega_0^2 + \lambda) \langle \xi(t)x \rangle - \sigma_1 \langle x \rangle - \Lambda_1 \langle \xi(t)x \rangle . \end{aligned} \quad (2.54)$$

Equation (2.54) together with Eq. (2.52a) form a system of equations for the two unknowns $\langle x \rangle$ and $\langle \xi(t)x \rangle$. In the limit $t \rightarrow \infty$

$$\langle x \rangle = \frac{f}{f_3} (f_1 \cos \omega t + f_2 \sin \omega t) , \quad (2.55a)$$

where

$$f_1 = \omega_0^2 \omega^2 + (\omega_0^2 + \Lambda_1 + \lambda) b_1 b_2 , \quad (2.55b)$$

$$f_2 = \omega \left[\omega^2 + (\omega_0^2 + \Lambda_1 + \lambda)^2 + \sigma_1 \right] , \quad (2.55c)$$

$$f_3 = (\omega^2 + b_1^2) (\omega^2 + b_2^2) , \quad (2.55d)$$

$$b_{1,2} = \omega_0^2 + \epsilon_{1,2} = \omega_0^2 + \frac{\lambda + \Lambda_1}{2} \pm \sqrt{\frac{(\lambda + \Lambda_1)^2}{4} + \sigma_1} . \quad (2.55e)$$

In a similar manner one can obtain the stationary second moment as

$$\begin{aligned} \langle x^2 \rangle_{\text{st}} = & \left\{ \frac{\sigma_2 [2(\omega_0^2 + \Lambda_1 + \lambda)^2 + (\omega_0^2 + \Lambda_1)\lambda + 2\sigma_1]}{(\omega_0^2 + \lambda)(\omega_0^2 + \Lambda_1 + 2\lambda) - \sigma_1} \right. \\ & \left. + \frac{f^2}{2f_3} [f_1(2\omega_0^2 + 2\Lambda_1 + \lambda) - 2(\omega^2 - b_1 b_2)\sigma_1] \right\} \\ & \times [\omega_0^2(2\omega_0^2 + 2\Lambda_1 + \lambda) - 2\sigma_1]^{-1} . \end{aligned} \quad (2.56)$$

Integration of Eq. (2.49) gives

$$\begin{aligned} x(t + \tau) = & x(t)g(\tau)e^{-\omega_0^2 \tau} + f \int_0^\tau e^{-\omega_0^2 v} g(v) \cos [\omega(t + \tau - v)] dv \\ & + \int_0^\tau e^{-\omega_0^2 v} h(v) dv , \end{aligned} \quad (2.57a)$$

where

$$g(v) = \langle e^{-\int_0^v \xi(u) du} \rangle , \quad h(t - v) = \langle \eta(v) e^{-\int_v^t \xi(u) du} \rangle . \quad (2.57b)$$

Expanding the exponentials in Eq. (2.57b) in series and evaluating the integrals results in

$$g(v) = \frac{1}{\epsilon_1 - \epsilon_2} (\epsilon_1 e^{-\epsilon_2 v} - \epsilon_2 e^{-\epsilon_1 v}) , \quad h(t - v) = 0 . \quad (2.58a)$$

Now, multiplication of Eq. (2.57a) by $x(t)$ and averaging gives

$$\langle x(t + \tau)x(t) \rangle = \langle x^2 \rangle_{\text{st}} g(\tau) e^{-\omega_0^2 \tau} + \frac{f \langle x \rangle}{\epsilon_1 - \epsilon_2} [f_4 \sin \omega t + f_5 \cos \omega t], \quad (2.59a)$$

where

$$f_4 = \frac{\epsilon_2 b_1 \sin \omega \tau - \epsilon_2 \omega f_6}{b_1^2 + \omega^2} + \frac{\epsilon_1 \omega f_7 - \epsilon_1 b_2 \sin \omega \tau}{b_2^2 + \omega^2}, \quad (2.59b)$$

$$f_5 = -\frac{\epsilon_2 \omega \sin \omega \tau + \epsilon_2 b_1 f_6}{b_1^2 + \omega^2} + \frac{\epsilon_1 b_2 f_7 + \epsilon_1 \omega \sin \omega \tau}{b_2^2 + \omega^2}, \quad (2.59c)$$

$$f_{6,7} = \cos \omega \tau - e^{-b_{1,2} \tau}. \quad (2.59d)$$

Averaging of $\langle x(t + \tau)x(t) \rangle$ over the period $2\pi/\omega$ of the external force $f \cos \omega t$ gives

$$\langle x(t + \tau)x(t) \rangle_{\text{st}} = \langle x^2 \rangle_{\text{st}} g(\tau) e^{-\omega_0^2 \tau} + \frac{f^2 (f_1 f_5 + f_2 f_4)}{2f_3 (\epsilon_1 - \epsilon_2)}. \quad (2.60)$$

Then the power spectrum is $S(\Omega) = S_s(\Omega) \delta(\Omega - \omega) + S_n(\Omega, \omega)$ where

$$S_s = \frac{\pi f^2 (u_2 l_1 - u_1 l_2)}{2f_3 (\epsilon_1 - \epsilon_2)}, \quad (2.61a)$$

$$S_n = \frac{2 \langle x^2 \rangle_{\text{st}} (b_2 l_1 - b_1 l_2)}{(\epsilon_1 - \epsilon_2)} + \frac{f^2}{f_3 (\epsilon_1 - \epsilon_2)} \left[\frac{b_1 l_2 u_1}{b_1^2 + \Omega^2} - \frac{b_2 l_1 u_2}{b_2^2 + \Omega^2} \right] \quad (2.61b)$$

with

$$l_{1,2} = \frac{b_{1,2} - \omega_0^2}{b_{2,1}^2 + \Omega^2}, \quad u_{1,2} = f_2 \Omega + f_1 b_{1,2}. \quad (2.61c)$$

Then $SNR = S_s(\Omega = \omega)/S_n(\Omega = \omega)$.

Figure 2.17a shows SNR versus ω for three values of f for fixed values of other parameters. For $f = 0.5, 0.6$ and 0.7 SNR becomes maximum at $\omega = 1.14, 1.32$ and 1.48 , respectively. For each fixed value of ω the value of SNR increases with f . In Fig. 2.17b SNR profile is shown for three values of Λ_1 . For $\Lambda_1 = 0.001, 0.01$ and 0.02 resonance is found at $0.36, 0.366$ and 0.38 , respectively. As Λ_1 increases the SNR value also increases.

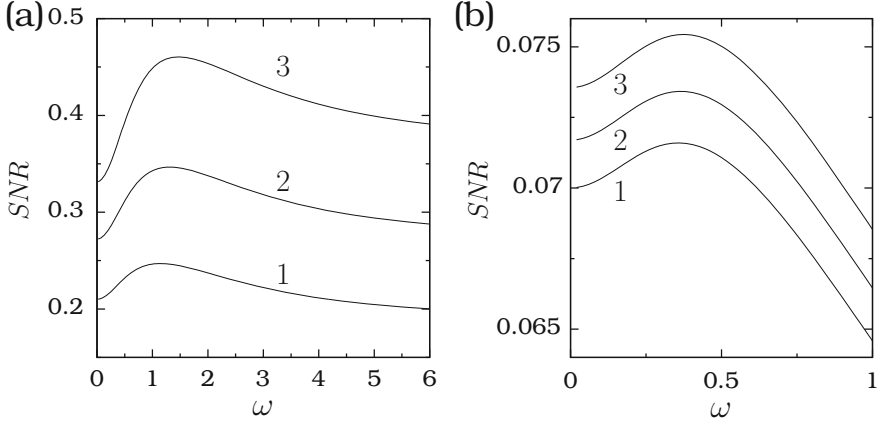


Fig. 2.17 Variation of SNR versus ω in the linear system with $\omega_0^2 = 1$, $\lambda = \sigma_1 = \sigma_2 = 1$ and $\Lambda_1 = \Lambda_2 = 1$. The curves 1, 2 and 3 in the subplot (a) are for $f = 0.5, 0.6$ and 0.7 , respectively. The curves 1, 2 and 3 in the subplot (b) are for $\Lambda_1 = 0.001, 0.01$ and 0.02 , respectively, with $f = 0.5$ and $\Lambda_2 = 1$

2.8 Stochastic Resonance in Quantum Systems

Stochastic resonance has been studied in certain quantum systems [44, 71–74]. Two-state theory and Feynman path integral approach were used to obtain SNR . This section considers two quantum systems and theoretically point out the occurrence of stochastic resonance.

2.8.1 A Particle in a Double-Well Potential

Let us consider a quantum mechanical particle in a double-well potential and assume that the depth of the left-well and right-well are unequal, $h_- = h_+ + \epsilon$ [71]. The system is subjected to a periodic forcing with frequency ω and a random noise characterized by a temperature T .

Expanding $W_{\pm}(t)$ in Eq. (2.9) as $W_{\pm}(t) = W_{\pm 0} + w_{\pm} \cos \omega t + \dots$ the rate equation can be solved and the correlation function to the order $(w_{\pm})^2$ can be determined. For the case of transition rates obeying the detailed balance

$$W_+/W_- = e^{(\epsilon_0 + \delta \epsilon \cos \omega t)/(k_B T)}, \quad (2.62)$$

where k_B is the Boltzmann constant, SNR is obtained as [71]

$$SNR = \frac{\pi W_{+0}}{4(1 + e^{\epsilon_0/(k_B T_0)})} \left[\delta \left(\frac{\epsilon}{k_B T} \right) \right]^2. \quad (2.63)$$

For a sinusoidally modulated asymmetry energy, the quantity $\epsilon = \epsilon_0 + \delta_\epsilon \cos \omega t$ is $\delta(\epsilon/k_B T) = \delta_\epsilon/k_B T_0$. Then the SNR given by Eq. (2.63) becomes

$$SNR = \frac{\alpha T'^2}{1 + e^{T'}} , \quad \alpha = \frac{\pi W_{\pm 0} \delta_\epsilon^2}{4\epsilon_0^2} , \quad T' = \frac{\epsilon_0}{k_B T_0} . \quad (2.64)$$

Lofstedt and Coppersmith pointed out the difference between the classical and quantum stochastic resonances. Quantum stochastic resonance does not occur in a system with symmetric potential. This is because in the symmetric well, $\epsilon/k_B T \ll 1$ the transition rates $W_\pm$ have power-law dependence on T . In the asymmetric case when $\epsilon_0/k_B T_0 \gg 1$, due to the detailed balance factor the transition rate W_- is exponentially small and the particle is confined to the lower well. The signal is thus suppressed. For $k_B T_0 \sim \epsilon_0$ the relative occupation in the upper well is sensitive to the temperature. On the other hand, for $k_B T_0 \gg \epsilon_0$, the relative occupations in the two wells are almost equal and the signal decreases.

2.8.2 A Double Quantum Dot System

Joshi [74] considered a system of a double quantum dot with only two nondegenerate and weakly coupled energy levels with energies E_1 and E_2 . The level 1 is occupied while the level 2 is empty in the absence of any external perturbation. There is no tunnelling of the particles between the two quantum dots since they are weakly coupled. Such a system can represent two energy states in two different wells. The Hamiltonian of the system is

$$H(t) = H_0(t) + H_{\text{int}}(t) , \quad (2.65a)$$

where

$$H_0 = \frac{1}{2} E(t) (|2\rangle\langle 2| - |1\rangle\langle 1|) , \quad H_{\text{int}} = \zeta (|2\rangle\langle 1| + |1\rangle\langle 2|) , \quad (2.65b)$$

$$E(t) = E_0 + \alpha \mathcal{E}(t) , \quad \mathcal{E}(t) = \mathcal{E}_0 \cos \omega t , \quad (2.65c)$$

where α is a constant, ζ is the coupling strength between the two dots and $\mathcal{E}(t)$ is the external field (which can be applied to the dots via gate electrodes). The applied field leads to an oscillation of the two energy levels. Note that in the usual stochastic

resonance phenomenon the wells of the potential are tilted periodically by means of external periodic force.

The master equation for the statistical operator ρ is given by

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H(t), \rho] + \mathcal{L}_r \rho + \mathcal{L}_w \rho, \quad (2.66)$$

where $\mathcal{L}_r \rho$ is the Liouville operator for reservoir tunnelling and $\mathcal{L}_w \rho$ governs inter dot particle relaxation. At low bias voltage the tunnelling of electrons through the double dot is suppressed by Coulomb blockade. Such conditions are appropriate to investigate stochastic resonance in the system. Using the master Eq. (2.66) evolution equation for density matrix elements ρ_{11} , ρ_{12} , ρ_{21} and ρ_{22} of the density matrix can be obtained. Here ρ_{11} and ρ_{22} are the probabilities for the electron to be in the left-dot and right-dot, respectively. ρ_{12} and ρ_{21} are the off-diagonal matrix elements. Assume that the potential of the double dot has local minima at $\pm x_m$ and the height of the potential barrier measured from the lower state $|1\rangle$ is ΔV and D is the intensity of the Gaussian white noise with zero mean. The applied periodic signal changes the relative separation of the states. When the noise intensity is varied, then at an optimum value the random switching frequency becomes close to the signal frequency leading to a higher probability for transition between the states.

Following the two-state theory described in Sect. 2.3 and under the assumption that the interaction between the dots is small and hence its contribution to SNR is negligible, Joshi finally obtained the SNR as [74]

$$SNR = \frac{\pi r_K f_s^2}{2(D/\Delta V)^2} \left(1 - \frac{2r_K^2 f_s^2}{(D/\Delta V)^2 (\omega^2 + 4r_K^2)} \right)^{-1}, \quad (2.67a)$$

where

$$r_K = \frac{1}{\sqrt{2}\pi} e^{-\Delta V/D}, \quad f_s = \frac{\mathcal{E}_0 x_m}{\Delta V}. \quad (2.67b)$$

For very small D the exponential term falls to zero rapidly so SNR is almost zero. For very large D , the exponential term reaches 1 but the term D^2 in the denominator makes SNR again to zero. For moderate values of D , there is the maximum near $D \sim \Delta V$.

2.9 Applications of Stochastic Resonance

The features of stochastic resonance have been studied experimentally in psychophysics, electrophysiology, human vision, hearing and tactile, animal behaviour and single and multiunit activity recording. Role of stochastic resonance in brain function including detection of weak signals, synchronization and coherence among

neuronal assemblies, phase resetting, carrier signals, animal avoidance and feeding behaviours were also investigated. A weak signal refers to a signal with amplitude much lower than the amplitude of background noise. Stochastic resonance is quite interesting for biological systems particularly in neurobiological systems, since it may provide a mechanism for such systems to detect and process weak signals. In neural systems, stochastic resonance has been shown in peripheral sensory neuron [75], neurotransmitter quantum or spike processing neuron [76], neural parallel array and network [77–83]. Collins group [84] has shown experimentally that the postural sway of both young and elderly individuals during quiet standing could be significantly reduced by applying subsensory mechanical noise to the feet. An excellent review on application of stochastic resonance in sensory information process was presented by Moss and his coworkers in [85]. This section points out some of the promising applications of stochastic resonance.

2.9.1 Vibration Energy Harvesting

Gammaitoni et al. [86] proposed a way of converting vibrational energy to electrical energy making use of stochastic dynamics. They considered a piezoelectric oscillator represented by the equations

$$\ddot{x} = \omega_0^2 x - \beta x^3 - d\dot{x} + K_v V + D\xi(t) , \quad (2.68a)$$

$$\dot{V} = K_c \dot{x} - \frac{1}{\tau_p} V , \quad (2.68b)$$

where V is the voltage drop and x is the relevant observable of the oscillator dynamics, d is the damping coefficient, K_v is the coupling coefficient relating the oscillations to the voltage and K_c is the coupling constant of the piezoelectric sample. τ_p is related to the coupling capacitance C and to the resistive load R ($\tau_p = RC$). $\xi(t)$ is a random force, a stochastic process with Gaussian distribution, zero mean and unit variance. V_{rms} versus ω_0^2 exhibits a nonmonotonic variation. V_{rms} is found to be maximum at an optimum value of ω_0^2 for fixed values of the other parameters.

2.9.2 Stochastic Encoding [87]

Stochastic encoding or transmission has been found in many systems ranging from molecular level to the nervous activity [88–91]. At molecular level molecules undergo continuous changes and in thermal equilibrium with the medium. It is possible to achieve communication between a cell and its external world making use

of a channel that open and close in a different situation (monostable ion channel). There are two consequences with this:

- The modulation of ion current by external stimuli would be difficult and less flexible.
- Economy of the cell.

In voltage-gated channels the membrane potential modulates the chances of opening and closing. Consequently, if the dependence of the rate constants on the potential is steep, the number of channels ($N(t)$) which are open can vary enormously in a stimulus cycle. There is no current when a channel is closed and the current is V/R , where R being the resistance of the channel when it is opened. The jump between 0 and V/R is much higher than the peak-to-peak amplitude of the time-varying component of V/R . Consequently, when the rate constants for the $0 - V/R$ transitions depend very steeply on the potential, the current through the membrane can be substantially modulated than in the case of channels opened at a regular rate. In the first case the current is $N(t)V/R$ while in the second case it is $N(t) \pm (V/R)$. Now, the peak-to-peak amplitude is greater. In this way, the stochastic mechanism acts as an amplifier enhancing the amplitude of the current passing through the membrane. That is, stochastically switching ion channels amplify the signal. Proteins have a lot of energetically nearly equivalent states and they switch thermally between these states so stochastic transmission is readily at hand for cell communication. Stochastic switching is present in neural activity also and it can serve as a tool to transmit information about high frequencies using nerve fibers. Further, stochastic encoding could be simpler to decode at the cortical level than the patterns transmitted particularly when deterministic phase-locking takes place.

2.9.3 *Weak Signal Detection*

The measurement of very weak forces is of great important. A gravitational wave detector is a typical example for a measurement system with very high precision. The combined optomechanical system exhibiting different characteristics and physical phenomena such as the optical spring effect and multistability become important. Here stochastic resonance can occur when noise is added and one may realize the enhancement of SNR in precision measurement. Signal detection employing stochastic resonance has been studied in [92, 93].

2.9.4 *Detection of Weak Visual and Brain Signals*

In human information processing, noise-enhanced performance is well established. In active sensory systems, however, an additional source of noise is self-generated [94, 95]. For example, consider the human visual system. When one looks at

stationary scenes, the oculomotor system moves an object of interest into the foveal part of the retina. However, one can notice a rapid adaptation of the visual system to a constant input. This adaptation leads to perceptual fading when the retinal image is experimentally stabilized in the laboratory paradigm of retinal stabilization. As a consequence, a built-in mechanism exists in the visual system. One has to fixate an object for the visual analysis of bleaching. It is to be noted that our eyes are never at a fixed state and undergo continuous motion. Such involuntarily and unconsciously miniature eye movements are produced when one fixate a stationary target. These fixational eye movements represent self-generated noise which serves significant perceptual functions. The fixational eye movements are traditionally interpreted as oculomotor noise [96].

Starzynski and Engbert [97] investigated fixational eye movements of 19 participants (with mean age 23.3 years) under the influence of external noise. In the experiment, the target stimulus performed a random walk with varying noise intensity. The random motion of the stimulus was implemented on a computer with a constant distribution of spatial increments but with rates of position change which is characterized by the diffusion constant. Noise-enhanced target discrimination was observed. Particularly, response times versus the diffusion constant exhibited U shape curve. That is, response time is minimum for an optimum value of diffusion constant. Starzynski and Engbert have pointed out that the noise artificially applied in their laboratory study was very likely to occur in natural settings where postural fluctuations [98] act as a noise source which is external with respect to the oculomotor system.

Human threshold for detection of luminance variations across space has been studied in the presence of noise. Ward and his co-workers [99] performed an experiment where observers were requested to recognize striped and nonstriped visual stimuli. Contrast was changed according to an adaptive technique and depending on the correct and incorrect answer of an observer. Noise (random amounts from Gaussian distribution) was implemented in the form of randomly changing the grey level of each pixel in the stimulus. The presence of noise was found to decrease the contrast threshold of detection of weak spatial modulations in luminance.

Stochastic resonance in more complex systems, such as human tactile sensing neural networks of mammalian brains and the blood pressure control system in the human brain system have been well established [100–104]. In the human, there is a large variety of chaotic firing of neuron networks and columns that function as complex oscillators, as well as spontaneous electrochemical noises. Such noisy signals provide internal noise sources to trigger stochastic resonance in brain. Stochastic resonance has been observed in the human brain's visual processing area [105].

The hippocampus is a brain tissue essential for learning and short-term memory. A hippocampal network model consists of two layers CA1 and CA3. Pyramidal cells in CA3 are connected to pyramidal cells in CA1. The CA3 network causes spontaneous irregular activity while the CA1 network does not. The activity of CA3 causes membrane potential fluctuations in CA1 pyramid cells. The CA1 network

also receives a subthreshold signal through the perforant path. The subthreshold perforant path signals can fire CA1 pyramid cells in cooperation with the membrane potential fluctuations that work as a noise. The firing of the CA1 network shows typical features of stochastic resonance [106]. It has also been shown that the stochastic resonance improves subthreshold detection in a single hippocampal CA1 cell [107].

2.9.5 Stochastic Resonance in Sensory and Animal Behaviour

Experiments on the feeding behaviour of the juvenile paddle fish have shown the enhancement of perception of sensory information and also affecting animal behaviour. Animal can perceive the enhancement of information available in the peripheral sensory system with external noise and they can make use of this noise-improved information for feeding or predator avoidance. For example, in an experiment the tail-fan of crayfish was covered with hydrodynamically sensitive hairs, each one innervated by sensory neurons that converge on inter-neurons in the sixth ganglion [108, 109]. Extra-cellular recordings were made from the sensory neurons in the root. *SNR* was measured from the power spectra of the recorded spike trains. It became maximum at an optimal noise intensity. This behaviour was also observed in an analog simulation of a model neuron [110].

Another experiment was done with paddle fish. An electrosensitive paddle fish has a long anterior rostrum, which is covered with few thousands of electrosensitive organs and are capable of detecting and tracking the weak electric fields generated by the swimming and feeding motions of its favourite prey zooplankton *Daphnia*. The generated electric field around the *Daphnia* is dipole-shaped. Consequently, when an individual plankton is farther from the rostrum, the weaker the electrical signal at the rostral surface. The result is dropping beneath the perceptive ability of the animal at certain limiting distance. When an external noise was added in the form of a random electric field applied parallel to the fish long axis, the perceptive ability was improved. The fish was able to extend the capture distance range in the presence of optimal noise [111–113].

2.9.6 Human Psychophysics Experiments

In psychophysics, noise usually interferes with detection and identification of a signal—a process called *masking*. Research on masking in vision, touch and audition indicated that at very low levels of signal and mask intensity, it is easier to detect the signal in the presence of the mask than alone when the two are added in-phase with each other [114]. Detection of weak signal is found to be more difficult for higher intensities of the mask. This phenomenon is explained by the energy addition of the signal and masking which makes the signal+mask discriminable

from the mask alone when signal+mask is near threshold but not when the mask is very intense or very weak. Threshold for human visual perception, effects of noise and stochastic resonance in the perception of gratings, ambiguous figures and in three-dimensional perception of autostereograms were studied through many psychophysics experiments [115–121].

2.9.7 Noise in Human Muscle Spindles and Hearing

It has been shown that the sensitivity of muscle-spindle receptors to a weak movement signal would be enhanced by adding noise through the tendon of the parent muscle. Cordo et al. [122] recorded firing activity of individual muscle-spindle afferents from the radial nerve in healthy human subjects. When an afferent is isolated, the subject's right-wrist is passively rotated by a manipulandum, with a small amplitude sinusoidal wave form. They applied a noise input by a tendon stimulator to stretch the appropriate muscle. In most of the examined afferents the calculated *SNR* rapidly increased to a peak and then slowly decreased. That is, the presence of a particular nonzero level of noise enhanced the sensitivity of the muscle-spindle receptor to the weak input signal.

In a hearing system, noise-enhanced peripheral sensory response has been demonstrated experimentally and theoretically [123–125]. The enhanced effect of noise on hearing sensitivity indicates that hearing performance is involved in a nonlinear mechanism in which fluctuation and noise are important and even beneficial. The source of the nonlinearity is the sensory hair cell. It has been shown that inner hair cells response in temporal pattern to sinusoidal signal was enhanced by optimal noise. Experiments [123–125] have shown that by adding noise of certain level, human can detect near threshold pure tone better than without noise.

A conceptual model has shown that stochastic resonance can optimize the bone remodelling process where the mechanical noisy stimulus is related to the stochastic nature of the activation of osteoblasts and osteoclasts through the connection between bone remodelling and external mechanical stimuli [126].

2.9.8 Electrophysiological Signals

The spontaneous background signals in an EEG are usually considered as noise with respect to stimulus or event related electrophysiological events. Information about neuronal interaction and changes in brain functional states are contained in the spontaneous brain signals. The analysis of these signals is important in sensory information processing. The occurrence of stochastic resonance in central neural system can be explored with the help of electrophysiological techniques. The added noise signal was found to improve the cortical somatosensory response to mechanical tactile simulation. Response amplitude versus noise intensity appeared

as an inverted U shape indicating enhancement of response at certain noise level [127]. Stochastic resonance was observed in the spinal and cortical stages of the sensory encoding. This was demonstrated in anesthetized cats with the coherence between spinal and cortical responses to tactile stimulation. Increase in the response amplitude and power of the first even harmonics of the cortical response to steady state contrast stimulation have been observed. This effect might be from activation of complex cells in striate visual cortex based on the nonlinear properties that these cells and steady-state contrast stimulation share [128].

2.9.9 Stochastic Resonance in Raman and X-Ray Spectra

Sometimes due to weak signal the intensity of Raman spectrum and X-ray diffraction of certain solutions corresponding to certain frequencies may not be clearly visible. They can be amplified by adding impurities as noise. An example is found in the case of alcohol solutions of CCl_4 . Increase in the intensity at a particular wave number has been realized when the concentration of CCl_4 is varied.

2.10 Noise-Induced Stochastic Resonance Versus Noise-Induced Synchronization

When the time variation of state variables $\mathbf{X}_1$ of one system follows with the state variables $\mathbf{X}_2$ of another identical system (coupled or uncoupled) then the two systems are said to be *synchronized*. There are different types of synchronization [129–132]. When $\mathbf{X}_1 = \mathbf{X}_2$ then the synchronization is termed as *general* or *complete*. Synchronization of two or more identical systems (with different initial conditions) exhibiting chaotic dynamics can be achieved by different kinds of deterministic coupling of the systems. Interestingly, noise can be used to synchronize two identical uncoupled systems started with different initial conditions [10, 133]. That is, two identical chaotic systems driven by the same noise forget their initial states and evolve to an identical state after transient.

It is to be noted that noise-induced resonance refers to a phenomenon wherein a maximum response of a system (not necessarily coupled systems) at an optimal value of noise intensity is realised and can be detected by measuring the statistical measure *SNR*. Noise-induced synchronization can be identified by direct comparison of the state variables $\mathbf{X}_1$ and $\mathbf{X}_2$ of two chaotic systems. To realize noise-induced resonance the system under consideration must have certain kind of bistability and it can occur in a single degree of freedom nonlinear system. Synchronization occurs in coupled systems or uncoupled more than one systems.

Synchronization is found to play an important role in information transmission and processing [134, 135]. It can be related to cognition and movement control

in brain function. It has been pointed out that in the human perception process synchronization of gamma wave in space and time occurs [136]. On the other hand, occurrence of synchronization may results in epilepsy [137] and Parkinson diseases [138]. Chaos synchronization is investigated with applications to secure communication and neural activity [132].

2.11 Concluding Remarks

In this chapter, the occurrence of stochastic resonance in systems like excitable, spatially periodic, maps and networks have not been considered. Such systems will be taken up in later chapters. Certain variant forms of stochastic resonance have been reported in the literature. For example, when the input signal is an aperiodic, like a frequency modulated one, the noise-induced resonance is called *aperiodic stochastic resonance*. Such a resonance was demonstrated by Collins et al. [139] on rat slowly adapting a type of afferents. In the experiment, each neuron was subjected to a perithreshold aperiodic stimulus and a noise. When the noise intensity was varied, a typical stochastic resonance profile was observed which indicates that external noise into sensory neurons could improve their abilities to detect weak signals. Aperiodic resonance was found in the theoretical FitzHugh–Nagumo model equations [140] and overdamped bistable systems [141]. In the double stochastic resonance [142] an extended system is subjected to both a multiplicative noise and an additive noise. Bimodality of the mean field of the network is induced by the multiplicative noise while the change in the response is governed by the additive noise. Splitting of the stochastic resonance peak thereby resulting in double stochastic resonance is observed in an optomechanical read-out device [52] and in an asymmetric barrier system [143]. Other extension of the concept of stochastic resonance such as stochastic giant resonance [144], stochastic multi-resonance [145, 146], autonomous stochastic resonance [147], geometric stochastic resonance [148] and control of stochastic resonance [149] were reported. Signal detection using residence time statistics was analysed in [150, 151].

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