Chapter 2
Mean-Field Regime for Bosonic Systems

One of the simplest non-trivial regimes in which it is possible to approximate the many-body dynamics by an effective equation is the mean-field limit for bosonic systems. In the mean-field regime, particles experience a large number of weak collisions, whose cumulative effect can be approximated by an average mean-field potential. To realize the mean-field regime, we consider a system of N bosons with a Hamilton operator of the form (1.7), in the limit of large N and small coupling constant λ, with $N\lambda$ of order one. This last condition guarantees that the total force on each particle is of order one, and therefore comparable with the inertia. In other words, we consider the dynamics generated by the mean-field Hamiltonian

$$H_N = \sum_{j=1}^{N} \left(-\Delta_{x_j} + V_{\text{ext}}(x_j) \right) + \frac{1}{N} \sum_{i<j}^{N} V(x_i - x_j) \tag{2.1}$$

acting on the Hilbert space $L_s^2(\mathbb{R}^{3N})$, the subspace of $L^2(\mathbb{R}^{3N})$ consisting of permutation symmetric functions, in the limit of large N. (This is of course an idealization since in physical systems N is finite, though its value ranges from the order 10^3 in very dilute Bose-Einstein condensates to the order 10^{23} in chemical systems.)

Initial data. The choice of the initial data is dictated by physics. In typical experiments a Bose gas is initially trapped by an external confining potential. To study the dynamics of the gas out of equilibrium, we consider the reaction of the system to a change of the external field. In other words, we are going to consider initial data given by equilibrium states of a Hamiltonian of the form (2.1), with V_{ext} modeling the external traps. In particular, at zero temperature, we are interested in initial data close to the ground state of (2.1). Under appropriate assumptions on the interaction potential V it is known that the ground state ψ_N^{gs} of (2.1) can be approximated, in the limit of large N, by a factorized wave function; i. e. $\psi_N^{\text{gs}} \simeq \varphi^{\otimes N}$, for an appropriate $\varphi \in L^2(\mathbb{R}^3)$ (φ is the minimizer of the Hartree energy functional). We are interested, therefore, in the solution of the Schrödinger equation

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$$i\partial_t \psi_{N,t} = H_N \psi_{N,t} \tag{2.2}$$

for approximately factorized initial data $\psi_{N,0} \simeq \varphi^{\otimes N}$. Note that the external potential in the operator H_N appearing on the r.h.s. of (2.2) is typically different from the external potential in the trapping Hamiltonian, whose ground state is approximated by $\varphi^{\otimes N}$ (otherwise, the dynamics would be trivial). For example the initial data would be taken as an approximation to the ground state in a harmonic trap, and the evolution of this initial data would be studied after switching off the trap, i.e. $V_{\text{ext}} = 0$.

Notice also that here we describe a high density situation: the decay of φ defines the length (and volume) scale of order one (w.r.t. to N); in the state $\varphi^{\otimes N}$ this volume is filled with N particles.

Hartree equation. Of course, since H_N is an interacting Hamiltonian, the dynamics does not preserve the factorization of the many-body wave function. Still, since the interaction is weak, we can expect factorization to be approximately (in a sense to be specified later) preserved, in the limit of large N. In other words, we can expect that for $N \gg 1$

$$\psi_{N,t}(x_1, \ldots, x_N) \simeq \prod_{j=1}^{N} \varphi_t(x_j) \tag{2.3}$$

for an evolved one-particle wave function φ_t. Assuming (2.3), it is easy to derive a self-consistent equation for the evolution of the one-particle wave function φ_t. In fact, factorization of the N-particle wave function means, in probabilistic terms, that the particles are distributed in space according to the density $|\varphi_t|^2$, independently of each other. The law of large numbers then suggests that the total potential experienced, say, by the jth particle can be approximated by

$$\frac{1}{N} \sum_{i \neq j} V(x_i - x_j) \simeq \int V(x_j - y)|\varphi_t(y)|^2 dy = (V * |\varphi_t|^2)(x_j).$$

Hence, φ_t must satisfy the Hartree equation

$$i\partial_t \varphi_t = (-\Delta + V_{\text{ext}})\varphi_t + (V * |\varphi_t|^2)\varphi_t \tag{2.4}$$

where the many-body interaction has been replaced by the effective one-particle potential $V * |\varphi_t|^2$, making (2.4) a nonlinear equation. Despite the nonlinearity, the Hartree equation for many purposes is much easier to treat than the original Schrödinger equation (2.2), because φ_t depends only on 3 rather than $3N$ spatial coordinates. In particular, it is numerically more tractable.

Reduced densities. To explain in which sense we can expect (2.3) to hold true, we introduce the notion of reduced density matrices (also known as reduced densities). The one-particle reduced density associated with the wave function $\psi_{N,t}$ is defined by

$$\gamma^{(1)}_{N,t} = N \,\text{Tr}_{2,3,\ldots,N} |\psi_{N,t}\rangle\langle\psi_{N,t}|,$$

where $|\psi_{N,t}\rangle\langle\psi_{N,t}|$ denotes the orthogonal projection onto $\psi_{N,t}$ and $\mathrm{Tr}_{2,\dots,N}$ is the partial trace over the last $(N-1)$ particles. In other words, the one-particle reduced density $\gamma^{(1)}_{N,t}$ is defined as the non-negative trace-class operator on $L^2(\mathbb{R}^3)$ with integral kernel

$$\gamma^{(1)}_{N,t}(x;y) = N \int dx_2 \dots dx_N \, \psi_{N,t}(x, x_2, \dots, x_N)\overline{\psi}_{N,t}(y, x_2, \dots, x_N).$$

Notice that we chose the normalization $\mathrm{Tr}\, \gamma^{(1)}_{N,t} = N$.

Analogously, for $k = 2, 3, \dots, N$, we can define the k-particle reduced density associated with $\psi_{N,t}$ by

$$\gamma^{(k)}_{N,t} = \binom{N}{k} \mathrm{Tr}_{k+1,\dots,N} |\psi_{N,t}\rangle\langle\psi_{N,t}|.$$

The integral kernel of the k-particle density matrix is given by

$$\begin{aligned} &\gamma^{(k)}_{N,t}(x_1, \dots, x_k; y_1, \dots, y_k) \\ &= \binom{N}{k} \int dx_{k+1} \dots dx_N \, \psi_{N,t}(x_1, \dots, x_k, x_{k+1}, \dots, x_N) \\ &\qquad\qquad\qquad\qquad \times \overline{\psi}_{N,t}(y_1, \dots, y_k, x_{k+1}, \dots, x_N). \end{aligned} \tag{2.5}$$

The normalization is such that $\mathrm{Tr}\, \gamma^{(k)}_{N,t} = \binom{N}{k}$.

Clearly, for $k < N$, the k-particle reduced density $\gamma^{(k)}_{N,t}$ does not contain the full information about the system. Still, $\gamma^{(k)}_{N,t}$ is enough to compute the expectation of any k-particle observable: Let $J^{(1)}$ be an operator on the one-particle space $L^2(\mathbb{R}^3)$, and denote by $J^{(1)}_i = 1 \otimes \cdots \otimes J^{(1)} \otimes \cdots \otimes 1$ the operator on $L^2(\mathbb{R}^{3N})$ acting like $J^{(1)}$ on the ith particle and trivially on the other $(N-1)$ particles. We write $d\Gamma(J^{(1)}) = \sum_{i=1}^N J^{(1)}_i$. Then

$$\begin{aligned} \langle \psi_{N,t}, d\Gamma(J^{(1)})\psi_{N,t}\rangle &= \sum_{i=1}^N \langle \psi_{N,t}, J^{(1)}_i \psi_{N,t}\rangle \\ &= N \mathrm{Tr} J^{(1)}_i |\psi_{N,t}\rangle\langle\psi_{N,t}| = \mathrm{Tr}\, J^{(1)} \gamma^{(1)}_{N,t}. \end{aligned}$$

Similarly, if $J^{(k)}$ is an operator on the k-particle space $L^2(\mathbb{R}^{3k})$ and if we denote by $J^{(k)}_{i_1,\dots,i_k}$ the operator acting like $J^{(k)}$ on the k particles $i_1, \dots, i_k$, we have

$$\langle \psi_{N,t}, \sum_{\{i_1,\dots,i_k\}} J^{(k)}_{i_1,\dots,i_k} \psi_{N,t}\rangle = \mathrm{Tr}\, J^{(k)} \gamma^{(k)}_{N,t}$$

where the sum on the l.h.s. runs over all sets of k different indices $\{i_1, \dots, i_k\}$ chosen among $\{1, \dots, N\}$.

Convergence of reduced densities. It turns out that reduced densities provide the appropriate language to describe the convergence of the many-body quantum evolution towards the Hartree dynamics. Under appropriate assumptions on the external potential V_{ext} and, more importantly, on the interaction potential V, one can show the convergence of the reduced densities associated with the solution $\psi_{N,t}$ of the Schrödinger equation (2.2) towards orthogonal projections onto products of solutions of the Hartree equation (2.4). More precisely, consider a sequence of wave functions $\psi_N \in L^2_s(\mathbb{R}^{3N})$ with reduced density $\gamma_N^{(1)}$ satisfying

$$\frac{1}{N}\gamma_N^{(1)} \to |\varphi\rangle\langle\varphi| \quad (N \to \infty) \tag{2.6}$$

for a $\varphi \in L^2(\mathbb{R}^3)$ (such ψ_N are said to exhibit complete Bose-Einstein condensation). Let $\psi_{N,t} = e^{-iH_N t}\psi_N$ be the solution of the Schrödinger equation (2.2) with initial data ψ_N. Then we expect, and under appropriate assumptions on V_{ext} and V we can show that

$$\frac{1}{N}\gamma_{N,t}^{(1)} \to |\varphi_t\rangle\langle\varphi_t| \quad (N \to \infty). \tag{2.7}$$

Here φ_t denotes the solution of the Hartree equation (2.4) with initial data $\varphi_0 = \varphi$. The convergence in (2.7) can be understood in the trace-class topology. In fact, since the limit is a rank-one projection, weak convergence implies convergence in the trace norm.[1] Moreover, notice that convergence of the one-particle reduced density towards a rank-one orthogonal projection also implies convergence of higher order reduced densities in the limit $N \to \infty$ (the argument is outlined in [1], after Theorem 1),

$$\frac{1}{\binom{N}{k}}\gamma_{N,t}^{(k)} \to |\varphi_t\rangle\langle\varphi_t|^{\otimes k}. \tag{2.8}$$

The convergence here (and in (2.7)) is for fixed t and k. Equations (2.7) and (2.8) explain in which sense one should understand the approximate factorization (2.3). (For any one-particle operator A on $L^2(\mathbb{R}^3)$ we use the notation $A^{\otimes k} = \bigotimes_{i=1}^k A$ for its k-fold tensor product acting on $L^2(\mathbb{R}^{3k})$.)

[1]First, by testing the difference $N^{-1}\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|$ against $|\varphi_t\rangle\langle\varphi_t|$, weak convergence implies Hilbert-Schmidt convergence. Then, since $|\varphi_t\rangle\langle\varphi_t|$ is a rank-one projection, the operator $N^{-1}\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t|$ has exactly one negative eigenvalue. (If there were two linearly independent eigenvectors ξ_1, ξ_2 with negative eigenvalue, one could find a linear combination ξ such that $\langle\xi, \gamma_{N,t}^{(1)}\xi\rangle < 0$.) Since $\operatorname{Tr}\gamma_{N,t}^{(1)} - |\varphi_t\rangle\langle\varphi_t| = 0$, its absolute value is equal to the sum of all positive eigenvalues, and therefore the trace norm is twice the operator norm.

Some results. The first rigorous results of the form (2.7) have been obtained for smooth interactions by Hepp [2] and for singular potentials by Ginibre and Velo [3]. For bounded potential, a proof of the convergence (2.7), based on the analysis of the so-called BBGKY hierarchy, has been given by Spohn [4] (see next paragraph). The BBGKY technique has been later extended to potentials with Coulomb singularity in [5, 6] and, for semi-relativistic bosons, in [7]. A new approach giving a precise estimate on the rate of the convergence towards the Hartree dynamics has been developed in [8] for potentials with Coulomb singularities (and further improved in [9]); this approach, based on the ideas of [2, 3], will be presented in Chap. 3. Other bounds on the rate of convergence towards the Hartree evolution have been established in [10] (based on ideas proposed in [11]) and recently in [12]. In [13, 14] the convergence of the many-body evolution has been interpreted as a Egorov-type theorem. In [15, 16] the authors study the propagation of the Wigner measure in the bosonic mean-field limit. Next order corrections to the Hartree dynamics have been considered in [17, 18], leading to a better approximation of the many-body evolution. (A related problem is the study of the fluctuations around the Hartree evolution, which will be discussed in Chap. 3.) Instead of a fixed interaction V, it is also interesting to consider N-dependent potentials, scaling like $V_N(x) = N^{3\alpha} V(N^\alpha x)$ (in the three dimensional case) and converging towards a delta-function in the limit of large N. In this case (assuming $\alpha < 1$; for $\alpha = 1$, one recovers instead the Gross-Pitaevskii regime, which will be discussed in Chap. 5), the many-body evolution can be approximated by a nonlinear Schrödinger equation with a local cubic nonlinearity. Results in this direction have been obtained in [19–21] in the one-dimensional setting, in [22] in the two-dimensional case and in [23] in three dimensions. It is also possible to start from Hamiltonians with three-body interactions; in this case, the evolution can be approximated by a quintic nonlinear Schrödinger equation; see [24, 25].

The BBGKY approach. The main idea of the BBGKY approach, which was first applied to many-body quantum systems in the mean-field regime in [4], is to study directly the evolution of the reduced densities defined in (2.5). To explain this idea it is convenient to normalize the reduced densities associated with the solution $\psi_{N,t}$ of the Schrödinger equation, defining, for $k = 1, \dots, N$,

$$\widetilde{\gamma}^{(k)}_{N,t} = \frac{1}{\binom{N}{k}} \gamma^{(k)}_{N,t}.$$

The new density matrices are normalized so that $\operatorname{Tr} \widetilde{\gamma}^{(k)}_{N,t} = 1$ for all $N \in \mathbb{N}$ and all $k = 1, \dots, N$. From the Schrödinger equation for $\psi_{N,t}$ it is easy to derive evolution equations for the family $\{\widetilde{\gamma}^{(k)}\}_{k=1}^N$. It turns out that the evolution of the N reduced densities is governed by a hierarchy[2] of N coupled equations, known as the BBGKY hierarchy (BBGKY stands for Bogoliubov-Born-Green-Kirkwood-Yvon):

[2] i.e. the equation for $\widetilde{\gamma}^{(k)}_{N,t}$ depends on $\widetilde{\gamma}^{(k+1)}_{N,t}$.

$$\begin{aligned} i\partial_t \widetilde{\gamma}_{N,t}^{(k)} &= \sum_{j=1}^{k} \left[-\Delta_{x_j}, \widetilde{\gamma}_{N,t}^{(k)}\right] + \frac{1}{N} \sum_{i<j}^{k} \left[V(x_i - x_j), \widetilde{\gamma}_{N,t}^{(k)}\right] \\ &\quad + \frac{N-k}{N} \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \left[V(x_j - x_{k+1}), \widetilde{\gamma}_{N,t}^{(k+1)}\right] \end{aligned} \tag{2.9}$$

where we use the convention that $\widetilde{\gamma}_{N,t}^{(N+1)} = 0$ and where Tr_{k+1} denotes the partial trace over the degrees of freedom of the $(k+1)$-st particle. Here we also introduced the commutator, defined by $[A, B] = AB - BA$ on a domain suited to the operators A and B. Notice that the second term on the r.h.s. of (2.9) describes the interaction among the first k particles in the system. The last term on the r.h.s. of (2.9), on the other hand, corresponds to the interaction of these k particles with the other $(N-k)$ particles. At least formally, the BBGKY hierarchy (2.9) converges, in the limit of large N, towards the infinite hierarchy

$$i\partial_t \widetilde{\gamma}_{\infty,t}^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \widetilde{\gamma}_{\infty,t}^{(k)}\right] + \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \left[V(x_j - x_{k+1}), \widetilde{\gamma}_{\infty,t}^{(k+1)}\right]. \tag{2.10}$$

It is simple to check that this infinite hierarchy has a factorized solutions $\widetilde{\gamma}_{\infty,t}^{(k)} = |\varphi_t\rangle\langle\varphi_t|^{\otimes k}$, given by products of the solution of the Hartree equation (2.4). This observation suggests a general strategy to show the convergence (2.8) of the reduced densities towards projections onto products of solutions of the Hartree equation. The strategy consists of three steps:

- *Compactness*. First, one needs to prove the compactness of the sequence (in N) of families $\Gamma_{N,t} = \{\widetilde{\gamma}_{N,t}^{(k)}\}_{k=1}^{N}$ with respect to an appropriate weak topology. Compactness implies in particular the existence of at least one limit point $\Gamma_{\infty,t} = \{\widetilde{\gamma}_{\infty,t}^{(k)}\}_{k\geq 1}$.
- *Convergence*. Secondly, one needs to characterize limit points of the sequence $\Gamma_{N,t}$ as solutions of the infinite hierarchy (2.10). In other words, one has to show that any limit point $\Gamma_{\infty,t}$ of the sequence $\Gamma_{N,t}$ satisfies (2.10).
- *Uniqueness*. Finally, one has to prove the uniqueness of the solution of the infinite hierarchy. This implies immediately that the sequence $\Gamma_{N,t}$ converges, since every compact sequence with at most one limit point converges. Moreover, since we know that the factorized densities $\widetilde{\gamma}_{\infty,t}^{(k)} = |\varphi_t\rangle\langle\varphi_t|^{\otimes k}$ are a solution of (2.10), uniqueness also implies that $\widetilde{\gamma}_{N,t} \to |\varphi_t\rangle\langle\varphi_t|^{\otimes k}$ for all $k \in \mathbb{N}$ (the argument proves convergence with respect to the weak topology with respect to which one showed compactness in the first step; however, since the limit is a rank-one projection, weak convergence immediately implies convergence in the trace norm).

The most difficult of the three steps is the proof of the uniqueness of the solution of the infinite hierarchy. Let us illustrate how to prove uniqueness in the case of a bounded interaction potential $V \in L^\infty(\mathbb{R}^3)$. To this end, we rewrite the infinite hierarchy (2.10) in integral form as

$$\widetilde{\gamma}^{(k)}_{\infty,t} = \mathcal{U}^{(k)}(t)\widetilde{\gamma}^{(k)}_{\infty,0} + \frac{1}{i}\int_0^t ds\, \mathcal{U}^{(k)}(t-s)B^{(k)}\widetilde{\gamma}^{(k+1)}_{\infty,s}. \tag{2.11}$$

Here we defined the action of the free evolution $\mathcal{U}^{(k)}(t)$ on a k-particle density $\gamma^{(k)}$ by

$$\mathcal{U}^{(k)}(t)\gamma^{(k)} = e^{-i\sum_{j=1}^{k}\Delta_{x_j}t}\gamma^{(k)}e^{i\sum_{j=1}^{k}\Delta_{x_j}t}$$

and the action of the collision operator $B^{(k)}$ on a $(k+1)$-particle density $\gamma^{(k+1)}$ by

$$B^{(k)}\gamma^{(k+1)} = \sum_{j=1}^{k}\mathrm{Tr}_{k+1}\left[V(x_j - x_{k+1}), \gamma^{(k+1)}\right]. \tag{2.12}$$

Notice that the collision operator maps $(k+1)$-particle density matrices into k-particle density matrices. It is important to observe how the free evolution and the collision operator affect the trace norm. On the one hand, we clearly have

$$\|\mathcal{U}^{(k)}\gamma^{(k)}\|_{\mathrm{tr}} = \|\gamma^{(k)}\|_{\mathrm{tr}}\,. \tag{2.13}$$

On the other hand, for bounded interaction potentials, we find

$$\begin{aligned}\|B^{(k)}\gamma^{(k+1)}\|_{\mathrm{tr}} &\le \sum_{j=1}^{k}\left[\mathrm{Tr}\left|V(x_j - x_{k+1})\gamma^{(k+1)}\right| + \mathrm{Tr}\left|\gamma^{(k+1)}V(x_j - x_{k+1})\right|\right] \\ &\le 2k\|V\|_\infty \mathrm{Tr}|\gamma^{(k+1)}| \\ &= 2k\|V\|_\infty\|\gamma^{(k+1)}\|_{\mathrm{tr}}\,.\end{aligned} \tag{2.14}$$

Here we used that $\mathrm{Tr}|AB| \le \|A\|\mathrm{Tr}|B|$ for any bounded operator A and any trace-class operator B. (In the same way $\mathrm{Tr}|AB| \le \|B\|\mathrm{Tr}|A|$ if B is bounded and A trace class.)

Iterating (2.11), we obtain the nth order Dyson series

$$\begin{aligned}\widetilde{\gamma}^{(k)}_{\infty,t} &= \mathcal{U}^{(k)}(t)\widetilde{\gamma}^{(k)}_{\infty,0} \\ &\quad + \sum_{m=1}^{n-1}\frac{1}{i^m}\int_0^t ds_1 \ldots \int_0^{s_{m-1}} ds_m \mathcal{U}^{(k)}(t-s_1)B^{(k)}\cdots \\ &\qquad \times \mathcal{U}^{(k+m-1)}(s_{m-1}-s_m)B^{(k+m-1)}\mathcal{U}^{(k+m)}(s_m)\widetilde{\gamma}^{(k+m)}_{\infty,0} \\ &\quad + \frac{1}{i^n}\int_0^t ds_1 \ldots \int_0^{s_{n-1}} ds_n \mathcal{U}^{(k)}(t-s_1)B^{(k)}\mathcal{U}^{(k+1)}(s_1-s_2)\cdots \\ &\qquad \times \mathcal{U}^{(k+n-1)}(s_{n-1}-s_n)B^{(k+n-1)}\widetilde{\gamma}^{(k+n)}_{\infty,s_n}.\end{aligned} \tag{2.15}$$

Suppose now that the families $\Gamma_{1,\infty,t} = \{\widetilde{\gamma}^{(k)}_{1,\infty,t}\}_{k\geq 0}$ and $\Gamma_{2,\infty,t} = \{\widetilde{\gamma}^{(k)}_{2,\infty,t}\}_{k\geq 0}$ are two solutions of the infinite hierarchy, written in the integral form (2.11), with the same initial data. Expanding both solutions as in (2.15), and taking the difference (so that all fully expanded terms cancel), we find

$$\begin{aligned}
&\operatorname{Tr}\left|\widetilde{\gamma}^{(k)}_{1,\infty,t} - \widetilde{\gamma}^{(k)}_{2,\infty,t}\right| \\
&\leq \int_0^t ds_1 \ldots \int_0^{s_{n-1}} ds_n \, \|\mathcal{U}^{(k)}(t-s_1)B^{(k)} \cdots B^{(k+n-1)}\widetilde{\gamma}^{(k+n)}_{1,\infty,s_n}\|_{\mathrm{tr}} \\
&\quad + \int_0^t ds_1 \ldots \int_0^{s_{n-1}} ds_n \, \|\mathcal{U}^{(k)}(t-s_1)B^{(k)} \cdots B^{(k+n-1)}\widetilde{\gamma}^{(k+n)}_{2,\infty,s_n}\|_{\mathrm{tr}}.
\end{aligned}$$

Applying iteratively the bounds (2.13) and (2.14), we conclude that

$$\operatorname{Tr}\left|\gamma^{(k)}_{1,\infty,t} - \gamma^{(k)}_{2,\infty,t}\right| \leq 2\frac{|t|^n}{n!}k(k+1)\cdots(k+n-1)(2\|V\|_\infty)^n \leq 2^k(4\|V\|_\infty|t|)^n,$$

where we used the normalization $\operatorname{Tr}\gamma^{(k+n)}_{1,\infty,t} = \operatorname{Tr}\gamma^{(k+n)}_{2,\infty,t} = 1$ and the bound

$$\binom{n+k-1}{k-1} \leq 2^{k+n-1}.$$

For $|t| \leq (8\|V\|_\infty)^{-1}$, we obtain that for all $n \in \mathbb{N}$

$$\operatorname{Tr}\left|\widetilde{\gamma}^{(k)}_{1,\infty,t} - \widetilde{\gamma}^{(k)}_{2,\infty,t}\right| \leq 2^{k-n}.$$

Since the l.h.s. is independent of n, it must vanish. This proves that $\widetilde{\gamma}^{(k)}_{1,\infty,t} = \widetilde{\gamma}^{(k)}_{2,\infty,t}$ for all $|t| \leq (8\|V\|_\infty)^{-1}$. Since this argument only uses the normalization $\operatorname{Tr}\widetilde{\gamma}^{(k+n)}_{1,\infty,t} = \operatorname{Tr}\widetilde{\gamma}^{(k+n)}_{2,\infty,t} = 1$, which holds for all $t \in \mathbb{R}$, it can be iterated to prove uniqueness of the solution of the infinite hierarchy for all $t \in \mathbb{R}$.

For a Coulomb potential $V(x) = \pm 1/|x|$, the proof we outlined above can be modified by introducing a different norm for density matrices (this approach was first used in [5]). For a k-particle density $\widetilde{\gamma}^{(k)}$, acting on $L^2(\mathbb{R}^{3k})$, we define the Sobolev-type norm

$$\|\widetilde{\gamma}^{(k)}\|_{H_1^{(k)}} = \operatorname{Tr}\left|S_1 \ldots S_k \widetilde{\gamma}^{(k)} S_k \ldots S_1\right| \tag{2.16}$$

where $S_j = (1-\Delta_{x_j})^{1/2}$. Since the Coulomb potential is bounded with respect to the kinetic energy in the sense that as operators

$$\pm\frac{1}{|x|} \leq C(1-\Delta)$$

one can show that the collision operator $B^{(k)}$ defined as in (2.12), but now with $V(x) = \pm 1/|x|$, satisfies

$$\|B^{(k)}\widetilde{\gamma}^{(k+1)}\|_{H_1^{(k)}} \leq Ck\|\widetilde{\gamma}^{(k+1)}\|_{H_1^{(k+1)}}.$$

This bound replaces (2.14). Using this new bound one can prove, similarly as explained above for the case of bounded potentials, that any two solutions $\Gamma_{1,\infty,t} = \{\widetilde{\gamma}^{(k)}_{1,\infty,t}\}_{k\geq 1}$ and $\Gamma_{2,\infty,t} = \{\widetilde{\gamma}^{(k)}_{2,\infty,t}\}_{k\geq 1}$ of the infinite hierarchy with the same initial data satisfy

$$\left\|\widetilde{\gamma}^{(k)}_{1,\infty,t} - \widetilde{\gamma}^{(k)}_{2,\infty,t}\right\|_{H_1^{(k)}} \leq C^{k-n}\left[\|\widetilde{\gamma}^{(k+n)}_{1,\infty,t}\|_{H_1^{(k+n)}} + \|\widetilde{\gamma}^{(k+n)}_{2,\infty,t}\|_{H_1^{(k+n)}}\right].$$

To conclude uniqueness, here we need to show a-priori estimates of the form

$$\left\|\widetilde{\gamma}^{(k)}_{\infty,t}\right\|_{H_1^{(k)}} \leq C^k \tag{2.17}$$

for all $t \in \mathbb{R}$ and $k \in \mathbb{N}$, valid for any limit point $\Gamma_{\infty,t} = \{\widetilde{\gamma}^{(k)}_{\infty,t}\}_{k\geq 1}$ of the sequence of families $\Gamma_{N,t} = \{\widetilde{\gamma}^{(k)}_{N,t}\}_{k=1}^N$ of densities associated with the solution of the Schrödinger equation. (This step was not needed in the case of bounded potentials because the trace norm trivially remains uniformly bounded.) To obtain a-priori bounds of the form (2.17), one can use energy conservation; for details, see [5].

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