

## Chapter 2

# Complex Dynamical Networks

The term complex networks refers to graphs with non-trivial topological features.<sup>1</sup> This chapter gives a recapitulation on the theory of complex networks, discusses the quantities used to describe networks, and introduces the most important network models namely random (Solomonoff and Rapoport 1951; Rapoport 1957; Erdős and Rényi 1959, 1960), small-world (Watts and Strogatz 1998; Monasson 1999), and scale-free networks (Barabasi and Bonabeau 2003). For reviews and textbooks on complex networks see Boccaletti et al. (2006b), Albert and Barabasi (2002), Newman (2003), Newman et al. (2006), Newman (2010), Barabasi (2012), Boccaletti et al. (2014), Kivela et al. (2014).

A prominent and on the first view surprising feature of most real-world networks is that they are characterized by a small shortest path length which means that it takes only a few steps to get from any node of the network to any other node. This is known as the small-world phenomenon and has been found in such different networks as networks of film actors, brain networks, and the power grid (Watts and Strogatz 1998). As also discussed in Chap. 1, further examples include the metabolic network (Jeong et al. 2000), the co-authorship of mathematicians (Barahona and Pecora 2002), the World Wide Web (Adamic 1999), food webs (Montoya and Solé 2002), and the co-occurrence of words (two words are considered as linked if they are neighbors in a sentence) (Ferrer i Cancho and Solé 2001).

The small-world characteristic is well reflected in an Erdős-Rényi random network, which was the first network model studied in detail and which allows, in many respects, for an analytic treatment. Unfortunately, the model lacks many other features found in real networks. Most prominently, no well connected neighborhoods exist, or, in other words, random networks have a small clustering coefficient, while most real-world networks are characterized by a high clustering coefficient. With the small-world network, Watts and Strogatz (1998) presented a model which shows

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<sup>1</sup>We use the terms graph and network synonymously.

both characteristics; it has a small shortest path lengths, and at the same time a high clustering coefficient.

The chapter is structured as follows: In the first section, I discuss the mathematical descriptions of networks in terms of the adjacency matrix and the coupling matrix. Section 2.2 introduces the different network quantities used to measure network features. An overview of the different network types is given in Sect. 2.3. In Sect. 2.4 dynamics on networks are discussed. The chapter is summarized in Sect. 2.4.

## 2.1 Mathematical Description: From Nodes and Links to the Coupling Matrix

A network consists of  $N$  nodes and  $L$  links connecting these nodes. We distinguish *directed* and *undirected* networks. In a directed network, a link has a direction, i.e., it points from node  $i$  to node  $j$ , while in an undirected network the coupling is always bidirectional, meaning that for each link from node  $i$  to node  $j$  another link with the same strength exists which points from node  $j$  to node  $i$ . Synonyms for node and link are vertex and edge, respectively.

A network can be represented by its  $N \times N$  adjacency matrix  $\mathbf{A}$  which is defined as

$$A_{ij} = \begin{cases} 1, & \text{if there is a link from node } j \text{ to node } i, \\ 0, & \text{otherwise.} \end{cases} \quad (2.1)$$

The adjacency matrix is very useful for quantifying networks properties. Most of the network quantities discussed in the next section make use of the adjacency matrix.

The adjacency matrix is a good description for unweighted networks or for problems where weights are negligible. However, real-world networks often have links with different strengths. For example, consider traffic networks, where different roads are used with very different frequencies, or the brain, where neurons are connected via synapses which vary in strength. The different weights are particularly important when it comes to dynamics on networks, the focus of this thesis. In Chap. 10, we will see that changing the weights in a network has a dramatic effect on the dynamics even if the adjacency matrix is kept constant.

In contrast to the adjacency matrix, the coupling matrix takes the different link weights into account: The element  $G_{ij}$  of the coupling matrix  $\mathbf{G}$  describes a link from node  $j$  to node  $i$  with weight  $G_{ij}$ . Note that the entries of  $\mathbf{G}$  can be positive and negative. In the context of the brain, this corresponds to excitatory and inhibitory coupling, respectively. The adjacency matrix  $\mathbf{A}$  can be obtained from  $\mathbf{G}$  by element-wise use of the Heaviside-function  $\Theta$ :

$$A_{ij} = \Theta(|G_{ij}|), \quad (2.2)$$

where  $\Theta(x) = 1$  if  $x > 0$ , and  $\Theta(x) = 0$  if  $x \leq 0$ .

Obviously, adjacency and coupling matrix are symmetric for undirected networks and unsymmetric for directed networks.

## 2.2 Network Quantities

Different aspect of networks can be characterized by a variety of different network quantities (Boccaletti et al. 2006b; Albert and Barabasi 2002; Newman 2003; Newman et al. 2006; Newman 2010). Here, I briefly introduce the most important quantities and terms to describe a network of  $N$  nodes. In parts, the presentation follows Lehnert (2010).

- Node degree:** The degree  $\mathcal{D}_j$  of node  $j$  is defined as the number of links attached to this node. The mean degree  $\langle \mathcal{D} \rangle$  is given as the average over all nodes, i.e.,  $\langle \mathcal{D} \rangle = \sum_j \mathcal{D}_j / N$ . In weighted networks, in- and out-degree are distinguished, where the in-degree counts the number of incoming links and the out-degree the number of outgoing ones.
- Degree distribution:** The degree distribution  $P(\mathcal{D})$  is the probability that a randomly chosen node has  $\mathcal{D}$  links. Its first moment is the mean degree  $\langle \mathcal{D} \rangle$ .
- Assortativity:** The term assortativity describes the tendency of nodes to connect more often to nodes which are in some way similar than to other nodes. Most often the term is used in the context of degree and is then understood synonymously to degree correlation. The assortativity regarding the degree can be quantified by calculating the average degree of the nearest neighbors of nodes with degree  $\mathcal{D}$ , denoted as  $\mathcal{D}_{nn}(\mathcal{D})$ :

$$\mathcal{D}_{nn}(\mathcal{D}) = \sum_{\mathcal{D}'} \mathcal{D}' P(\mathcal{D}' | \mathcal{D}), \quad (2.3)$$

where  $P(\mathcal{D}' | \mathcal{D})$  is the conditional probability that a node of degree  $\mathcal{D}$  is linked to a node with degree  $\mathcal{D}'$ . If  $\mathcal{D}_{nn}(\mathcal{D})$  increases with  $\mathcal{D}$  the network is assortative, otherwise we will call it disassortative. In the case that  $\mathcal{D}_{nn}(\mathcal{D})$  is independent of  $\mathcal{D}$  no degree correlations exist.

- Community structure:** A community is a subgraph whose nodes are more tightly connected to each other than to the rest of the network. Social networks often exhibit a community structure. A variety of different approaches have been developed to define and detect communities; for a review see Fortunato (2010).

**Component:**

A network is considered as connected if a path exists between every node pair  $i$  and  $j$ . Otherwise it will be referred to as disconnected. A disconnected network is composed of several components where the components are the largest possible, connected subgraphs. A connected network consists of one single component comprising all nodes. In this thesis, I will only consider connected networks.

In Erdős-Rényi random networks, a critical connection probability  $q_c = 1/N$  exists above which a giant component emerges, meaning that almost all nodes are connected within one component and only a few are part of other components. For details see Sect. 2.3.2.

**Shortest path length:**

$\mathcal{L}$  is the shortest distance between two nodes, averaged over all nodes:

$$\mathcal{L} = \frac{1}{N^2} \sum_{i,j} g_{ij}. \quad (2.4)$$

$g_{ij}$  is the shortest distance (or geodesic distance) between node  $i$  and node  $j$ , defined as the minimal number of links one has to traverse to get from node  $i$  to node  $j$ .

Note that definition (2.4) includes the diagonal terms  $g_{ii}$ . In a bidirectional network without self-coupling,  $g_{ii}$  is equal to 2: One step is needed to reach a nearest neighbor, the second one to get back to the original node. Thus, including the diagonal terms in the definition (2.4) contributes a constant term of  $2/N$  in networks without self-coupling. With self-coupling,  $g_{ii} = 1$  since a link exists which starts and ends at the  $i$ th node. As a consequence, the shortest path length is reduced in networks including self-coupling compared to ones without self-coupling.

Disconnected networks call for another definition since  $g_{ij}$  is infinite if node  $i$  and  $j$  are not in the same component. A definition using the reciprocal values of the geodesic distances reads

$$\mathcal{L}^{-1} = \frac{1}{N^2} \sum_{i,j} g_{ij}^{-1}. \quad (2.5)$$

**Clustering coefficient:**

In real-world networks, two nodes connected to the same node are often more likely to be linked than two arbitrary nodes. As an example, consider a social network. Friends of a given person are much more likely to be mutual friends than two randomly picked people. The clustering

coefficient  $\mathcal{C}$  allows for quantifying this phenomenon. A high clustering coefficient means that with a higher probability a link exists between nodes that are adjacent to the same nodes than between two randomly picked nodes.

Formally,  $\mathcal{C}$  is defined as follows: Let  $\ell_i$  denote the number of links between the neighbors of the  $i$ th node. Then the clustering coefficient of the  $i$ th node is defined as

$$\mathcal{C}_i = \frac{2\ell_i}{\mathcal{D}_i(\mathcal{D}_i - 1)}, \quad (2.6)$$

where  $\mathcal{D}_i$  is the degree of the  $i$ th node, i.e., the number of neighbors node  $i$  has. Thus, the clustering coefficient is defined as the ratio between the actual existing links  $\ell_i$  and the maximum possible number of links  $\mathcal{D}_i(\mathcal{D}_i - 1)/2$  between the neighbors of the  $i$ th node. The clustering coefficient of the network is obtained by averaging over all nodes:

$$\mathcal{C} = \frac{1}{N} \sum_i \mathcal{C}_i. \quad (2.7)$$

### Betweenness:

The betweenness measures the number of shortest paths going through a given node. Thus, the betweenness is helpful for quantifying how important a node is for the communication in a social network or transport in a railway network, for example.

The betweenness of the  $i$ th node is defined as

$$\mathcal{B}_i = \sum_{k,j} \frac{n_{jk}(i)}{n_{jk}}, \quad (2.8)$$

where  $n_{jk}$  is the number of all shortest paths  $g_{jk}$  between the  $j$ th and the  $k$ th node, and  $n_{jk}(i)$  is the number of these shortest paths which pass through the  $i$ th node.

### Closeness:

The closeness  $\mu_i$  of node  $i$  is the inverse of the average distance of the  $i$ th node from all other nodes:

$$\mu_i = \frac{N}{\sum_j g_{ij}}, \quad (2.9)$$

where  $g_{ij}$  is the shortest distance between node  $i$  and  $j$ .

**Spectral density:** The spectral density of the adjacency matrix  $\mathbf{A}$  or the coupling matrix  $\mathbf{G}$  of dimension  $N$  is given by

$$\rho(v) = \frac{1}{N} \sum_{j=0}^{N-1} \delta(v - v_j) \quad (2.10)$$

where  $v_j, j = 0, \dots, N-1$ , are the eigenvalues of  $\mathbf{A}$  or  $\mathbf{G}$ , respectively. In the limit of  $N \rightarrow \infty$ ,  $\rho(v)$  becomes a continuous function.

## 2.3 Network Topologies

At first glance, real-world networks often appear to be completely in random structure. However, most are characterized by distinct topological features e.g., they might have a particularly high clustering coefficient, a small shortest path-length, or might be characterized by the existence of a few hubs, e.g., nodes of a particularly high degree. Several models including random, small-world, and scale-free networks have been developed which reflect these topological features. Furthermore, ideal network structures which allow for calculating the eigenvalue spectrum analytically have been considered. In Chap. 3, the importance of the eigenvalue spectrum for the synchronizability of a network will be discussed. The current section introduces different topologies, their constructing methods and discusses their eigenvalue spectrum.

### 2.3.1 Regular Networks with Circulant Matrices

An  $N \times N$  circulant matrix  $\mathbf{C}$  is given by the condition  $C_{ij} = c_{(j-i+N) \bmod N}$ ,  $i, j = 0, \dots, N-1$ . Thus,  $\mathbf{C}$  can be written as

$$\mathbf{C} = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}. \quad (2.11)$$

Circulant matrices are of particular interest as their eigenvalues and eigenvectors are analytically accessible. The eigenvectors are given by Gray (2005)

$$\mathbf{e}_j = \frac{1}{\sqrt{N}} (1, \omega_j, \omega_j^2, \dots, \omega_j^{N-1})^T, \quad j = 0, \dots, N-1, \quad (2.12)$$

where  $\omega_j = \exp\left(\frac{2\pi ij}{N}\right)$  is the  $N$ th roots of unity. The corresponding eigenvalues can be calculated as

$$v_j = \sum_{k=0}^{N-1} c_k \omega_j^k. \quad (2.13)$$

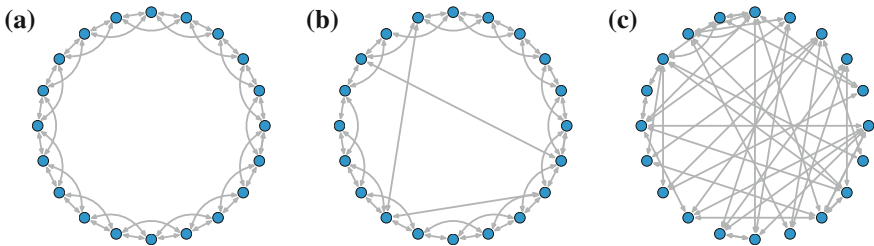
A subclass of networks with circulant matrices are regular ring networks. In a regular ring network, each node is with equal strength connected to its  $k$  nearest neighbors to the right and to the left. Thus, the normalized coupling matrix is given by

$$C_{ij} = \frac{1}{2k} \begin{cases} 1, & \text{if } |(i - j + N) \bmod N| \leq k \text{ and } i \neq j, \\ 0, & \text{otherwise.} \end{cases} \quad (2.14)$$

With Eq. (2.13), the eigenvalues can be calculated as

$$v_j = \frac{1}{k} \sum_{l=1}^k \cos\left(\frac{2\pi jl}{N}\right). \quad (2.15)$$

Figure 2.1a shows a regular ring network with  $k = 2$ , i.e., nearest and next-nearest neighbors are linked. (Panel (b) and (c) will be explained in Sects. 2.3.3 and 2.3.2, respectively.) Regular ring are of particular interest as they exhibit chimera states under certain conditions (Kuramoto and Battogtokh 2002; Abrams and Strogatz 2004; Sethia et al. 2008; Wolfrum and Omel'chenko 2011; Wolfrum et al. 2011; Omelchenko et al. 2011, 2012; Hagerstrom et al. 2012; Omelchenko et al. 2013; Vüllings et al. 2014; Zakharova et al. 2014; Omelchenko et al. 2015). In a chimera state, the nodes in the network separate into two groups with distinctly different behavior, e.g., spatially coherent and incoherent behavior or coherent and incoherent oscillations.



**Fig. 2.1** Rewiring procedure in the Watts-Strogatz model. **a**  $p = 0$  (regular network). **b**  $p = 0.08$  (small-world network). **c**  $p = 1$  (random network). Parameters:  $N = 20$ ,  $k = 2$ . Figure modified from Lehnert (2010) (color figure online)

The unidirectional ring network is another special case of a circulant matrix network. It is given by the coupling matrix

$$\mathbf{C} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (2.16)$$

and its eigenvalues read

$$v_j = e^{\frac{2\pi i}{N} j}, \quad j = 0, \dots, N-1. \quad (2.17)$$

Another important example of a circulant matrix network is the all-to-all topology, i.e., a network where each node is connected to all other nodes. If we normalize each row to one, i.e.,  $\sum_j C_{ij} = 1$ ,  $j = 0, \dots, N-1$ ,  $\mathbf{C}$  is described by  $C_{ij} = 1/(N-1)$  for  $i \neq j$  assuming that we exclude self-coupling. A normalization to one is useful when considering dynamics on networks, as will be discussed in Sect. 3.1. The eigenvalues are given by

$$v_j = \begin{cases} 0, & \text{if } j = 0, \\ \frac{-1}{N-1}, & \text{if } j > 0. \end{cases} \quad (2.18)$$

### 2.3.2 Erdős-Rényi Random Networks

Random networks have been among the first to be studied in network science. They were independently proposed by Rapoport (1957), Solomonoff and Rapoport (1951), and most prominently by Erdős and Rényi (1959, 1960). Two different possibilities to construct a random network exist:

- In a directed (undirected) graph,  $N^2$  ( $N(N-1)/2$ ) links can exist. In the first model,  $L(N)$  links are randomly distributed on these possible positions (Erdős and Rényi 1959).
- In the second model, each of the possible position is occupied with the probability  $q$ . The expectation value of the number of links is then  $qN^2$  (directed network) or  $qN(N-1)/2$  (undirected network) (Erdős and Rényi 1960).

Obviously, in the limit of large  $N$  and  $L$ , both construction methods coincide.

In the following discussion, a directed, unweighted network is assumed. For such a network, Erdős and Rényi (1959, 1960) showed that a critical probability  $q_c = 1/N$  exists separating the following regimes:



- $q < q_c$ : For  $N \rightarrow \infty$ , with probability one the graph has no component larger than  $\mathcal{O}(\ln N)$ , where  $\mathcal{O}$  denotes “of the order of”.
- $q = q_c$ : The largest component has the size  $\mathcal{O}(N^{2/3})$ .
- $q > q_c$ : A giant component of size  $\mathcal{O}(N)$  arises and no other component greater than  $\mathcal{O}(\ln N)$  exists.

For  $q > q_c$  and large  $N$ , the spectral density  $\rho(v)$  of an undirected graph converges to

$$\rho(v) = \begin{cases} \frac{\sqrt{4Nq(1-q)-v^2}}{2\pi Nq(1-q)}, & \text{if } |v| < 2\sqrt{Nq(q-1)}, \\ 0, & \text{otherwise.} \end{cases} \quad (2.19)$$

Equation (2.19) is known as Wigner’s law or semicircle law (Wigner 1958). The largest eigenvalue  $v_0$  is separated from the bulk and increases with the network size as  $pN$ .

At the beginning of this chapter, I have discussed the small-world phenomenon, i.e., the fact that many real-world networks are characterized by a small shortest path length. More formally, we speak of the small-world effect if the shortest path length of a network grows slower than  $\log(N)$ , i.e.,  $\mathcal{L}(N) \leq \alpha \log(N)$  with  $\alpha \in \mathbb{R}$ . The shortest path length of a random network is given by  $\mathcal{L} = \log N / \log \mathcal{D}$  where  $\mathcal{D} = L(N)/N = qN$  is the mean degree (Newman 2003). Obviously, this fulfills the criteria of the small-world effect if we increase the network size while keeping the average degree fixed. However, in most other aspects the random network is a poor model for real-world networks. Its clustering coefficient is given by  $\mathcal{C} = q$  which tends to  $1/N$  if we let  $N$  go to infinity for constant  $\mathcal{D}$ , while real-world network often exhibit high clustering coefficients. Furthermore, in contrast to real-world networks, the random network has a Poissonian degree distribution but no community structure, and no assortativity (Newman 2003).

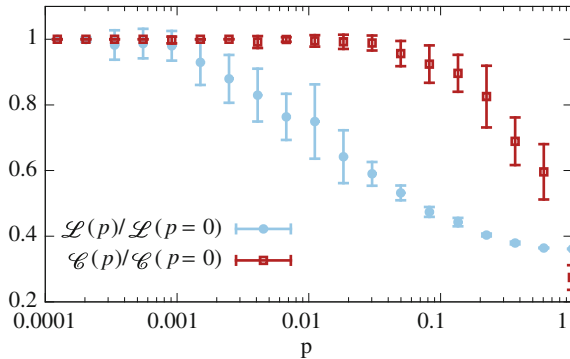
### 2.3.3 Small-World Networks

The shortcoming of the random network to reflect real-world network properties motivated Watts and Strogatz (1998) to their seminal work on small-world networks. The small-world network is characterized by both, a small shortest path length and a high clustering coefficient.

The Watt-Strogatz model relies on a one-dimensional interpolation between a regular network and a random network as it is depicted in Fig. 2.1. This interpolation involves the following steps (for details see Watts and Strogatz 1998):

1. Start with a regular network of  $N$  nodes where every node is linked to its  $2k$  neighbors (see Sect. 2.3.1 and Fig. 2.1a). Watts and Strogatz required  $N \gg k \gg \ln(N) \gg 1$  ensuring that the regular ring is sparse but remains connected during the following procedure.
2. Then, rewired one end of each link with probability  $p$  to a new node, where duplicated links are forbidden; see Fig. 2.1b.

**Fig. 2.2** Small-world network. Shortest path length  $\mathcal{L}/\mathcal{L}(p=0)$  (light blue circles) and clustering coefficient  $\mathcal{C}/\mathcal{C}(p=0)$  (dark red squares) versus rewiring probability  $p$ . Parameters:  $N = 100$ ,  $k = 4$ . Number of realizations: 70. (Reproduction of Fig. 2 in Watts and Strogatz(1998)) (color figure online)



In the limiting cases of  $p = 0$  and  $p = 1$ , this procedure recovers the regular ring and the random network, respectively; for a schematic view see Fig. 2.1a, c, respectively.

More interesting is the case of intermediate  $p$ . In this regime, which is also called small-world regime, the regular ring is still undamaged in large parts, while a few long-range connections exist; see Fig. 2.1b. Watts and Strogatz (1998) showed that this configuration yields a high clustering coefficient and a small shortest path length. Figure 2.2 shows the normalized shortest path length  $\mathcal{L}/\mathcal{L}(p=0)$  (blue symbols) and the normalized clustering coefficient  $\mathcal{C}/\mathcal{C}(p=0)$  (red symbols) as a function of the rewiring probability  $p$ . The small-world regimes, where  $\mathcal{L}$  is small and  $\mathcal{C}$  is large, is approximately located between  $p = 0.001$  and  $p = 0.01$ .

### 2.3.4 Scale-Free Networks

In a scale-free network, the degree distribution  $P(\mathcal{D})$  follows a power law,  $P(\mathcal{D}) \sim \mathcal{D}^{-\beta}$ ,  $\beta > 0$ , for large  $\mathcal{D}$ , where  $\mathcal{D}$  is the degree of a given node. This implies the existence of *hubs*: Nodes that have a very large number of links attached to them, while the majority of nodes have only a few links. These hubs are of great importance for the behavior of the whole network. For example, a scale-free network is very sensitive to the failure of a hub.

Famous examples of scale-free networks include the World Wide Web (Albert and Barabasi 2002), citation networks (de Solla Price 1965), the collaboration of mathematicians (Grossman and Ion 1995), the Internet at the level of autonomous systems (Chen et al. 2002), and the interaction network of proteins (Jeong et al. 2001).

## 2.4 Dynamics on Networks

So far I considered the structure of networks, i.e., how the nodes in a network are connected via links. In many applications, the structure is not sufficient to describe the complexity of the system but it has to be taken into account that the nodes are dynamical systems interacting through links. This approach has been used in very different fields. For example, a node can represent a laser which interacts via its light output with other lasers (Dahms 2011; Flunkert 2011). In epidemiology, SIR models are used to describe the spread of diseases: The nodes are individuals which can be susceptible (S), infected (I), or recovered (R). The links correspond to contacts between the individuals (Anderson and May 1992; Hethcote 2000; Belik et al. 2011; Lentz et al. 2012). Neural networks gained a lot of attention in the last decades. Here, the node dynamics are described by neural models and the links represents the interaction of the neurons via action potentials; see Ernst et al. (1998), Dayan and Abbott (2005), Timme et al. (2006), Jahnke et al. (2008), Vogels and Abbott (2009), Hövel (2010), Vogels et al. (2011), Popovych et al. (2011), (2013) and many more. Furthermore, many authors investigated the interplay between dynamical nodes and networks structure on a fundamental level in particular with a focus on synchronization (Pecora and Carroll 1998; Chavez et al. 2006; Arenas et al. 2006; Boccaletti et al. 2006a; Hunt et al. 2010; Flunkert et al. 2010; Sorrentino 2014; Geffert 2015).

For all examples mentioned above, the dynamics on a network can be described as a set of coupled differential equations,  $i = 0, \dots, N - 1$ ,

$$\dot{\mathbf{x}}_i = \mathbf{f}_i[\mathbf{x}_i(t)] + \sigma \sum_{j=0}^{N-1} G_{ij} \mathbf{h}_{ij}(\mathbf{x}_i, \mathbf{x}_j), \quad (2.20)$$

where  $\mathbf{x}_i \in \mathbb{C}^d$  is the state of the  $i$ th node and  $\mathbf{f}_i$  the local dynamics which describes the uncoupled node.  $\sigma = K \exp(i\beta)$ ,  $K, \beta \in \mathbb{R}$ , is the complex coupling strength. The function  $\mathbf{h}_{ij}$  models the coupling between node  $i$  and node  $j$ .  $\mathbf{G}$  is the coupling matrix as introduced in Sect. 2.3. For simplicity, we assumed that the states of all nodes are of dimension  $d$ . From Sect. 3.1 it will be clear that this is one of several necessary conditions for the existence of a zero-lag synchrony solution. However, group synchrony can exist in networks where the nodes are not of the same dimension as will be discussed in Chap. 5.

Often and throughout this thesis it is assumed that the coupling is linear and diffusive. In this case, Eq. (2.20) reads

$$\dot{\mathbf{x}}_i = \mathbf{f}_i[\mathbf{x}_i(t)] + \sigma \sum_{j=0}^{N-1} G_{ij} \mathbf{H}_{ij}(\mathbf{x}_j - \mathbf{x}_i) \quad (2.21)$$

where  $\mathbf{H}_{ij}$  is a  $d \times d$  matrix called the coupling scheme.

### 2.4.1 Delayed Coupling

As discussed in Chap. 1, delay is a ubiquitous phenomenon in real-world systems. Two types of delay can be distinguished: Propagation and processing delays. Here we consider the first type, i.e., a time delay which arises due to the finite propagation speed between node  $i$  and  $j$ . Including this delay, Eq. (2.21) becomes

$$\dot{\mathbf{x}}_i = \mathbf{f}_i[\mathbf{x}_i(t)] + \sigma \sum_{j=0}^{N-1} G_{ij} \mathbf{H}_{ij} [\mathbf{x}_j(t - \tau_{ij}) - \mathbf{x}_i(t)], \quad i = 0, \dots, N-1. \quad (2.22)$$

Equation (2.21) is a delay differential equation (DDE). DDEs are mathematically challenging since their phase spaces are infinite. In order to find a solution of Eq. (2.22) for  $t > 0$  the value of  $\mathbf{X} \equiv (\mathbf{x}_0, \dots, \mathbf{x}_{N-1})$  has to be known on the interval  $[-\tau_{\max}, 0]$  where  $\tau_{\max}$  is the maximum time delay, i.e.,  $\tau_{\max} = \max_{ij} \tau_{ij}$ . In other words, we have to specify the initial value or history function

$$\mathbf{X}(s) = \Phi(s), \quad s \in [-\tau_{\max}, 0]. \quad (2.23)$$

This is in contrast to ordinary differential equation where defining  $\mathbf{X}(0)$  is sufficient.

Only in rare cases, DDEs are analytically solvable by employing the methods of steps in which case the DDE is integrated stepwise over an interval of length  $\tau_{\max}$ , i.e., for step  $j$  over the interval  $[j\tau_{\max}, (j+1)\tau_{\max}]$ , where  $j = 0, \dots$  (Fridman 2014). Linear DDEs can be solved semi-analytically by considering the roots of their characteristic equation (see Appendix A). However, in most cases DDEs have to be solved numerically. Details can be found in Farmer (1982). For reviews on time delayed systems see Erneux (2009), Atay (2010), Fridman (2014).

## 2.5 Summary

This chapter has given an introduction to networks. I have discussed the mathematical representation of networks and different network quantities. Network types and their eigenvalue spectra have been addressed. The development of these network types has been driven by the observations that many real networks from very different areas exhibit similar characteristics. In particular, they often have a high clustering coefficient and a small shortest path length. These features are very well reflected by the small-world model by Watts and Strogatz (1998). This model uses a rewiring procedure to interpolate between a regular network and random graph. For intermediate rewiring rates—the small world regime—the model displays a high clustering coefficient and a small shortest paths length.

Besides describing the topology of real-world networks, networks can be used to model the dynamics and interactions of many systems or agents. In this case, the network is represented by a set of coupled ordinary differential equations or, if delay

in the coupling is considered, by a set of delay differential equations. The aim of this thesis is to explore the interplay between the topology of networks and the dynamics on the network. Furthermore, in Chap. 9 it will be shown how the topology can be changed in targeted manner to control the dynamics on the network. A particularly interesting dynamical state on a network is when all nodes follow the same dynamics or, in other word, are synchronized. In the next chapter, this state will be discussed in detail.

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Lehnert, J.

2016, XV, 203 p., Hardcover

ISBN: 978-3-319-25113-4