

Chapter 2

Prolegomena

2.1 Inertial Frames of Reference

The Special Theory of Relativity describes the physical laws that are valid in inertial frames of reference, i.e. frames of reference in which Newton's first law holds. Classical Mechanics deals mostly with inertial frames of reference. In these frames, a body on which the total external force applied is equal to zero is either stationary or moves with constant speed. There are, however, frames of reference in which, in order for Newton's first law to hold, it is necessary to include so called fictitious or d'Alembert forces, which have no apparent physical source of origin. To apply Newton's laws of motion on a rotating frame of reference, for example, we must assume the existence of fictitious forces, namely the centrifugal force and the Coriolis force [1]. These forces have no obvious physical source of origin, such as another body that exerts them, but depend entirely on the rotation of the frame of reference with respect to the rest of the matter in the universe.

Acceleration, in contrast to velocity, is an absolute magnitude. If we move with a uniform velocity, being acted on by zero forces, we are unable to sense this fact. Acceleration, on the other hand, is perceptible, due to the inertial properties of matter. If we try to accelerate a mass, we will feel a reaction to this effort of ours. Even in a closed room, we would be able to sense that the Earth rotates about its axis, either by the use of a Foucault pendulum or by studying the motion of a gyroscope. However, Galileo's principle of relativity, which we have mentioned in Sect. 1.2, considers the detection of absolute speed impossible.

If a body was the only body in the universe, its motion or its rotation would have no meaning, since these would not be able to be defined with reference to some other body. We are necessarily led to suspect that the inertial properties of matter are due to the other bodies present in the universe. These opinions were put forward by Mach, who disagreed radically with the concept of absolute space of Aristotle and Newton. We find ourselves in the apparently difficult position to need the rest of the matter in the universe for matter to have inertial properties, but have a body

on which no net external force is exerted, which would accelerate the body and make it a non-inertial frame of reference. We will assume that the matter in the universe is enough and isotropically distributed, so that the total force on a frame of reference we might choose is practically zero and the frame is inertial, but this matter imparts to the material bodies their inertial properties. It appears that mass in the universe is so sparsely distributed that we are able to regard a frame of reference as almost isolated if it is far enough from other bodies. We will see below to what degree this is possible, by examining various frames of reference which we might consider as even approximately inertial.

2.1.1 The Earth, the Sun and the Galaxy as Inertial Frames of Reference

The Earth is only approximately an inertial frame of reference, since it rotates about its axis and revolves around the Sun, a fact that leads to the appearance of fictitious forces when the Earth is assumed to be an inertial frame of reference. The centrifugal force is a fictitious force, which leads, in the case of the Earth, to a decrease of a body's weight as its geographical latitude is decreased. The Coriolis force is another fictitious force, to which cyclones and anticyclones are due in the motions of masses of air in the atmosphere. Depending on geographical latitude, the centrifugal acceleration at the surface of the Earth varies from zero at the poles up to a maximum value at the Equator. For a point on the Equator (Fig. 2.1), which is at a distance of $R_E = 6.38 \times 10^6$ m from the axis of rotation of the Earth and has an angular velocity equal to that of the Earth, $\omega = 7.3 \times 10^{-5}$ rad/s, the centrifugal acceleration is $a_1 = 0.034$ m/s². The value of the centrifugal acceleration varies, therefore, from zero at the poles to this value at the Equator, which is equal to 0.4 % of the mean acceleration due to gravity, g , on the surface of the Earth, and is easily observable.

The Earth revolves about the Sun, at a mean distance of $D_E = 1.50 \times 10^{11}$ m, with angular velocity equal to $\omega_E = 2 \times 10^{-7}$ rad/s. This means that the centrifugal acceleration of the Earth relative to the Sun is equal to $a_E = 6 \times 10^{-3}$ m/s².

The solar system revolves about the center of mass of the Galaxy, which is at a distance of $D_S = 2.7 \times 10^{20}$ m, approximately. The speed of the solar system in its motion about the center of the Galaxy is about $v_S = 220$ km/s. The angular velocity of the solar system about the center of the Galaxy is, therefore, $\omega_S = 10^{-15}$ rad/s and its centrifugal acceleration, due to this motion, is of the order of $a_S = 1.8 \times 10^{-10}$ m/s².

The Galaxy, in its turn, revolves about the center of mass of the group of galaxies to which it belongs. The centrifugal acceleration due to this motion is not known, but we may assume that it is much smaller than the last acceleration.

The conclusion we reach is that the solar system is a satisfactory inertial frame of reference for most applications. An even better inertial system is that defined by the

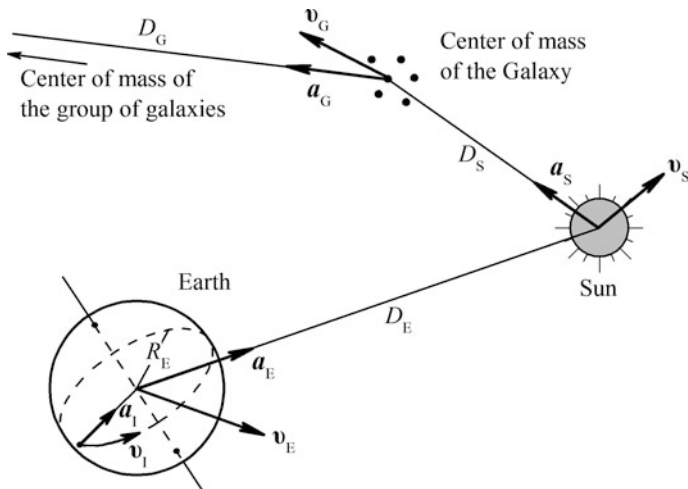


Fig. 2.1 The accelerations due to the rotation of the Earth around its axis, a_I , the revolution of the Earth about the Sun, a_E , the revolution of the Sun about the center of mass of our galaxy, a_S , and the revolution of our galaxy about the center of mass of the local group of galaxies to which it belongs, a_G

‘distant fixed stars’, i.e. the mean position of the distant galaxies. As we have already said, this is due to the fact that the distances between the galaxies are large enough and their distribution in space is isotropic enough.

In what will follow we will assume that it is possible for frames of reference to exist which are inertial to a satisfactory degree for most practical applications. Such would be frames of reference in interstellar space, at large distances from stars, and the infinite other frames of reference which move with constant velocities relative to these frames of reference.

2.2 The Calibration of a Frame of Reference and the Synchronization of Its Clocks

Something which occurs in a small enough volume and lasts for a short enough time we will call an *event*. The localization of an event in space and time requires giving its four coordinates (x, y, z, t) . The Special Theory of Relativity deals with events in various frames of reference. It must be possible to measure lengths in every inertial frame of reference and for this we need a common length standard. In contrast to Classical Physics, time is not absolute and so we must also have the ability to measure time at all points of a frame of reference, with identical and

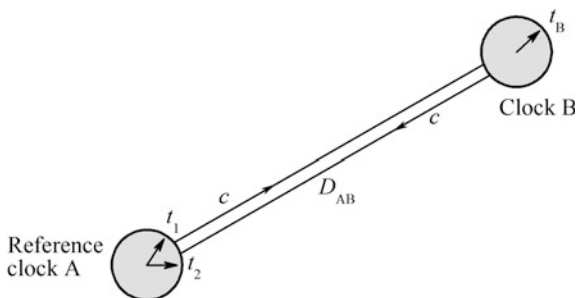
synchronized clocks. The solution to these two problems is given by the two postulates on which the Special Theory of Relativity is based:

- (a) The laws of Physics are the same in all inertial frames of reference, and
- (b) The speed of light in vacuum is invariant in all inertial frames of reference and does not depend on the speed of the source.

As a standard of length we may use the wavelength of the light from a particular transition in a certain atom. Equivalently, defining the unit of time as a given number of vibrations of the light of a certain spectral line of a particular atom, the distance travelled by light in a specified interval of time may be used as the length standard. We suppose that the atoms obey identical laws of Physics and so our definition gives identical results in all inertial frames of reference. Thus, we have standards of length and time in all inertial frames of reference. In addition, the units of time and of length are defined today in such a way that the speed of light in vacuum is exactly equal to $c = 299\,792\,458$ m/s.

We will imagine that there is a clock at every point of a frame of reference, which shows the correct time of that particular frame of reference. This presupposes the *synchronization* of these clocks for every frame of reference. The method proposed by Einstein for the synchronization of the clocks of a system is the following: We define one of the clocks of the frame of reference as the reference clock, A (Fig. 2.2). From the point at which clock A is situated, a light pulse is emitted at the moment this clock's reading is t_1 . The pulse propagates in vacuum to the point where clock B, which is to be synchronized with the reference clock A, is situated. The reading of clock B, t_B , at the moment of arrival of the pulse at B, is noted. The light is reflected, without delay, back towards A, where it arrives when clock A indicates time t_2 . We suppose, in addition, that the speed of light from A to B is the same as the speed of light from B to A. Thus, we conclude that the time of arrival of the light at B is $t_1 + \frac{t_2 - t_1}{2} = \frac{t_1 + t_2}{2}$, when clock B had an indication t_B . The values of t_1 and t_2 may be sent to B for the necessary adjustment to be made in order that the two clocks are synchronized. The distance of clock B from clock A may be calculated from these measurements as being equal to $D_{AB} = (t_2 - t_1)c/2$. In the same way, all the clocks of the frame of reference may be synchronized and their positions found.

Fig. 2.2 The synchronization of the clocks and the calibration of distances in a frame of reference



The existence of these calibrated frames of reference is necessary in the Special Theory of Relativity. It will be found that the difference from Classical Physics is that every frame of reference has its own synchronized clocks which show the time of the frame of reference, and which may differ from those of other frames of reference.

2.3 The Relativity of Simultaneity

We will now examine one of the important consequences of the invariance of the speed of light: the collapse of the concept of simultaneity as this is known in Classical Physics.

Let us assume that a train is moving relative to an inertial observer O with constant speed V . We will assume that the train moves with a speed which is smaller than that of light in vacuum. Exactly at the middle of the train, and stationary relative to it, stands another observer O' . Two light sources, A and B , are situated at the two ends of the train. At the exact moment when O' passes in front of O , two pulses of light arrive simultaneously at the two observers, having been emitted, respectively, one from source A and the other from source B (Fig. 2.3).

We examine first the train(!) of thought of observer O' , who is on the train and at its middle (Fig. 2.4):

'Points A and B are at equal distances $L_0/2$ from me, where L_0 is the length of the train. The speed of light is the same for both directions from A and B .

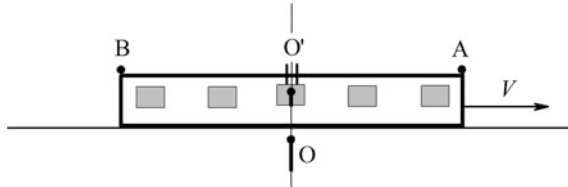


Fig. 2.3 A train moves with speed V relative to an observer O . At the middle of the train stands another observer, O' . When the two observers coincide, two pulses of light reach both of them, having been emitted, respectively, from the light sources A and B at the train's two ends

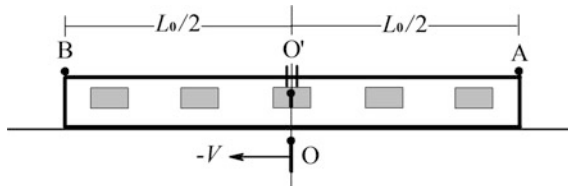


Fig. 2.4 Observer O' , on the train, concludes that the two pulses of light were emitted simultaneously from the two sources, A and B , which lie at equal distances from him

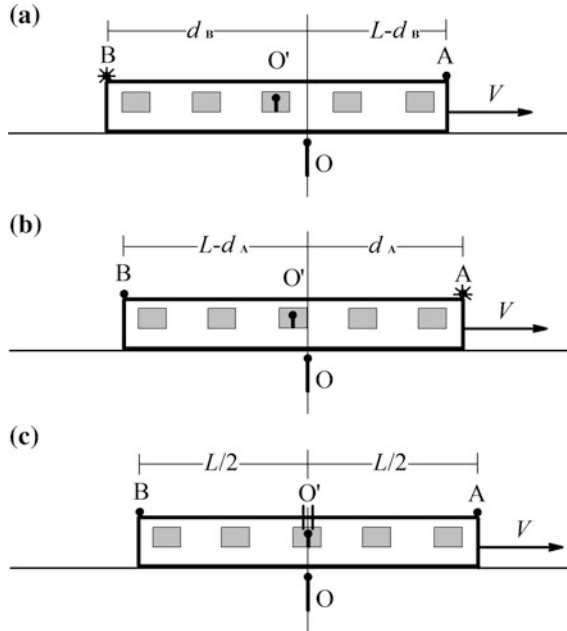


Fig. 2.5 Observer O, on the platform, concludes that the two light pulses were not emitted simultaneously from the two sources A and B. **a** When the pulse is emitted by source B, observer O' has not yet reached observer O, and the distance of source B from O must be $|d_B| > L/2$, L being the length of the train as far as O is concerned. **b** When source A emits its pulse, observer O' has still not reached observer O and source A must be at a distance from O which is $|d_A| < L/2$. It is, therefore, $|d_B| > |d_A|$. Thus, pulse B must have been emitted earlier than pulse A, since both arrive simultaneously at O, (c)

Therefore, the two pulses took equal times to arrive to me and, in order to arrive at the same time, they must have been emitted simultaneously from the two sources.'

Observer O, who is standing on the platform (Fig. 2.5), reasons as follows:

'The light from points A and B takes some time to reach me, from the moment it is emitted. When the pulse from source B is emitted, Fig. 2.5a, observer O' has not yet reached me. Therefore, the distance from me of source B, when it emits its pulse, is $|d_B| > L/2$, where L is the length of the train. When the pulse from source A is emitted, Fig. 2.5b, observer O' has still not reached me. Therefore, the distance from me of source A, when it emits its pulse, is $|d_A| < L/2$. Thus it is $|d_B| > |d_A|$. The speed of light is the same in both directions, being independent of the speeds of the sources or the observer. For the two pulses to arrive to me at the same time, Fig. 2.5c, the pulse from source B must have been emitted before the pulse from source A.'

We conclude, therefore, that two events which are simultaneous in one inertial frame of reference are not necessarily simultaneous in another inertial frame which is moving relative to the first. The invariance of the speed of light obliges us to abandon the concept of simultaneity which we take for granted in Classical Physics.

The reader must have noticed that, in the figures, the length of the train was taken as being different for the two observers: L_0 for the observer on the train and L for the observer on the platform. This is not accidental, as we will see in what follows. However, our ignorance, at present, of the exact lengths, does not influence the arguments developed above.

2.4 The Relativity of Time and Length

The invariance of the speed of light has very important consequences on time and length, as we will see in this section. We will begin by examining a quantity which remains invariant—any transverse dimension of a moving body. We will then examine the changes that have to be made to our understanding of time and length. We will return to these subjects in the next chapter, for a more general examination which will be based on the Lorentz transformation.

2.4.1 The Invariance of the Dimensions of a Body Which Are Perpendicular to Its Velocity

We examine two hollow cylinders, A and B (Fig. 2.6), which, when at rest, are identical, with radius r_0 and length L_0 . Let the two cylinders be in relative motion with relative speed V , along their common axis. We will assume that to an observer a moving cylinder shrinks in directions perpendicular to its velocity, so that the radius of the moving cylinder is $r(V) < r_0$. The arguments that will follow would still hold had we assumed that the moving cylinder expands in directions perpendicular to its velocity. It will not affect our arguments, but we may assume that the

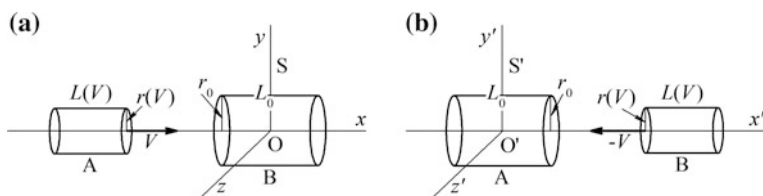


Fig. 2.6 Two identical *hollow cylinders*, A and B, in relative motion to each other, with their axes being coincident. The two cylinders, when at rest, have a radius equal to r_0 . We will assume that when a cylinder is moving with speed V along its axis, its radius changes to $r(V)$ and its length to $L(V)$. Without loss of generality, we will take $r(V) < r_0$. In the reference frame of B, S, cylinder A moves with a velocity $\mathbf{V} = V\hat{x}$ and its radius shrinks to $r(V)$. Cylinder A will, therefore, pass through cylinder B. In the reference frame of A, S', cylinder B moves with a velocity $\mathbf{V} = -V\hat{x}$ and its radius shrinks to $r(V)$. In the frame of reference S', cylinder B will be expected to pass through cylinder A

length of the moving cylinder changes to $L(V)$. In the frame of reference S of cylinder B, cylinder A moves with a velocity $\mathbf{V} = V\hat{\mathbf{x}}$ and its radius is $r(V) < r_0$. Cylinder A will, therefore, pass through cylinder B (Fig. 2.6a).

In the frame of reference S' of cylinder A, cylinder B moves with a velocity $\mathbf{V} = -V\hat{\mathbf{x}}$. Since space is isotropic, the radius of the moving cylinder will depend on its speed and so the radius of cylinder B will be $r(V) < r_0$. In this frame of reference, cylinder B is expected to pass through cylinder A (Fig. 2.6b).

Since it is impossible for both these predictions to be true, we must conclude that the transverse dimensions of a moving body do not change. This conclusion is purely a consequence of the non-existence of a privileged inertial frame of reference and the equivalence of all the inertial frames of reference.

2.4.2 The Relativity of Time

We will examine the rates of counting time by two clocks in relative motion to each other with a constant speed. The clocks we will consider work in the following way: Two mirrors, A and B in Fig. 2.7a, are at a distance D from each other. A very short light pulse (or a photon!) travels between the two mirrors. An observer, who is at rest relative to such a clock, will consider one unit of time, as measured by the clock, to be the time $T_0 = 2D/c$ needed for the pulse to travel from A to B and back to A, Fig. 2.7a.

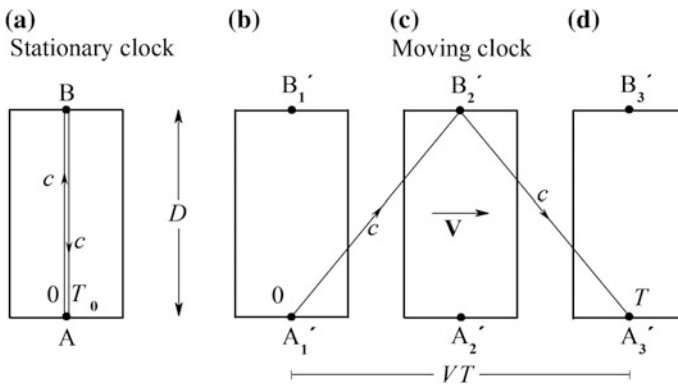


Fig. 2.7 *The dilation of time.* Two clocks, one at rest relative to an observer and one moving with a speed V relative to him. The operation of the clocks is based on a light pulse traveling between two mirrors, A and B (a). A unit of time (between two successive beats) is measured by the clock when the light pulse travels from A to B and then back to A. The moving clock moves in a direction perpendicular to the line joining the two mirrors of the clock at rest. The observer sees that the light in the moving clock has to cover a greater distance between successive beats than the light in the clock at rest relative to him (b–d). He therefore observes that more time passes between the successive beats of the moving clock as compared to the corresponding time in the clock at rest relative to him

Let now the observer who is at rest relative to a clock, observe another identical clock moving with speed V relative to himself in a direction perpendicular to the line AB, as shown in Figs. 2.7b, c, d. If for this observer the light needs a time T to travel from A to B and then back to A, the observer will see the moving clock shift by a distance $A'_1A'_3 = VT$. The distance this observer will see as travelled by the light, always with a speed c , will be

$$cT = A'_1B'_2A'_3 = 2\sqrt{D^2 + (A'_1A'_2)^2} = 2\sqrt{D^2 + (VT/2)^2}, \quad (2.1)$$

where we have taken into account the fact we have already proved, that the distance D does not change due to the motion of the clock. But $D = cT_0/2$ and, hence, substituting, we have

$$c^2T^2 = c^2T_0^2 + V^2T^2, \quad (2.2)$$

from which we find that

$$T = \frac{T_0}{\sqrt{1 - V^2/c^2}}, \quad (2.3)$$

which is a time greater than T_0 .

It follows that the observer sees the moving clock count time at a slower rate than that of the clock at rest relative to himself. A clock at rest registers that the time needed for light to travel the path ABA, from one of its mirrors to the other and back (*proper time*), is smaller by a factor of $\sqrt{1 - V^2/c^2}$ compared to the time that the clock at rest registers as needed for the light to travel from one mirror of the moving clock to the other and back. This phenomenon is known as *dilation of time*. We will return to it in the next chapter and in the one after, where we will also examine its experimental verification.

As a numerical example, consider two identical clocks, one at rest in a frame of reference S and one at rest in a frame of reference S'. S' moves relative to S with a speed $V = (4/5)c$, for which it is $\sqrt{1 - V^2/c^2} = 3/5$. Let the clock in S' count a time interval $T_0 = 1$ unit of time, as its light pulse moves from mirror A to mirror B and back. To an observer relative to which this clock is moving, the time taken for this process to be completed in the moving clock will be $T = T_0 / \sqrt{1 - V^2/c^2} = 1/(3/5) = 5/3 = 1.67$ units of time. The observer in S sees that the moving clock needs 1.67 units of time to complete a procedure that corresponds to the counting of 1 unit of time. He, therefore, concludes that the moving clock is slow in counting time.

One may claim that the dilation of time was proved to apply only to a clock in which the light travels in a direction normal to the clock's velocity $\mathbf{V}$. We will now show that the same result holds for a clock at any orientation relative to $\mathbf{V}$. In Fig. 2.8 we have drawn two identical clocks, at rest in the same frame of reference,

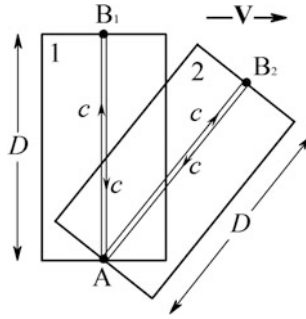


Fig. 2.8 Two identical clocks, similar to those of Fig. 2.7. The clocks are at rest in the same frame of reference and are oriented one (1) normal to the velocity $\mathbf{V}$ of an outside observer and the other (2) at some other angle. The two clocks emit their pulses of light at a common point, A

one (1) with its axis normal to $\mathbf{V}$ and the other (2) inclined at another angle. The two clocks have a common point of emission of their photons, A.

In the frame of reference of the clocks, if two pulses start simultaneously in the two clocks from point A, they will return to the same point simultaneously, because they have the same distance to travel, $2D$, and the speed of light is the same in all directions. In the frame of reference of the clocks, point A and the two pulses coincide initially and they will also coincide again when the two pulses return. It is obvious that the two pulses will also coincide at point A when viewed from any other frame of reference. If this did not happen, we might have a phenomenon, such as an interaction, in one frame of reference and not in the other. This would naturally be physically unacceptable. The two pulses will therefore coincide at point A in all other frames of reference as well. The rate at which the two clocks measure time will be the same and what is true for clock 1 will also be true for clock 2. The dilation of time is, therefore, the same, independent of the clock's orientation relative to the velocity $\mathbf{V}$.

2.4.3 The Relativity of Length

We will now consider two clocks, similar to those of the previous section, one of which is at rest relative to some observer, and another that is moving with speed V , in a direction which is now parallel to the straight line AB between the two mirrors, as shown in Fig. 2.9. We must leave open the possibility that the length of the moving clock in the direction of its motion may differ from that of a stationary clock. If the distance measured by the observer between the two mirrors is L_0 for the clock at rest, let it be equal to L for the clock that moves relative to the observer (Fig. 2.9b, c, d).

In the clock which is at rest relative to the observer, light travels the distance ABA in time $T_0 = 2L_0/c$. We have already proved that the observer will measure

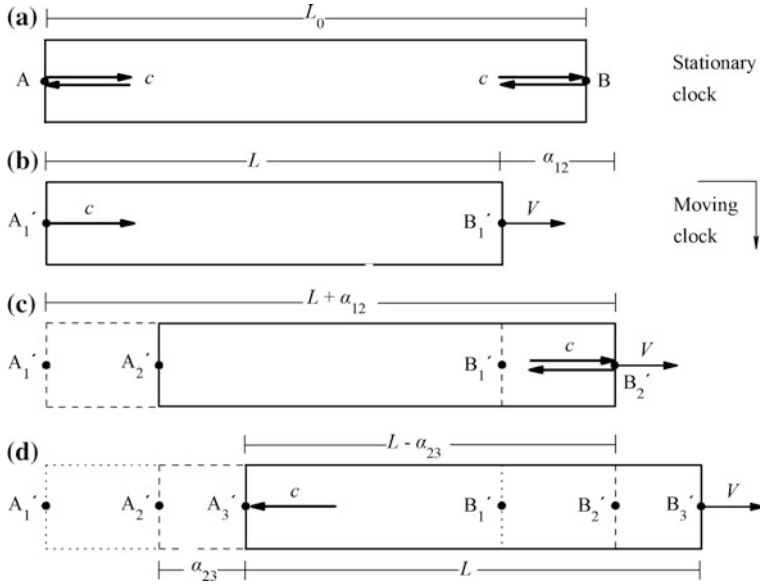


Fig. 2.9 *The contraction of length.* Two clocks, one at rest relative to an observer and one moving with a speed V relative to him. The operation of the clocks is based on a light pulse traveling between two mirrors, A and B. The moving clock moves in a direction parallel to the line AB. The distances travelled by the light in the one direction and in the other, as measured by the observer, are different in the moving clock

that for the moving clock the time needed for the light to travel from one mirror to the other and back is

$$T = \frac{T_0}{\sqrt{1 - V^2/c^2}}. \quad (2.4)$$

Alternatively, the observer sees light travel in the moving clock from mirror A to mirror B in time ΔT_+ , covering a distance $A_1'B_2'$ (Fig. 2.9b, c) which, due to the motion of the clock with speed V , will be equal to

$$A_1'B_2' = c\Delta T_+ = L + \alpha_{12} = L + V\Delta T_+, \quad (2.5)$$

from which we find,

$$\Delta T_+ = \frac{L}{c - V}. \quad (2.6)$$

In the motion from mirror B to mirror A, light travels in time ΔT_- the distance $B'_2A'_3$ (Fig. 2.9c, d) which is equal to

$$B'_2A'_3 = c\Delta T_- = L - \alpha_{23} = L - V\Delta T_-, \quad (2.7)$$

from which we find,

$$\Delta T_- = \frac{L}{c+V}. \quad (2.8)$$

The total time needed for light in the moving clock to travel from mirror A to mirror B and back is, according to the external observer,

$$T = \Delta T_+ + \Delta T_- = \frac{L}{c-V} + \frac{L}{c+V} = \frac{2cL}{c^2 - V^2} = \frac{2L/c}{1 - V^2/c^2}. \quad (2.9)$$

It follows from Eqs. (2.4) and (2.9) that

$$T = \frac{T_0}{\sqrt{1 - V^2/c^2}} = \frac{2L/c}{1 - V^2/c^2}. \quad (2.10)$$

However, it is $T_0 = 2L_0/c$, so that

$$\frac{2L_0/c}{\sqrt{1 - V^2/c^2}} = \frac{2L/c}{1 - V^2/c^2} \quad (2.11)$$

and, finally,

$$L = L_0 \sqrt{1 - V^2/c^2}. \quad (2.12)$$

This equation describes the phenomenon of the *contraction of length*, according to which an observer finds that the length of a body that is moving with a speed V relative to him is, in the direction of motion of the body, smaller by a factor $\sqrt{1 - V^2/c^2}$ compared to the length of the same body in its own frame of reference, in which it is stationary (*rest length* or *proper length*). We will return to this effect in the next chapter.

2.5 The Inevitability of the Special Theory of Relativity

It is natural that someone learning about the Special Theory of Relativity for the first time will react to the proposition that he or she should accept the consequences of the theory. The conclusions of the theory are in direct contrast to our everyday experiences, given the limited means we have for acquiring them with our senses.

It must, however, be understood that the results produced by the Theory of Relativity are not a consequence of some choice we have made at some stage of the development of the theory. On the contrary, it is the way the natural world is and the only thing we do is to discover these physical laws, based on experiment and logical reasoning. So long as our knowledge was limited to the study of mechanical phenomena at low energies and low speeds, these laws remained unknown to us. When, however, we studied the laws of electromagnetism and of light, which is pre-eminently relativistic, the laws of relativity became inevitable. It is almost certain that, sooner or later, the theory will be modified to take into account new experimental results. At today's limits of energy, speed and phenomena observed, however, this improved theory must give the same results as today's theory does, for today's accuracy of observations. This is exactly what the Theory of Relativity does with respect to Classical Mechanics, with which it agrees at the limit of low energies and speeds. In the same way, Quantum Mechanics has Classical Mechanics as its limit macroscopically, and the General Theory of Relativity agrees, in the limit of weak gravitational fields, with the Newtonian theory of universal attraction.

To the question of whether things could have been different, the answer seems to be negative. Once it would have been unreasonable for an absolute frame of reference to exist in the universe, the Theory of Relativity is a necessity. It would have been impossible for Nature to describe which frame of reference should be privileged. It is, therefore impossible for such a privileged frame of reference to exist and all inertial frames of reference are equivalent. The Theory of Relativity necessarily follows.

With these facts in mind, perhaps it will be easier for the reader to accept the conclusions in what will follow, even if they radically disagree with our everyday experiences.

Reference

1. For a good introduction to the subject of fictitious forces and non-inertial frames of reference, see e.g. C. Kittel, W.D. Knight, M.A. Ruderman, A.C. Helmholz, B.J. Moyer, *Mechanics* (Berkeley Physics Course, vol. 1), (McGraw-Hill Book Company, 2nd ed., 1973), Chap. 4. For a more advanced analysis see e.g. K.R. Symon, *Mechanics* (Addison-Wesley, 3rd ed., 1971)

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