

Chapter 1

Introduction to Modern Physics

This chapter provides an introduction to modern physics and covers basic elements of atomic, nuclear, relativistic, and quantum physics as well as electromagnetic theory. These elements form the background knowledge that is required for a study of medical radiation physics. The first few pages of this chapter present lists of basic physical constants, of important derived physical constants, and of milestones in modern physics and medical physics. These lists would normally be relegated to appendices at the end of the book; however, in this textbook they are given a prominent place at the beginning of the book to stress their importance to modern physics as well as to medical physics.



After introducing the basic physical constants and the derived physical constants of importance in modern physics and medical physics, the chapter spells out the rules governing physical quantities and units and introduces the classification of natural forces, fundamental particles, ionizing radiation in general as well as directly and indirectly ionizing radiation. Next the basic definitions for atomic and nuclear structure are given and the concepts of the physics of small dimensions (quantum physics) as well as large velocities (relativistic physics) are briefly reviewed.

A short introduction to particle-wave duality is given, wave mechanics is briefly discussed, and Fermi second golden rule and the Born collision formula are introduced. After a brief discussion of Maxwell equations and the Poynting theorem, the chapter concludes with a discussion of the normal probability distribution described by the continuous probability density function.

Medical physics is intimately related to modern physics and most milestone discoveries in modern physics were rapidly translated into medical physics and medicine, as evident from the list of milestones in medical physics provided in Sect. 1.3. Medical physics is a perfect and long-standing example of translational research where basic experimental and theoretical discoveries are rapidly implemented into benefiting humanity through improved procedures in diagnosis and treatment of disease.

A thorough understanding of the basics presented in this chapter will facilitate the readers' study of subsequent chapters and enhance their appreciation of the nature, importance, and history of medical physics as it relates to the use of ionizing radiation in diagnosis and treatment of human disease.

1.1 Fundamental Physical Constants

Currently the best source of data on fundamental physical constants is the *Committee on Data for Science and Technology* (CODATA), an inter-disciplinary scientific committee of the International Council for Science (ICSU) with headquarters in Paris, France. The ICSU's membership comprises 121 national organizations and 32 international scientific unions.

The CODATA Task Group on Fundamental Constants (www.codata.org) was established in 1969 and its purpose is to periodically provide the scientific and technological communities with an internationally accepted set of values of fundamental physical constants for worldwide use. The mission of CODATA is: *"To strengthen international science for the benefit of society by promoting improved scientific and technical data management and use"* and its membership comprises 23 full national members, 2 associate members, and 17 international scientific unions. The committee publishes *"The DATA Science Journal"*, a peer-reviewed, open access electronic journal featuring papers on management of data and databases in science and technology.

The data below (rounded off to four significant figures) were taken from the recent CODATA set of values issued in 2010 and easily available from the web-site supported by the National Institute of Science and Technology (NIST) in Washington, DC, USA (<http://physics.nist.gov/cuu/Constants/>)

Avogadro constant	$N_A = 6.022 \times 10^{23} \text{ mol}^{-1}$
Speed of light in vacuum	$c = 2.998 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s}$
Atomic mass constant	$u = 1.661 \times 10^{-27} \text{ kg} = 931.5 \text{ MeV}/c^2$
Elementary charge	$e = 1.602 \times 10^{-19} \text{ C}$
Electron rest mass	$m_e = 9.109 \times 10^{-31} \text{ kg} = 0.5110 \text{ MeV}/c^2$
Positron rest mass	$m_e = 9.109 \times 10^{-31} \text{ kg} = 0.5110 \text{ MeV}/c^2$
Proton rest mass	$m_p = 1.673 \times 10^{-27} \text{ kg} = 1.007 u = 938.3 \text{ MeV}/c^2$
Neutron rest mass	$m_n = 1.675 \times 10^{-27} \text{ kg} = 1.009 u = 939.6 \text{ MeV}/c^2$
Planck constant	$h = 6.626 \times 10^{-34} \text{ J}\cdot\text{s} = 4.136 \times 10^{-15} \text{ eV}\cdot\text{s}$
Reduced Planck constant	$\hbar = \frac{1}{2\pi} h = 1.055 \times 10^{-34} \text{ J}\cdot\text{s}$
Boltzmann constant	$k = 1.381 \times 10^{-23} \text{ J}\cdot\text{K}^{-1} = 0.8617 \times 10^{-4} \text{ eV}\cdot\text{K}^{-1}$
Electric constant	$\epsilon_0 = 8.854 \times 10^{-12} \text{ F}\cdot\text{m}^{-1} = 8.854 \times 10^{-12} \text{ C}/(\text{V}\cdot\text{m})$
Magnetic constant	$\mu_0 = 4\pi \times 10^{-7} \text{ N}\cdot\text{A}^{-2}/(\text{A}\cdot\text{m}) \approx 12.57 \text{ (V}\cdot\text{s)} / (\text{A}\cdot\text{m})$
Newtonian gravitation constant	$G = 6.674 \times 10^{-11} \text{ m}^3\cdot\text{kg}^{-1}\cdot\text{s}^{-2}$
Proton/electron mass	$m_p/m_e = 1836.1$
Specific charge of electron	$e/m_e = 1.759 \times 10^{11} \text{ C}\cdot\text{kg}^{-1}$
Planck constant/electron charge	$h/e = 4.136 \times 10^{-15} \text{ V}\cdot\text{s}$
Alpha particle mass	$m_\alpha = 6.645 \times 10^{-27} \text{ kg} = 4.0015 u = 3727.4 \text{ MeV}/c^2$
Elementary charge/Planck constant	$e/h = 2.418 \times 10^{14} \text{ A}\cdot\text{J}^{-1}$

1.2 Derived Physical Constants and Relationships

- **Speed of light** in vacuum:

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \approx 3 \times 10^8 \text{ m/s} \quad (1.1)$$

- **Reduced Planck constant \times speed of light in vacuum:**

$$\hbar c = \frac{h}{2\pi} c = 197.3 \text{ MeV}\cdot\text{fm} = 197.3 \text{ eV}\cdot\text{nm} \approx 200 \text{ MeV}\cdot\text{fm} \quad (1.2)$$

- **Bohr radius constant** [see (3.4)]

$$a_0 = \frac{\hbar c}{\alpha m_e c^2} = \frac{4\pi\varepsilon_0}{e^2} \frac{(\hbar c)^2}{m_e c^2} = 0.5292 \text{ \AA} = 0.5292 \times 10^{-10} \text{ m} \quad (1.3)$$

- **Fine structure constant** [see (3.6)]

$$\alpha = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{\hbar c} = \frac{\hbar c}{a_0 m_e c^2} = 7.297 \times 10^{-3} \approx \frac{1}{137} \quad (1.4)$$

- **Rydberg energy** [see (3.8)]

$$E_R = \frac{1}{2} m_e c^2 \alpha^2 = \frac{1}{2} \left[\frac{e^2}{4\pi\varepsilon_0} \right]^2 \frac{m_e c^2}{(\hbar c)^2} = 13.61 \text{ eV} \quad (1.5)$$

- **Rydberg constant** [see (3.11)]

$$R_\infty = \frac{E_R}{2\pi\hbar c} = \frac{m_e c^2 \alpha^2}{4\pi\hbar c} = \frac{1}{4\pi} \left[\frac{e^2}{4\pi\varepsilon_0} \right]^2 \frac{m_e c^2}{(\hbar c)^3} = 109737 \text{ cm}^{-1} \quad (1.6)$$

- **Classical electron radius** [see (6.43), (6.60), (7.31), (7.41), and (7.89)]

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2} = 2.818 \text{ fm} = 2.818 \times 10^{-15} \text{ m} \quad (1.7)$$

- **Compton wavelength** of the electron [see (7.44)]

$$\lambda_c = \frac{h}{m_e c} = \frac{2\pi\hbar c}{m_e c^2} = 0.02426 \text{ \AA} = 2.426 \times 10^{-12} \text{ m} \quad (1.8)$$

- **Thomson classical cross section** for free electron [see (7.41)]

$$\sigma_{\text{Th}} = \frac{8\pi}{3} r_e^2 = 0.6653 \text{ b} = 0.6653 \times 10^{-24} \text{ cm}^2 \quad (1.9)$$

- **Collision stopping power constant** [see (6.43)]

$$C_0 = 4\pi N_A \left(\frac{e^2}{4\pi\varepsilon_0} \right)^2 \frac{1}{m_e c^2} = 4\pi N_A r_e^2 m_e c^2 = 0.3071 \text{ MeV}\cdot\text{cm}^2\cdot\text{mol}^{-1} \quad (1.10)$$

1.3 Milestones in Modern Physics and Medical Physics

<i>x-rays</i>	Wilhelm Konrad Röntgen	1895
<i>Natural radioactivity</i>	Antoine-Henri Becquerel	1896
<i>Electron</i>	Joseph John Thomson	1897
<i>Radium-226</i>	Pierre Curie , Marie Skłodowska-Curie	1898
<i>Alpha particle</i>	Ernest Rutherford	1899
<i>Energy quantization</i>	Max Planck	1900
<i>Special theory of relativity</i>	Albert Einstein	1905
<i>Photoelectric effect</i>	Albert Einstein	1905
<i>Characteristic x-rays</i>	Charles Barkla	1906
<i>Alpha particle scattering</i>	Hans Geiger , Ernest Marsden	1909
<i>Atomic model</i>	Ernest Rutherford	1911
<i>Thermionic emission</i>	Owen W. Richardson	1911
<i>Electron charge</i>	Robert Millikan	1911
<i>Model of hydrogen atom</i>	Neils Bohr	1913
<i>Tungsten filament for x-ray tubes</i>	William D. Coolidge	1913
<i>Energy quantization</i>	James Franck , Gustav Hertz	1914
<i>Proton</i>	Ernest Rutherford	1919
<i>x-ray scattering (Compton effect)</i>	Arthur H. Compton	1922
<i>Exclusion principle</i>	Wolfgang Pauli	1925
<i>Quantum wave mechanics</i>	Erwin Schrödinger	1926
<i>Wave nature of the electron</i>	Clinton J. Davisson , Lester H. Germer	1927
<i>Cyclotron</i>	Ernest O. Lawrence	1931
<i>Neutron</i>	James Chadwick	1932
<i>Positron</i>	Carl D. Anderson	1932
<i>Artificial radioactivity</i>	Irène Joliot-Curie , Frédéric Joliot	1934
<i>Čerenkov radiation</i>	Pavel A. Čerenkov , Sergei I. Vavilov	1934
<i>Uranium fission</i>	Meitner , Frisch , Hahn , Strassmann	1939
<i>Betatron</i>	Donald W. Kerst	1940
<i>Spontaneous fission</i>	Georgij N. Flerov , Konstantin A. Petržak	1940
<i>Nuclear magnetic resonance</i>	Felix Bloch , Edward Purcell	1946
<i>Cobalt-60 machine</i>	Harold E. Johns	1951
<i>Recoil-less nuclear transition</i>	Rudolf L. Mössbauer	1957
<i>Gamma Knife</i>	Lars Leksell	1968
<i>Computerized tomography (CT)</i>	Godfrey N. Hounsfield , Alan M. Cormack	1971
<i>Magnetic resonance imaging (MRI)</i>	Paul C. Lauterbur , Peter Mansfield	1973
<i>Positron emission tomography (PET)</i>	Michael Phelps	1973

1.4 Physical Quantities and Units

1.4.1 Rules Governing Physical Quantities and Units

A physical quantity is defined as quantity that can be used in mathematical equations of science and technology. It is characterized by its numerical value (magnitude) and associated unit. The following rules apply in general:

- Symbols for physical quantities are set in *italic (sloping)* type, while symbols for units are set in roman (upright) type.

For example: $m = 21$ kg; $E = 15$ MeV; $K = 180$ cGy.

- The numerical value and the unit of a physical quantity must be separated by space.

For example: 21 kg, *not* 21kg; 15 MeV, *not* 15MeV.

- Superscripts and subscripts used with physical quantities are in italic type if they represent variables, quantities, or running numbers; they are in roman type if they are descriptive.

For example: N_X (exposure calibration coefficient with X a quantity), *not* N_X ; $\sum_{i=0}^n X_i$, where i and n represent running numbers, *not* $\sum_{i=0}^n X_i$; ${}_a\mu_{tr}$ where a and tr are descriptive subscripts, *not* ${}_a\mu_{tr}$; U_{max} *not* U_{max} .

- A space or half-high dot is used to signify multiplication of units.

For example: 15 m/s or 15 m·s⁻¹ or 15 m s⁻¹, *not* 15 ms⁻¹.

- It must be clear to which unit a numerical value belongs and which mathematical operation applies to the specific quantity.

For example:

10 cm × 15 cm, *not* 10 × 15 cm;

1 MeV to 10 MeV or (1 to 10) MeV, *not* 1 MeV–10 MeV and *not* 1 to 10 MeV;

100 cGy ± 2 cGy or (100 ± 2) cGy, *not* 100 ± 2 cGy;

80% ± 10% or (80 ± 10) %, *not* 80 ± 10%;

210 × (1 ± 10%) cGy, *not* 210 cGy ± 10%.

1.4.2 The SI System of Units

The currently used metric system of units is known as the *Système International d'Unités* (International System of Units) with the international abbreviation SI. The system is founded on base units for *seven basic physical quantities*. All other physical quantities and units are derived from the seven base quantities and units. The system also defines standard prefixes to the unit names and symbols to form decimal

multiples of fundamental and derived units with special names. Currently 20 agreed upon prefixes are in use.

The seven base quantities and their units are:

<i>Length</i> ℓ	meter (m)
<i>Mass</i> m	kilogram (kg)
<i>Time</i> t	second (s)
<i>Electric current</i> I	ampere (A)
<i>Temperature</i> T	kelvin (K)
<i>Amount of substance</i>	mole (mol)
<i>Luminous intensity</i>	candela (cd)

Examples of basic and derived physical quantities and their units are given in Table 1.1. The Système International obtains its international authority from the Meter Convention that was endorsed in 1875 by 17 countries; the current membership stands at 48 countries.

While six of the seven basic physical quantities and their units seem straightforward, the quantity “amount of substance” and its unit the mole (mol) cause many conceptual difficulties for students. The SI definition of the mole is as follows: “*One mole is the amount of substance of a system which contains as many elementary entities as there are atoms (unbound, at rest, and in ground state) in 0.012 kg of carbon-12*”.

Table 1.1 The main basic physical quantities and several derived physical quantities with their units in Système International (SI) and in radiation physics

Physical quantity	Symbol	Unit in SI	Units commonly used in physics	Conversion
Length	ℓ	m	nm, Å, fm	$1 \text{ m} = 10^9 \text{ nm} = 10^{10} \text{ Å} = 10^{15} \text{ fm}$
Mass	m	kg	MeV/ c^2	$1 \text{ MeV}/c^2 = 1.778 \times 10^{-30} \text{ kg}$
Time	t	s	ms, μs , ns, ps	$1 \text{ s} = 10^3 \text{ ms} = 10^6 \mu\text{s} = 10^9 \text{ ns}$ $= 10^{12} \text{ ps}$
Current	I	A	mA, μA , nA, pA	$1 \text{ A} = 10^3 \text{ mA} = 10^6 \mu\text{A} = 10^9 \text{ nA}$
Temperature	T	K		$T (\text{in K}) = T (\text{in } ^\circ\text{C}) + 273.16$
Mass density	ρ	kg/m ³	g/cm ³	$1 \text{ kg}/\text{m}^3 = 10^{-3} \text{ g}/\text{cm}^3$
Current density	j	A/m ²		
Velocity	v	m/s		
Acceleration	a	m/s ²		
Frequency	ν	Hz		$1 \text{ Hz} = 1 \text{ s}^{-1}$
Electric charge	q	C	e	$1 \text{ e} = 1.602 \times 10^{-19} \text{ C}$
Force	F	N		$1 \text{ N} = 1 \text{ kg}\cdot\text{m}\cdot\text{s}^{-2}$
Pressure	p	Pa	760 torr = 101.3 kPa	$1 \text{ Pa} = 1 \text{ N}\cdot\text{m}^{-2} = 7.5 \times 10^{-3} \text{ torr}$
Momentum	p	N·s		$1 \text{ N}\cdot\text{s} = 1 \text{ kg}\cdot\text{m}\cdot\text{s}^{-1}$
Energy	E	J	eV, keV, MeV	$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} = 10^{-3} \text{ keV}$
Power	P	W		$1 \text{ W} = 1 \text{ J}/\text{s} = 1 \text{ V}\cdot\text{A}$

Table 1.2 Non-SI unit of quantities of importance to general physics as well as medical physics

Physical quantity	Unit	Symbol	Value in SI units
Time	Day	d	1 d = 24 h = 86400 s
	Hour	h	1 h = 60 min = 3600 s
	Minute	min	1 min = 60 s
Angle	Degree	°	1° = ($\pi/180$) rad
Energy	Electron volt ^a	eV	1 eV = 1.6×10^{-19} J
Mass	Unified atomic mass unit ^b	u	1 u = 931.5 MeV/c ²

^aThe electron volt (eV) is defined as the kinetic energy acquired by an electron with charge $e = 1.602 \times 10^{-19}$ C passing through a potential difference of 1 V in vacuum

^bThe unified atomic mass unit (u) or dalton (Da) is a unit of atomic or molecular mass. It is equal to 1/12 of the mass of an unbound carbon-12 atom, at rest in its ground state. (for more detail see Sect. 1.13)

The following additional features should be noted:

- The term “elementary entity” is defined as atom, molecule, ion, electron, or some other particle and represents the smallest component of a substance which cannot be broken down further without altering the nature of the substance.
- When referring to “mole of a substance” it is important to specify the elementary entity under consideration, such as atom, molecule, ion, other particle, etc.
- The number of elementary entities in a mole is by definition the same for all substances and is equal to a universal constant called the Avogadro constant N_A with unit mol^{-1} , as discussed in Sect. 1.13.2.

1.4.3 Non-SI Units

Certain units are not part of the SI system of units but, despite being outside the SI, are important and widely used with the SI system. Some of the important non-SI units are listed in Table 1.2.

1.5 Classification of Forces in Nature

Four distinct forces are observed in the interaction between various types of particles. These forces, in decreasing order of strength, are the *strong force*, *electromagnetic (EM) force*, *weak force*, and *gravitational force* with relative strengths of 1, 1/137, 10^{-6} , and 10^{-39} , respectively. The four fundamental forces, their source, and their transmitted particle are listed in Table 1.3. As far as the range of the four fundamental forces is concerned, the forces are divided into two groups: two forces are infinite range force and two are very short-range force:

Table 1.3 The four fundamental forces in nature, their source, their transmitted particle, and their relative strength normalized to 1 for the strong force

Force	Source	Transmitted particle	Relative strength
Strong	Strong charge	Gluon	1
EM	Electric charge	Photon	1/137
Weak	Weak charge	W^+ , W^- , and Z^0	10^{-6}
Gravitational	Energy	Graviton	10^{-39}

1. The range of the EM and gravitational force is infinite ($1/r^2$ dependence where r is the separation between two interacting particles).
2. The range of the strong and weak force is extremely short (of the order of a few femtometers).

Each force results from a particular intrinsic property of the particles, such as strong charge for the strong force, electric charge for the EM force, weak charge for the weak force, and energy for the gravitational force:

- *Strong charge* enables the strong force transmitted by mass-less particles called gluons and resides in particles referred to as quarks.
- *Electric charge* enables the electromagnetic force transmitted by photons and resides in charged particles such as electrons, positrons, protons, etc.
- *Weak charge* enables the weak force transmitted by particles called W and Z^0 , and resides in particles called quarks and leptons.
- *Energy* enables the gravitational force transmitted by a hypothetical particle called graviton.

1.6 Classification of Fundamental Particles

Two classes of fundamental particles are known: *hadrons* and *leptons*.

1. *Hadrons* are particles that exhibit strong interactions. They are composed of quarks with a fractional electric charge ($\frac{2}{3}$ or $-\frac{1}{3}$) and characterized by one of three types of strong charge called color (red, blue, green). There are six known quarks: up, down, strange, charm, top, and bottom. Two classes of hadrons are known: mesons and baryons.
 - Mesons are unstable subatomic particles consisting of a quark and antiquark bound together by strong force. They have rest masses that fall between the rest mass of the electron ($m_e = 0.511 \text{ MeV}/c^2$) and the rest mass of the proton ($m_p = 938.3 \text{ MeV}/c^2$).
 - Baryons have a rest mass equal to or greater than the proton rest mass. Proton and neutron as well as many more exotic heavy particles belong to the baryon

group. All baryons with the exception of proton are unstable and decay into products that include a proton as the end product.

- 2. *Leptons* are particles that do not interact strongly. Electron e , muon μ , tau τ and their corresponding neutrinos ν_e , ν_μ , and ν_τ are in this category.

1.7 Classification of Radiation

Radiation is classified into two main categories, as shown in Fig. 1.1: *non-ionizing* and *ionizing*, depending on its ability to ionize matter. The ionization energy (IE), also known as ionization potential (IP), of atoms is defined as the minimum energy required for ionizing an atom and is typically specified in electron volts (eV). In nature IE ranges from a few electron volts (~ 4 eV) for alkali elements to 24.6 eV for helium (noble gas) with IE for all other atoms lying between the two extremes.

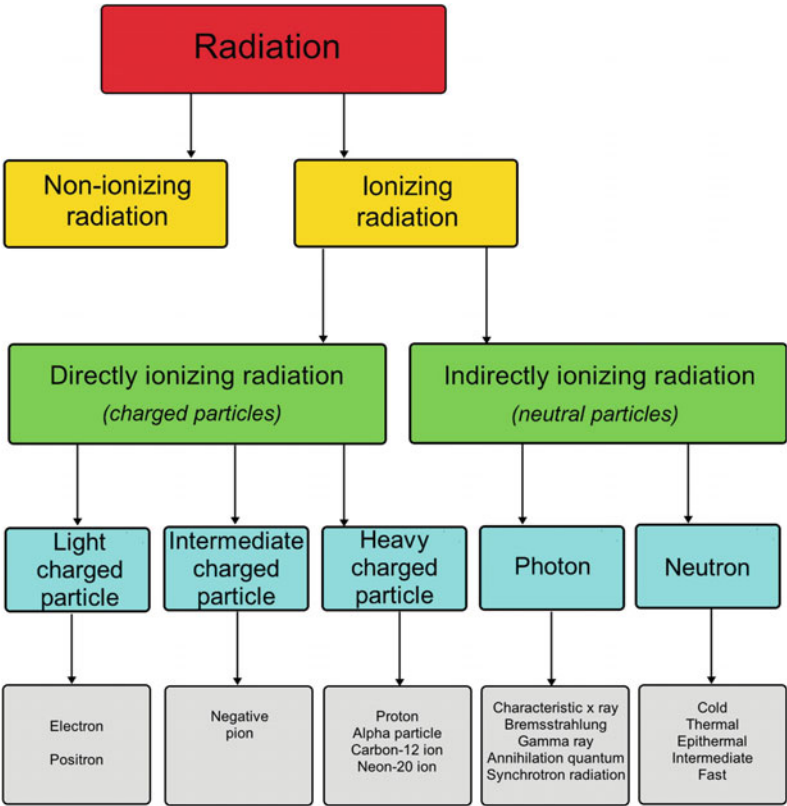


Fig. 1.1 Classification of radiation

- *Non-ionizing radiation* cannot ionize matter because its energy is lower than the ionization energy of atoms or molecules of the absorber. The term non-ionizing radiation thus refers to all types of electromagnetic radiation that do not carry enough energy per quantum to ionize atoms or molecules of the absorber. Near ultraviolet radiation, visible light, infrared photons, microwaves, and radio waves are examples of non-ionizing radiation.
- *Ionizing radiation* can ionize matter either directly or indirectly because its quantum energy exceeds the ionization potential of atoms and molecules of the absorber. Ionizing radiation has many practical uses (see Sect. 1.8.3) in industry, agriculture, and medicine but it also presents a health hazard when used carelessly or improperly. Medical physics is largely, but not exclusively, based on the study and use of ionizing radiation in medicine; health physics deals with health hazards posed by ionizing radiation and with safety issues related to use of ionizing radiation.

1.8 Classification of Ionizing Radiation

Ionizing radiation is classified into two distinct categories according to the mode of ionization and also into two categories according to the density of ionization it produces in the absorber.

1.8.1 Directly and Indirectly Ionizing Radiation

According to the mode of ionization there are two categories of ionizing radiation, directly ionizing and indirectly ionizing:

1. *Directly ionizing radiation*: Comprises charged particles (electrons, protons, α -particles, heavy ions) that deposit energy in the absorber through a direct one-step process involving Coulomb interactions between the directly ionizing charged particle and orbital electrons of the atoms in the absorber.
2. *Indirectly ionizing radiation*: Comprises neutral particles (photons such as x-rays and γ -rays, neutrons) that deposit energy in the absorber through a two-step process as follows:
 - In the first step a charged particle is released in the absorber (photons release either electrons or electron/positron pairs, neutrons release protons or heavier ions).
 - In the second step, the released charged particles deposit energy to the absorber through direct Coulomb interactions with orbital electrons of the atoms in the absorber.

Both directly and indirectly ionizing radiations are used in diagnosis and treatment of disease. The branch of medicine that uses ionizing radiation in treatment of disease is called radiotherapy, therapeutic radiology or radiation oncology. The branch of medicine that uses ionizing radiation in diagnosis of disease is called medical imaging and is usually divided into two categories: diagnostic radiology based on use of x-rays for imaging and nuclear medicine now often referred to as molecular imaging based on use of radionuclides for imaging.

1.8.2 Low LET and High LET Radiation

The ionization density produced by ionizing radiation in tissue depends on the linear energy transfer (LET) of the ionizing radiation beam. The LET is defined as the mean amount of energy that a given ionizing radiation imparts to absorbing medium (such as tissue) per unit path length and is used in radiobiology and radiation protection to specify the quality of an ionizing radiation beam. According to the density of ionization produced in the absorber there are two distinct categories of ionizing radiation:

1. Low LET (also referred to as *sparsely ionizing*) radiation.
2. High LET (also referred to as *densely ionizing*) radiation.

In contrast to stopping power (see Chap. 6) which focuses attention on the energy loss by an energetic charged particle moving through an absorber, the LET focuses on the linear rate of energy absorption by the absorbing medium as the charged particle traverses the absorber. The LET is measured in $\text{keV}/\mu\text{m}$ with $10 \text{ keV}/\mu\text{m}$ separating the low LET (sparsely ionizing) radiation from the high LET (densely ionizing) radiation. Table 1.4 gives a list of various low LET and high LET ionizing radiation beams and the LETs they produce in tissue.

Table 1.4 LET values for various low LET radiation beams (left hand side of table) and high LET radiation beams (right hand side of table)

Low LET radiation	LET ($\text{keV}/\mu\text{m}$)	High LET	LET ($\text{keV}/\mu\text{m}$)
x-rays: 250 kVp	2	Electrons: 1 keV	12.3
γ -rays: Co-60	0.3	Neutrons: 14 MeV	12
x-rays: 3 MeV	0.3	Protons: 2 MeV	17
Electrons: 10 keV	2.3	Carbon ions: 100 MeV	160
Electrons: 1 MeV	0.25	Heavy ions	100–2000

LET of $10 \text{ keV}/\mu\text{m}$ separates low LET radiation from high LET radiation

1.8.3 Use of Ionizing Radiation

The study and use of ionizing radiation started with the discovery of x-rays by Wilhelm Röntgen in 1895 and the discovery of natural radioactivity by Henri Becquerel in 1896. Since then, ionizing radiation played an important role in atomic and nuclear physics where it ushered in the era of modern physics and in many diverse areas of human endeavor, such as medicine, industry, power generation, weapon production, waste management, and security services. Concurrently with the development of new practical uses of ionizing radiation, it became apparent that ionizing radiation can cause somatic and genetic damage to biologic material including human tissue. For safe use of ionizing radiation it is thus imperative that the users have not only a clear understanding of the underlying physics but also of the biological hazards posed by ionizing radiation.

Ionizing radiation is used in the following areas:

1. In *medicine* where it is used for: (i) imaging in diagnostic radiology and nuclear medicine; (ii) treatment of cancer in radiotherapy; (iii) blood irradiation to prevent transfusion-associated graft versus host disease; and (iv) sterilization of single use medical devices.
2. In *nuclear reactors* where it is used for: (i) basic nuclear physics research; (ii) production of radionuclides used in medicine and industry; and (iii) electric power generation.
3. In *industrial radiography* where it is used for nondestructive inspection of welds in airplane manufacturing as well as inspection of welds in gas and oil pipelines.
4. In *well logging* where it is used to obtain information about the geologic media and recoverable hydrocarbon zones through which a borehole has been drilled.
5. In *insect pest control* where insects made sterile by a high radiation dose are released into the wild to control and eradicate insect pests.
6. In *security services* where it is used for screening of cargo and luggage as well as for mail sanitation mainly against the anthrax bacterium.
7. In *food production* where it is used for irradiation of foods such as meat, poultry, fish, spices, fresh fruit, vegetables, and grains to: (i) kill bacteria, viruses, parasites, and mold; (ii) slow the ripening process; (iii) prevent sprouting; and (iv) extend shelf life.
8. In *waste management* where hospital waste and domestic sewage sludge are irradiated with the objective to kill pathogenic microorganisms and disease-causing bacteria before release into the environment.
9. In *chemical industry* where radiation processing produces a chemical modification of industrial materials such as polymers (polyethylene) and crude rubber used in vulcanized tires.
10. In *production of weapons* based on fission and fusion for military purpose.

1.9 Classification of Directly Ionizing Radiation

Most directly ionizing radiations have been found suitable for use in external beam radiotherapy; however, their usage varies significantly from one particle to another, as result of physical and economic considerations. Generally, with regard to radiotherapy, directly ionizing radiations are divided into two categories: (i) electron therapy with megavoltage electron beams and (ii) hadron therapy with hadron beams.

- Electrons interact with absorber atoms mainly through Coulomb interactions with atomic orbital electrons experiencing collision (ionization) loss and with atomic nuclei experiencing radiation loss sometimes also referred to as bremsstrahlung loss.
- Hadrons, with the exception of neutrons that fall into the indirectly ionizing radiation category, interact with absorber atoms through Coulomb interactions with atomic orbital electrons experiencing collision loss as well as through strong interactions with atomic nuclei (referred to as nuclear reactions).

Electrons have been used in routine radiotherapy for treatment of superficial lesions for the past 50 years, while proton beams, the most common hadron beams used in external beam radiotherapy, have only recently been used on a wider, albeit still limited, scale. Heavier hadrons, such as carbon-12, are still considered an experimental modality available in only a few institutions around the world.

Electron beams are produced relatively inexpensively in clinical linear accelerators (linacs). Proton beams, on the other hand, are produced in a cyclotron or synchrotron and these machines are significantly more sophisticated and expensive in comparison with linacs. Much work is currently being done on alternative means for proton beam generation with the goal to design compact machines that will fit into a treatment bunker similarly to the current experience with clinical linacs. Laser-based proton generating methods currently hold most promise for eventual use in compact, inexpensive, and practical proton machines.

1.9.1 Electrons

Electrons play an important role in medical physics and, because of their relatively small mass are considered light charged particles. *Joseph J. Thomson* discovered electrons in 1897 while studying the electric discharge in a partially evacuated Crookes tube (see Sect. 14.4.1). They are used directly as beams for cancer therapy, are responsible for the dose deposition in media by photon and electron beams, and they govern the experimental and theoretical aspects of radiation dosimetry. With regard to their mode of production, electrons fall into the following categories:

- Electrons released in medium by photoelectric effect are referred to as *photoelectrons*.

- Electrons released in medium by Compton effect are referred to as *Compton* or recoil *electrons*.
- Electrons produced in medium by pair production interactions in the field of the nucleus or in the field of an orbital electron are referred to as *pair production electrons*.
- Electrons emitted from nuclei by β^- decay are referred to as *beta particles* or *beta rays*.
- Electrons produced by linear accelerators (linacs), betatrons or microtrons for use in radiotherapy with kinetic energies typically in the range from 4 MeV to 30 MeV are referred to as *megavoltage electrons*.
- Electrons produced through Auger effect are referred to as *Auger electrons*, *Coster–Kronig electrons*, or *super Coster–Kronig electrons*.
- Electrons produced through internal conversion are referred to as *internal conversion electrons*.
- Electrons produced by charged particle collisions are of interest in radiation dosimetry and are referred to as *delta* (δ) *rays*.
- Electrons released from metallic surface in thermionic emission are referred to as *thermions*.

1.9.2 Positrons

The positron or antielectron is the antiparticle of an electron with same mass $0.511 \text{ MeV}/c^2$ and spin $(1/2)$ as the electron and charge $(1.602 \times 10^{-19} \text{ C})$ equal in magnitude but opposite in sign to that of the electron. In 1928 *Paul Dirac* was the first to postulate positron's existence and in 1932 *Carl D. Anderson* discovered it as the first evidence of antimatter in his experimental study of cosmic rays. There are three ways for generating positrons: (1) positron emission beta decay, (2) nuclear pair production, and (3) triplet production:

1. Positrons emitted from nuclei by β^+ radioactive decay are used in positron emission tomography (PET) and referred to as *beta particles* or *beta rays*.
2. Positrons produced by nuclear pair production and triplet production are referred to as *pair production positrons* and play an important role in interactions of high-energy photons with absorbing medium.

1.9.3 Heavy Charged Particles

For use in radiotherapy heavy charged particles are defined as particles such as proton and heavier ions with mass exceeding the electron mass. They are produced through acceleration of nuclei or ions in cyclotrons, synchrotrons or heavy particle linacs.

Heavy charged particles of importance in nuclear physics and also potentially useful in medicine for treatment of disease are:

- *Proton* is the nucleus of a hydrogen-1 (${}^1_1\text{H}$) atom. The hydrogen-1 atom, a stable isotope of hydrogen with a natural abundance of 99.985%, is called protium or light hydrogen and consists of a nucleus (proton) and one electron.
- *Deuteron*, the nucleus of a hydrogen-2 (${}^2_1\text{H}$) atom, consists of one proton and one neutron bound together with a total binding energy of 2.225 MeV or 1.1125 MeV/nucleon. The hydrogen-2 atom, a stable isotope of hydrogen with a natural abundance of 0.015%, is called deuterium or heavy hydrogen and consists of a nucleus (deuteron) and one electron.
- *Triton*, the nucleus of a hydrogen-3 (${}^3_1\text{H}$) atom, consists of one proton and two neutrons bound together with a total binding energy of 8.48 MeV or 2.83 MeV/nucleon. The hydrogen-3 atom, a radioactive isotope of hydrogen with a half-life $t_{1/2}$ of 12.32 years, is called tritium and consists of a nucleus (triton) and one electron.
- *Helion*, the nucleus of a helium-3 (${}^3_2\text{He}$) atom, consists of two protons and one neutron bound together with a total binding energy of 7.72 MeV or 2.57 MeV/nucleon. The helium-3 atom, a stable isotope of helium with a natural abundance of $\sim 0.00014\%$, consists of a nucleus (helion) and two electrons.
- *Alpha particle*, the nucleus of a helium-4 (${}^4_2\text{He}$) atom, consists of two protons and two neutrons bound together with a total binding energy of 28.3 MeV or 7.075 MeV/nucleon. The helium-4 atom, a stable isotope of helium with a natural abundance of $\sim 99.99986\%$, consists of a nucleus (alpha particle) and two electrons.

The basic atomic and nuclear properties of heavy charged particles and atoms listed above are summarized in Appendix A and in Table 1.5. Appendix A also lists the atomic mass, nuclear rest energy, nuclear total binding energy, and binding energy per nucleon.

Table 1.5 Basic properties of heavy charged particles used in nuclear physics and medicine

ATOM				NUCLEUS			
Designation	Symbol	Natural abundance (%)	Name	Name	Protons	Neutrons	Nuclear stability
Hydrogen-1	${}^1_1\text{H}$	99.985	Protium	Proton	1	0	Stable
Hydrogen-2	${}^2_1\text{H}$	0.015	Deuterium	Deuteron	1	1	Stable
Hydrogen-3	${}^3_1\text{H}$	–	Tritium	Triton	1	2	Radioactive
Helium-3	${}^3_2\text{He}$	0.00014	Helium-3	Helion	2	1	Stable
Helium-4	${}^4_2\text{He}$	99.99986	Helium-4	Alpha particle	2	2	Stable

For the nuclear structures, the table lists the special name as well as the number of protons and neutrons; for associated atomic structures the table lists the symbol, natural abundance and the special name

As they penetrate into an absorber, energetic heavy charged particles lose energy through Coulomb interactions with orbital electrons of the absorber. Just before the heavy charged particle has expended all of its kinetic energy, its energy loss per unit distance traveled increases drastically and this results in a high dose deposition at that depth in the absorber. This high dose region appears close to the particle's range in the absorber and is referred to as the *Bragg peak*. The depth of the Bragg peak in tissue depends on the mass and incident energy of the charged particle (see Sect. 1.12.4).

In contrast to heavy charged particles listed above, heavier charged particles are nuclei or ions of heavier atoms such as carbon-12 ($^{12}_6\text{C}$), nitrogen-14 ($^{14}_7\text{N}$), or neon-20 ($^{20}_{10}\text{Ne}$). They are generated with cyclotrons and synchrotrons for general use in nuclear and high-energy physics but are also used for radiotherapy in a few highly specialized institutions around the world. They offer some advantages over charged particle radiotherapy with proton beams; however, equipment for their production is very expensive to build and operate, and advantages of their use for general radiotherapy are still not clearly established.

1.9.4 Pions

Pions π also called π mesons belong to a group of short-lived subatomic particles called mesons. They are either neutral (π^0) or come with positive (π^+) or negative (π^-) electron charge and their rest mass is about $273m_e$ with $m_e = 0.511 \text{ MeV}/c^2$ the rest mass of the electron. Pions do not exist in free state in nature; they reside inside the nuclei of atoms and, based on their mass, were identified as the quanta of the strong interaction. They can be ejected from the nucleus in nuclear reactions by bombarding target nuclei with energetic electrons or protons.

Of the three pion types, the negative π mesons (π^-) have been used for radiotherapy, since, by virtue of their negative charge, they produce the so-called “pion stars” in irradiated nuclei. As negative pions penetrate an absorber, they lose energy, similarly to heavy charged particles, through Coulomb interactions with orbital electrons of the absorber. However, close to their range in the absorber, they not only exhibit a Bragg peak, they are also drawn into a nucleus of the absorber. This nuclear penetration makes the absorbing nucleus unstable and causes it to break up violently into smaller energetic fragments. These fragments fly apart and deposit a significant amount of energy within a short distance from the point of the nuclear reaction. The effect is called a “pion star” and it accentuates the normal Bragg peak dose distribution. In the past pions showed great promise for use in radiotherapy; however, during recent years the studies of pions were largely abandoned in favor of heavy charged particles such as protons.

1.10 Classification of Indirectly Ionizing Photon Radiation

Indirectly ionizing photon radiation consists of three categories of photon: ultraviolet (UV), x-ray, and γ -ray. While UV photons are of some limited use in medicine, imaging and treatment of disease are carried out with photons of higher energy such as x-rays and γ -rays. With regard to their origin, these photons fall into five categories, all of them discussed in detail in subsequent chapters of this book. These five photon categories are:

- *Gamma rays*: photons resulting from nuclear shell transitions (see Sect. 11.7).
- *Annihilation quanta*: photons resulting from positron–electron annihilation (see Sect. 7.6.10).
- *Characteristic (fluorescence) x-rays*: photons resulting from electron transitions between atomic shells (see Sect. 4.1).
- *Bremsstrahlung x-rays*: photons resulting from Coulomb interaction between energetic electrons and positrons with atomic nuclei of absorber (see Sect. 4.2).
- *Synchrotron radiation* (also known as *magnetic bremsstrahlung* and *cyclotron radiation*): photons resulting from charged particles (electrons, positrons, protons, etc.) moving through a magnetic field (e.g., in storage rings, see Sect. 4.3).

1.11 Radiation Quantities and Units

Accurate measurement of radiation is very important in any medical use of radiation, be it for diagnosis or treatment of disease. In diagnosis one must optimize the image quality so as to obtain the best possible image quality with the lowest possible radiation dose to the patient to minimize the risk of radiation induced morbidity. In radiotherapy the prescribed dose must be delivered accurately and precisely to maximize the tumor control probability (TCP) and to minimize the normal tissue complication probability (NTCP). In both instances the risk of morbidity includes acute radiation effects (radiation injury) as well as late effects such as induction of cancer and genetic damage.

Several quantities and units were introduced for the purpose of quantifying radiation and the most important of these are listed in Table 1.6. Also listed in Table 1.6 are the definitions for the various quantities and the relationships between the old units and the SI units for these quantities.

- *Exposure X* is related to the ability of photons to ionize air. Its unit roentgen (R) is defined as charge of 2.58×10^{-4} C of either sign produced per kilogram of air.
- *Kerma K* (acronym for kinetic energy released in matter) is defined for indirectly ionizing radiations (photons and neutrons) as energy transferred to charged particles per unit mass of the absorber.
- *Dose D* is defined as energy absorbed per unit mass of absorbing medium. Its SI unit gray (Gy) is defined as 1 J of energy absorbed per kilogram of absorbing medium.

Table 1.6 Radiation quantities, radiation units, and conversion between old and SI units

Quantity	Definition	SI unit	Old unit	Conversion
Exposure X	$X = \frac{\Delta Q}{\Delta m_{\text{air}}}$	$2.58 \times \frac{10^{-4} \text{ C}}{\text{kg air}}$	$1 \text{ R} = \frac{1 \text{ esu}}{\text{cm}^3 \text{ air}_{\text{STP}}}$	$1 \text{ R} = 2.58 \times \frac{10^{-4} \text{ C}}{\text{kg air}}$
Kerma K	$K = \frac{\Delta E_{\text{tr}}}{\Delta m}$	$1 \text{ Gy} = 1 \frac{\text{J}}{\text{kg}}$	–	–
Dose D	$D = \frac{\Delta E_{\text{ab}}}{\Delta m}$	$1 \text{ Gy} = 1 \frac{\text{J}}{\text{kg}}$	$1 \text{ rad} = 100 \frac{\text{erg}}{\text{g}}$	$1 \text{ Gy} = 100 \text{ rad}$
Equiv. dose H	$H = Dw_{\text{R}}$	1 Sv	1 rem	$1 \text{ Sv} = 100 \text{ rem}$
Activity \mathcal{A}	$\mathcal{A} = \lambda N$	$1 \text{ Bq} = 1 \text{ s}^{-1}$	$1 \text{ Ci} = 3.7 \times 10^{10} \text{ s}^{-1}$	$1 \text{ Bq} = \frac{1 \text{ Ci}}{3.7 \times 10^{10}}$

where

- ΔQ is the charge of either sign collected.
- Δm_{air} is the mass of air.
- ΔE_{tr} is energy transferred from indirectly ionizing particles to charged particles in absorber.
- ΔE_{ab} is absorbed energy.
- Δm is the mass of medium.
- w_{R} is the radiation weighting factor.
- λ is the decay constant.
- N is the number of radioactive atoms.
- R stands for roentgen.
- Gy stands for gray.
- Sv stands for sievert.
- Bq stands for becquerel.
- Ci stands for curie.
- STP stands for standard temperature (273.2 K) and standard pressure (101.3 kPa)

- *Equivalent dose H* is defined as the dose multiplied by a radiation-weighting factor w_{R} . The SI unit of equivalent dose is sievert (Sv).
- *Activity \mathcal{A}* of a radioactive substance is defined as the number of nuclear decays per time. Its SI unit is becquerel (Bq) corresponding to one decay per second.

1.12 Dose Distribution in Water for Various Radiation Beams

Dose deposition in water is one of the most important characteristics of the interaction of radiation beams with matter. This is true in general radiation physics and even more so in medical physics, where the dose deposition properties in tissue govern both the diagnosis of disease with radiation (*imaging physics*) as well as treatment of disease with radiation (*radiotherapy physics*).

Imaging with ionizing radiation is limited to the use of x-ray beams in *diagnostic radiology* and γ -ray beams in *nuclear medicine*, while in *radiotherapy* the use of radiation is broader and covers essentially all ionizing radiation types ranging from x-rays and γ -rays through electrons to neutrons, protons and heavier charged particles.

In diagnostic radiology imaging one is interested in the radiation beam that propagates and is transmitted through the patient, while in nuclear medicine imaging (now referred to as molecular imaging) one is interested in the radiation beam that emanates from the patient. In radiotherapy, on the other hand, one is interested in the energy deposited in the patient by a radiation source that is located either outside of the patient (*external beam radiotherapy*) or inside the tumor (*brachytherapy*).

When considering the dose deposition in tissue by radiation beams, four beam categories are usually defined: two categories (*photons* and *neutrons*) for indirectly ionizing radiation and two categories (*electrons* and *heavy charged particles*) for directly ionizing radiation. Typical dose distributions in water for the four categories are displayed in Fig. 1.2, normalized to 100% at the dose maximum (percentage depth doses) for various radiation types and energies: for indirectly ionizing radiation in (a) for *photons* and in (b) for *neutrons* and for directly ionizing radiation in (c) for

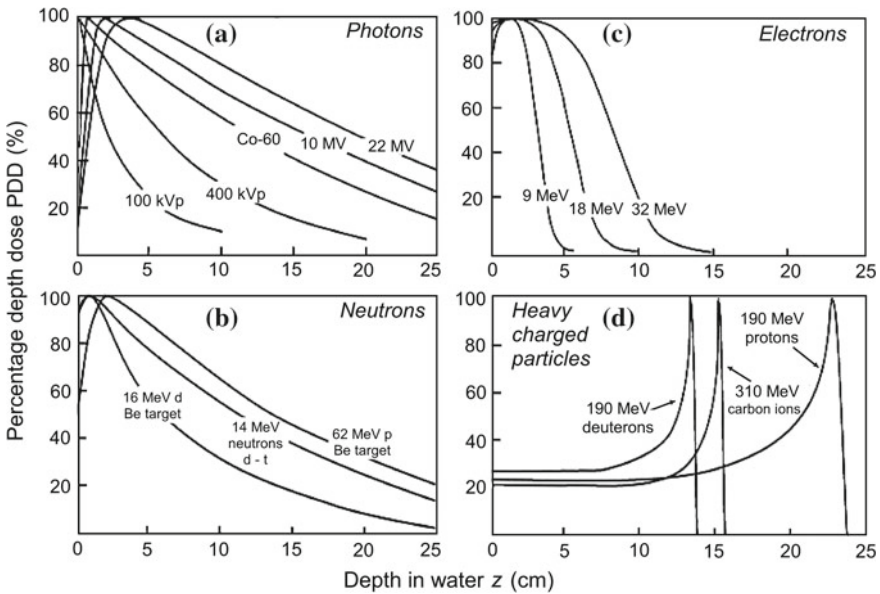


Fig. 1.2 Percentage depth dose against depth in water for radiation beams of various types and energies. Parts **a** and **b** are for *indirectly ionizing radiation*: in **(a)** for photon beams in the range from 100 kVp to 22 MV and in **(b)** for various neutron beams. Parts **c** and **d** are for *directly ionizing radiation*: in **(c)** for electron beams in the range from 9 MeV to 32 MeV and in **(d)** for heavy charged particle beams (190 MeV protons, 190 MeV deuterons, and 310 MeV carbon ions)

electrons and in (d) for *heavy charged particles* (protons, deuterons, and carbon-12 ions). It is evident that the depth dose characteristics of radiation beams depend strongly upon beam type and energy. However, they also depend in a complex fashion on other beam parameters, such as field size, source–patient distance, etc. In general, indirectly ionizing radiation exhibits exponential-like attenuation in absorbing media, while directly ionizing radiation exhibits a well-defined range in absorbing media.

Of the four beam categories of Fig. 1.2, photon beams in the indirectly ionizing radiation category and electron beams in the directly ionizing radiation category are considered conventional beams, well understood and readily available for radiotherapy in all major medical institutions around the world. On the other hand, neutron beams in the indirectly ionizing radiation category and heavy ions including protons in the directly ionizing radiation category remain in the category of special beams, available in only a limited number of institutions around the world, despite having been in use for the past five decades. These beams offer some advantages in treatment of certain malignant diseases; however, in comparison with conventional beams, they are significantly more complicated to use as well as to maintain and their infrastructure and operating costs are also considerably higher, currently precluding a widespread clinical use.

The special beams (neutrons and protons) provide certain advantages when used in treatment of selected tumor types; however, their choice and prescribed dose must account not only for the physical beam characteristics but also for the biological effects associated with radiation beams: the relative biological effectiveness (RBE) and the oxygen enhancement ratio (OER).

Since the biological effect of a dose of radiation depends on its LET, knowing the LET of a given radiation beam is important when prescribing a tumoricidal dose in radiotherapy. As the LET of radiation increases, the ability of the radiation to produce biological damage also increases. The relative biological effectiveness (RBE) is used for comparison of the dose of test radiation to the dose of standard radiation to produce the same biological effect. The RBE varies not only with the type of radiation but also with the type of cell or tissue, biologic effect under investigation, dose, dose rate and fractionation. In general, the RBE increases with LET to reach a maximum of 3–8 at very high LET of $\sim 200 \text{ keV}/\mu\text{m}$ and then it decreases with further increase in LET.

The presence or absence of molecular oxygen within a cell influences the biological effect of ionizing radiation: the larger is the cell oxygenation above anoxia, the larger is the biological effect of ionizing radiation. The effect is quite dramatic for low LET (sparsely ionizing) radiations, while for high LET (densely ionizing) radiations it is much less pronounced. The oxygen enhancement ratio (OER) is defined as the ratio of doses without and with oxygen (hypoxic vs. well oxygenated cells) to produce the same biological effect. For low LET radiations, such as x-rays, γ -rays and electrons, OER equals about 3, while for high LET radiations such as neutrons it is about 1.5.

1.12.1 Dose Distribution in Water for Photon Beams

A photon beam propagating through air or vacuum is governed by the inverse-square law; a photon beam propagating through a patient, on the other hand, is not only affected by the inverse-square law but also by attenuation and scattering of the photon beam inside the patient. The three effects make the dose deposition in a patient a complicated process and its determination a complex task.

Typical dose distributions for several photon beams in the energy range from 100 kVp to 22 MV are shown in Fig. 1.2a. Several important points and regions of the absorbed dose curves may be identified. The beam enters the patient on the surface where it delivers a certain surface dose D_s . Beneath the surface, the dose first rises rapidly, reaches a maximum value at a depth z_{\max} , and then decreases almost exponentially until it reaches an exit dose value at the patient's exit point. The depth of dose maximum z_{\max} is proportional to the beam energy and amounts to 0 for superficial (50 kVp to 100 kVp) and orthovoltage (100 kVp to 300 kVp) beams; 0.5 cm for cobalt-60 γ -rays; 1.5 cm for 6 MV beams; 2.5 cm for 10 MV beams; and 4 cm for 22 MV beams.

The relatively low surface dose for high-energy photon beams (referred to as the *skin sparing effect*) is of great importance in radiotherapy for treatment of deep-seated lesions without involvement of the skin. The tumor dose can be concentrated at large depths in the patient concurrently with delivering a low dose to patient's skin that is highly sensitive to radiation and must be spared as much as possible when it is not involved in the disease.

The dose region between the surface and the depth of dose maximum z_{\max} is called the dose *build-up region* and represents the region in the patient in which the dose deposition rises with depth as a result of the range of secondary electrons released in tissue by photon interactions with the atoms of tissue. It is these secondary electrons released by photons that deposit energy in tissue (indirect ionization). The larger is the photon energy, the larger are the energy and the range of secondary electrons and, consequently, the larger is the depth of dose maximum.

It is often assumed that at depths of dose maximum and beyond electronic equilibrium is achieved; however, a better term is transient electronic equilibrium because of the effects of photon beam attenuation as the photon beam penetrates into a patient. Electronic equilibrium or, more generally, charged particle equilibrium (CPE) exist for a volume if each charged particle of a given type and energy leaving the volume is replaced by an identical particle of the same type and energy entering the volume.

1.12.2 Dose Distribution in Water for Neutron Beams

Neutron beams belong to the group of indirectly ionizing radiation, but rather than releasing electrons like photons do, they release protons or heavier ions that

subsequently deposit their energy in absorbing medium through Coulomb interactions with electrons and nuclei of the absorber.

As shown in Fig. 1.2b, the dose deposition characteristics in water by neutrons are similar to those of photon beams. Neutron beams exhibit a relatively low surface dose although the skin sparing effect is less pronounced than that for photon beams. They also exhibit a dose maximum beneath the skin surface and an almost exponential decrease in dose beyond the depth of dose maximum. Similarly to photons, the dose build up region depends on neutron beam energy; the larger is the energy, the larger is the depth of dose maximum.

For use in radiotherapy, neutron beams are produced either with cyclotrons or neutron generators (see Sect. 9.6). In a cyclotron, protons or deuterons are accelerated to kinetic energies of 50 MeV to 80 MeV and strike a thick beryllium target to produce fast neutrons that are collimated into a clinical neutron beam. The neutron beams produced in the beryllium target have beam penetration and build up characteristics similar to those of 4 MV to 10 MV megavoltage x-ray beams.

Less common in clinical use are neutron generators (see Sect. 9.6.3) in which deuterons are accelerated to 250 keV and strike a tritium target to produce a 14 MeV neutron beam which exhibits penetration characteristics similar to those obtained for a cobalt-60 teletherapy γ -ray beam.

Producing physical depth dose characteristics that are similar to those produced by megavoltage photon beams, neutrons offer no advantage over photons in this area. However, neutrons are high LET (densely ionizing) particles in comparison with low LET (sparsely ionizing) photon radiation and produce more efficient cell kill per unit dose. The high LET of neutron beams produces $RBE > 1$ which means that, to achieve the same biological effect, a lower neutron dose is required compared to the photon dose.

Moreover, in comparison with photons, neutrons by virtue of their high LET are more efficient in killing hypoxic cells in comparison to well-oxygenated cells. The oxygen enhancement ratio (OER) of neutrons is 1.5 compared to an OER of 3 for photons. Thus, neutrons offer no physical advantage over photons and it is the biological advantage of neutron beams that makes neutrons attractive for use in radiotherapy despite the increased complexity of producing and using them clinically.

1.12.3 Dose Distribution in Water for Electron Beams

Electrons are directly ionizing radiations that deposit their energy in tissue through Coulomb interactions with orbital electrons and nuclei of the absorber atoms. Megavoltage electron beams represent an important treatment modality in modern radiotherapy, often providing a unique option for treatment of superficial tumors that are less than 5 cm deep. Electrons have been used in radiotherapy since the early 1950s, first produced by betatrons and then by linear accelerators. Modern high-energy linear accelerators used in radiotherapy typically provide, in addition to two megavoltage x-ray beam energies, several electron beams with energies from 4 MeV to 25 MeV.

As shown in Fig. 1.2c, the electron beam percentage depth dose curve plotted against depth in patient exhibits a relatively high surface dose (of the order of 80%) and then builds up to a maximum dose at a certain depth referred to as the electron beam depth dose maximum z_{\max} . Beyond z_{\max} the dose drops off rapidly, and levels off at a small low-level dose component referred to as the *bremsstrahlung* tail. The *bremsstrahlung* component of the electron beam is the photon contamination of the beam that results from radiation loss experienced by incident electrons as they penetrate the various machine components, air, and the patient. The higher is the energy of the incident electrons, the higher is the *bremsstrahlung* contamination of the electron beam.

Several parameters are used to describe clinical electron beams, such as the most probable energy on the patient's skin surface; mean electron energy on the patient's skin surface; and the depth at which the absorbed dose falls to 50% of the maximum dose. The depth of dose maximum generally does not depend on beam energy, contrary to the case for photon beams; rather, it is a function of machine design. On the other hand, the beam penetration into tissue clearly depends on incident electron beam energy; the higher is the energy, the more penetrating is the electron beam, as is evident from Fig. 1.2c.

1.12.4 Dose Distribution in Water for Heavy Charged Particle Beams

Heavy charged particle beams fall into the category of directly ionizing radiation depositing their energy in tissue through Coulomb interactions with orbital electrons of the absorber. As they penetrate into tissue, heavy charged particles lose energy but, in contrast to electrons, do not diverge appreciably from their direction of motion and therefore exhibit a distinct range in tissue. This range depends on the incident particle's kinetic energy and mass.

Just before the heavy charged particle reaches its range in the absorbing medium, its energy loss per unit distance traveled increases dramatically and this results in a high dose deposition referred to as Bragg peak (see Sect. 1.9.3). As indicated in Fig. 1.2d, the Bragg peak appears close to particle's range, is very narrow, defines the maximum dose deposited in tissue, and characterizes all heavy charged particle dose distributions.

Because of their large mass compared to the electron mass, heavy charged particles lose their kinetic energy only interacting with orbital electrons of the absorber; they do not lose any appreciable amount of energy through *bremsstrahlung* interactions with absorber nuclei.

1.12.5 Choice of Radiation Beam and Prescribed Target Dose

The choice of radiation beam and dose prescription in treatment of disease with radiation depends on many factors such as:

1. Medical patient-related and physician-related factors:
 - Tumor type and histology.
 - Tumor location in the patient.
 - Location of sensitive structures and healthy tissues in the vicinity of the target.
 - Patient's tolerance of treatment.
 - Any previous radiation treatment.
 - Physician's training and experience.
2. Availability of equipment for diagnostic imaging and dose delivery.
3. Physical parameters of the radiation beam to be used in treatment:
 - Depth dose characteristics, governed by machine design, beam energy, field size, and other machine parameters.
 - Density of ionization produced in tissue by the radiation beam to be used in treatment.
4. Biological factors produced in tissue by the radiation beam:
 - Relative biological effectiveness (RBE).
 - Oxygen enhancement ratio (OER).

Based on machine-related physical factors, *superficial tumors* are treated, depending on their size, with single beams of:

1. Superficial x-rays (50 kVp to 80 kVp) produced in superficial x-ray machines.
2. Orthovoltage x-rays (100 kVp to 300 kVp) produced in orthovoltage x-ray machines.
3. Electrons in the energy range from 4 MeV to 25 MeV produced in clinical linear accelerators.

Deep-seated tumors, on the other hand, are treated with multiple megavoltage beams from cobalt-60 teletherapy machine or linear accelerator in the energy range from 4 MV to 25 MV. Over the past decade there have been significant advances in technology and techniques used to plan and deliver precision radiotherapy. The patient's path through a radiotherapy department involves several steps, some of them not involving the patient directly but all of them important for a favorable treatment outcome. In short, the steps are as follows:

1. Definition of target and collection of patient data with diagnostic imaging techniques.
2. Treatment planning and, if required, fabrication of treatment accessories.
3. Prescription of target dose and dose fractionation.

4. Dose delivery, typically with multiple fractions at one fraction per day for a duration of several weeks.
5. Follow up at regular intervals.

The aim of modern dose delivery is to optimize the tumor control probability (TCP) by delivering as high as possible dose to the target and causing as little as possible morbidity for the patient by keeping the normal tissue complication probability (NTCP) as low as possible. Therefore, accurate knowledge of the tumor (target) location in the body, as well as the target's shape and volume are very important parameters of modern radiotherapy. This knowledge is usually acquired with diagnostic imaging which involves appropriate fusion of patient data collected with two or more imaging modalities.

Traditional treatment planning is carried out by matching radiation fields with target dimensions and subsequently calculating the resulting dose distribution, while modern treatment planning is carried out by prescribing a suitable dose distribution and subsequently calculating the intensity modulated fields required to achieve the prescribed dose distribution. The traditional treatment planning is now referred to as forward planning; planning with beam intensity modulation is called inverse planning and dose delivery using intensity modulated fields is called intensity modulated radiotherapy (IMRT). In principle, the IMRT optimizes the dose delivery to the patient by conforming the prescribed dose distribution to the target volume; however, in this process one assumes that the target location is accurately known and that the position and shape of the target do not change during the treatment (intra-treatment motion) and from one fractionated treatment to another (inter-treatment motion).

To have a better control of the target motion problem recent development in dose delivery technology introduced the so-called image guided radiotherapy (IGRT) which merged imaging and dose delivery into one machine, allowing accurate positioning of the target into the radiation beam. The most recent development is adaptive radiotherapy (ART) which enables target shape and position verification not only before and after treatment but also during the dose delivery process.

1.13 Basic Definitions for Atomic Structure

The constituent particles forming an atom are *protons*, *neutrons* and *electrons*. Protons and neutrons are known as *nucleons* and form the nucleus of the atom. The following definitions apply for atomic structure:

- *Atomic number Z* : number of protons and number of electrons in an atom.
- *Atomic mass number A* : number of nucleons in an atom, i.e., number of protons Z plus number of neutrons N in an atom; i.e., $A = Z + N$.
- *Atomic mass \mathcal{M}* : expressed in unified atomic mass units u , where $1 u$ is equal to one twelfth of the mass of the carbon-12 atom (unbound, at rest, and in ground state) or $931.5 \text{ MeV}/c^2$. The atomic mass \mathcal{M} is smaller than the sum of individual masses of constituent particles because of the intrinsic energy associated with

binding the particles (nucleons) within the nucleus (see Sect. 1.15). On the other hand, the atomic mass \mathcal{M} is larger than the nuclear mass M because the atomic mass \mathcal{M} includes the mass contribution by Z orbital electrons while the nuclear mass M does not. The binding energy of orbital electrons to the nucleus is ignored in the definition of the atomic mass \mathcal{M} .

- While for carbon-12 the atomic mass \mathcal{M} is exactly 12 u, for all other atoms \mathcal{M} in u does not exactly match the atomic mass number A . However, for all atomic entities A (an integer) and \mathcal{M} are very similar to one another and often the same symbol (A) is used for the designation of both.
- Number of atoms N_a per mass of an element is given as

$$\frac{N_a}{m} = \frac{N_A}{A}, \quad (1.11)$$

where N_A is the Avogadro number discussed in Sect. 1.13.2.

- Number of electrons per volume of an element is

$$Z \frac{N_a}{V} = \rho Z \frac{N_a}{m} = \rho Z \frac{N_A}{A}. \quad (1.12)$$

- Number of electrons per mass of an element is

$$Z \frac{N_a}{m} = Z \frac{N_A}{A}. \quad (1.13)$$

Note that $(Z/A) \approx 0.5$ for all elements with one notable exception of hydrogen for which $(Z/A) = 1$. Actually, Z/A slowly decreases from 0.5 for low Z elements to 0.4 for high Z elements. *For example:* Z/A for helium-4 is 0.5, for cobalt-60 it is 0.45, for uranium-235 it is 0.39.

1.13.1 Mean Atomic Mass (Standard Atomic Weight)

Most of the naturally occurring elements are mixtures of several stable isotopes, each isotope with its own relative natural abundance. For a given chemical element one stable isotope usually predominates; however, natural elements generally consist of atoms of same atomic number Z but of various different atomic mass numbers A as a result of different numbers of neutrons N . The *mean atomic mass* $\bar{\mathcal{M}}$ of an element is often referred to as the *standard atomic weight* of an element and is given as the mean atomic mass of all stable isotopes of the element, accounting for the isotopes' natural relative abundance and relative atomic mass.

For example:

- Natural carbon ($Z = 6$) consists of two stable isotopes, carbon-12 with a natural abundance of 98.93% and relative atomic mass of 12.0000 u and carbon-13 with a natural abundance of 1.07% and relative atomic mass of 13.003355 u. The mean atomic mass (standard atomic weight) of carbon $\bar{M}(\text{C})$ is determined as follows

$$\bar{M}(\text{C}) = 0.9893 \times 12.0000 \text{ u} + 0.0107 \times 13.003355 \text{ u} = 12.0107 \text{ u}. \quad (1.14)$$

- Natural iridium ($Z = 77$) consists of two stable isotopes, iridium-191 with a natural abundance of 37.3% and relative atomic mass of 190.960591 u and iridium-193 with a natural abundance of 62.7% and relative atomic mass of 192.962924 u. The mean atomic mass (standard atomic weight) of iridium $\bar{M}(\text{Ir})$ is determined as follows

$$\bar{M}(\text{Ir}) = 0.373 \times 190.960591 \text{ u} + 0.627 \times 192.962924 \text{ u} = 192.216 \text{ u}. \quad (1.15)$$

- Natural iron is a slightly more complicated example containing four stable isotopes with the following relative abundances and relative atomic masses:

Iron-54: 5.845% and 53.9396148 u

Iron-56: 91.754% and 55.9349421 u

Iron-57: 2.119% and 56.9353987 u

Iron-58: 0.282% and 57.9332805 u

After accounting for the relative abundances and atomic masses for the four iron isotopes we get the following mean atomic mass (standard atomic weight) for natural iron: $\bar{M}(\text{Fe}) = 55.845 \text{ u}$.

1.13.2 Atomic Mass Constant and the Mole

Atomic mass constant u or Dalton (Da) previously known as *unified atomic mass unit* (amu) is related to the macroscopic SI base unit of mass, the kilogram (kg), through the Avogadro constant (Avogadro number) N_A , defined as the number of atoms ($6.022141293 \times 10^{23}$) at rest and in their ground state contained in exactly 12 g (12×10^{-3} kg) of carbon-12. Since 12 g of carbon-12 is also defined as a mole (mol) of carbon-12, we can state that $N_A = 6.022141293 \times 10^{23} \text{ mol}^{-1}$.

Since 1 u by definition equals to the mass of 1/12 of the carbon-12 atom and since 12 g of carbon-12 by definition contains Avogadro number ($6.022141293 \times 10^{23}$) of atoms, we conclude that the mass of one carbon-12 atom equals to $(12 \text{ g})/N_A$ and that the relationship between the *atomic mass constant* u and the SI mass unit *kilogram* is

$$1 \text{ u} = \frac{1}{12} \times \frac{12 \text{ g} \cdot \text{mol}^{-1}}{N_A} = \frac{10^{-3} \text{ kg}}{\text{mol}} \frac{\text{mol}}{6.022141293 \times 10^{23}} = 1.660538922 \times 10^{-27} \text{ kg} \quad (1.16)$$

In terms of energy we can express the mass 1 u in MeV/c^2 to get

$$\begin{aligned} 1 \text{ u} &= \frac{1 \text{ u} c^2}{c^2} \\ &= \frac{(1.660538922 \times 10^{-27} \text{ kg}) \times (2.99792458 \times 10^8 \text{ m} \cdot \text{s}^{-1})^2}{c^2 \times 1.602176565 \times 10^{-13}} \times \frac{\text{MeV}}{\text{J}} \\ &= 931.494060 \text{ MeV}/c^2 \approx 931.5 \text{ MeV}/c^2, \end{aligned} \quad (1.17)$$

where we use the following 2010 CODATA constants available from the NIST:

$$\text{Avogadro constant:} \quad N_A = 6.022141293 \times 10^{23} \text{ mol}^{-1} \quad (1.18)$$

$$\text{Atomic mass constant:} \quad \text{u} = 1.660538922 \times 10^{-27} \text{ kg} \quad (1.19)$$

$$\text{Speed of light in vacuum:} \quad c = 2.9979458 \times 10^8 \text{ m} \cdot \text{s}^{-1} \quad (1.20)$$

$$\text{Elementary charge (electron):} \quad e = 1.602176565 \times 10^{-19} \text{ C} \quad (1.21)$$

The mass in grams of a chemical element equal to the mean atomic mass (standard atomic weight) $\bar{M}(\text{X})$ of an element X is referred to as a *mole* of the element and contains exactly Avogadro number of atoms. In general, Avogadro constant N_A is given as the number of entities per mole, where the entity can be atoms per mole, molecules per mole, ions per mole, electrons per mole, etc. The mean atomic mass number \bar{M} of all elements is thus defined such that a mass of \bar{M} grams of the element contains exactly Avogadro number of atoms.

For example: Cobalt has only one stable isotope, cobalt-59, so that the mean atomic mass $\bar{M}(\text{Co})$ of natural cobalt is the atomic mass of cobalt-59 at 58.9332 u. Thus:

- 1 mol of natural cobalt is 58.9332 g of natural cobalt.
- A mass of 58.9332 g of natural cobalt contains 6.022×10^{23} cobalt atoms.

As far as the unit of atomic mass is concerned, we have three related options. The atomic mass can be expressed in one of the following three formats:

- Without a unit when it represents the ratio between the mass of a given element and the unified atomic mass unit u ($1/12$ of the mass of the carbon-12 atom). In this case the atomic mass is dimensionless and expresses the magnitude of the atomic mass relative to the standard mass that is $1/12$ of the mass of the carbon-12 atom.
- In units of the unified atomic mass unit u where u represents $1/12$ of the mass of the carbon-12 atom.
- In units of g/mol when the mean atomic mass (standard atomic weight) is multiplied by the molar mass constant $M_u = 1 \text{ g/mol}$.

For example, for the carbon-12 atom we can state that its relative atomic mass is 12.000, or that its atomic mass is 12.000 u, or that its molar mass is 12.000 g/mol. For elemental carbon we can state that its mean atomic mass (standard atomic weight) is 12.0107, or that its mean atomic mass (standard atomic weight) is 12.0107 u, or that its molar atomic weight is 12.0107 g/mol.

1.13.3 Mean Molecular Mass (Standard Molecular Weight)

If we assume that the mass of a molecule is equal to the sum of the masses of the atoms that make up the molecule, then for any molecular compound there are N_A molecules per mole of the compound where the *mole* in grams is defined as the sum of the mean atomic masses of the atoms making up the molecule. Moreover, a mole of a molecular compound contains Avogadro number N_A of molecules.

The standard molecular weight of a molecular compound is calculated from the molecule's chemical formula and the standard atomic weights for the atoms constituting the molecule. For example:

- Water molecule contains two atoms of hydrogen [$\bar{M}(\text{H}) = 1.00794$] and one atom of oxygen [$\bar{M}(\text{O}) = 15.9994$]. The standard molecular weight of water is:

$$\bar{M}(\text{H}_2\text{O}) = 2 \times \bar{M}(\text{H}) + \bar{M}(\text{O}) = 2 \times 1.00794 \text{ u} + 15.9994 \text{ u} = 18.0153 \text{ u} \quad (1.22)$$

and one mole of water that contains N_A molecules is 18.0153 g of water.

- The molecule of carbon dioxide contains one atom of carbon [$\bar{M}(\text{C}) = 12.0107$] and two atoms of oxygen [$\bar{M}(\text{O}) = 15.9994$]. The standard molecular weight of carbon dioxide CO_2 is:

$$\bar{M}(\text{CO}_2) = \bar{M}(\text{C}) + 2 \times \bar{M}(\text{O}) = 12.0107 \text{ u} + 2 \times 15.9994 \text{ u} = 44.0095 \text{ u} \quad (1.23)$$

and one mole of carbon dioxide that contains N_A molecules is 44.0095 g of carbon dioxide.

In the calculation of the standard molecular weight of a water molecule above we used the mean values for the standard atomic weights to account for traces of deuterium, oxygen-17, and oxygen-18 in natural water molecules and found 18.0153 for the molecular weight of water. However, the most common water molecule will contain hydrogen-1 (protium) and oxygen-16 and will thus have a slightly lower molecular weight amounting to 18.0106 as a result of protium atomic mass of 1.00783 and oxygen-16 atomic mass of 15.9949.

Similarly, we get a molecular weight of carbon dioxide as 43.9898 for a typical molecule containing carbon-12 and oxygen-16 in contrast to 44.0095 that we

calculated as the mean value after accounting for traces of carbon-13, oxygen-17 and oxygen-18 in natural carbon dioxide.

In the first approximation assuming that atomic mass equals to the atomic mass number A , we get 18 g for the mole of water and 44 g for the mole of carbon dioxide, very similar to the values obtained in (1.22) and (1.23), respectively.

1.14 Basic Definitions for Nuclear Structure

Most of the atomic mass is concentrated in the atomic nucleus consisting of Z protons and $(A - Z)$ neutrons, where Z is the atomic number and A the atomic mass number of a given nucleus. Proton and neutron have nearly identical rest masses; the proton has positive charge, identical in magnitude to the negative electron charge and the neutron has no charge and is thus neutral.

In nuclear physics the convention is to designate a nucleus with symbol X as A_ZX , where A is the atomic mass number and Z the atomic number. *For example:* The cobalt-60 nucleus is identified as ${}^{60}_{27}\text{Co}$, the radium-226 nucleus as ${}^{226}_{88}\text{Ra}$, and the uranium-235 nucleus as ${}^{235}_{92}\text{U}$.

There is no basic relation between the atomic mass number A and the atomic number Z in a nucleus, but the empirical relationship

$$Z = \frac{A}{1.98 + 0.0155A^{2/3}} \quad (1.24)$$

provides a good approximation for stable nuclei. Protons and neutrons are commonly referred to as *nucleons*, have identical strong attractive interactions, and are bound in the nucleus with the strong force. As discussed in Sect. 1.5, in contrast to electrostatic and gravitational forces that are inversely proportional to the square of the distance between two particles, the strong force between two nucleons is a very short-range force, active only at distances of the order of a few femtometers. At these short distances the strong force is the predominant force exceeding other forces by many orders of magnitude, as shown in Table 1.3. With regard to relative values of atomic number Z and atomic mass number A of nuclei, the following conventions apply:

- An element may be composed of atoms that all have the same number of protons, i.e., have the same atomic number Z , but have a different number of neutrons, i.e., have different atomic mass numbers A . Such atoms of identical Z but differing A are called *isotopes* of a given element.
- The term *isotope* is often misused to designate nuclear species. For example, cobalt-60, cesium-137, and radium-226 are not isotopes, since they do not belong to the same element. Rather than isotopes, they should be referred to as *nuclides*. On the other hand, it is correct to state that deuterium (with nucleus called deuteron) and tritium (with nucleus called triton) are heavy isotopes of hydrogen or that cobalt-59 and cobalt-60 are isotopes of cobalt. Thus, the term *radionuclide* should

be used to designate radioactive species; however, the term radioisotope is often used for this purpose.

- The term *nuclide* refers to all atomic forms of all elements. The term *isotope* is narrower and only refers to various atomic forms of a single chemical element.
- In addition to being classified into isotopic groups (common atomic number Z), nuclides are also classified into groups with common atomic mass number A (*isobars*) and common number of neutrons (*isotones*). For example, cobalt-60 and nickel-60 are isobars with 60 nucleons each ($A = 60$); hydrogen-3 (tritium) and helium-4 are isotones with two neutrons each ($A - Z = 2$).
- If a nucleus exists in an excited state for some time, it is said to be in an isomeric (metastable) state. *Isomers* are thus nuclear species that have common atomic number Z and atomic mass number A . For example, technetium-99 m is an isomeric state of technetium-99 and cobalt-60 m is an isomeric state of cobalt-60.

1.15 Nuclear Binding Energies

The sum of masses of the individual components of a nucleus that contains Z protons and $(A - Z)$ neutrons is larger than the actual mass of the nucleus. This difference in mass is called the mass defect (deficit) Δm and its energy equivalent Δmc^2 is called the total binding energy E_B of the nucleus. The *total binding energy* E_B of a nucleus can thus be defined as:

1. The positive work required to disassemble a nucleus into its individual components: Z protons and $(A - Z)$ neutrons.

or

2. The energy liberated when Z protons and $(A - Z)$ neutrons are brought together to form the nucleus.

The *binding energy per nucleon* (E_B/A) in a nucleus (i.e., the total binding energy of a nucleus divided by the number of nucleons) varies with the number of nucleons A and is of the order of ~ 8 MeV/nucleon. It may be calculated from the energy equivalent of the mass deficit Δm as follows:

$$\frac{E_B}{A} = \frac{\Delta mc^2}{A} = \frac{Zm_p c^2 + (A - Z)m_n c^2 - Mc^2}{A}, \quad (1.25)$$

where

- A is the atomic mass number.
- M is the nuclear mass in atomic mass units u .
- $m_p c^2$ is the proton rest energy (938.3 MeV).
- $m_n c^2$ is the neutron rest energy (939.6 MeV).

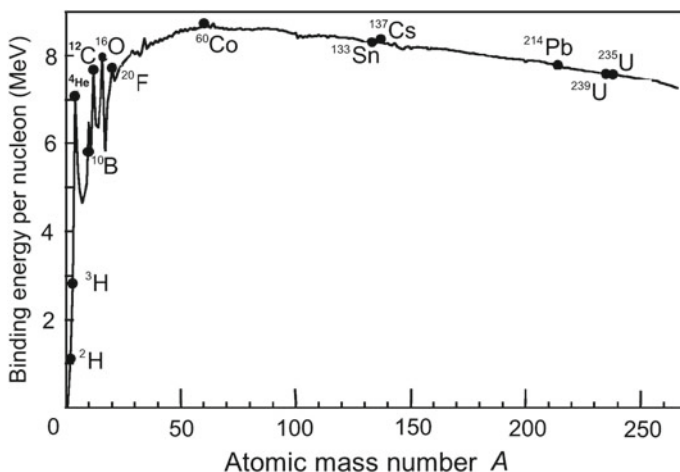


Fig. 1.3 Binding energy per nucleon in MeV/nucleon against atomic mass number A

As shown in Fig. 1.3, the binding energy per nucleon E_B/A against the atomic mass number A exhibits the following characteristics:

1. For $1 \leq A \leq 4$ the binding energy per nucleon rises rapidly from 1.1 MeV per nucleon for deuteron (^2_1H) through 2.8 MeV and 2.6 MeV per nucleon for triton (^3_1H) and helium-3 (^3_2He), respectively, to 7.1 MeV per nucleon for helium-4 (^4_2He). The nucleus of the helium-4 atom is the α -particle.
2. For $4 \leq A \leq 28$, E_B/A fluctuates and exhibits peaks for nuclides in which A is a multiple of four.
3. For $28 < A < 60$, E_B/A rises slowly with increasing A to reach a peak value of 8.8 MeV per nucleon for $A \approx 60$ (iron, cobalt, nickel).
4. For A exceeding 60, E_B/A falls monotonically from 8.8 MeV/nucleon to reach 7.5 MeV per nucleon for uranium-238.

The larger is the binding energy per nucleon (E_B/A) of an atom, the larger is the stability of the atom. Thus the most stable nuclei in nature are the ones with $A \approx 60$. Nuclei of light elements (small A) are generally less stable than the nuclei with $A \approx 60$ and the heaviest nuclei (large A) are also less stable than the nuclei with $A \approx 60$.

The peculiar shape of the E_B/A against A curve suggests two methods for converting mass into energy: (1) *fusion* of nuclei at low A and (2) *fission* of nuclei at large A , as discussed in greater detail in Sect. 12.8.

- Fusion of two nuclei of very small mass, e.g., $^2_1\text{H} + ^3_1\text{H} \rightarrow ^4_2\text{He} + \text{n}$, will create a more massive nucleus and release a certain amount of energy. Experiments using controlled nuclear fusion for production of energy have so far not been successful; however, steady progress in fusion research is being made in various laboratories around the world. It is reasonable to expect that in the future controlled fusion

will become possible and will result in a relatively clean and abundant means for sustainable power generation.

- Fission of elements of large mass, e.g., $^{235}_{92}\text{U} + \text{n}$, will create two lower mass and more stable nuclei and lose some mass in the form of kinetic energy. Nuclear fission was observed first in 1934 by *Enrico Fermi* and described correctly by *Otto Hahn*, *Fritz Strassman*, *Lise Meitner*, and *Otto Frisch* in 1939. In 1942 at the University of Chicago Enrico Fermi and colleagues carried out the first controlled chain reaction based on nuclear fission (see Sect. 12.7).

1.16 Nuclear Models

Several models of the nucleus have been proposed; all phenomenological and none of them capable of explaining completely the complex nature of the nucleus, such as its binding energy, stability, radioactive decay, etc. The two most successful models are the *liquid-drop model* that accounts for the nuclear binding energy and the *shell model* that explains nuclear stability.

1.16.1 Liquid-Drop Nuclear Model

The *liquid-drop* nuclear model, proposed by *Niels Bohr* in 1936, treats the nucleons as if they were molecules in a spherical drop of liquid. Scattering experiments with various particles such as electrons, nucleons and α -particles reveal that to a first approximation nuclei can be considered spherical with essentially constant density.

The radius R of a nucleus with atomic mass number A is estimated from the following expression

$$R = R_0 \sqrt[3]{A}, \quad (1.26)$$

where R_0 is the nuclear radius constant equal to 1.25 fm.

Using (1.26) we estimate the density of the nucleus with mass M and volume \mathcal{V} as

$$\rho = \frac{M}{\mathcal{V}} \approx \frac{Am_p}{(4/3)\pi R^3} = \frac{m_p}{(4/3)\pi R_0^3} \approx 1.5 \times 10^{14} \text{ g}\cdot\text{cm}^{-3}, \quad (1.27)$$

where m_p is the rest mass of the proton (938.3 MeV/ c^2).

Based on the liquid drop model of the nucleus the nuclear binding energy was split into various components, each with its own dependence on the atomic number Z and atomic mass number A . Four of the most important components of the nuclear binding energy are:

1. *Volume correction* Since the binding energy per nucleon E_B/A is essentially constant, as shown in Fig. 1.3, the total nuclear binding energy is linearly proportional to A .
2. *Surface correction* Nucleons on the surface of the liquid-drop have fewer neighbors than those in the interior of the drop. The surface nucleons will reduce the

total binding energy by an amount proportional to R^2 , where R is the nuclear radius proportional to $A^{1/3}$, as given in (1.26). Thus the surface effect correction is proportional to $A^{2/3}$.

3. *Coulomb repulsion correction* accounts for the Coulomb repulsion among protons in the nucleus. The repulsive energy reduces the total binding energy and is proportional to $Z(Z - 1)$, the number of proton pairs in the nucleus, and inversely proportional to R , i.e., inversely proportional to $A^{1/3}$.
4. *Neutron excess correction* reduces the total binding energy and is proportional to $(A - 2Z)^2$ and inversely proportional to A .

The total nuclear binding energy E_B is then written as follows

$$E_B = C_1 A - C_2 A^{2/3} - C_3 \frac{Z(Z - 1)}{A^{1/3}} - C_4 \frac{(A - 2Z)^2}{A}. \quad (1.28)$$

Equation (1.28) is referred to as the *Weizsäcker semi-empirical binding energy formula* in which the various components are deduced theoretically but their relative magnitudes are determined empirically to match the calculated results with experimental data. The constants in (1.28) were determined empirically and are given as follows:

$$C_1 \approx 16 \text{ MeV}; \quad C_2 \approx 18 \text{ MeV}; \quad C_3 \approx 0.7 \text{ MeV}; \quad \text{and} \quad C_4 \approx 24 \text{ MeV}.$$

1.16.2 Shell Structure Nuclear Model

Experiments have shown that the number of nucleons the nucleus contains affects the stability of nuclei. The general trend in binding energy per nucleon E_B/A , as shown in Fig. 1.3, provides the E_B/A maximum at around $A = 60$ and then drops for smaller and larger A . However, there are also considerable variations in stability of nuclei depending on the parity in the number of protons and neutrons forming a nucleus.

In nature there are 280 nuclides that are considered stable with respect to radioactive decay. Some 60% of these stable nuclei have an even number of protons and an even number of neutrons (even-even nuclei); some 20% have an even-odd configuration and a further 20% have an odd-even configuration. Only four stable nuclei are known to have an odd-odd proton/neutron configuration. A conclusion may thus be made that an even number of protons or even number of neutrons promotes stability of nuclear configurations.

When the number of protons is: 2, 8, 20, 28, 50, 82 or the number of neutrons is: 2, 8, 20, 28, 50, 82, 126, the nucleus is observed particularly stable and these numbers are referred to as *magic numbers*. Nuclei in which the number of protons as well as the number of neutrons is equal to a magic number belong to the most stable group of nuclei.

The existence of magic numbers stimulated development of a nuclear model containing a nuclear shell structure in analogy with the atomic shell structure configuration of electrons. In the nuclear shell model, often also called the independent particle model, the nucleons are assumed to move in well-defined orbits within the nucleus in a field produced by all other nucleons. The nucleons exist in quantized energy states of discrete energy that can be described by a set of quantum numbers, similarly to the situation with electronic states in atoms.

The ground state of a nucleus constitutes the lowest of the entire set of energy levels and, in contrast to atomic physics where electronic energy levels are negative, in nuclear physics the nuclear ground state is set at zero and the excitation energies of the respective higher bound states are shown positive with respect to the ground state.

To raise the nucleus to an excited state an appropriate amount of energy must be supplied. On de-excitation of a nucleus from an excited state back to the ground state a discrete amount of energy will be emitted.

1.17 Physics of Small Dimensions and Large Velocities

At the end of the nineteenth century physics was considered a completed discipline within which most of the natural physical phenomena were satisfactorily explained. However, as physicists broadened their interests and refined their experimental techniques, it became apparent that classical physics suffered severe limitations in two areas:

1. Dealing with dimensions comparable to small atomic dimensions.
2. Dealing with velocities comparable to the speed of light.

Modern physics handles these limitations in two distinct, yet related, sub-specialties: *quantum physics* and *relativistic physics*, respectively:

1. *Quantum physics* extends the range of application of physical laws to small atomic dimensions of the order of 10^{-10} m (radius a of atom), includes classical laws as special cases when dimension $\gg a$, and introduces the Planck constant h as a universal constant of fundamental significance. *Erwin Schrödinger*, *Werner Heisenberg*, and *Max Born* are credited with developing quantum physics in the mid 1920s.
2. *Relativistic physics* extends the range of application of physical laws to large velocities v of the order of the speed of light in vacuum c (3×10^8 m/s), includes classical laws as special cases when $v \ll c$, and introduces c as a universal physical constant of fundamental significance. The protagonist of relativistic physics was *Albert Einstein* who formulated the special theory of relativity in 1905.

1.18 Planck Energy Quantization

Modern physics was born in 1900 when *Max Planck* presented his revolutionary idea of energy quantization of physical systems that undergo simple harmonic oscillations. Planck energy ε quantization is expressed as

$$\varepsilon = nh\nu, \quad (1.29)$$

where

- n is the quantum number ($n = 0, 1, 2, 3 \dots$).
- h is a universal constant referred to as the Planck constant.
- ν is the frequency of oscillation.

The allowed energy states in a system oscillating harmonically are continuous in classical models, while in the Planck model they consist of discrete allowed quantum states with values $nh\nu$, where n is a non-negative integer quantum number. Planck used his model to explain the spectral distribution of thermal radiation emitted by a blackbody defined as an entity that absorbs all radiant energy incident upon it. All bodies emit thermal radiation to their surroundings and absorb thermal radiation from their surroundings; in thermal equilibrium the rates of thermal emission and thermal absorption are equal.

Planck assumed that sources of thermal radiation are harmonically oscillating atoms possessing discrete vibrational energy states. When an oscillator jumps from one discrete quantum energy state E_1 to another energy state E_2 where $E_1 > E_2$, the energy difference $\Delta E = E_1 - E_2$ is emitted in the form of a photon with energy $h\nu$, i.e.,

$$\Delta E = E_1 - E_2 = h\nu = \frac{hc}{\lambda}, \quad (1.30)$$

where

- h is the Planck constant.
- ν is the frequency of the photon.
- c is the speed of light in vacuum.
- λ is the wavelength of the photon.

Thus, according to Planck law, radiation such as light is emitted, transmitted, and absorbed in discrete energy quanta characterized by the product of frequency ν and Planck constant h . Planck's postulate of energy quantization leads to the atomic model with its angular momentum quantization introduced by *Niels Bohr* in 1913 and to quantum wave mechanics developed by *Erwin Schrödinger* in 1926. The so-called *Schrödinger equation*, used extensively in atomic, nuclear, and solid-state physics, is a wave equation describing probability waves (wave functions) that govern the motion of small atomic particles. The equation has the same fundamental importance to quantum mechanics as Newton laws have for large dimension phenomena in classical mechanics.

1.19 Quantization of Electromagnetic Radiation

Electromagnetic (EM) radiation incident on metallic surface may eject charged particles from the surface, as first observed by *Heinrich Hertz* in 1887. *Joseph Thomson* proved that the emitted charged particles were electrons and *Albert Einstein* in 1905 explained the effect by proposing that EM radiation was quantized similarly to the quantization of oscillator levels in matter introduced by *Max Planck* in 1900 (see Sect. 1.26).

A quantum of EM radiation, called a *photon*, has the following properties:

- It is characterized by frequency ν and wavelength $\lambda = c/\nu$ where c is the speed of light in vacuum.
- It carries energy $h\nu$ and momentum $p_\nu = h/\lambda$ where h is Planck constant.
- It has zero rest mass (see Sect. 1.21).

In a metal the outer electrons move freely from atom to atom and behave like a gas with a continuous spectrum of energy levels. To release an electron from a metal a minimum energy, characteristic of the given metal and referred to as the work function $e\phi$, must be supplied to the electron. Einstein postulated that the maximum kinetic energy $(E_K)_{\max}$ of the electron ejected from the surface of a metal by a photon with energy $h\nu$ is given by the following expression

$$(E_K)_{\max} = h\nu - e\phi. \quad (1.31)$$

The maximum kinetic energy of the ejected electrons depends on the incident photon energy $h\nu$ and the work function $e\phi$ of the metal but does not depend on the incident radiation intensity. The effect of electron emission from metallic surfaces was named the photoelectric effect and its explanation by Einstein on the basis of quantization of EM radiation is an important contribution to modern physics. Notable features of the surface photoelectric effect are as follows:

- Electrons can be ejected from a metallic surface by the photoelectric effect only when the incident photon energy $h\nu$ exceeds the work function $e\phi$ of the metal, i.e., $h\nu > e\phi$.
- The photoelectric effect is a quantum phenomenon: a single electron absorbs a single photon; the photon disappears and the electron is ejected with a certain kinetic energy.
- The typical magnitude of the work function $e\phi$ for metals is of the order of a few electron volts (e.g., aluminum: 4.3 eV; cesium: 2.1 eV; cobalt: 5.0 eV; copper: 4.7 eV; iron: 4.5 eV; lead: 4.3 eV; uranium: 3.6 eV), as given in the *Handbook of Chemistry and Physics*. The work function is thus of the order of energy $h\nu$ of visible photons ranging from 1.8 eV (700 nm) to 3 eV (400 nm) and near ultraviolet photons ranging in energy from 3 eV (400 nm) to 10 eV (80 nm).
- The surface photoelectric effect is related to the atomic photoelectric effect in which high-energy photons with energies exceeding the binding energy of orbital electrons eject electrons from atomic shells (see Sect. 7.5) rather than from metallic surfaces.

1.20 Special Theory of Relativity

The special theory of relativity, introduced in 1905 by *Albert Einstein*, extends the range of physical laws to large velocities and deals with transformations of physical quantities from one inertial frame of reference to another.

An *inertial frame of reference* implies motion with uniform velocity. The two postulates of the special relativity theory are:

1. The laws of physics are identical in all inertial frames of reference.
2. The speed of light in vacuum c is a universal constant independent of the motion of the source.

Albert A. Michelson and *Edward W. Morley* in 1887 showed that the speed of light c is a universal constant independent of the state of motion of the source or observer. Einstein, with his special theory of relativity, explained the results of the Michelson–Morley experiment and introduced, in contrast to classical *Galilean transformations*, special transformations referred to as *Lorentz transformations* to relate measurements in one inertial frame of reference to measurements in another inertial frame of reference.

When the velocities involved are very small, the Lorentzian transformations simplify to the classical Galilean transformations, and the relativistic relationships for physical quantities transform into classical Newtonian relationships. Galilean and Lorentzian transformations have the following basic characteristics:

- Galilean and Lorentzian transformations relate the spatial and time coordinates x , y , z , and t in a stationary frame of reference to coordinates x' , y' , z' , and t' in a reference frame moving with a uniform speed v in the x direction as follows:

Galilean transformation

$$x' = x - vt \quad (1.32)$$

$$y' = y \quad (1.34)$$

$$z' = z \quad (1.36)$$

$$t' = t \quad (1.38)$$

$$\text{where } \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Lorentzian transformation

$$x' = \gamma(x - vt) \quad (1.33)$$

$$y' = y \quad (1.35)$$

$$z' = z \quad (1.37)$$

$$t' = \gamma \left(t - \frac{xv}{c^2} \right) \quad (1.39)$$

- For $v \ll c$ the Lorentzian transformation reduces to the Galilean transformation since $\gamma \approx 1$. The specific form of the Lorentzian transformation is a direct consequence of $c = \text{const}$ in all frames of reference.

- Einstein also showed that atomic and subatomic particles, as they are accelerated to a significant fraction of the speed of light c , exhibit another relativistic effect, an increase in mass as a result of the mass-energy equivalence stated as $E = mc^2$, where m and E are the mass and energy, respectively, of the particle. A corollary to the second postulate of relativity is that no particle can move faster than the speed of light c in vacuum.
- *Conservation of energy and momentum:*
 - In *classical mechanics* where $v \ll c$, the momentum given as $p = m_0v$ and the kinetic energy given as $E_K = \frac{1}{2}m_0v^2$ are conserved in all collisions (m_0 is the mass of the particle at $v = 0$).
 - In *relativistic mechanics* where $v \approx c$, the momentum $p = mv = \gamma m_0v$ and the total energy $E = m_0c^2 + E_K$ are conserved in all collisions.

1.21 Important Relativistic Relations

In relativistic mechanics the mass of a particle is not a conserved quantity, since it depends on the velocity of the particle and may be converted into kinetic energy. The reverse transformation is also possible and energy may be converted into matter.

1.21.1 Relativistic Mass

Newton classical equation of motion is preserved in relativistic mechanics, i.e.,

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}, \quad (1.41)$$

where \mathbf{p} is the momentum of a particle acted upon by force \mathbf{F} . The momentum \mathbf{p} is proportional to the velocity v of a particle through the relationship

$$\mathbf{p} = m\mathbf{v}, \quad (1.42)$$

where m is the mass of the particle, dependent on the magnitude of the particle velocity v , i.e., $m = m(v)$.

The mass $m(v)$ is referred to as the *relativistic mass* of a particle and is given by Einstein's expression (see Fig. 1.4 and Table 1.7) as follows

$$m(v) = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1 - \beta^2}} = \gamma m_0 \quad (1.43)$$

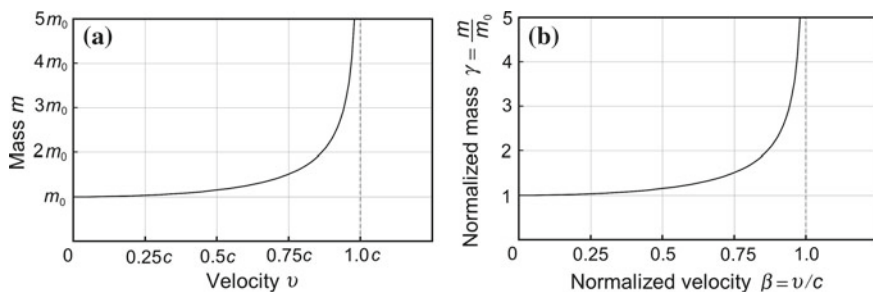


Fig. 1.4 Particle mass m as a function of its velocity v . A plot of m against v of (1.43) in **a** and a plot of γ against β of (1.44) in **b**

Table 1.7 Mass against velocity according to (1.44)

$(v/c) = \beta$	0	0.1	0.25	0.5	0.75	0.9	0.99	0.999	0.9999
$(m/m_0) = \gamma$	1.000	1.005	1.033	1.155	1.512	2.294	7.089	22.37	70.71

or

$$\frac{m(v)}{m_0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}} = \gamma, \quad (1.44)$$

where

m_0 is the mass of a particle at $v = 0$, referred to as the particle *rest mass*.

c is the speed of light in vacuum, a universal constant.

β is v/c .

γ is the Lorentz factor expressed as $(1 - \beta^2)^{-1/2}$ or $[1 - (v/c)^2]^{-1/2}$.

1.21.2 Relativistic Force and Relativistic Acceleration

In *classical physics* the Newton second law of mechanics is given as follows

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = m_0 \frac{d\mathbf{v}}{dt} = m_0 \mathbf{a}, \quad (1.45)$$

indicating that the acceleration \mathbf{a} is parallel to force \mathbf{F} , and that mass m_0 is constant.

In *relativistic physics* the acceleration \mathbf{a} is not parallel to the force \mathbf{F} at large velocities because the speed of a particle cannot exceed c , the speed of light in

vacuum. The force \mathbf{F} , with the mass m a function of particle velocity v as given in (1.43), can be written as

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m \frac{d\mathbf{v}}{dt} + \mathbf{v} \frac{dm}{dt} \quad (1.46)$$

and

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = \frac{d(\gamma m_0 \mathbf{v})}{dt} = \gamma m_0 \frac{d\mathbf{v}}{dt} + m_0 \mathbf{v} \frac{d\gamma}{dt} = \gamma m_0 \frac{d\mathbf{v}}{dt} + m_0 \mathbf{v} \frac{\gamma^3 v}{c^2} \frac{d\mathbf{v}}{dt}, \quad (1.47)$$

where

$$\frac{d\gamma}{dt} = \frac{1}{\left[1 - \frac{v^2}{c^2}\right]^{3/2}} \frac{v}{c^2} \frac{d\mathbf{v}}{dt} = \frac{\gamma^3 v}{c^2} \frac{d\mathbf{v}}{dt}. \quad (1.48)$$

The acceleration $\mathbf{a} = d\mathbf{v}/dt$ will be determined by obtaining a dot product of the force \mathbf{F} and velocity \mathbf{v} as follows

$$\mathbf{F} \cdot \mathbf{v} = \gamma m_0 \mathbf{v} \frac{d\mathbf{v}}{dt} + \frac{m_0 \gamma^3 v^3}{c^2} \frac{d\mathbf{v}}{dt} = \gamma m_0 \mathbf{v} \frac{d\mathbf{v}}{dt} (1 + \gamma^2 \beta^2) = \gamma^3 m_0 \mathbf{v} \frac{d\mathbf{v}}{dt}. \quad (1.49)$$

Inserting (1.49) into (1.47) gives the following result for the relativistic force \mathbf{F}

$$\mathbf{F} = \gamma m_0 \frac{d\mathbf{v}}{dt} + \frac{\mathbf{F} \cdot \mathbf{v}}{c^2} \mathbf{v} = \gamma m_0 \frac{d\mathbf{v}}{dt} + (\mathbf{F} \cdot \boldsymbol{\beta}) \boldsymbol{\beta}. \quad (1.50)$$

Solving (1.50) for $\mathbf{a} = d\mathbf{v}/dt$ gives the relativistic relationship for the acceleration \mathbf{a}

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\beta}) \boldsymbol{\beta}}{\gamma m_0}. \quad (1.51)$$

For velocities $v \ll c$, where $\beta \rightarrow 0$ and $\gamma \rightarrow 1$, the relativistic expression (1.51) for acceleration \mathbf{a} transforms into Newton's classical result $\mathbf{a} = d\mathbf{v}/dt = \mathbf{F}/m_0$ with \mathbf{a} parallel to \mathbf{F} .

1.21.3 Relativistic Kinetic Energy

The expression for the relativistic kinetic energy $E_K = E - E_0$, where $E = mc^2$ is the total energy of the particle and $E_0 = m_0 c^2$ is its rest energy, is derived below.

The particle of rest mass m_0 is initially at rest at the initial position x_i and moves under the influence of force F to its final position x_f . The work done by force F is

the kinetic energy E_K of the particle calculated using the integration of (1.46) and the following steps:

1.

$$E_K = \int_{x_i}^{x_f} F dx = \int_{x_i}^{x_f} \left(m \frac{dv}{dt} + v \frac{dm}{dt} \right) dx \quad (1.52)$$

2. Multiply (1.43) by c , square the result, and rearrange the terms to obtain

$$m^2 c^2 - m^2 v^2 = m_0^2 c^2. \quad (1.53)$$

3. Differentiate (1.53) with respect to time t to obtain

$$c^2 \frac{d(m^2)}{dt} - \frac{d}{dt}(m^2 v^2) = 0 \quad (1.54)$$

4. Equation (1.54), after completing the derivatives, gives

$$2c^2 m \frac{dm}{dt} - 2m^2 v \frac{dv}{dt} - 2v^2 m \frac{dm}{dt} = 0. \quad (1.55)$$

5. After dividing (1.55) by $2mv$ we obtain the following expression

$$\frac{c^2}{v} \frac{dm}{dt} = m \frac{dv}{dt} + v \frac{dm}{dt}. \quad (1.56)$$

The expression for E_K in (1.52) using (1.56) can now be written as follows

$$E_K = c^2 \int_{x_i}^{x_f} \frac{1}{v} \frac{dm}{dt} dx = c^2 \int_{m_0}^m dm = mc^2 - m_0 c^2 = E - E_0, \quad (1.57)$$

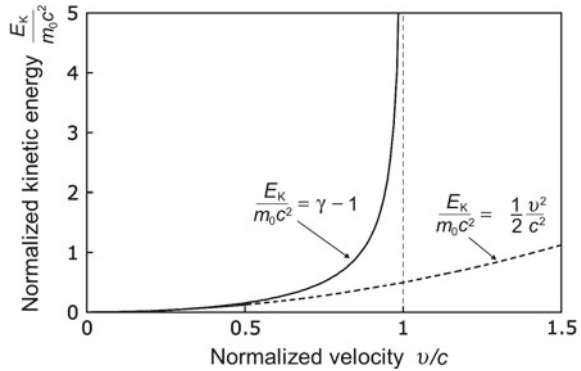
since dx/dt is the particle velocity v by definition and the masses m_0 and m correspond to particle positions x_i and x_f , respectively.

Inserting (1.43) into (1.57) results in the following expression for the relativistic kinetic energy E_K

$$\begin{aligned} E_K &= mc^2 - m_0 c^2 = \gamma m_0 c^2 - m_0 c^2 = (\gamma - 1) m_0 c^2 \\ &= (\gamma - 1) E_0 = \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) E_0, \end{aligned} \quad (1.58)$$

in contrast with the well known classical expression

Fig. 1.5 Classical expression expression for kinetic energy (1.59) and relativistic expression for kinetic energy (1.58) both normalized to electron rest mass $m_0 c^2$ and plotted against normalized velocity v/c



$$E_K = \frac{m_0 v^2}{2}. \quad (1.59)$$

The relativistic expression (1.58) and the classical expression (1.59) are plotted in Fig. 1.5, with the kinetic energy normalized to rest energy $[E_K/(m_0 c^2)]$ on the ordinate (y) axis and the velocity normalized to the speed of light (v/c) on the abscissa (x) axis. At low particle velocity where ($v \ll c$), the two expressions coincide; however, at high velocities v the relativistic expression (1.58), as a result of increase in mass m with velocity v , increases rapidly to attain an infinite value at $v = c$ or at $v/c = 1$, while, for the classical expression (1.59) in which the mass remains constant with increasing velocity v , the ratio $E_K/(m_0 c^2)$ attains a value of 0.5 for $v = c$.

1.21.4 Total Relativistic Energy as a Function of Momentum

The expression for the total relativistic energy E as a function of momentum p is as follows

$$E = \sqrt{E_0^2 + p^2 c^2}. \quad (1.60)$$

Equation (1.60) is obtained from Einstein expression for the relativistic mass given in (1.43) as follows:

1. Square the relationship for the relativistic mass m of (1.43), multiply the result by c^4 , and rearrange the terms to obtain

$$m^2 c^4 - m^2 c^2 v^2 = m_0^2 c^4. \quad (1.61)$$

2. Equation (1.61) can be written as

$$E^2 - p^2 c^2 = E_0^2 \quad (1.62)$$

or

$$E = \sqrt{E_0^2 + p^2 c^2}, \quad (1.63)$$

using the common relativistic relationships for the total energy E , rest energy E_0 , and momentum p , i.e., $E = mc^2$, $E_0 = m_0 c^2$, and $p = mv$.

The following two relationships are also often used in relativistic mechanics:

1. The particle momentum p using (1.57) and (1.63) for the kinetic energy E_K and total energy E , respectively, can be expressed as

$$p = \frac{1}{c} \sqrt{E^2 - E_0^2} = \frac{1}{c} \sqrt{E_K^2 + 2E_K E_0} = \frac{E_K}{c} \sqrt{1 + \frac{2E_0}{E_K}}. \quad (1.64)$$

2. The particle speed v is, in terms of its total energy E and momentum p , given as

$$\frac{v}{c} = \frac{mvc}{mc^2} = \frac{pc}{E}. \quad (1.65)$$

It is easy to show that the two expressions for momentum p , given in (1.42) and (1.64) and restated below, are equivalent to one another. From (1.42) and (1.64) we get

$$p = mv = \gamma m_0 v = \frac{m_0 v}{\sqrt{1 - \beta^2}} \quad (1.66)$$

and

$$p = \frac{1}{c} \sqrt{E^2 - E_0^2} = \frac{1}{c} \sqrt{\gamma^2 m_0^2 c^4 - m_0^2 c^4} = m_0 c \sqrt{\gamma^2 - 1}, \quad (1.67)$$

respectively, and since

$$\gamma^2 - 1 = \frac{1}{1 - \beta^2} - 1 = \frac{\beta^2}{1 - \beta^2}, \quad (1.68)$$

we can express the momentum p of (1.67) as

$$p = m_0 c \sqrt{\gamma^2 - 1} = \frac{m_0 c \beta}{\sqrt{1 - \beta^2}} = \gamma m_0 v = mv, \quad (1.69)$$

proving the equivalence of the two expressions (1.42) and (1.64) for momentum p .

1.21.5 Taylor Expansion and Classical Approximations for Kinetic Energy and Momentum

The Taylor expansion of a function $f(x)$ about $x = a$ is given as follows

$$f(x) = f(a) + (x - a) \left. \frac{df}{dx} \right|_{x=a} + \frac{(x - a)^2}{2!} \left. \frac{d^2f}{dx^2} \right|_{x=a} + \cdots + \frac{(x - a)^n}{n!} \left. \frac{d^n f}{dx^n} \right|_{x=a}. \quad (1.70)$$

The Taylor expansion into a series given by (1.70) is particularly useful when one can neglect all but the first two terms of the series. For example, the first two terms of the Taylor expansion of the function $f(x) = (1 \pm x)^n$ about $x = 0$ for $x \ll 1$ are given as follows

$$f(x) = (1 \pm x)^n \approx 1 \pm nx. \quad (1.71)$$

- The approximation of (1.71) is used in showing that, for small velocities where $v \ll c$ or $v/c \ll 1$, the relativistic kinetic energy E_K of (1.57) transforms into the well-known classical relationship $E_K = \frac{1}{2}m_0v^2$

$$\begin{aligned} E_K &= E - E_0 = m_0c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) \\ &\approx m_0c^2 \left\{ 1 + \frac{1}{2} \frac{v^2}{c^2} - \cdots - 1 \right\} = \frac{m_0v^2}{2}. \end{aligned} \quad (1.72)$$

- Another example of the use of Taylor expansion of (1.71) is the classical relationship for the momentum $p = m_0v$ that, for $v \ll c$, i.e., $v/c \ll 1$, is obtained from the relativistic relationship of (1.64) as follows

$$\begin{aligned} p &= \frac{1}{c} \sqrt{E^2 - E_0^2} = \frac{1}{c} \sqrt{m_0^2c^4 \left(\frac{1}{1 - \frac{v^2}{c^2}} - 1 \right)} = \frac{m_0c^2}{c} \sqrt{\left(1 - \frac{v^2}{c^2} \right)^{-1} - 1} \\ &\approx m_0c \sqrt{1 + \frac{v^2}{c^2} - 1} = m_0v. \end{aligned} \quad (1.73)$$

1.21.6 Relativistic Doppler Shift

The speed of light emitted from a moving source is equal to c , a universal constant, irrespective of the source velocity. While the speed of the emitted photons equals c , the energy, wavelength, and frequency of the emitted photons all depend on the velocity of the moving source. The energy shift resulting from a moving source in comparison with the stationary source is referred to as the Doppler shift and the following conditions apply:

- When the source is moving toward the observer, the measured photon frequency and energy increase while the wavelength decreases (*blue Doppler shift*).
- When the source is moving away from the observer, the measured photon frequency and energy decrease while the wavelength increases (*red Doppler shift*).

1.22 Particle–Wave Duality

For electromagnetic radiation, the energy E_ν of a photon is given by Planck law as

$$E_\nu = h\nu = h\frac{c}{\lambda}, \quad (1.74)$$

where

- h is Planck constant.
- c is the speed of light in vacuum.
- ν is photon frequency.
- λ is the wavelength of the photon.

The photon energy E_ν can be also written in terms of the photon momentum p_ν as

$$E_\nu = p_\nu c, \quad (1.75)$$

using (1.63) and recognizing that the rest mass of the photon is zero. Merging (1.74) and (1.75), the photon momentum p_ν is given as

$$p_\nu = \frac{E_\nu}{c} = \frac{h\nu}{c} = \frac{h}{\lambda}, \quad (1.76)$$

highlighting the particle–wave duality of electromagnetic radiation, since both (1.74) and (1.76) contain within their structure a wave concept through wavelength λ and frequency ν as well as a particle concept through energy E_ν and momentum p_ν .

1.22.1 De Broglie Equation and de Broglie Wavelength

Following the recognition of particle–wave duality of electromagnetic radiation, *Louis de Broglie* in 1924 postulated a similar property for particles in particular and matter in general, namely a characterization with wavelength λ and momentum p related to one another through the following expression, already applicable to electromagnetic radiation, as shown in (1.74) and (1.75)

$$\lambda = \frac{h}{p}. \quad (1.77)$$

In relation to particles, (1.77) is referred to as de Broglie equation and λ is referred to as de Broglie wavelength of a particle. Using (1.64) and (1.77) we can calculate de Broglie wavelength λ of a particle as a function of its kinetic energy E_K to get

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar c}{E_K \sqrt{1 + \frac{2E_0}{E_K}}}. \quad (1.78)$$

De Broglie wavelength λ of a particle can also be expressed as a function of its velocity v or normalized velocity β using (1.42) and (1.43) to get

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{2\pi\hbar c}{\gamma m_0 c^2 \beta} = \frac{2\pi\hbar c}{m_0 c^2} \frac{\sqrt{1 - \beta^2}}{\beta}. \quad (1.79)$$

In Fig. 1.6 we plot (1.78) representing de Broglie wavelength λ for a given particle against the particle's kinetic energy E_K for an electron, proton and α -particle with solid curves and for a photon of energy $h\nu$ with the dashed line. Figure 1.7 displays (1.79), a plot of de Broglie wavelength λ as a function of the normalized velocity β for an electron, proton, and α -particle.

Fig. 1.6 Plot of (1.78) representing de Broglie wavelength λ against kinetic energy E_K for electron, proton, and alpha particle with solid curves and for photon with a dashed curve

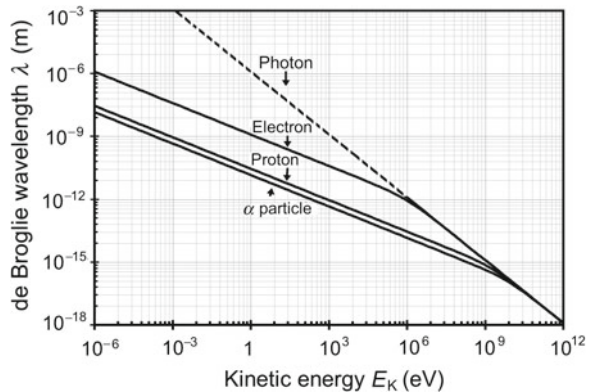
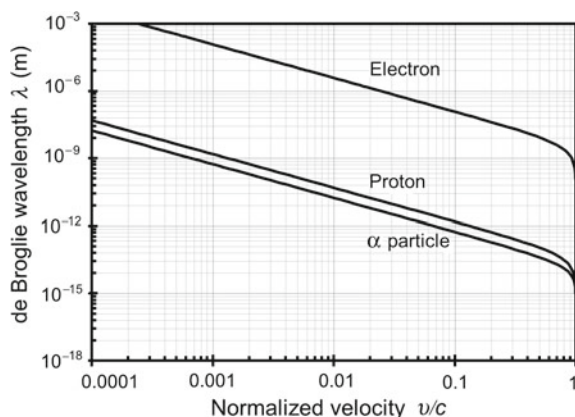


Fig. 1.7 Plot of (1.79) representing de Broglie wavelength λ against the normalized velocity v/c for electron, proton, and alpha particle



The following observations can be made from Figs. 1.6 and 1.7:

- At large kinetic energies, where $E_K \gg E_0$ and total energy $E \approx E_K$, de Broglie wavelength λ converges to $2\pi\hbar c/E_K$ coinciding with the photon line, so that all particles have the same de Broglie wavelength at a given energy.
- At low kinetic energies, as $E_K \rightarrow 0$, de Broglie wavelength λ approaches ∞ as $(2\pi\hbar c)/\sqrt{2E_0E_K}$; the larger is E_0 , the slower is the approach of λ to ∞ with a decreasing kinetic energy.
- For macroscopic objects that are moving with practical speed of the order of 250 m/s (airplane) or less de Broglie wavelength λ is extremely small with an order of magnitude of 10^{-34} m, some 16 orders of magnitude smaller than the highest resolution achievable experimentally.

1.22.2 Davisson–Germer Experiment

In 1927 Clinton J. Davisson and Lester H. Germer confirmed experimentally the wave nature of electrons by bombarding a nickel target with electrons and measuring the intensity of electrons scattered from the target. The target was in the form of a regular crystalline alloy that was formed through a special annealing process. The beam of electrons was produced by thermionic emission from a heated tungsten filament. The electrons were accelerated through a relatively low variable potential difference U that enabled the selection of the incident electron kinetic energy E_K . The scattered electrons were collected with a Faraday cup and their intensity was measured with a galvanometer.

Davisson and Germer discovered that for certain combinations of electron kinetic energy E_K and angle of incidence ϕ the intensity of scattered electrons exhibited maxima, similar to the scattering of x-rays from a crystal with a crystalline plane separation d that follows the Bragg relationship. Bragg has shown that for x-rays from two successive planes to interfere constructively their path lengths must differ

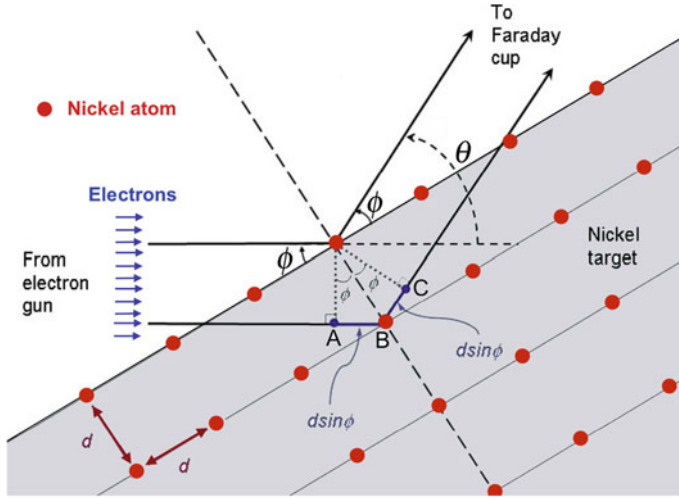


Fig. 1.8 Schematic representation of the Davisson–Germer experiment of elastic electron scattering on a nickel single crystal target. The electrons are produced in an electron gun and scattered by the nickel crystalline structure that has atom spacing d and acts as a reflection grating. The maximum intensity of scattered electrons occurs at Bragg angle of incidence ϕ as a result of constructive interference from electron matter waves following the Bragg relationship of (1.80). The scattering angle θ is the angle between the incident and scattered wave and equals to 2ϕ

by an integral number m of wavelengths λ . The first order diffraction maximum for $m = 1$ is the most intense.

Figure 1.8 shows the similarity between the Bragg experiment in x-ray scattering on a crystal with plane separation d and Davisson–Germer experiment of electron wave scattering on nickel atoms also arranged in crystalline planes with separation d . Constructive interference between waves scattered from two planes occurs when

$$\overline{AB} + \overline{BC} = 2d \sin \phi = m\lambda, \quad (1.80)$$

where

ϕ is the angle of incidence (Bragg angle) of the incident wave equal to the angle of reflection.

λ is the wavelength of the incident wave that may be a monoenergetic x-ray wave in Bragg experiment or matter (electron) wave in Davisson–Germer experiment.

Davisson and Germer determined the wavelength λ of electrons from the known atomic separation d and the measured Bragg angle ϕ at which the electron intensity exhibited a maximum. They found that the wavelength λ calculated with (1.80) agreed well with electron wavelength λ_e calculated from de Broglie relationship

$$\lambda_e = \frac{h}{p} = \frac{h}{m_e v} = \frac{2\pi\hbar c}{\sqrt{2m_e c^2 eU}}, \quad (1.81)$$

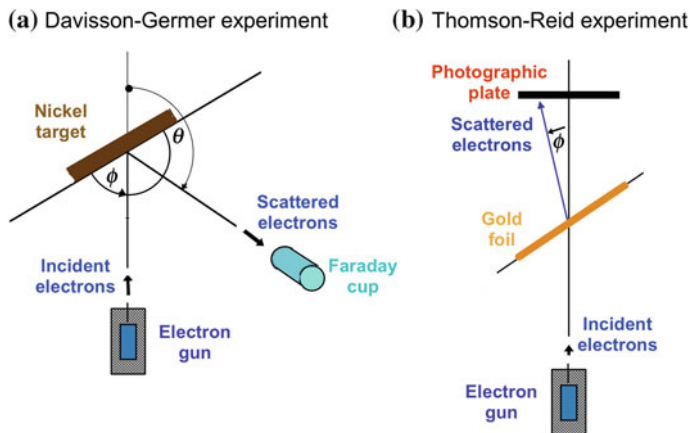


Fig. 1.9 Schematic representation of two seminal experiments to confirm the wave nature of electrons with diffraction patterns: **a** Davisson–Germer experiment using electron scattering on nickel crystalline structure and **b** Thomson–Reid experiment using electron scattering on a gold foil. In (a) electrons are collected with a Faraday cup, in (b) they are collected on a photographic plate

where v is the velocity of electrons determined from the classical kinetic energy relationship $E_K = \frac{1}{2}m_e v^2 = eU$ with U the applied potential of the order of 10 V to 100 V. The Davisson–Germer experiment, shown schematically in Fig. 1.9a, unequivocally demonstrated the diffraction of electrons and with it the wave nature of the electron.

1.22.3 Thomson–Reid Experiment

At about the same time as Davisson and Germer, *George P. Thomson* and his assistant A. Reid carried out an experiment that also confirmed de Broglie contention that matter can behave as waves. In contrast to Davisson and Germer who measured with a Faraday cup the intensity of electrons scattered from a nickel absorber, Thomson and Reid passed a collimated electron beam with kinetic energy of the order of 50 keV through a thin gold or aluminum foil (thickness of about 1000 Å and measured the transmitted electron intensity with a photographic plate.

Because of the crystalline structure of the metallic foil and the wave nature of the electrons, the intensity of electrons measured with film appeared in the form of concentric circles governed by the Bragg condition (1.80). The diameter of each ring is proportional to de Broglie wavelength of the electron and inversely proportional to the speed of the electron, similar to the results found in Davisson–Germer experiment. The Thomson–Reid experiment is shown schematically in Fig. 1.9b.

1.22.4 General Confirmation of Particle–Wave Duality

Davisson–Germer and Thomson–Reid experiments confirmed that electrons behave as waves under certain conditions, and other experimentalists have subsequently tested and confirmed the universal character of the de Broglie postulate by observing similar diffraction results for hydrogen and helium atoms as well as for neutron beams.

The experimental discovery of electron and neutron diffraction was a very important finding in support of quantum mechanics. The wavelength λ associated with a particle is called its de Broglie wavelength and is defined as the Planck constant divided by the particle momentum p , as shown in (1.77).

The experimentally determined *particle–wave duality* suggests that both models can be used for particles as well as for photon radiation. However, for a given measurement only one of the two models will apply. For example, in the case of photon radiation, the Compton effect is explained with the particle model, while the diffraction of x-rays is explained with the wave model. On the other hand, the charge-to-mass ratio e/m_e of the electron implies a particle phenomenon, while the electron diffraction suggests wave-like behavior.

Both Davisson and Thomson shared the 1937 Nobel Prize in Physics for their discovery of the diffraction of electrons by crystals. It is interesting to note that Joseph J. Thomson, the father of George Thomson, discovered the electron as particle in 1897, while George Thomson confirmed the wave nature of the electron in 1927.

1.23 Matter Waves

1.23.1 Introduction to Wave Mechanics

Associated with any particle is a matter wave, as suggested by the de Broglie relationship of (1.77). This matter wave is referred to as the particle’s wave function $\Psi(z, t)$ for one-dimensional problems or $\Psi(x, y, z, t)$ for three-dimensional problems and contains all the relevant information about the particle. Quantum mechanics, developed by *Erwin Schrödinger* as wave mechanics and *Werner Heisenberg* as matrix mechanics between 1925 and 1929, is a branch of physics that deals with the properties of wave functions as they pertain to particles, nuclei, atoms, molecules, and solids.

The main characteristics of wave mechanics are as follows:

- The theory has general application to microscopic systems and includes Newton theory of macroscopic particle motion as a special case in the macroscopic limit.
- The theory specifies the laws of wave motion that the particles of any microscopic system follow.

- The theory provides techniques for obtaining the wave functions for a given microscopic system.
- It offers means to extract information about a particle from its wave function.

The main attributes of wave functions $\Psi(z, t)$ are:

- Wave functions are generally but not necessarily complex and contain the imaginary number i .
- Wave functions cannot be measured with any physical instrument.
- Wave functions serve in the context of the Schrödinger wave theory but contain physical information about the particle they describe.
- Wave functions must be single-valued and continuous functions of z and t to avoid ambiguities in predictions of the theory.

The information on a particle can be extracted from a complex wave function $\Psi(z, t)$ through a postulate proposed by *Max Born* in 1926 relating the probability density $dP(z, t)/dz$ in one dimension and $dP(x, y, z, t)/dV$ in three dimensions with the wave functions $\Psi(z, t)$ and $\Psi(x, y, z, t)$, respectively, as follows

$$\frac{dP(z, t)}{dz} = \Psi^*(z, t) \cdot \Psi(z, t) \quad (1.82)$$

and

$$\frac{dP(x, y, z, t)}{dV} = \Psi^*(x, y, z, t) \cdot \Psi(x, y, z, t), \quad (1.83)$$

where

Ψ^* is the complex conjugate of the wave function Ψ .

V stands for volume.

The probability density is real, non-negative and measurable. In one-dimensional wave mechanics, the total probability of finding the particle somewhere along the z axis in the entire range of the z axis is equal to one, if the particle exists. We can use this fact to define the following *normalization condition*

$$\int_{-\infty}^{+\infty} \frac{dP(z, t)}{dz} dz = \int_{-\infty}^{+\infty} \Psi^*(z, t) \Psi(z, t) dz = 1. \quad (1.84)$$

Similarly, in three-dimensional wave mechanics, the normalization expression is written as

$$\int_{-\infty}^{+\infty} \frac{dP(x, y, z, t)}{dV} dV = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi^*(x, y, z, t) \Psi(x, y, z, t) dV = 1, \quad (1.85)$$

where the volume integral extends over all space and represents a certainty that the particle will be found somewhere (unit probability). Any one-dimensional wave function $\Psi(z, t)$ that satisfies (1.84) is by definition normalized. Similarly, any three-dimensional wave function $\Psi(x, y, z, t)$ that satisfies (1.85) is also normalized.

While the normalization condition expresses certainty that a particle, if it exists, will be found somewhere, the probability that the particle will be found in any interval $a \leq z \leq b$ is obtained by integrating the probability density $\Psi^* \cdot \Psi$ from a to b as follows

$$P = \int_a^b \Psi^* \cdot \Psi dV. \quad (1.86)$$

1.23.2 Quantum Mechanical Wave Equation

The particulate nature of photons and the wave nature of matter are referred to as the wave–particle duality of nature. The waves associated with matter are represented by the wave function $\Psi(x, y, z, t)$ that is a solution to a quantum mechanical wave equation. This wave equation cannot be derived directly from first principles of classical mechanics; however, it must honor the following four conditions:

1. It should respect the *de Broglie postulate* relating the wavelength λ of the wave function with the momentum p of the associated particle

$$p = \frac{h}{\lambda} = \hbar k, \quad (1.87)$$

where k is the wave number defined as $k = 2\pi/\lambda$.

2. It should respect *Planck law* relating the frequency ν of the wave function with the total energy E of the particle

$$E = h\nu = \hbar\omega, \quad (1.88)$$

where ω is the angular frequency of the wave function.

3. It should respect the relationship expressing the total energy E of a particle of mass m as a sum of the particle's kinetic energy $E_K = p^2 / (2m)$ and potential energy V , i.e.,

$$E = \frac{p^2}{2m} + V. \quad (1.89)$$

4. It should be linear in $\Psi(z, t)$ which means that any arbitrary linear combination of two solutions for a given potential energy V is also a solution to the wave equation.

While the wave equation cannot be derived directly, we can determine it for a free particle in a constant potential and then generalize the result to other systems and other potential energies. The free particle wave function in one dimension $\Psi(z, t)$ can be expressed as follows

$$\Psi(z, t) = C e^{i(kz - \omega t)}, \quad (1.90)$$

where $(kz - \omega t)$ is the phase of the wave with $k = 2\pi/\lambda$ the wave number and $\omega = 2\pi\nu$ the angular frequency of the wave.

We now determine the partial derivatives $\partial/\partial z$ and $\partial/\partial t$ of the wave function to obtain

$$\frac{\partial \Psi(z, t)}{\partial z} = ikC e^{i(kz - \omega t)} = ik\Psi(z, t) = i\frac{p}{\hbar}\Psi(z, t) \quad (1.91)$$

and

$$\frac{\partial \Psi(z, t)}{\partial t} = -i\omega C e^{i(kz - \omega t)} = -i\omega\Psi(z, t) = i\frac{E}{\hbar}\Psi(z, t). \quad (1.92)$$

Equations (1.91) and (1.92) can now be written as follows

$$p\Psi(z, t) = -i\hbar \frac{\partial}{\partial z}\Psi(z, t) \quad (1.93)$$

and

$$E\Psi(z, t) = i\hbar \frac{\partial}{\partial t}\Psi(z, t), \quad (1.94)$$

where $(-i\hbar\partial/\partial z)$ and $(i\hbar\partial/\partial t)$ are differential operators for the momentum p and total energy E , respectively.

Equations (1.93) and (1.94) suggest that multiplying the wave function $\Psi(z, t)$ by a given physical quantity, such as p and E in (1.93) and (1.94), respectively, has the same effect as operating on $\Psi(z, t)$ with an operator that is associated with the given physical quantity. As given in (1.89), the total energy E of the particle with mass m is the sum of the particle's kinetic and potential energies. If we now replace p and E in (1.89) with respective operators, given in (1.93) and (1.94), respectively, we obtain

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V = i\hbar \frac{\partial}{\partial t}. \quad (1.95)$$

Equation (1.95) represents two new differential operators; the left hand side operator is referred to as the hamiltonian operator $[H]$, the right hand side operator is the operator for the total energy E . When the two operators of (1.95) are applied to a free particle wave function $\Psi(z, t)$ we get

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(z, t)}{\partial z^2} + V \Psi(z, t) = i\hbar \frac{\partial \Psi(z, t)}{\partial t}. \quad (1.96)$$

Equation (1.96) was derived for a free particle moving in a constant potential V ; however, it turns out that the equation is valid in general for any potential energy $V(z, t)$ and is referred to as the time-dependent Schrödinger equation with $V(z, t)$ the potential energy describing the spatial and temporal dependence of forces acting on the particle of interest. The time-dependent Schrödinger equation is thus in the most general three dimensional form written as follows

$$-\frac{\hbar^2}{2m} \nabla^2 \Psi(x, y, z, t) + V(x, y, z, t) \Psi(x, y, z, t) = i\hbar \frac{\partial \Psi(x, y, z, t)}{\partial t}. \quad (1.97)$$

1.23.3 Time-Independent Schrödinger Equation

In most physical situations the potential energy $V(z, t)$ only depends on the spatial coordinate z , i.e., $V(z, t) = V(z)$ and then the time-dependent Schrödinger equation can be solved with the method of separation of variables. The wave function $\Psi(z, t)$ is written as a product of two functions $\psi(z)$ and $T(t)$, one depending on the spatial coordinate z only and the other depending on the temporal coordinate t only, i.e.,

$$\Psi(z, t) = \psi(z) T(t). \quad (1.98)$$

Inserting (1.98) into the time-dependent wave equation given in (1.96) and dividing by $\psi(z)T(t)$ we get

$$-\frac{\hbar^2}{2m} \frac{1}{\psi(z)} \frac{\partial^2 \psi(z)}{\partial z^2} + V(z) = i\hbar \frac{1}{T(t)} \frac{\partial T(t)}{\partial t}. \quad (1.99)$$

Equation (1.99) can be valid in general only if both sides, the left hand side that depends on z alone and the right hand side that depends on t alone, are equal to a constant, referred to as the separation constant Λ . We now have two ordinary differential equations: one for the spatial coordinate z and the other for the temporal coordinate t given as follows

$$-\frac{\hbar^2}{2m} \frac{d^2 \psi(z)}{dz^2} + V(z) \psi(z) = \Lambda \psi(z) \quad (1.100)$$

and

$$\frac{dT(t)}{dt} = -\frac{i\Lambda}{\hbar} T(t). \quad (1.101)$$

The solution to the temporal equation (1.101) is

$$T(t) = \exp\left(-i\frac{\Lambda}{\hbar}t\right), \quad (1.102)$$

representing simple oscillatory function of time with angular frequency $\omega = \Lambda/\hbar$. According to de Broglie and Planck, the angular frequency must also be given as E/\hbar , where E is the total energy of the particle.

We can now conclude that the separation constant Λ equals the total particle energy E and obtain from (1.101) the following solution to the temporal equation

$$T(t) = \exp\left(-i\frac{E}{\hbar}t\right) = e^{-i\omega t}. \quad (1.103)$$

Recognizing that $\Lambda = E$ we can write (1.100) as

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(z)}{dz^2} + V(z)\psi(z) = E\psi(z) \quad (1.104)$$

and obtain the so-called *time-independent Schrödinger wave equation* for the potential $V(z)$.

The essential problem in quantum mechanics is to find solutions to the time-independent Schrödinger equation for a given potential energy V , generally only depending on spatial coordinates. The solutions are given in the form of:

1. Physical wave functions $\psi(x, y, z)$ referred to as *eigenfunctions*.
2. Allowed energy states E referred to as *eigenvalues*.

The time independent Schrödinger equation does not include the imaginary number i and its solutions, the eigenfunctions, are generally not complex. Since only certain functions (eigenfunctions) provide physical solutions to the time-independent Schrödinger equation, it follows that only certain values of E , referred to as *eigenvalues*, are allowed. This results in discrete energy values for physical systems and it also results in energy quantization.

Many mathematical solutions are available as solutions to wave equations. However, to serve as a physical solution, an eigenfunction $\psi(z)$ and its derivative $d\psi/dz$ must be:

1. *Finite*
2. *Single valued*
3. *Continuous*

Corresponding to each eigenvalue E_n is an eigenfunction $\psi_n(z)$ that is a solution to the time-independent Schrödinger equation for the potential $V_n(z)$. Each eigenvalue

is also associated with a corresponding wave function $\Psi(z, t)$ that is a solution to the time-dependent Schrödinger equation and can be expressed as

$$\Psi(z, t) = \psi(z)e^{-i\frac{E}{\hbar}t}. \quad (1.105)$$

1.23.4 Measurable Quantities and Operators

As the term implies, a measurable quantity is any physical quantity of a particle that can be measured. Examples of measurable physical quantities are: position z , momentum p , kinetic energy E_K , potential energy V , total energy E , etc.

In quantum mechanics an operator is associated with each measurable quantity. The operator allows for a calculation of the average (expectation) value of the measurable quantity, provided that the wave function of the particle is known.

The *expectation value* (also referred to as the average or mean value) \bar{Q} of a physical quantity Q , such as position z , momentum p , potential energy V , and total energy E of a particle, is determined as follows provided that the particle's wave function $\Psi(z, t)$ is known

$$\bar{Q} = \int \Psi^*(z, t)[Q]\Psi(z, t)dz, \quad (1.106)$$

where $[Q]$ is the operator associated with the physical quantity Q . A listing of most common measurable quantities in quantum mechanics and their associated operators is given in Table 1.8. Two entries are given for the momentum, kinetic energy, and Hamiltonian operators: in one dimension and in three dimensions.

The quantum uncertainty ΔQ for any measurable quantity Q is given as

$$\Delta Q = \sqrt{\bar{Q}^2 - \bar{Q}^2}, \quad (1.113)$$

where \bar{Q}^2 is the square of the expectation value of the quantity Q and \bar{Q}^2 is the expectation value of Q^2 . The following conditions apply for ΔQ :

- When $\Delta Q = 0$, the measurable quantity Q is said to be sharp and all measurements of Q yield identical results.
- In general, $\Delta Q > 0$ and repeated measurements result in a distribution of measured points.

1.23.5 Transition Rate and the Fermi Second Golden Rule

Many physical interactions that particles undergo can be described and evaluated with the help of the transition rate between the initial state i and the final state f of

Table 1.8 Several measurable quantities and their associated operators used in quantum mechanics

Measurable	Symbol	Associated operator	Symbol	Equation
Position	z	z	$[z]$	(1.107)
Momentum	p	$-i\hbar \frac{\partial}{\partial z}$ or $-i\hbar \nabla$	$[p]$	(1.108)
Potential energy	V	V	$[V]$	(1.109)
Kinetic energy	E_K	$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2}$ or $-\frac{\hbar^2}{2m} \nabla^2$	$[E_K]$	(1.110)
Hamiltonian	H	$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V$ or $-\frac{\hbar^2}{2m} \nabla^2 + V$	$[H]$	(1.111)
Total energy	E	$i\hbar \frac{\partial}{\partial t}$	$[E]$	(1.112)

Two entries are given for the momentum, kinetic energy, and Hamiltonian operators, one-dimensional on the left and 3-dimensional on the right

the particular system consisting of the particle and the potential operator acting on it. In general, the transition rate (also called reaction rate) W_{if} is equal to the transition probability per unit time and depends upon:

1. Coupling between the initial and final states.
2. Density of the final states $\rho(E_f)$ defined as the number of levels per energy interval.

The transition (reaction) rate W_{if} is usually expressed with the following relationship referred to as the *Fermi second golden rule*

$$W_{if} = \frac{2\pi}{\hbar} |\mathbf{M}_{if}|^2 \rho(E_f), \quad (1.114)$$

with \mathbf{M}_{if} denoting the amplitude of the transition matrix element (probability amplitude) and describing the dynamics of the particular interaction. \mathbf{M}_{if} is usually expressed in an integral form as

$$\mathbf{M}_{if} = \iiint \psi_f^* V \psi_i dV, \quad (1.115)$$

where

- ψ_i is the wave function of the initial state.
- ψ_f^* is the complex conjugate of the wave function of the final state.
- V is the potential which operates on the initial state wave function ψ_i , i.e., is the potential operator which couples the initial and final eigenfunctions of the system.

Fermi second golden rule is derived from time dependent perturbation theory and is used widely in atomic, nuclear, and particle physics to deal with a wide variety of interactions, such as atomic transitions, beta decay, gamma decay, particle scattering (Rutherford, Mott) and nuclear reactions. At relatively low energies, all these interactions also play an important role in medical physics and radiation dosimetry.

1.23.6 Particle Scattering and Born Collision Formula

Much of the current knowledge in atomic, nuclear, and particle physics comes from analyses of scattering experiments carried out with various particles as projectile interacting with a target represented by atoms, nuclei, or other particles. An important parameter for describing a given scattering interaction is the differential cross section $d\sigma/d\Omega$ for the scattering interaction. The differential cross section is defined as the probability of scattering into a given solid angle $d\Omega$ and in quantum mechanics it is expressed as a transition rate, highlighting the most important use of the Fermi second golden rule stated in (1.114)

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{\hbar} |\mathbf{M}_{if}|^2 \rho(E_f), \quad (1.116)$$

where the matrix element \mathbf{M}_{if} for the scattering interaction is given in (1.115).

The differential cross section for a scattering process can be calculated with many degrees of sophistication, and the result of the calculation is accepted as valid when it agrees with the experimental result. Some of the issues to be considered and accounted for in the calculations are:

1. Classical versus relativistic mechanics.
2. Elastic versus inelastic scattering.
3. Point-like versus finite projectile.
4. Point-like versus finite scatterer.
5. No spin versus spin of projectile.
6. No spin versus spin of scatterer.
7. Shape of the coupling potential.

The matrix element \mathbf{M}_{if} is usually calculated using the Born approximation under the following assumptions:

1. Only a single scattering event occurs.
2. Initial and final states of the particle undergoing the scattering event are described by plane waves.

The wave function ψ_i of the plane wave for the initial particle to undergo scattering is given by

$$\psi_i(\mathbf{r}) = C \exp(i\mathbf{k}_i \cdot \mathbf{r}) = C \exp\left[\frac{i\mathbf{p}_i \cdot \mathbf{r}}{\hbar}\right], \quad (1.117)$$

where

\mathbf{k}_i is the initial wave vector of the particle (projectile) before interaction, related to the incident particle momentum \mathbf{p}_i through $\mathbf{k}_i = \mathbf{p}_i/\hbar$.

\mathbf{p}_i is the initial momentum of the particle before scattering.

C is a normalization constant obtained from the normalization equation (see (1.85)) $\int \psi_i^* \psi_i dV = 1$.

Similarly, the wave function ψ_f of the final plane wave is given as

$$\psi_f(\mathbf{r}) = C \exp(i\mathbf{k}_f \mathbf{r}) = C \exp\left[\frac{i\mathbf{p}_f \mathbf{r}}{\hbar}\right], \quad (1.118)$$

where

\mathbf{k}_f is the final wave vector of the particle (projectile) after the scattering event, related to the final particle momentum \mathbf{p}_f through $\mathbf{k}_f = \mathbf{p}_f/\hbar$.

\mathbf{p}_f is the final momentum of the particle after scattering.

C is a normalization constant obtained from the normalization equation (see (1.85)) $\int \psi_f^* \psi_f dV = 1$.

Figure 1.10 shows the general relationship between the initial momentum of the particle $\mathbf{p}_i = \hbar\mathbf{k}_i$ before scattering and the final momentum of the particle $\mathbf{p}_f = \hbar\mathbf{k}_f$ after scattering. The momentum transfer $\Delta\mathbf{p}$ from the particle to the scatterer is given as

$$\Delta\mathbf{p} = \mathbf{p}_i - \mathbf{p}_f \quad (1.119)$$

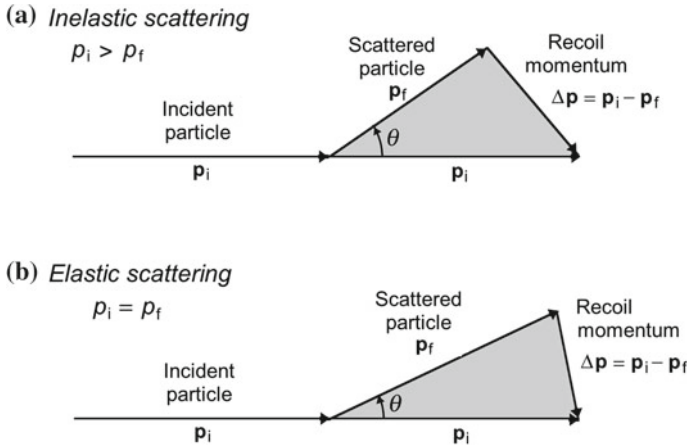


Fig. 1.10 General relationship between the initial momentum of the particle \mathbf{p}_i before scattering and the final momentum of the particle \mathbf{p}_f after scattering. $\Delta\mathbf{p}$ is the momentum transfer from the incident particle to the scatterer and θ is the scattering angle. Part **a** is for general inelastic scattering with Δp given by (1.121); part **b** is for elastic scattering with Δp given by (1.123)

and we define the wave vector \mathbf{K} as

$$\mathbf{K} = \frac{\Delta \mathbf{p}}{\hbar} = \frac{\mathbf{p}_i}{\hbar} - \frac{\mathbf{p}_f}{\hbar} = \mathbf{k}_i - \mathbf{k}_f. \quad (1.120)$$

Using the law of cosines we express the magnitudes of $\Delta \mathbf{p}$ and \mathbf{K} as follows

$$\Delta p = |\Delta \mathbf{p}| = \sqrt{|\mathbf{p}_i|^2 + |\mathbf{p}_f|^2 - 2 |\mathbf{p}_i| |\mathbf{p}_f| \cos \theta} = \sqrt{p_i^2 + p_f^2 - 2 p_i p_f \cos \theta} \quad (1.121)$$

and

$$K = |\mathbf{K}| = \frac{1}{\hbar} \sqrt{|\mathbf{p}_i|^2 + |\mathbf{p}_f|^2 - 2 |\mathbf{p}_i| |\mathbf{p}_f| \cos \theta} = \frac{1}{\hbar} \sqrt{p_i^2 + p_f^2 - 2 p_i p_f \cos \theta}, \quad (1.122)$$

where θ , as shown in Fig. 1.10, is the scattering angle.

When the scattering process is elastic, the magnitude of the initial momentum is equal to the magnitude of the final momentum, i.e., $|\mathbf{p}_i| = |\mathbf{p}_f| = p$ and (1.121) and (1.122) yield

$$\Delta p = |\Delta \mathbf{p}| = \sqrt{2p^2 - 2p^2 \cos \theta} = \sqrt{2p^2 (1 - \cos \theta)} = 2p \sin \frac{\theta}{2}. \quad (1.123)$$

and

$$K = |\mathbf{K}| = \frac{\Delta p}{\hbar} = \frac{2p}{\hbar} \sin \frac{\theta}{2} = \frac{2}{\lambda} \sin \frac{\theta}{2}. \quad (1.124)$$

The matrix element (scattering amplitude) \mathbf{M}_{if} for spherically symmetrical central scattering potential $V(r)$ is in general written as

$$\mathbf{M}_{if} = C^2 \iiint e^{-i\mathbf{k}_f \cdot \mathbf{r}} V e^{+i\mathbf{k}_i \cdot \mathbf{r}} d\mathcal{V} = C^2 \iiint V(r) e^{+i\mathbf{K} \cdot \mathbf{r}} d\mathcal{V} = V(\mathbf{K}) \quad (1.125)$$

using (1.120) for \mathbf{K} . The right hand integral in (1.125) is the Fourier transform of the central potential $V(r)$ representing $V(r)$ in the momentum space. For the central potential $V(r)$ the matrix element \mathbf{M}_{if} can now be simplified to get

$$\begin{aligned} \mathbf{M}_{if} &= C^2 \iiint V(r) e^{+i\mathbf{K} \cdot \mathbf{r}} d\mathcal{V} = C^2 \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} V(r) e^{i\mathbf{K} \cdot \mathbf{r}} r^2 \sin \Theta dr d\Theta d\phi \\ &= 2\pi C^2 \int_0^{\infty} r^2 V(r) \left\{ \int_{-1}^{+1} e^{iKr \cos \Theta} d \cos \Theta \right\} dr. \end{aligned} \quad (1.126)$$

First, we deal with the integral inside the curly bracket of (1.126) to get

$$\int_{-1}^{+1} e^{iKr \cos \Theta} d(\cos \Theta) = \left[\frac{e^{iKr \cos \Theta}}{iKr} \right]_{\cos \Theta = -1}^{\cos \Theta = 1} = \left[\frac{e^{iKr} - e^{-iKr}}{iKr} \right] = \frac{2 \sin(Kr)}{Kr}, \quad (1.127)$$

and finally we have the following expression for the matrix element \mathbf{M}_{if}

$$\mathbf{M}_{if} = 4\pi C^2 \int_0^\infty r^2 V(r) \frac{\sin(Kr)}{Kr} dr. \quad (1.128)$$

The differential scattering cross section $d\sigma/d\Omega$ (1.116) is now given as

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{2\pi}{\hbar} |\mathbf{M}_{if}|^2 \rho(E_f) = \left\{ \frac{2\pi}{\hbar} (4\pi C^2)^2 \right\} \rho(E_f) \left| \int_0^\infty r^2 V(r) \frac{\sin(Kr)}{Kr} dr \right|^2 \\ &= \left| \frac{2m}{\hbar^2} \int_0^\infty r^2 V(r) \frac{\sin(Kr)}{Kr} dr \right|^2. \end{aligned} \quad (1.129)$$

Equation (1.129) is referred to as the *Born collision formula* and is valid for elastic scattering brought about by a spherically symmetrical central scattering potential $V(r)$. The derivation of the density of final states $\rho(E_f)$ is cumbersome but can be found in standard quantum mechanics and nuclear physics texts.

1.24 Uncertainty Principle

In classical mechanics the act of measuring the value of a measurable quantity does not disturb the quantity; therefore, the position and momentum of an object can be determined simultaneously and precisely. However, when the size of the object diminishes and approaches the dimensions of microscopic particles, it becomes impossible to determine with great precision at the same instant both the position and momentum of particles or radiation nor is it possible to determine the energy of a system in an arbitrarily short time interval.

Werner Heisenberg in 1927 proposed the *uncertainty principle* that limits the attainable precision of measurement results. The uncertainty principle covers two distinct components:

One component (*momentum-position uncertainty principle*) deals with the simultaneous measurement of the position z and momentum p_z of a particle and limits the attainable precision of z and p_z measurement to the following product

$$\Delta z \Delta p_z \geq \frac{1}{2} \hbar, \quad (1.130)$$

where

Δz is the uncertainty on z .

Δp_z is the uncertainty on p_z .

Thus, there are no limits on the precision of individual z and p_z measurements. However, in a simultaneous measurement of z and p_z the product of the two uncertainties cannot be smaller than $\frac{1}{2} \hbar$ where \hbar is the reduced Planck constant ($\hbar = \frac{1}{2\pi} h$). Thus, if z is known precisely ($\Delta z = 0$), then we cannot know p_z , since ($\Delta p_z = \infty$). The reverse is also true: if p_z is known exactly ($\Delta p_z = 0$), then we cannot know z , since $\Delta z = \infty$.

The other component (*energy-time uncertainty principle*) deals with the measurement of the energy E of a system and the time interval Δt required for the measurement. Similarly to the (Δz , Δp_z) situation, the Heisenberg uncertainty principle states the following

$$\Delta E \Delta t \geq \frac{1}{2} \hbar, \quad (1.131)$$

where

ΔE is the uncertainty in the energy determination.

Δt is the time interval taken for the measurement.

Classical mechanics sets no limits on the precision of measurement results and allows a deterministic prediction of the behavior of a system in the future. Quantum mechanics, on the other hand, limits the precision of measurement results and thus allows only probabilistic predictions of the system's behavior in the future.

1.25 Complementarity Principle

In 1928 *Niels Bohr* proposed the *principle of complementarity* postulating that any atomic scale phenomenon for its full and complete description requires that both its wave and particle properties be considered and determined. This is in contrast to macroscopic scale phenomena where particle and wave characteristics (e.g., billiard ball as compared to water wave) of the same macroscopic phenomenon are mutually incompatible rather than complementary.

The Bohr principle of complementarity is thus valid only for atomic size processes and asserts that these processes can manifest themselves either as waves or as particles (corpuscles) during a given experiment, but never as both during the same experiment. However, to understand and describe fully an atomic scale physical

process the two types of properties must be investigated with different experiments, since both properties complement rather than exclude each other.

The most important example of this *particle–wave duality* is the photon, a massless particle characterized with energy, frequency and wavelength. However, in certain experiments such as in Compton effect the photon behaves like a particle; in other experiments such as double-slit diffraction it behaves like a wave.

Another example of the particle–wave duality are the wave-like properties of electrons as well as heavy charged particles and neutrons that manifest themselves through diffraction experiments (Davisson–Germer and Thomson–Reid experiments; see Sect. 1.22.2 and 1.22.3, respectively).

1.26 Emission of Electrons from Material Surface: Work Function

Emission of electrons from the surface of a solid material into vacuum is an important phenomenon governed by the so-called work function $e\phi$ defined as the minimum energy that must be supplied to an electron to remove it from the surface of a given material. For condensed matter $e\phi$ is of the order of a few electron volts and presents an effective surface barrier preventing electrons from leaving the material under normal circumstances. However, electrons can be liberated from the material surface into vacuum through various effects such as, for example:

1. Energy equal to or exceeding the work function $e\phi$ can be supplied to surface electrons by photons of energy $h\nu$ larger than $e\phi$, typically in the visible or near ultraviolet region. The electrons obtain sufficient energy to overcome the surface potential barrier and can leave the metal surface. The effect is referred to as the surface photoelectric effect or photoemission and Albert Einstein is credited with explaining the effect theoretically in 1905 on the basis of quantization of electromagnetic radiation.
2. Heating a metal to temperature above 1000°C increases the kinetic energy of electrons and enables these electrons to overcome the potential barrier and leave the metal surface. Emission of electrons under this condition is referred to as thermionic emission (see Sect. 1.27) and forms the basis for production of electrons with hot cathode in Coolidge x-ray tubes, electron guns of linear accelerators, and many other modern sources of electrons.
3. Placing a material into a very strong electric field may deform the material potential barrier and allow unexcited electrons to escape through the surface barrier from the condensed material into vacuum. This leakage or tunneling of electrons through the potential barrier is referred to as field emission (FE) and, as discussed in Sect. 1.28, has found use in electron microscopes, flat panel displays and, more recently has shown promise in electron sources based on cold cathode manufactured with carbon nanotubes.

4. In radiation dosimetry, weak electron emission from pre-irradiated condensed matter dosimeters (phosphors) is referred to as exoelectron emission (EE) and can be stimulated by heating of the phosphor to get thermally stimulated EE or by exposing the phosphor with visible or ultraviolet light to get optically stimulated EE.

1.27 Thermionic Emission

Thermionic emission, a very important phenomenon in medical physics, is defined as the flow of charge carriers from the surface of a solid or over some kind of potential barrier, facilitated by supplying thermal energy to the solid. Charge carriers so released from the solid are called thermions and the science dealing with the phenomenon is called thermionics. The most common practical example of thermionic emission is the emission of electrons from a hot metal cathode into vacuum, as used in filaments of Coolidge x-ray tubes, electron guns of linear accelerators, and so-called thermionic diodes.

Many physicists have contributed to the science of thermionics, most notably *Owen W. Richardson* who in 1928 received the Nobel Prize in Physics for his work and the discovery of the law governing the phenomenon. Saul Dushman demonstrated the modern form of the law governing the thermionic emission and the law is now referred to as the Richardson–Dushman equation. It expresses the relationship between the current density j in ampere per meter of electrons emitted from the metal and the absolute temperature T in kelvin of the metal as

$$j = A_R T^2 e^{-\frac{e\phi}{kT}}, \quad (1.132)$$

where

A_R is the Richardson constant with theoretical value of $1.2 \times 10^6 \text{ A} \cdot \text{m}^{-2} \cdot \text{K}^{-2}$.

$e\phi$ is the work function of the metal.

k is Boltzmann constant ($1.381 \times 10^{-23} \text{ J} \cdot \text{K}^{-1} = 0.8617 \times 10^{-4} \text{ eV} \cdot \text{K}^{-1}$).

From (1.132) it is evident that thermionic emission is controlled by three characteristics of the emitter: (1) its temperature; (2) its material composition (work function); and (3) its surface area. The current density j rises rapidly with temperature T of the emitter and decreases with an increase in the work function $e\phi$. The work function is defined, similarly to the definition of the atomic ionization energy, as:

- Ionization of a free atom involves removing one of its outer shell electrons by giving it an energy which is equal or exceeds its binding energy and is referred to as the ionization energy of the atom. The range of atomic ionization energy in nature is from a few electron volts for alkali elements to 24.6 eV for the noble gas helium.

- To remove an electron from a solid consisting of an array of atoms an electron in the conduction band must be supplied a minimum energy referred to as the work function typically ranging from 2 eV to 5 eV for most elements. The magnitude of the work function for a metal is usually about a half of the ionization potential of the free atom of the metal.

In the absence of external electric field electrons leaving a heated metal in vacuum form a cloud surrounding the emitter. The cloud, referred to as a space charge, represents an electronic equilibrium in which the number of electrons governed by (1.132) that leave the emitter and enter the cloud is equal to the number of electrons that are attracted back to the emitter. As the temperature of the emitter increases, the number of electrons in the space charge also increases because of the increased electron emission, as predicted by (1.132).

In practical use of thermionic emission, the thermionic emitter is not only heated but is also immersed in an external electric field. This field enhances the emission current density j if it has the same sign as the emitted thermion charge and diminishes it if the signs are opposite. In practice, one is interested in enhancing j with external electric field. This corresponds to lowering the work function $e\phi$ by $e\Delta\phi$ where $\Delta\phi$ is proportional to $\mathcal{E}^{1/2}$ where \mathcal{E} is the electric field applied externally, such as, for example, the field applied between the cathode and anode of a Coolidge x-ray tube. The Richardson–Dushman equation (1.132) then reads

$$j = A_R T^2 e^{-\frac{e\phi - e\Delta\phi}{kT}}. \quad (1.133)$$

The effect of lowering the work function of a metal by applying an external electric field is called the Schottky effect and (1.133) is referred to as the Schottky equation. In typical x-ray tubes the external electric field is such that all electrons emitted from the filament are attracted to the anode and no space charge around the filament is present. At very high external electric fields of the order of 10^8 V/m quantum tunneling begins to contribute to the emission current and the effect is referred to as field emission (see Sect. 1.28.2).

1.28 Tunneling

The particle–wave duality is highlighted in discussions of potential wells and potential barriers, both important phenomena in quantum and wave mechanics; the potential wells attract and trap particles, potential barriers reflect or transmit them. While medical physics and clinical physics rarely deal with quantum and wave mechanics, there are several physical phenomena of importance to radiation physics and, by extension, to medical physics that can only be explained through wave-mechanical reasoning. *Tunneling*, for example, is a purely wave-mechanical phenomenon that is used in explaining α decay and field emission: two important effects in radiation physics.

In addition, there are several other phenomena of importance in electronics that can be explained invoking tunneling such as, for example, in the periodic inversion of the ammonia molecule NH_3 , used as standard in atomic clocks, and in a semiconductor device called tunnel diode that is used for fast switching in electronic circuits.

A *classical particle* incident on a square barrier will pass the barrier only if its kinetic energy E_K exceeds the barrier potential E_P . If $E_P > E_K$, the classical particle is reflected at the barrier and no transmission occurs.

A *quantum-mechanical particle* incident on a square barrier has access to regions on both sides of the barrier, irrespective of the relative magnitudes of kinetic energy E_K and the barrier potential E_P . A matter wave is associated with the particle and it has a non-zero magnitude on both sides of the barrier as well as inside the barrier. The wave penetrates and traverses the barrier even when $E_P > E_K$, clearly contravening classical physics but conforming to the rules of wave mechanics. The non-zero probability for finding the particle on the opposite side of the barrier indicates that the particle may tunnel through the barrier or one may say that the particle undergoes the tunneling effect. In tunneling through a barrier, the particle behaves as a pure wave inside the barrier and as a pure particle outside the barrier.

1.28.1 Alpha Decay Tunneling

Alpha decay is considered a tunneling phenomenon in which α -particles with kinetic energies between 4 MeV and 9 MeV tunnel through a potential barrier of the order of 30 MeV. The tunneling theory of the α decay was proposed by *George Gamow* in 1928 (see Sect. 11.2). Inside the parent nucleus (atomic number Z) the α -particle is free yet confined to the nuclear potential well by the strong nuclear force. The dimension of the well is of the order of few femtometers; once the α -particle is beyond this distance from the center of the parent nucleus, it only experiences Coulomb repulsion between its charge $2e$ and the charge of the daughter nucleus $(Z - 2)e$.

A classical α -particle with $E_K < 9$ MeV cannot overcome a potential barrier with $E_P > 30$ MeV. On the other hand, a α -particle with wave-like attributes may tunnel through the potential barrier and escape the parent nucleus through this purely quantum-mechanical phenomenon.

1.28.2 Field Emission Tunneling

Emission of electrons from a solid into vacuum under the influence of a strong electric field aimed in a direction to accelerate electrons away from the surface is referred to as field emission (FE). Unlike thermionic emission (see Sect. 1.27), field emission does not depend on temperature of the material. The effect is purely quantum-mechanical and is attributed to wave-mechanical tunneling of electrons through the surface potential barrier which is affected by the strong electric field

so as to facilitate the tunneling process. Since electric field rather than heat is used to induce FE, the effect is often referred to as cold cathode emission, in contrast to thermionic emission which is stimulated by heat and consequently referred to as hot cathode emission.

In FE as well as in the Schottky effect the surface potential barrier is effectively lowered by the applied electric field. However, in Schottky effect the electrons surmount the barrier while in TE they tunnel through it.

Field emission has been known from the early days of quantum mechanics as a clear example of electron tunneling through a sufficiently thin potential barrier. The effect is also known as Fowler–Nordheim tunneling in honor of the physicists who in 1928 were the first to study it experimentally and theoretically. The tunneling probability shows exponential dependence on the tunneling distance which is inversely proportional to the electric field. The Fowler–Nordheim expression for the current density j in FE exhibits a functional dependence on electric field that is similar to the functional dependence of current density j on temperature T for thermionic emission given in (1.132).

Field emission occurs at surface points where the local electric field is extremely high, typically of the order of 1 V/nm or 10^9 V/m. These high electric fields are generated by applying relatively low voltages to needlelike metal tips with minute radius of curvature r of the order of 100 nm or less. Since the electric field equals $\sim V/r$, its magnitude for small r is very large.

Field emission has found practical application in solid-state electronic components such as tunnel diodes, high-resolution electron microscopy, high-resolution flat panel display, and many other electronic devices using an electron source. During recent years, cold cathodes based on carbon nanotubes (CNT) have shown promise in x-ray tube technology where they could serve as an alternative to Coolidge's hot cathode design (see Sect. 14.4.3).

1.29 Maxwell Equations

The basic laws of electricity and magnetism can be summarized by four Maxwell equations. The equations may be expressed in integral form or in differential form and the two forms are linked through two theorems of vector calculus: *Gauss divergence theorem* and *Stokes theorem*. In radiation physics and medical physics Maxwell equations play an important role in the understanding of bremsstrahlung production, in waveguide theory of particle acceleration, and in the theory of ionization chamber operation.

For a vector function \mathbf{A} and volume \mathcal{V} bounded by a surface \mathcal{S} the two vector calculus theorems are given as follows:

Gauss theorem

$$\iiint_V \nabla \cdot \mathbf{A} \, dV = \iiint_V \operatorname{div} \mathbf{A} \, dV = \oiint_S \mathbf{A} \cdot d\mathbf{S} \quad (1.134)$$

Stokes theorem

$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \iint_S \operatorname{curl} \mathbf{A} \cdot d\mathbf{S} = \oint_\ell \mathbf{A} \cdot d\boldsymbol{\ell} \quad (1.135)$$

The four Maxwell equations (integral form on the right, differential form on the left) are given as follows:

1. *Maxwell–Gauss equation for electricity* (also known as Gauss law for electricity)

$$\nabla \cdot \boldsymbol{\mathcal{E}} = \frac{\rho}{\varepsilon_0} \quad \oiint_S \boldsymbol{\mathcal{E}} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \iiint_V \rho \, dV = \frac{q}{\varepsilon_0} \quad (1.136)$$

2. *Maxwell–Gauss equation for magnetism* (also known as Gauss law for magnetism)

$$\nabla \cdot \mathbf{B} = 0 \quad \oiint_S \mathbf{B} \cdot d\mathbf{S} = 0. \quad (1.137)$$

3. *Maxwell–Faraday equation* (also known as Faraday law of induction)

$$\nabla \times \boldsymbol{\mathcal{E}} = -\frac{\partial \mathbf{B}}{\partial t} \quad \oint_\ell \boldsymbol{\mathcal{E}} \cdot d\boldsymbol{\ell} = -\frac{\partial}{\partial t} \iint_S \mathbf{B} \cdot d\mathbf{S} = -\frac{\partial \phi_{\text{mag}}}{\partial t} \quad (1.138)$$

4. *Maxwell–Ampère equation* (also known as Ampère circuital law)

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \boldsymbol{\mathcal{E}}}{\partial t} \quad \oint_\ell \mathbf{B} \cdot d\boldsymbol{\ell} = \mu_0 I + \frac{1}{c^2} \int_S \boldsymbol{\mathcal{E}} \cdot d\mathbf{S}, \quad (1.139)$$

where

- $\boldsymbol{\mathcal{E}}$ is the electric field in volt per meter (V/m).
- \mathbf{B} is the magnetic field in tesla T where $1 \text{ T} = 1 \text{ V} \cdot \text{s} / \text{m}^2$.
- ρ is the total charge density in coulomb per cubic meter (C / m^3).
- \mathbf{j} is the total current density in ampère per square meter (A / m^2).
- ε_0 is the permeability of vacuum (electric constant).
- μ_0 is are the permittivity of vacuum (magnetic constant).
- q is the total charge enclosed by the Gaussian surface S in coulombs (C).
- I is the total current passing through the surface S in ampères (A).

The four Maxwell equations combined with the *Lorentz force* and *Newton second law of motion* provide a complete description of the classical dynamics of interacting charged particles and electromagnetic fields. The Lorentz force \mathbf{F}_L for charge q moving in electric field \mathcal{E} and magnetic field \mathcal{B} with velocity \mathbf{v} is given as follows

$$\mathbf{F}_L = q(\mathcal{E} + \mathbf{v} \times \mathcal{B}). \quad (1.140)$$

Maxwell equations boosted the theory of electromagnetic fields in a similar fashion to the boost classical mechanics received from Newton laws. However, classical mechanics has subsequently been shown deficient at small dimensions on atomic and nuclear scale where quantum physics applies and at large velocities of the order of the speed of light in vacuum where relativistic physics applies. Maxwell equations, on the other hand, survived subsequent developments in physics related to quantum and relativistic mechanics and remain as valid today as they were when Maxwell introduced them some 150 years ago. With the theory of electromagnetic field, Maxwell equations succeeded in unifying electricity, magnetism, and photons on a broad spectrum ranging in frequency from radio waves to gamma rays.

1.30 Poynting Theorem and Poynting Vector

In 1884 English physicist *John Henry Poynting* used the Lorentz equation for a moving charge in an electromagnetic (EM) field and Maxwell equations for electromagnetism to derive a theorem that expresses the conservation of energy for EM fields. The theorem relates the rate of change of the energy u stored in the EM field and the energy flow expressed by the Poynting vector \mathbf{S} .

An electromagnetic field interacts with a charged particle q traveling with velocity \mathbf{v} through the Lorentz force \mathbf{F}_L

$$\mathbf{F}_L = \frac{d(m\mathbf{v})}{dt} = q(\mathcal{E} + \mathbf{v} \times \mathcal{B}). \quad (1.141)$$

Multiply (1.141) with velocity \mathbf{v} to obtain an energy relationship

$$\mathbf{v} \cdot \mathbf{F}_L = \mathbf{v} \cdot \frac{d(m\mathbf{v})}{dt} = \frac{1}{2} \frac{dmv^2}{dt} = q\mathbf{v} \cdot \mathcal{E}, \quad (1.142)$$

where only the electric field contributes to the particle's energy, since for the magnetic field $\mathbf{v} \cdot (\mathbf{v} \times \mathcal{B}) = 0$. We now multiply (1.142) with the particle density n to get

$$n \frac{d}{dt} \frac{mv^2}{2} = \frac{dE_K}{dt} = nq\mathbf{v} \cdot \mathcal{E} = \mathbf{j} \cdot \mathcal{E}, \quad (1.143)$$

where

E_K is the kinetic energy of the ensemble of charged particles.

\mathbf{j} is the current density $\mathbf{j} = nq\mathbf{v}$.

Next, we use the Ampère–Maxwell equation (1.139) to express the current density \mathbf{j} in terms of electric field \mathcal{E} and magnetic field \mathcal{B} .

$$\mathbf{j} \cdot \mathcal{E} = \frac{1}{\mu_0} \mathcal{E} \cdot \nabla \times \mathcal{B} - \varepsilon_0 \frac{\partial}{\partial t} \frac{\mathcal{E}^2}{2}, \quad (1.144)$$

where ε_0 and μ_0 are the electric permittivity of vacuum and magnetic permeability of vacuum, respectively. Using the vector identity

$$\nabla \cdot (\mathcal{E} \times \mathcal{B}) = \mathcal{B} \cdot \nabla \times \mathcal{E} - \mathcal{E} \cdot \nabla \times \mathcal{B} \quad (1.145)$$

we express (1.144) as

$$\mathbf{j} \cdot \mathcal{E} = -\frac{1}{\mu_0} \nabla \cdot (\mathcal{E} \times \mathcal{B}) - \frac{1}{\mu_0} \mathcal{B} \cdot \nabla \times \mathcal{E} - \varepsilon_0 \frac{\partial}{\partial t} \frac{\mathcal{E}^2}{2}. \quad (1.146)$$

Inserting (1.138), the Faraday–Maxwell equation, into (1.146) we get

$$\mathbf{j} \cdot \mathcal{E} = -\nabla \cdot \left(\mathcal{E} \times \frac{\mathcal{B}}{\mu_0} \right) - \frac{\partial}{\partial t} \left\{ \frac{\varepsilon_0 \mathcal{E}^2}{2} + \frac{\mathcal{B}^2}{2\mu_0} \right\} = -\nabla \cdot \mathcal{S} - \frac{\partial u}{\partial t}, \quad (1.147)$$

where

u is the electromagnetic field energy density

$$u = \varepsilon_0 \frac{\mathcal{E}^2}{2} + \frac{\mathcal{B}^2}{2\mu_0}. \quad (1.148)$$

\mathcal{S} is the Poynting vector

$$\mathcal{S} = \mathcal{E} \times \frac{\mathcal{B}}{\mu_0}, \quad (1.149)$$

representing the energy flow with dimensions energy/(area \times time) or power/area.

The Poynting theorem can also be expressed in integral form where the integration is carried out over an arbitrary volume \mathcal{V}

$$-\int_{\mathcal{V}} \mathbf{j} \cdot \mathcal{E} d\mathcal{V} = \int_{\mathcal{V}} \left[\frac{du}{dt} + \nabla \cdot \mathcal{S} \right] d\mathcal{V}. \quad (1.150)$$

The conservation of energy is with Poynting theorem thus expressed as follows: the time rate of change of electromagnetic (EM) energy within a given volume \mathcal{V} added to the energy leaving the given volume per unit time equals to the negative value of the total work done by the EM fields on the ensemble of charged particles encompassed by the given volume.

1.31 Normal Probability Distribution

Random variation in natural processes most commonly follows the probability distribution generally known in mathematics as the normal probability distribution but also referred to as Gaussian distribution in physics and “bell curve” in social science. The function describing the normal distribution has a long tradition in mathematics and physics. De Moivre used it in 18th century as an approximation to the binomial distribution, Laplace used it to study measurement errors, and Gauss used it in his analysis of astronomical data.

In general, the normal probability distribution is described by the following continuous *probability density function* $P(x)$

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}}, \quad (1.151)$$

where

σ is the standard deviation related to the width of the “bell curve”.

\bar{x} is the mean value of x also called the expectation value of x .

The following general characteristics apply to the probability distribution function $P(x)$ given by (1.151)

1. $P(x)$ has a peak at \bar{x} and is symmetric about \bar{x} .
2. $P(x)$ is unimodal.
3. $P(x)$ extends from $-\infty$ to $+\infty$.
4. The area under $P(x)$ equals to 1. i.e., $\int_{-\infty}^{+\infty} P(x)dx = 1$.
5. $P(x)$ is completely specified by the two parameters \bar{x} and σ .
6. $P(x)$ follows the empirical rule which states that (see Fig. 1.11):
 - 68.3% of data will fall within 1 standard deviation σ of the mean \bar{x} .
 - 95.5% of the data will fall within 2σ of the mean \bar{x} .
 - 99.7% of the data will fall within 3σ of the mean \bar{x} .
7. The inflection points on the $P(x)$ curve occur one standard deviation from the mean, i.e., at $\bar{x} - \sigma$ and $\bar{x} + \sigma$.

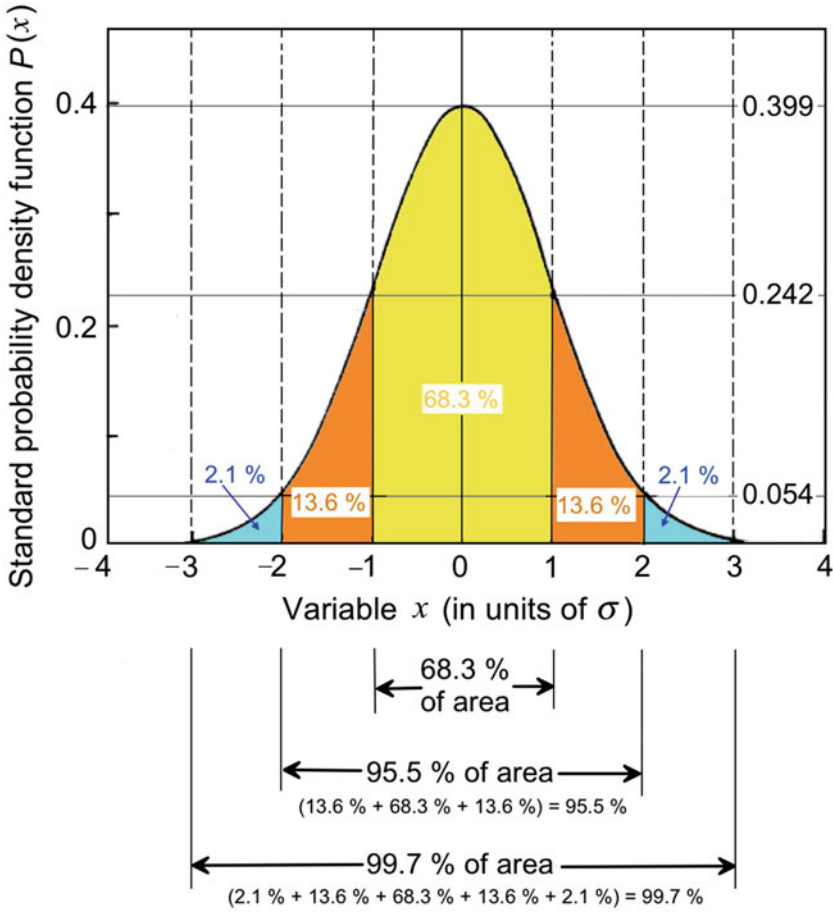


Fig. 1.11 Standard probability density function $P(x)$ of (1.152) with the variable x plotted in units of σ , the standard deviation of the mean. The total area under the curve from $x = -\infty$ to $x = +\infty$ is equal to 1. The *yellow* area under the probability curve represents 68.3% of the total area, i.e., $\frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} e^{-\frac{x^2}{2}} dx = 0.683$; the two *brown* areas represent 13.6% of the total area each, i.e.,

$$\frac{1}{\sqrt{2\pi}} \int_{-2}^{+2} e^{-\frac{x^2}{2}} dx = 2 \times 0.136 + 0.683 = 0.955$$

1.31.1 Standard Probability Density Function

For the special case of mean value $\bar{x} = 0$ and standard deviation $\sigma = 1$, the probability distribution is called the standard normal distribution and the probability density function of (1.151) simplifies into the *standard probability density function*

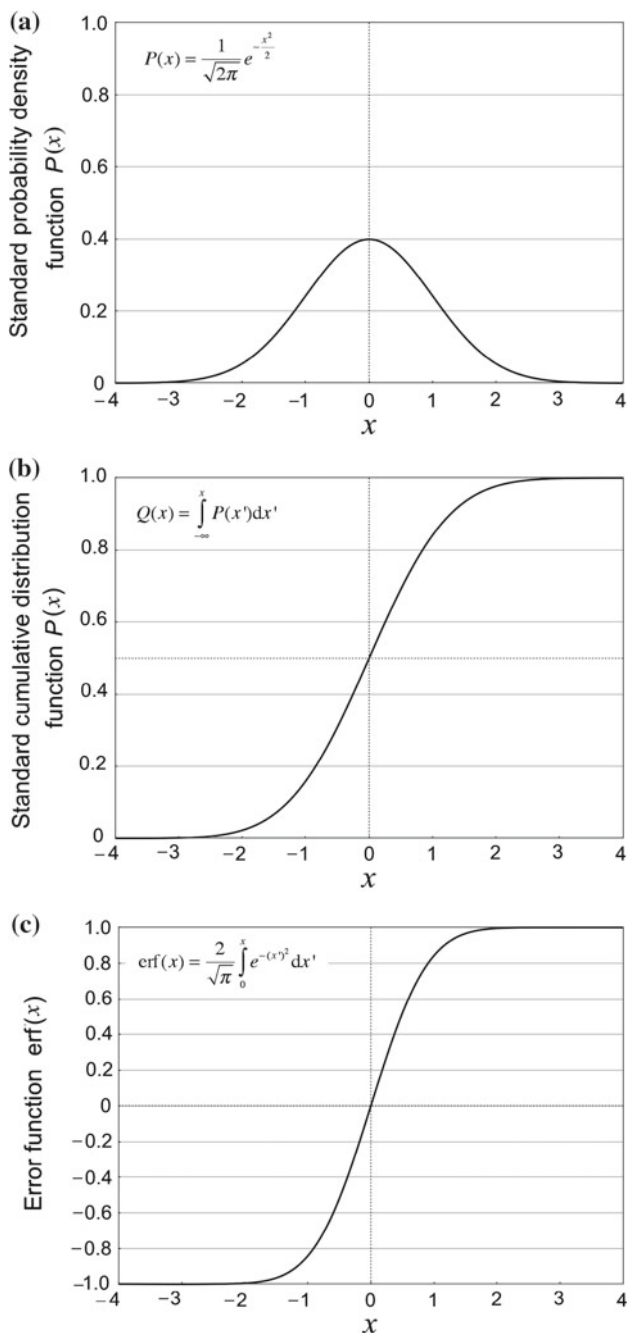


Fig. 1.12 Part **a** Standard probability density function $P(x)$ of (1.152) for the mean or expectation value $\bar{x} = 0$; standard deviation $\sigma = 1$, and variance $\nu = 1$. Part **b** Standard cumulative distribution function $Q(x)$ of (1.158) for the mean or expectation value $\bar{x} = 0$; standard deviation $\sigma = 1$, and variance $\nu = 1$. Part **c** Error function $\text{erf}(x)$ of (1.159)

$$P(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \quad (1.152)$$

The standard probability density function $P(x)$ of (1.152) is plotted in Figs. 1.11 and 1.12a. It exhibits the following notable characteristics:

1. For $x = 0$, $P(0) = 1/\sqrt{2\pi} = 0.399$.
 For $x = 1$, $P(1) = (1/\sqrt{2\pi})e^{-0.5} = 0.242$.
 For $x = 2$, $P(2) = (1/\sqrt{2\pi})e^{-2} = 0.054$.
2. *Total area* under the $P(x)$ curve from $x = -\infty$ to $x = +\infty$

$$\int_{-\infty}^{+\infty} P(x) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \int_0^{+\infty} e^{-\frac{x^2}{2}} dx = 1. \quad (1.153)$$

3. *Mean value or expectation value* of x

$$\bar{x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x e^{-\frac{x^2}{2}} dx = 0. \quad (1.154)$$

4. *Variance* v of x is a parameter giving a measure of the dispersion of a set of data points around the mean value. For the normal probability distribution the variance is defined as follows

$$v(x) = \overline{x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^2 e^{-\frac{x^2}{2}} dx = \frac{4}{\sqrt{\pi}} \int_0^{+\infty} u^2 e^{-u^2} du = 1. \quad (1.155)$$

5. *Standard deviation* σ of the mean \bar{x} is the square root of the variance v

$$\sigma = \sqrt{v} = 1. \quad (1.156)$$

1.31.2 Cumulative Distribution Function

A probability distribution can also be characterized with the general cumulative distribution function $Q(x)$ expressed as follows

$$Q(x) = \int_{-\infty}^x P(x') dx' = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(x'-\bar{x})^2}{2\sigma^2}} dx' = \frac{1}{2} \left[1 + \operatorname{erf} \frac{x - \bar{x}}{\sigma\sqrt{2}} \right], \quad (1.157)$$

while the standard cumulative distribution function is the general cumulative distribution function of (1.157) evaluated with $\bar{x} = 0$ and $\sigma = 1$

$$\begin{aligned}
Q(x) &= \int_{-\infty}^x P(x') dx' = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{(x')^2}{2}} dx' = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{2}}} e^{-\left(\frac{x'}{\sqrt{2}}\right)^2} d\frac{x'}{\sqrt{2}} \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^0 e^{-\left(\frac{x'}{\sqrt{2}}\right)^2} d\frac{x'}{\sqrt{2}} + \frac{1}{2} \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{2}}} e^{-\left(\frac{x'}{\sqrt{2}}\right)^2} d\frac{x'}{\sqrt{2}} = \frac{1}{2} \left[1 + \operatorname{erf} \frac{x}{\sqrt{2}} \right].
\end{aligned}
\tag{1.158}$$

The standard cumulative distribution function is plotted in Fig. 1.12b and exhibits the following notable properties:

1. $Q(x) = 0$ for $x = -\infty$.
2. $Q(x) = 0.5$ for $x = 0$.
3. $Q(x) = 1$ for $x = \infty$.

1.31.3 Error Function

In (1.157) and (1.158) “erf” denotes a special function of sigmoid shape called the error function defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-(x')^2} dx', \tag{1.159}$$

with the following notable features: $-1 \leq \operatorname{erf}(x) \leq +1$; $\operatorname{erf}(0) = 0$; $\operatorname{erf}(\infty) = 1$; $\operatorname{erf}(-x) = -\operatorname{erf}(x)$; and $\operatorname{erf}(-\infty) = -1$.

The error function $\operatorname{erf}(x)$ is plotted in Fig. 1.12c and its tabulated values are readily available in standard mathematical tables.

Monte Carlo Treatment Planning in Radiotherapy

In modern societies cancer became the most significant health care problem surpassing heart disease as the leading cause of death and potential years of life lost. The cancer rate is slowly but steadily growing mainly because of population aging, and currently stands at about 4500 new cancer patients per million population per year.

Radiotherapy is an important modality of cancer therapy and over 50% of all cancer patients receive radiation treatment either as the primary component of their treatment or as adjuvant therapy. Many crucial steps affect the final outcome of treatment and physics plays a pivotal role in many of these steps, such as in: (1) Imaging for cancer diagnosis, target localization, virtual simulation, and treatment planning and (2) Delivery of the prescribed dose either with external beam radiotherapy or with brachytherapy involving sealed internal sources.

Currently, linear accelerators (linacs) are the most important high technology machines for dose delivery in modern radiotherapy. Algorithms for calculation of dose distribution in the irradiated patient (treatment planning) are an important component of radiotherapy. They evolved during the past few decades from calculations that relied heavily on analytic, semi-analytic and empirical algorithms based on dose measurements in water to the current stage, where we are on the verge of using patient-specific Monte Carlo algorithm-based dose distribution calculation as the most accurate method for predicting the dose distribution in the patient. Accurate and reliable Monte Carlo algorithms have been available for some time, the main problem, however, was adequate computing power to allow fast, reliable, and efficient dose calculations. With the current rapid increase in computing power one can expect that routine treatment planning for radiotherapy will be carried out with Monte Carlo-based dose calculation engines in the near future.

Dose distributions can be calculated accurately only if the calculation adequately models: (1) the radiation source, (2) beam geometry, and (3) patient anatomy. Future Monte Carlo engines are expected to allow a single simulation involving radiation transport through the linac as well as the patient; to save on calculation time current techniques all share the common approach of separating calculations into two parts: (1) source and beam geometry and (2) patient.

Figure on next page shows results of a Monte Carlo simulation of a typical linac head (Varian, Clinac 2100 C) operating in the 6 MeV clinical electron mode. About 100 histories are displayed in addition to the main beam-forming components of the linac. The 6 MeV electron pencil beam produced in the accelerator waveguide of the linac is transmitted through the beryllium exit window and strikes the scattering foil to increase the beam cross sectional area. The scattered beam then passes through the monitor chamber and the field mirror and is shaped into the clinical beam by the photon collimator and the electron applicator (cone).

Figure on next page: Courtesy of Dr. David W.O. Rogers, Carleton University, Ottawa, Canada. Reproduced with permission.



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