

Chapter 2

Presentations into Investigations

As we all recognize, gaining any habit such as learning to read, walk, tie our shoes, etc., begins with being awkward and, as importantly, takes time to secure. But developing productive habits is of course a really good idea despite the complexity of that learning experience. Habits allow us to do things efficiently without giving much of any thought to the behavior, and that allows more time to do other interesting things. Indeed, they would seem essential for becoming a capable and thoughtful mathematics problem-solver. That suggests we need to take seriously what habits we as mathematics teachers should promote, and how we go about doing so. These would seem to be of fundamental concern in our work.

William James, a psychologist concerned about the educational experience, wrote how preeminent a role habit development should have in schools. As he saw it, “Education, in short, cannot be better described than by calling it *the organization of acquired habits of conduct and tendencies to behavior*” (1899/2008, p. 25, italics in original). And his fellow traveler, John Dewey, also recognized their heightened importance: “We state emphatically that, *upon its intellectual side, education consists in the formation of wide-awake, careful, thorough habits of thinking*” (1933/1936, p. 78; italics in original).

Here, we mathematics educators find ourselves face-to-face with the problem mentioned earlier. Given the press of covering the mathematics curriculum, being able to demonstrate algorithms tend to be the practices that we want our students to develop as habits. Yet, in a number of instances, their efficient demonstration provides little evidence why they work. As a direct consequence when presented without discussion, they can leave students more numb than educated or with a false sense of their mathematical capacity. (Consider, e.g., “to solve a proportion, cross multiply” or “to divide by a fraction, invert and multiply.”) They often seem not much different than magic tricks. They work, but of course the question is why, for nothing has otherwise been learned. (Readers interested in seeing a collection of such poor representations see *Nix the Tricks* by Tina Cardone and the MTBoS, *NixtheTricks.pdf*, updated January 30, 2014.)

Yet, what some students really like about mathematics is having specific techniques that allow them to solve problems. Knowing the division algorithm or how

to factor a quadratic expression, for example, or in general being able to deal directly with a problem situation by applying a technique demonstrates how efficient one is. But with teachers and texts presenting mathematics where solution models and algorithms are the primary focus to solve sets of problems that students then practice, students will likely have a surface knowledge of mathematics and themselves as mathematics students.

Experienced mathematics educators know well “There is no guarantee in any amount of information, even if skillfully conveyed, that an intelligent attitude of mind will be formed” (Dewey 1937, p. 183). This is made poignantly clear in a National Assessment of Educational Progress question analysis where students were asked to determine the value of $(2/3) \times (2/5)$. The findings were that 70% of 13-year-olds and 74% of 17-year-olds could do the multiplication correctly. But when those same students were presented with, “Jane lives $2/5$ of a mile from school; when she has walked $2/3$ of the way to school, how far has she walked?”, students demonstrated very little understanding, with 20% of the 13-year-olds and 21% of the 17-year-olds responding correctly.

Surely, efficient means to solving problems should be practiced. In that way, more interesting mathematics problems can be considered. However, what the research says is that “Mathematics learning has often been more a matter of memorizing than understanding” (Kilpatrick et al. 2001, p. 16). And with regard to memorizing algorithms, because of their form being one of technical efficiency, the thinking that gives legitimacy to the procedure is often hidden. So students may memorize a procedure and do well on an exam and yet have no idea why it works. They come to believe “In Math you have to remember, in other subjects you can think about it” (a student quoted in Boaler 2008). In this way, mathematics textbooks and teacher presentations which share the same surface aesthetic can actually be an impediment to student learning and their gaining “deep understanding.” The rather exclusive focus on efficient approaches promotes a lack of thoughtful experiences by omitting the otherwise needed time for developing the habit of being able to stay with a problem, that essential quality that life will reward with it happening.

* * *

Fortunately, mathematics algorithms and practices obscuring their rationale can often be re-presented by introducing problem-clarifying strategies, which over time, with dedicated focus, will become mental habits. Such strategies give students a more significant role in shaping the conversation, and so promote their understanding. “A ‘habit of mind’ means having a disposition toward behaving intelligently when confronted with problems” (Kosta and Kallick 2009). What follow will be companion pieces—standard algorithms and practices lacking explanation presented along with mathematical habits of mind, *problem-clarifying* strategies—a comparison that illustrates the opportunity for students’ thoughtful agency rather than their passive acceptance.

The focus will begin with some of the earlier mathematics experiences students are likely to have. For if we want students to “question the teacher and one another; [and] try to convince themselves and one another of the validity of particular representations..., and answers” (NCTM *Professional Teaching Standards*, Standard 3, p. 45), then the place to begin must be with presentations associated with students learning arithmetic procedures.

2.1 The Long-Division Algorithm and *Take Things Apart*

The long-division algorithm is clearly efficient in determining the quotient of a division problem involving multi-digit numbers. Yet, students often have considerable difficulty working with it, so much so that long division tends to be omitted from classroom discussions and is considered by many mathematics educators as not worth the time. This of course does not go uncontested. The authors of “Ten Myths About Math Education And Why You Shouldn’t Believe Them” (Budd and Carson et al. 2005), some of whom are mathematicians, are clearly upset with “the snubbing or outright omission of the long division algorithm by NCTM-based curricula...”. The conversation continues, and is quite heated. For example, in a recent *Education Week* blog, “Welcome Back, Long Division?”, David Ginsburg shares thinking on both sides of this contentious issue and concludes that it should not be invited back (blogs.edweek.org/teachers/coach_gs_teaching_tips/2014/05/welcome_back_long_division).

Yet, there are clearly common instances where long division can be of value. For example: “How many buses will we need to take our 526 students to the museum if each bus holds 36 students?”; “We have seven dozen lollipops. Will we have to buy more so that everyone in our class of 29 students gets the same number?” and “The trip will take 1250 miles. If we can average 55 miles an hour driving, how long will it take us to get there?” These seem to be reasonable considerations where long division would be perfectly good in resolving those questions (even if you do not think a child having four lollipops is reasonable!)

What exactly is the difficulty? One mathematics text made it perfectly clear in presenting the problem of dividing 17 into 231, and asking and answering, “how many 17s are in 23?”, to begin the division algorithm. Hopefully, students would be baffled by the question—not the answer, but why *that* would be the question given the problem. If the problem was to determine how many 17s are in 231, why are they being asked to determine how many 17s are in 23? They may well be asking themselves “why is what I am thinking not being discussed?”, and in the absence of a response that acknowledges their legitimate confusion, cognitive dissonance becomes the real problem. Here we can see why students could develop an aversion to mathematics, having its beginning with some confounding experiences with arithmetic.

If one did not think about what was going on but just answered the question and continued, the power of the algorithm would be made clear. But blindly following instructions would not be a tenet of a democratic society—quite the contrary. And in this particular case, it apparently has caused more misguided attempts than successful calculations. The algorithm offers an efficient means that makes the problem simpler, but it is not clear why to many students for good reason.

Consider a different way, one where the student’s experience shapes the conversation and procedure. Given the problem of how many 17s are in 231, students could be asked “How might you make the problem simpler?” From their earlier experience with division problems, they could well decide to “take away ten sev-

enteens—170,” as 170 divided by 17 is clearly 10. (This is what the algorithm is doing, but instead of putting “10” over the 31 in 231, a “1” is put above the 3.) Now that leaves 61 to be divided by 17. From here to determining the final answer of 13 with a remainder of 10, students can subtract multiples of 17 from 61 as a function of the degree of comfort and memorization they have gained, with all arriving at the correct answer. Granted the mathematical actions here to *take things apart* are not as crisp as the explicit procedure of the long-division algorithm. Yet, introducing this *problem-clarifying* heuristic first, as some texts do, would likely make the introduction of the textbook algorithm easier to accommodate after a while. In his blog, Ginsburg quotes Evelyn Hines, a teacher in Georgia, who wrote “... we do teach students to make sense of the problem. For example, 3657 divided by 35—a student looks for a ‘friendly number’, so he/she might say that 35 times 100 equals 3500 which leaves 657.... By the time they get to 5th grade, they can choose between the strategy or use the standard algorithm after they understand WHY the algorithm works” (emphasis in original). Here the mathematical agency is in the hands of the students, not the procedure.

Unfortunately, the pressure mathematics teachers experience to cover the curriculum in preparation for standardized testing may result in short-circuiting the very conversations students need to develop their number sense, with the time instead being dedicated to practicing the algorithm. With “good” students being those who readily adopt the otherwise puzzling procedure, our work as mathematics educators is made even more perplexing—for, apparently, being insensitive to what appears to be a mathematical action lacking explanation comes to be seen as a desirable quality! While other students for fear of their being seen as ignorant in questioning the evidently successful albeit confounding practice remain silent naturally lose some of the positive energy needed to try to make sense of what’s going on. It is not hard to imagine that sometimes the effects would be long-standing.

It is not only this algorithmic procedure of long division that tends to short circuit students’ inclination to inquire. It can be also seen in all the basic presentations of addition, subtraction, and multiplication, where the procedures that make sense to a naïve understanding tend to be omitted and in their place are procedures bereft of explanation. (The reader might consider, for example, when adding three-digit numbers the counterintuitive practice that begins with acting on the *least* significant digits. Imagine standing in front of three piles of money—\$100, \$10, and \$1 bills—and being asked how much money was there in total. Which pile suggests it be counted first? Also confusing is the practice of multiplying multi-digit numbers by breaking up the partial products and writing one part of the number in one place and the other part of the number in a different place; surely it is visually and mentally quite perplexing for a number of students.)

Building students’ mathematical intuition is essential in their becoming a good problem-solver. In the face of counterintuitive demonstrations, the mathematics experience as a shared journey is lost and the valuable aim of enhancing student–teacher and student–student constructive conversations is diminished. Instead, the interior dialogue students have with themselves becomes, at best, one involving an effort to dismiss their own concerns so as to internalize some perplexing practice.

At such times “... there is no contradiction in their saying, ‘I know that such and such is considered to be true, but I do not believe it’” (Confrey 1990, p. 111). This naturally sets up the psychologically discomforting condition of students being supported in their doing what they do not believe in.

2.2 Combining Fractions and *Changing Representation*

It is rather a common classroom experience that when presented with a fraction addition problem such as $\frac{2}{3} + \frac{4}{5}$, students often believe $\frac{6}{8}$ is the sum. In response, some teachers tell students that “you can’t add apples and oranges,” but this tends to leave many students mystified. After all, they added numerator with numerator and denominator with denominator—in effect, they added apples with apples, and oranges with oranges. (And to compound the confusion, later on when multiplying fractions, students will learn that you can *multiply* “apples” and “oranges”!) Sometimes, students are just told “you can’t add that way” without being provided any explanation. But a simple explanation is available. For example, using the model of adding numerators to numerators and denominators to denominators, we would have that $\frac{1}{2} + \frac{1}{2} = \frac{2}{4}$. But that clearly does not make sense since the left-hand side represents a totality of 1, as physical demonstrations would make clear.

Yet, inasmuch as the problem is such a common enduring misunderstanding, there must be something deeper that promotes youngsters adding that way. It could be that the problem beneath the surface is the symbolic representation—or more directly, the absence of any. Namely, from experience, youngsters winning two of three games of checkers on Monday and four of five games on Tuesday know they have won six of eight games. And with that distinction going unmentioned, the confusion naturally persists. Students fall back on their experience outside the classroom, on their very reasonable “combining habit” developed earlier in another context.

When combining numerical fractions, however, the parts are relative to the same whole. This suggests that to help students put the addition of fractions operation into perspective, the teacher, when introducing some *change in representation* such as using pizzas or chocolate cake, make the distinction regarding parts of the same unit being combined a conversation. After a few more similar problems, students will come to appreciate that fractions are associated with some common unit (“two-thirds of a cup,” etc.). Then it can be made clear in the abstracted, efficient practice of combining fractions where the common unit has been omitted, including using the number line. In this way, students can appreciate the *change of representation* now that they see what was involved. Yet this is not the only time that working with fractions causes considerable confusion, and for good reason.

Later on in their mathematics education, students will be introduced to another algorithm involving fractions—to determine which of two fractions is greater by

cross multiplying. This practice also mystifies students, and rightly so. When we compare $\frac{3}{4}$ and $\frac{5}{8}$, and determine that the first fraction is greater on cross multiplying, we find that 24 is greater than 20, the “cross-multiplying” algorithm omits the critical understanding—what is going on beneath the surface—that makes clear why it is legitimate to do so. In the absence of a conversation, students naturally come to believe memorizing is what they must do and paradoxically can feel less capable for having done so. They of course deserve to see that the procedure is in effect, creating a comparison of two fractions with the same denominators and looking to see which numerator was greater.

Yet there are other situations where working with fractions can introduce more perplexity to the naïve viewer, as when negative numbers are included. Consider the mathematics statement $\frac{-3}{6} = \frac{4}{-8}$. Simplifying both fractions or cross multiplying can serve to demonstrate that both fractions have the same value. But it would not eliminate students’ consternation regarding how the ratio of a smaller number to a larger number could be equal to the ratio of a larger number to a smaller number! That perplexing relationship has a history that stretches back to the seventeenth century, when a number of mathematicians expressed considerable discomfort in accepting such a statement (Kline 1972, p. 252). Surely we would not call upon the field axioms of mathematics to convince youngsters why the ratio of a negative number to a positive equals the ratio of a positive number to a negative. How can we help them appreciate their consternation is indeed legitimate? It is to be appreciated that the discomfort negative numbers have had affected some of the finest mathematicians. (That consideration will come in a while.)

2.3 Invert and Multiply and *Make the Problem Simpler*

Rather than use cross multiplication to determine which of two fractions was greater, we could more reasonably choose to divide one fraction by the other, and if the quotient was greater than one, then the fraction in the numerator would be the greater. But dividing fractions can be quite challenging. In its stead, students are often introduced to the efficient algorithm of “inverting and multiplying.” This dual procedure is another mathematics classroom experience which, for many students, is even more challenging in its acceptance.

In being shown that algorithm, students tend to have one of two responses: “Great—an easy way to divide by a fraction!”, and “What’s going on here?” To help clarify the procedure for all students, there is a conversation that draws upon the habit of mind to *make the problem simpler*. As one of the finest mathematical problem-solvers of the twentieth century, George Polya wrote, “If you cannot solve the proposed problem, could you imagine a more accessible related problem?” (1965, p. 114). (Part III of his *How To Solve It* is a “short dictionary of heuristic” with over 50 brief articles.)

If we apply *make the problem simpler* and determine the result, for example, of dividing 8 by $\frac{3}{4}$, we can find a rationale for “invert and multiply,” one that students will understand and may well appreciate. (To begin the conversation, it would seem necessary to first make such a problem a legitimate concern. For example, asking how many $\frac{3}{4}$ cup servings would there be in an eight-cup recipe, or how many steps would it take to walk across a room 8 yards long if each step was $\frac{3}{4}$ yard.)

The reason this problem is chosen is that it is not too easy or too hard to solve by what the students know already. If we asked “what if 8 was divided by $\frac{1}{2}$?”, some students could yell out “the answer is 16,” and then grouse about why they have to sit through another procedure when they were able to solve the problem. So let us see what is involved when 8 is divided by $\frac{3}{4}$. Surely, it must be more than 8. (Since 8 divided by 1 is 8, dividing by a smaller number means there must be more left over—as the two physical instances suggested.) It must also be less than 16, since 16 would be the number of halves in 8.

Now some students may conjecture that the answer is 12. This is a common and to be appreciated misconception as students are thinking of additive differences; namely $\frac{3}{4}$ is right between $\frac{1}{2}$ and 1 on the number line. However, the conjecture is worth acknowledging but not as being wrong. Students are demonstrating that they are thinking about and connecting to the problem, sharing what does seem to be the case to an intuition that is developing. Rather than tell them their conjecture is incorrect, for *conjecturing* is another habit of mind that is worth promoting, we have a wonderful opportunity for them to test their guess, to determine its *plausibility*—another habit of mind that is worth promoting.

In this case, *suppose* 8 divided by $\frac{3}{4}$ did equal 12, then $\frac{3}{4} \times 12 = 8$. But the product equals 9. So 12 is too big. The answer must be more toward the middle of 12 and 8. Is it 10? Then $\frac{3}{4} \times 10$ would equal 8. It does not, but it is closer! While the students’ conjectures were “wrong,” there is no reason to make that the focus. Learning to test answers as being plausible is a mathematical habit of mind to check our thinking; it is a truly valuable strategy. And too, making conjectures is often how we learn, as we learn to test our assumptions.

At this point in time, students see the answer must be close to 10. Further inroads with dividing by a fraction can be made if we *make the problem simpler* by considering what the easiest number to divide by is. Students may initially offer 10 or 2, but after a moment more of reflection they often come upon 1, as with 1 the division is done! So with the focus on the valuable habit of mind to *make the problem simpler*, the problem of determining “how many $\frac{3}{4}$ s are in 8?” turns into the problem to replace $\frac{3}{4}$ with 1 in the denominator, but without changing the value of the fraction.

Two thoughts regarding how the denominator could be made 1 usually come to students’ minds: add $\frac{1}{4}$ to $\frac{3}{4}$, or multiply $\frac{3}{4}$ by $\frac{4}{3}$. Trying each method is instructive—and to be appreciated is their intuitive judgement that whatever is done to the denominator must be done to the numerator, for otherwise it does not *feel* right. (The power of “what feels right” is surely to be respected, especially the more

experience one has. Werner Heisenberg, a winner of the Nobel Prize in Physics, said that if he arrived at an equation that did not feel right, he reconsidered his approach as his developed intuition suggested something was problematic.)

When students try both approaches, they discover that adding the same value to both the numerator and denominator, while intuitively seeming a good solution, is actually a problem. In this instance, adding $1/4$ to both creates a new fraction where the numerator is $8\frac{1}{4}$ and the denominator is 1. They know the answer to the original problem has to be closer to 10, as discussed earlier. (At this juncture, the teacher may decide to stop and reinforce that the action of adding the same quantity to the numerator and denominator of a fraction is not generally successful. For example, by pointing out to students that $2/3$ would become $3/4$ by adding 1 to each the numerator and the denominator.) Now they can try their second idea: multiplying the numerator and denominator by the reciprocal of the denominator. Here the numerator becomes $8 \times \frac{4}{3} = 10\frac{2}{3}$, with the denominator 1.

This answer is indeed plausible, as it is more than $8\frac{1}{4}$ and close to 10. To reinforce that thinking, we can consider another problem to which they likely know the answer: 8 divided by $\frac{1}{2}$. Having students use the method of creating 1 in the denominator by multiplying by the reciprocal of the denominator, students can see that 16 is the answer, which provides supporting recognition. Conversation can then turn to helping students appreciate that when they multiplied the denominator *and* the numerator by $\frac{4}{3}$ in the original problem, they in effect were multiplying the original fraction by $\frac{4}{3} / \frac{4}{3}$ —in essence, by 1. This of course ensures the numerical value of the original problem has not been changed, just represented differently. A few more practice problems and students can see that the *change of representation* by “inverting and multiplying” is really efficient—and now makes sense.

In addition, those students who expressed their confusion can deservedly experience the respected nods of their classmates. It was their question of “what’s going on here?” that served to promote the worthwhile and needed investigation. Writ large, it is that kind of questioning generated by students’ concerned interest which, supported by their mathematics teacher, would be the natural and logical way a mathematics class experience evolves if sense-making was the object. With more such truly educational experiences, as future citizens of a democratic society, our students would become comfortable with offering their concerned responses in the face of a statement made by someone in authority that appeared to them as confusing. (Experience suggests that experience is very much part of their future.)

2.4 Mathematical Slope and *Visualize*

Typically, algebra textbooks define a straight line as “ $y = mx + b$.” This equation demonstrates clearly how mathematics is so much more dense than spoken language. For instance, students could get a 10–20 page reading assignment, but they would hardly ever get an assignment of that size in a mathematics textbook. In the given linear equation, we find six symbols—the same number as in the word “window.” The mathematical expression represents a relationship, actually multiple relationships, with one overriding consideration that could well appear opaque to many students. That is to say a lot is packed into that symbolic representation.

Helping students begin to understand that equality by *taking things apart* enables them to get a hold of its richness, and in that way helps them become more comfortable tinkering with the different quantities in the equation. For one, letting them know that by agreement with the exceptional mathematician Rene Descartes, mathematicians use letters at the front of the alphabet to represent constants, while those at the end are chosen to represent variables. This distinction helps begin to take apart the dense equality statement for students. Also to be appreciated is that the letters in “the middle” are used to represent parameters, quantities that vary given the particular situation. For example, physics equations use the parameter “ g ” as the constant of gravitational attraction which takes on the value of 32 near Earth, and approximately 5.3 near the moon, as the moon’s gravity is about one-sixth that of Earth’s.

In the equation “ $y = mx + b$,” students then see that x and y are variables, and together represent any point in the two-dimensional plane; and that m and b are parameters representing constants, with m the slope of the line, and b the value of the y -intercept as a consequence of $x=0$. Quite an intricate relationship! What is immediately intriguing to some and confusing to other students is why the symbol m ? Surely a fits so much more appropriately with b ! (One can share that s was not available as it is taken to represent *distance* in physics formulas, perhaps as a consequence of being the first letter of *stadia*, a unit of distance from earlier times.) And yet, the reason just shared regarding the letters in the middle of the alphabet representing parameters does not really fit the given equation. Inasmuch as a is replaced by m then it would seem that b should also be replaced with some middle-located letter as it also represents a parameter. (Being confused here seems to make perfectly good sense!)

Hopefully, the mathematics teacher can take a naive perspective and appreciate what students are experiencing when presented with the equation of the general line in the x - y plane. There is a lot to deal with. Apparently, it is not known how m became the symbol to represent the slope value. However, with students graphing different lines as a function of their numerical values, they would start to see that the coefficient of the x -term is greater the steeper the lines. Then they would likely come to decide that it would be a good idea to think of that coefficient as the measure of comparative steepness. What else do they associate with steepness? Mountains seem a reasonable response. So why not use m to represent slope?

Descartes was French and wrote in Latin; so using m does make sense as it is the first letter of “*mons*,” Latin for “mountain,” which as a physical presence is surely distinguished one from another in terms of the difficulty of ascent as a function of its slope. Some mathematics historians say this explanation lacks evidence. However, until a better rationale is found for choosing m , it seems quite reasonable, given the goal of promoting students’ mathematical intelligence to use the students’ thinking.

But there is more to discuss here. Part of the usual presentation students receive regarding straight lines is the definition of the slope as “the change in y over the change in x .” But why should the slope be defined *that way*? What about “the change in x divided by the change in y ?” This may well be a question in the minds of some “naïve” but thoughtful students who are reluctant to ask, and for those students who accept the definition as is, such a consideration helps them to understand that definitions are not chosen without consideration. The resolution will take just a bit of time, but clearly it is worth that as the goal is that students become educated, which requires their making sense of what puzzles them. They can determine the wiser choice. The better slope–ratio representation would be decided by *visualizing* how the contrasting definitions would actually connect or not to graphed lines. In this way, they can appreciate not only their good question, but its resolution. In this way, students develop more sophisticated means of valuing that go beyond impulses of likes and dislikes and the choice made by the authority of others, and promote their own valued and valuable (mathematics) education.

2.5 Arithmetic Series and *Tinkering*

Mathematics textbooks often begin the study and formulation of an arithmetic series by relating the story of the 10-year-old student Carl Gauss, and his teacher who, wanting to promote more student discipline, had his students sum the numbers from 1 to 100. This was extremely tedious and annoying, especially with the “endless” summing on small writing slates! But precocious Carl noted that if the terms of the series $1 + 2 + 3 + \dots + 98 + 99 + 100$ were written again right below those numbers in reverse order, the answer could be immediately determined. Below 1 he wrote 100, below 2 he wrote 99, 98 below 3, etc., and it was clear that at the end of the series 3 would be below 98, 2 below 99, and 1 below 100. With summing those vertical pairings, Carl saw he would have 100 pairs of 101. This being twice the sought-after sum required dividing the product in half, arriving at 5050. Problem solved!

This tends to be where mathematics texts end the story so that students can then practice the procedure, and then proceed to consider other arithmetic series where the common difference is other than 1. The popularity of introducing arithmetic series by this approach can surely be understood and also questioned, especially with regard to its pedagogical value.

We would imagine Carl’s teacher was in a state of disbelief, for it would be hours to do the problem in the straightforward manner, not minutes or less! He realized he was dealing with a most precocious 10-year-old mathematical mind. As such,

Enabling Students in Mathematics
A Three-Dimensional Perspective for Teaching
Mathematics in Grades 6-12

Gordon, M.

2016, XVI, 147 p. 19 illus. in color., Hardcover

ISBN: 978-3-319-25404-3