2

Infinite-Dimensional Vector Spaces and Sequences

After the introduction to frames in finite-dimensional vector spaces in Chapter 1, the rest of the book will deal with expansions in infinite-dimensional vector spaces. Here great care is needed: we need to replace finite sequences $\{f_k\}_{k=1}^n$ by infinite sequences $\{f_k\}_{k=1}^\infty$, and suddenly the question of convergence properties becomes a central issue. The vector space itself might also cause problems, e.g., in the sense that Cauchy sequences might not be convergent. We expect the reader to have a basic knowledge about these problems and the way to circumvent them, but for completeness we repeat the central definitions and results concerning Banach spaces and operators hereon in Sections 2.1–2.2. In Sections 2.3–2.4 we specialize to Hilbert spaces and their operators. Section 2.5 deals with pseudo-inverse operators; this subject is not expected to be known and is treated in more detail. Section 2.6 introduces the so-called moment problems in Hilbert spaces. In Sections 2.7–2.9, we discuss the Hilbert space $L^2(\mathbb{R})$ consisting of the square integrable functions on $\mathbb{R}$ and three classes of operators hereon, as well as the Fourier transform. The material in those sections is not needed for the study of frames and bases on abstract Hilbert spaces, but it forms the basis for all the constructions in Chapters 9–20.

2.1 Banach Spaces and Sequences

A central theme in this book is to find conditions on a sequence $\{f_k\}$ in a vector space X such that every $f \in X$ has a representation as a

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superposition of the vectors f_k. In most spaces appearing in functional analysis, this cannot be done with a finite sequence $\{f_k\}$. We are therefore forced to work with infinite sequences, say, $\{f_k\}_{k=1}^{\infty}$, and the representation of f in terms of $\{f_k\}_{k=1}^{\infty}$ will be via an infinite series. For this reason, the starting point must be a discussion of convergence of infinite series. We collect the basic definitions here together with some conventions.

Throughout the section, we let X denote a complex vector space. A *norm* on X is a function $||\cdot|| : X \to [0,\infty[$ satisfying the following three conditions:

(i) $||x|| = 0 \Leftrightarrow x = 0;$

(ii) $||\alpha x|| = |\alpha|\ ||x||,\ \forall x \in X,\ \alpha \in \mathbb{C};$

(iii) $||x+y|| \leq ||x|| + ||y||,\ \forall x, y \in X.$

In situations where more than one vector space appear, we will frequently denote the norm on X by $||\cdot||_X$. If X is equipped with a norm, we say that X is a *normed vector space*. The *opposite triangle inequality* is satisfied in any normed vector space:

$$||x-y|| \geq |\ ||x|| - ||y||\ |,\ \ x, y \in X. \tag{2.1}$$

We say that a sequence $\{x_k\}_{k=1}^{\infty}$ in X

(i) converges to $x \in X$ if

$$||x - x_k|| \to 0 \text{ for } k \to \infty;$$

(ii) is a *Cauchy sequence* if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$||x_k - x_\ell|| \leq \epsilon \text{ whenever } k, \ell \geq N.$$

A convergent sequence is automatically a Cauchy sequence, but the opposite is not true in general. There are, however, normed vector spaces in which a sequence is convergent if and only if it is a Cauchy sequence; a space X with this property is called a *Banach space.*

Imitating the finite-dimensional setting described in Chapter 1, we want to study sequences $\{f_k\}_{k=1}^{\infty}$ in X with the property that each $f \in X$ has a representation $f = \sum_{k=1}^{\infty} c_k f_k$ for some coefficients $c_k \in \mathbb{C}$. In order to do so, we have to explain exactly what we mean by convergence of an infinite series. There are, in fact, at least three different options; we will now discuss these options.

First, the notation $\{f_k\}_{k=1}^{\infty}$ indicates that we have chosen some ordering of the vectors f_k,

$$f_1, f_2, f_3, \dots, f_k, f_{k+1}, \dots \quad .$$

We say that an *infinite series* $\sum_{k=1}^{\infty} c_k f_k$ is convergent with sum $f \in X$ if

$$\left|\left| f - \sum_{k=1}^{n} c_k f_k \right|\right| \to 0 \text{ as } n \to \infty.$$

If this condition is satisfied, we write

$$f = \sum_{k=1}^{\infty} c_k f_k. \tag{2.2}$$

Thus, the definition of a convergent infinite series corresponds exactly to our definition of a convergent sequence with $x_n = \sum_{k=1}^{n} c_k f_k$.

Above we insisted on a fixed ordering of the sequence $\{f_k\}_{k=1}^{\infty}$. It is very important to notice that convergence properties of $\sum_{k=1}^{\infty} c_k f_k$ not only depend on the sequence $\{f_k\}_{k=1}^{\infty}$ and the coefficients $\{c_k\}_{k=1}^{\infty}$ but also on the ordering. Even if we consider a sequence in the simplest possible Banach space, i.e., a sequence $\{a_k\}_{k=1}^{\infty}$ in $\mathbb{C}$, it can happen that $\sum_{k=1}^{\infty} a_k$ is convergent but that $\sum_{k=1}^{\infty} a_{\sigma(k)}$ is divergent for a certain permutation σ of the natural numbers (Exercise 2.1). This observation leads to the second definition of convergence. If $\{f_k\}_{k=1}^{\infty}$ is a sequence in X and $\sum_{k=1}^{\infty} f_{\sigma(k)}$ is convergent for all permutations σ, we say that $\sum_{k=1}^{\infty} f_k$ is *unconditionally convergent*. In that case, the limit is the same regardless of the order of summation.

As soon as we have defined frames and Riesz bases in Hilbert spaces, it will become clear that they automatically lead to unconditionally convergent expansions. For this reason, we never need to prove by hand that a given series converges unconditionally. For the sake of completeness, we refer to [495] and [577] for a more detailed analysis of the different types of convergence and the proof of the following lemma.

Lemma 2.1.1 *Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in a Banach space X, and let $f \in X$. Then the following are equivalent:*

(i) $\sum_{k=1}^{\infty} f_k$ converges unconditionally to $f \in X$.

(ii) For every $\epsilon > 0$ there exists a finite set F such that

$$\left|\left| f - \sum_{k \in I} f_k \right|\right| \leq \epsilon$$

for all finite sets $I \subset \mathbb{N}$ containing F.

Finally, an infinite series $\sum_{k=1}^{\infty} f_k$ is said to be *absolutely convergent* if

$$\sum_{k=1}^{\infty} ||f_k|| < \infty.$$

In any Banach space, absolute convergence of $\sum_{k=1}^{\infty} f_k$ implies that the series converges unconditionally (Exercise 2.2), but the opposite does not hold in infinite-dimensional spaces (see page 51 in [401] or page 68 in [464] and the references therein). In finite-dimensional spaces, the two types of convergence are identical.

A subset $Z \subseteq X$ (countable or not) is said to be *dense* in X if for each $f \in X$ and each $\epsilon > 0$ there exists $g \in Z$ such that

$$||f - g|| \leq \epsilon.$$

In words, this means that elements in X can be approximated arbitrarily well by elements in Z.

For a given sequence $\{f_k\}_{k=1}^{\infty}$ in X, we let $\text{span}\{f_k\}_{k=1}^{\infty}$ denote the vector space consisting of all *finite* linear combinations of vectors f_k, i.e.,

$$\text{span}\,\{f_k\}_{k=1}^{\infty} = \{\alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_N f_N \,\big|\, N \in \mathbb{N}, \alpha_1, \alpha_2, \ldots, \alpha_N \in \mathbb{C}\}.$$

The definition of convergence shows that if each $f \in X$ has a representation of the type (2.2), then each $f \in X$ can be approximated arbitrarily well in norm by elements in $\text{span}\{f_k\}_{k=1}^{\infty}$, i.e.,

$$\overline{\text{span}}\{f_k\}_{k=1}^{\infty} = X. \tag{2.3}$$

A sequence $\{f_k\}_{k=1}^{\infty}$ having the property (2.3) is said to be *complete* or *total.* We note that there exist normed spaces where no sequence $\{f_k\}_{k=1}^{\infty}$ is complete. A normed vector space, in which a countable and dense family exists, is said to be *separable.*

When considering expansions of the form (2.2), the coefficients c_k are real or complex numbers. In case at most finitely many entries c_k are nonzero, we say that $\{c_k\}_{k=1}^{\infty}$ is a *finite sequence.*

2.2 Operators on Banach Spaces

Let X and Y denote Banach spaces. A linear map $U : X \to Y$ is called an *operator,* and U is *bounded* or *continuous* if there exists a constant $K > 0$ such that

$$||Ux||_Y \leq K\, ||x||_X, \ \forall x \in X. \tag{2.4}$$

Usually, it will be clear from the context which norm we use, so we will write $||\cdot||$ for both $||\cdot||_X$ and $||\cdot||_Y$. The *norm* of the operator U, denoted by $||U||$, is the smallest constant K that can be used in (2.4). Alternatively,

$$||U|| = \sup\left\{||Ux|| \,\big|\, x \in X, ||x|| = 1\right\}.$$

If U_1 and U_2 are operators for which the range of U_2 is contained in the domain of U_1, we can consider the composed operator U_1U_2; if U_1 and U_2 are bounded, then also U_1U_2 is bounded, and

$$||U_1U_2|| \leq ||U_1||\ ||U_2||. \tag{2.5}$$

Now consider a sequence of operators $U_n : X \to Y,\ n \in \mathbb{N}$, which converges pointwise to a mapping $U : X \to Y$, i.e.,

$$U_n x \to Ux, \ \text{as } n \to \infty, \ \forall x \in X.$$

We say that U_n converges to U in the *strong operator topology.* The *Banach–Steinhaus theorem*, also known as the *uniform boundedness principle,* states the following (see page 69 in [621]) or page 14 in [401]):

Theorem 2.2.1 *Let $U_n : X \to Y,\ n \in \mathbb{N}$, be a sequence of bounded operators, which converges pointwise to a mapping $U : X \to Y$. Then U is linear and bounded. Furthermore, the sequence of norms $||U_n||$ is bounded, and $||U|| \leq \liminf ||U_n||$.*

An operator $U : X \to Y$ is *invertible* if U is surjective and injective. For a bounded, invertible operator, the inverse operator is bounded; see, e.g., page 286 in [464]:

Theorem 2.2.2 *A bounded bijective operator between Banach spaces has a bounded inverse.*

In case $X = Y$, it makes sense to speak about the identity operator I on X. The *Neumann theorem* states that an operator $U : X \to X$ is invertible if it is close enough to the identity operator; a proof can be found on page 48 in [401].

Theorem 2.2.3 *If $U : X \to X$ is bounded and $||I - U|| < 1$, then U is invertible, and*

$$U^{-1} = \sum_{k=0}^{\infty} (I - U)^k. \tag{2.6}$$

Furthermore,

$$||U^{-1}|| \leq \frac{1}{1 - ||I - U||}.$$

Note that (2.6) should be interpreted in the sense of the operator norm, i.e., as

$$\left|\left| U^{-1} - \sum_{k=0}^{N} (I - U)^k \right|\right| \to 0 \text{ as } N \to \infty.$$

A special role is played by the continuous linear operators $\Phi : X \to \mathbb{C}$; they are called *functionals,* and the collection of all functionals is the *dual* space X^* of X. The dual X^* is itself a Banach space with respect to the norm

$$||\Phi|| = \sup\{|\Phi(x)| \,\big|\, x \in X, ||x|| = 1\}.$$

It is well known that X is isometrically isomorphic to a subspace of the double dual $X^{**} := (X^*)^*$ and thus can be identified with a subspace of X^{**}; in case $X = X^{**}$, we say that X is *reflexive.*

2.3 Hilbert Spaces

A special class of normed vector spaces is formed by *inner product spaces.* Recall that an inner product on a complex vector space X is a mapping $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{C}$ for which

(i) $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle,\ \forall x, y, z \in X,\ \alpha, \beta \in \mathbb{C};$

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle},\ \forall x, y \in X;$

(iii) $\langle x, x\rangle \geq 0,\ \forall x \in X$, and $\langle x, x\rangle = 0 \Leftrightarrow x = 0.$

Note that we have chosen to let the inner product be linear in the first entry. It implies that the inner product is conjugated linear in the second entry. Frequently, the opposite convention is used in the literature.

A vector space X with an inner product $\langle\cdot,\cdot\rangle$ can be equipped with the norm

$$||x|| := \sqrt{\langle x, x\rangle},\ x \in X.$$

If X is a Banach space with respect to this norm, then X is called a *Hilbert space.* We reserve the letter $\mathcal{H}$ for these spaces. We will always assume that $\mathcal{H}$ is *nontrivial*, i.e., that $\mathcal{H} \neq \{0\}$. The standard examples are the spaces $L^2(\mathbb{R})$ and $\ell^2(\mathbb{N})$ discussed in Section 2.7.

Two elements $x, y \in \mathcal{H}$ are *orthogonal* if $\langle x, y\rangle = 0$; and the *orthogonal complement* of a subspace U of $\mathcal{H}$ is

$$U^\perp = \{x \in \mathcal{H} :\ \langle x, y\rangle = 0,\ \forall y \in U\}.$$

The above definitions apply whether $\mathcal{H}$ is finite-dimensional or infinite-dimensional. Also note that norms and inner products are defined in a similar way on real vector spaces (just replace the scalars $\mathbb{C}$ by the real scalars $\mathbb{R}$).

We will now collect a few elementary results concerning Hilbert spaces that will be used repeatedly during the book.

Lemma 2.3.1 *Let $\mathcal{H}$ denote a Hilbert space. Then the following hold:*

(i) For any $x, y \in \mathcal{H}$,

$$|\langle x, y\rangle| \leq ||x||\ ||y||.$$

(ii) For any $x \in \mathcal{H}$,

$$||x|| = \sup_{||y||=1} |\langle x, y\rangle|.$$

(iii) If $\mathcal{H}$ is a complex Hilbert space, then for any $x, y \in \mathcal{H}$,

$$\langle x, y\rangle = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2 + i\ ||x+iy||^2 - i\ ||x-iy||^2\right); \quad (2.7)$$

in case $\mathcal{H}$ is a real Hilbert space,

$$\langle x, y\rangle \quad = \quad \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right).$$

(iv) Assume that $x, y \in \mathcal{H}$ satisfy that

$$\langle x, z\rangle = \langle y, z\rangle, \ \forall z \in \mathcal{H}.$$

Then $x = y$.

(v) For any sequence $\{x_k\}_{k=1}^{\infty}$ in $\mathcal{H}$, the following are equivalent:

(a) $\{x_k\}_{k=1}^{\infty}$ is complete.
(b) If $\langle x, x_k\rangle = 0$ for all $k \in \mathbb{N}$, then $x = 0$.

Note that Lemma 2.3.1 (i) is the *Cauchy–Schwarz inequality;* the classical proofs in $\mathbb{R}^n$ carries over to the Hilbert space setting. The result in (ii) shows that the norm in the Hilbert space can be recovered with knowledge of the inner product; we ask the reader to prove this in Exercise 2.3. On the other hand, the result in (iii) shows that we can also recover the inner product in $\mathcal{H}$ from the norm (Exercise 2.4); (iii) is known in the literature under the name *the polarization identity.* The proof of the results in (iv) and (v) are also left to the reader (Exercise 2.5 and Exercise 2.6).

Among the linear operators on a Hilbert space, a special role is played by the functionals, i.e., the continuous linear operators $\Phi : \mathcal{H} \to \mathbb{C}$. They are characterized in the well-known *Riesz' representation theorem* (see, e.g., page 81 in [565] for a proof):

Theorem 2.3.2 *Let $\Phi : \mathcal{H} \to \mathbb{C}$ be a continuous linear mapping. Then there exists a unique $y \in \mathcal{H}$ such that $\Phi x = \langle x, y\rangle$ for all $x \in \mathcal{H}$.*

Note that the uniqueness of the element $y \in \mathcal{H}$ associated with a given functional is a consequence of Lemma 2.3.1 (iv).

Corollary 2.3.3 *The dual of a Hilbert space $\mathcal{H}$ can be identified with $\mathcal{H}$.*

2.4 Operators on Hilbert Spaces

Let U be a bounded operator from the Hilbert space $(\mathcal{K}, \langle\cdot,\cdot\rangle_{\mathcal{K}})$ into the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\mathcal{H}})$. The *adjoint* operator is defined as the unique operator $U^* : \mathcal{H} \to \mathcal{K}$ satisfying that

$$\langle x, Uy\rangle_{\mathcal{H}} = \langle U^*x, y\rangle_{\mathcal{K}}, \ \forall x \in \mathcal{H}, y \in \mathcal{K}.$$

Usually, we will write $\langle\cdot,\cdot\rangle$ for both inner products; it will always be clear from the context in which space the inner product is taken.

We collect some relationships between U and U^*; the proofs can be found in, e.g., Theorem 4.14 and Theorem 4.15 in [566].

Lemma 2.4.1 *Let $U : \mathcal{K} \to \mathcal{H}$ be a bounded operator. Then the following hold:*

(i) $||U|| = ||U^*||$, *and* $||UU^*|| = ||U||^2$.

(ii) $\mathcal{R}_U$ *is closed in* $\mathcal{H}$ *if and only if* $\mathcal{R}_{U^*}$ *is closed in* $\mathcal{K}$.

(iii) U *is surjective if and only if there exists a constant* $C > 0$ *such that*

$$||U^*y|| \geq C \; ||y||, \quad \forall y \in \mathcal{H}.$$

An operator $U : \mathcal{K} \to \mathcal{H}$ is *compact* if $V := \overline{\{Ux \; : \; ||x|| \leq 1\}}$ is compact, i.e., if every sequence from V has a convergent subsequence. A compact operator is bounded. Among the compact operators, we find all operators having *finite rank*, i.e., a finite-dimensional range. We collect some of the most important properties of compact operators; the proofs are in [566].

Lemma 2.4.2 *Let $U : \mathcal{K} \to \mathcal{H}$ be a compact operator. Then*

(i) The composition of U and a bounded operator (from left or right) is a compact operator.

(ii) The adjoint operator U^* is compact.

(iii) If $\mathcal{K} = \mathcal{H}$ and $\lambda \neq 0$, then $U - \lambda I$ has closed range; here I denotes the identity operator on $\mathcal{H}$.

In the rest of this section, we consider the case $\mathcal{K} = \mathcal{H}$. A bounded operator $U : \mathcal{H} \to \mathcal{H}$ is *unitary* if $UU^* = U^*U = I$. If U is unitary, then

$$\langle Ux, Uy \rangle = \langle x, y \rangle, \; \forall x, y \in \mathcal{H}.$$

A bounded operator $U : \mathcal{H} \to \mathcal{H}$ is *self-adjoint* if $U = U^*$. When U is self-adjoint,

$$||U|| = \sup_{||x||=1} |\langle Ux, x \rangle| \,. \tag{2.8}$$

For a self-adjoint operator U, the inner product $\langle Ux, x \rangle$ is real for all $x \in \mathcal{H}$. One can introduce a partial order on the set of self-adjoint operators by

$$U_1 \leq U_2 \Leftrightarrow \langle U_1 x, x \rangle \leq \langle U_2 x, x \rangle, \; \forall x \in \mathcal{H}. \tag{2.9}$$

Using this order, one can work with self-adjoint operators almost as with real numbers. For example, under certain conditions, it is possible to "multiply" an operator inequality with a bounded operator. The precise statement below can be found in [401].

Theorem 2.4.3 *Let U_1, U_2, U_3 be self-adjoint operators. If $U_1 \leq U_2$, $U_3 \geq 0$, and U_3 commutes with U_1 and U_2, then $U_1U_3 \leq U_2U_3$.*

An important class of self-adjoint operators consists of the *orthogonal projections*. Given a closed subspace V of $\mathcal{H}$, the orthogonal projection of $\mathcal{H}$ onto V is the operator $P : \mathcal{H} \to \mathcal{H}$ for which

$$Px = x, \ x \in V, \ Px = 0, \ x \in V^{\perp}.$$

If $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for V (see Definition 3.4.1), the operator P is given explicitly by

$$Px = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k, \ x \in \mathcal{H}.$$

Lemma 2.4.4 *Let $U : \mathcal{H} \to \mathcal{H}$ be a bounded operator, and assume that $\langle Ux, x \rangle = 0$ for all $x \in \mathcal{H}$. Then the following hold:*

(i) If $\mathcal{H}$ is a complex Hilbert space, then $U = 0$.

(ii) If $\mathcal{H}$ is a real Hilbert space and U is self-adjoint, then $U = 0$.

Proof. If $\mathcal{H}$ is a complex Hilbert space, a direct calculation shows that

$$\begin{aligned} 4\langle Ux, y \rangle &= \langle U(x+y), x+y \rangle - \langle U(x-y), x-y \rangle \\ &\quad + i\langle U(x+iy), x+iy \rangle - i\langle U(x-iy), x-iy \rangle, \ \forall x, y \in \mathcal{H}. \end{aligned}$$

Thus, if $\langle Ux, x \rangle = 0$ for all $x \in \mathcal{H}$, then $\langle Ux, y \rangle = 0$ for all $x, y \in \mathcal{H}$, and therefore $U = 0$.

In case $\mathcal{H}$ is a real Hilbert space, we must use a different approach. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. Then, for arbitrary $j, k \in \mathbb{N}$,

$$\begin{aligned} 0 &= \langle U(e_k + e_j), e_k + e_j \rangle \\ &= \langle Ue_k, e_k \rangle + \langle Ue_j, e_j \rangle + \langle Ue_k, e_j \rangle + \langle Ue_j, e_k \rangle \\ &= \langle Ue_k, e_j \rangle + \langle e_j, Ue_k \rangle = 2\, \langle Ue_j, e_k \rangle; \end{aligned}$$

therefore, $U = 0$. □

Note that without the assumption $U = U^*$, Lemma 2.4.4 (ii) would fail; consider, e.g., the case where U is a rotation of $90°$ in $\mathbb{R}^2$.

A bounded operator $U : \mathcal{H} \to \mathcal{H}$ is *positive* if $\langle Ux, x \rangle \geq 0, \ \forall x \in \mathcal{H}$. On a complex Hilbert space, every bounded positive operator is self-adjoint. For a positive operator U, we will often use the following result about the existence of a *square root*, i.e., a bounded operator W such that $W^2 = U$; a proof can be found, e.g., at page 476 in [464].

Lemma 2.4.5 *Every bounded and positive operator $U : \mathcal{H} \to \mathcal{H}$ has a unique bounded and positive square root W. The operator W has the following properties:*

(i) If U is self-adjoint, then W is self-adjoint.

(ii) If U is invertible, then W is also invertible.

(iii) W can be expressed as a limit (in the strong operator topology) of a sequence of polynomials in U and commutes with U.

Frequently, the study of an operator is easier if it can be represented as a sum or product of "simple" operators. We mention a few examples of such representations:

Lemma 2.4.6 *Let $\mathcal{H}$ be a complex Hilbert space. Then the following hold:*

(i) Every bounded and invertible operator $U : \mathcal{H} \to \mathcal{H}$ has a unique representation $U = WP$, where W is unitary and P is positive.

(ii) Every positive operator P on $\mathcal{H}$ with $||P|| \leq 1$ can be written as an average of unitary operators, namely,

$$P = \frac{1}{2}(W + W^*) \text{ with } W = P + i\sqrt{I - P^2}.$$

The representation $U = WP$ in (i) is called the *polar decomposition;* a proof of this result can be found on page 315 in [566]. The representation in (ii) is probably less known, but it is proved by direct verification. That $W = P + i\sqrt{I - P^2}$ is unitary follows by calculating WW^* and WW^* using that the square root of $I - P^2$ can be considered as a limit of polynomials in $I - P^2$ and therefore commutes with P. Note that (ii) applies if P is an orthogonal projection.

2.5 The Pseudo-inverse Operator

For operators that are not invertible, various types of generalized inverses exist in the literature; see, e.g., the book [43]. Among these generalized inverses, we will focus on a particular one, which will be called the *pseudo-inverse.* We first prove that if an operator U from one Hilbert space to another has closed range, there exists a " right-inverse operator" $U^\dagger$ in the following sense:

Lemma 2.5.1 *Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, and suppose that $U : \mathcal{K} \to \mathcal{H}$ is a bounded operator with closed range $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger : \mathcal{H} \to \mathcal{K}$ for which*

$$UU^\dagger x = x, \ \forall x \in \mathcal{R}_U. \tag{2.10}$$

Proof. Consider the restriction of U to an operator on the orthogonal complement of the kernel of U, i.e., let

$$\widetilde{U} := U_{|\mathcal{N}_U^\perp} : \mathcal{N}_U^\perp \to \mathcal{H}.$$

Clearly, $\widetilde{U}$ is linear and bounded. $\widetilde{U}$ is also injective: if $\widetilde{U}x = 0$, it follows that $x \in \mathcal{N}_U^\perp \cap \mathcal{N}_U = \{0\}$. We now prove that the range of $\widetilde{U}$ equals the range of U. Given $y \in \mathcal{R}_U$, there exists $x \in \mathcal{K}$ such that $Ux = y$. By writing $x = x_1 + x_2$, where $x_1 \in \mathcal{N}_U^\perp,\ x_2 \in \mathcal{N}_U$, we obtain that

$$\widetilde{U}x_1 = Ux_1 = U(x_1 + x_2) = Ux = y.$$

It follows from Theorem 2.2.2 that $\widetilde{U}$ has a bounded inverse

$$\widetilde{U}^{-1} : \mathcal{R}_U \to \mathcal{N}_U^\perp.$$

Extending $\widetilde{U}^{-1}$ by zero on the orthogonal complement of $\mathcal{R}_U$, we obtain a bounded operator $U^\dagger : \mathcal{H} \to \mathcal{K}$ for which $UU^\dagger x = x$ for all $x \in \mathcal{R}_U$. □

The operator $U^\dagger$ constructed in the proof of Lemma 2.5.1 is called the *pseudo-inverse* of U. In the literature, one will often see the pseudo-inverse of an operator U with closed range defined as the unique operator $U^\dagger$ satisfying that

$$\mathcal{N}_{U^\dagger} = \mathcal{R}_U^\perp, \quad \mathcal{R}_{U^\dagger} = \mathcal{N}_U^\perp, \text{ and } UU^\dagger x = x, x \in \mathcal{R}_U; \tag{2.11}$$

this definition is equivalent to the above construction (Exercise 2.7). We collect some properties of $U^\dagger$ and its relationship to U.

Lemma 2.5.2 *Let $U : \mathcal{K} \to \mathcal{H}$ be a bounded operator with closed range. Then the following hold:*

(i) The orthogonal projection of $\mathcal{H}$ onto $\mathcal{R}_U$ is given by $UU^\dagger$.

(ii) The orthogonal projection of $\mathcal{K}$ onto $\mathcal{R}_{U^\dagger}$ is given by $U^\dagger U$.

(iii) U^* has closed range, and $(U^*)^\dagger = (U^\dagger)^*$.

(iv) On $\mathcal{R}_U$, the operator $U^\dagger$ is given explicitly by

$$U^\dagger = U^*(UU^*)^{-1}. \tag{2.12}$$

Proof. All statements follow from the characterization of $U^\dagger$ in (2.11). For example, it shows that

$$UU^\dagger = I \text{ on } \mathcal{R}_U \text{ and that } UU^\dagger = 0 \text{ on } \mathcal{N}_{U^\dagger} = \mathcal{R}_U^\perp;$$

this gives (i) by the definition of an orthogonal projection. The proof of (ii) is similar. That $\mathcal{R}_{U^*}$ is closed was stated already in Lemma 2.4.1; thus, $(U^*)^\dagger$ is well defined. That $(U^*)^\dagger$ equals $(U^\dagger)^*$ follows by verifying that $(U^\dagger)^*$ satisfies (2.11) with U replaced by U^*. Finally, UU^* is invertible as an operator on $\mathcal{R}_U$, and the operator given by

$$U^*(UU^*)^{-1} \text{ on } \mathcal{R}_U \ \text{ and } \ 0 \text{ on } \mathcal{R}_U^\perp$$

satisfies the conditions (2.11) characterizing $U^\dagger$. □

The pseudo-inverse gives the solution to an important optimization problem:

Theorem 2.5.3 *Let $U : \mathcal{K} \to \mathcal{H}$ be a bounded surjective operator. Given $y \in \mathcal{H}$, the equation $Ux = y$ has a unique solution of minimal norm, namely, $x = U^\dagger y$.*

Proof. The proof is identical with the proof of Theorem 1.6.2. Alternatively, if x is a solution to the equation $Ux = y$, then

$$x = (x - U^\dagger y) + U^\dagger y \in \mathcal{N}_U + \mathcal{N}_U^\perp;$$

thus, the norm of x is minimal precisely when $x = U^\dagger y$. $\square$

2.6 A Moment Problem

Before we leave the discussion of abstract Hilbert spaces, we mention a special class of equations, known as moment problems. The general version of a *moment problem* is as follows: given a collection of elements $\{x_k\}_{k=1}^\infty$ in a Hilbert space $\mathcal{H}$ and a sequence $\{a_k\}_{k=1}^\infty$ of complex numbers, can we find an element $x \in \mathcal{H}$ such that

$$\langle x, x_k \rangle = a_k, \text{ for all } k \in \mathbb{N}?$$

Many of the equations that will appear throughout the book will be formulated in terms of moment problems. We will need a special moment problem in Section 9.5:

Lemma 2.6.1 *Let $\{x_k\}_{k=1}^N$ be a collection of vectors in $\mathcal{H}$ and consider the moment problem*

$$\langle x, x_k \rangle = \begin{cases} 1 \text{ if } k = 1, \\ 0 \text{ if } k = 2, \dots, N. \end{cases} \tag{2.13}$$

Then the following are equivalent:

(i) The moment problem (2.13) has a solution x.

(ii) If $\sum_{k=1}^N c_k x_k = 0$ for some scalar coefficients c_k, then $c_1 = 0$.

(iii) $x_1 \notin span\{x_k\}_{k=2}^N$.

In case the moment problem (2.13) has a solution, it can be chosen of the form $x = \sum_{k=1}^N d_k x_k$ for some scalar coefficients d_k.

Proof. (i)$\Rightarrow$ (ii). Assume first that (i) is satisfied, i.e., (2.13) has a solution x. Then, if $\sum_{k=1}^N c_k x_k = 0$ for some coefficients $\{c_k\}_{k=1}^N$, we have that

$$0 = \langle x, \sum_{k=1}^N c_k x_k \rangle = \sum_{k=1}^N \overline{c_k} \langle x, x_k \rangle = \overline{c_1},$$

i.e., (ii) holds.
(ii) $\Rightarrow$(iii). This implication is clear.
(iii) $\Rightarrow$(i). Let P denote the orthogonal projection of $\mathcal{H}$ onto $\text{span}\{x_k\}_{k=2}^N$, and put $\varphi = x_1 - Px_1$. Then

$$\langle \varphi, x_1\rangle = \langle x_1 - Px_1, x_1 - Px_1\rangle + \langle x_1 - Px_1, Px_1\rangle = ||x_1 - Px_1||^2 \neq 0,$$

and $\langle \varphi, x_k\rangle = 0$ for $k = 2, \ldots, N$. Thus, the element

$$x := \frac{\varphi}{||x_1 - Px_1||^2} \tag{2.14}$$

solves the moment problem (2.13), i.e., (i) is satisfied.

In case the equivalent conditions are satisfied, the construction of x in (2.14) shows that $x \in \text{span}\{x_k\}_{k=1}^N$. $\square$

2.7 The Spaces $L^p(\mathbb{R})$, $L^2(\mathbb{R})$, $\ell^p(\mathbb{N})$, and $\ell^2(\mathbb{N})$

The most important class of Banach spaces is formed by the L^p-spaces, $1 \leq p \leq \infty$. We expect these spaces and their basic properties to be known, so we only provide a quick overview; proofs and further results can be found in any standard book on the subject, e.g., [565].

First, $L^\infty(\mathbb{R})$ is the space of essentially bounded (Lebesgue) measurable functions $f : \mathbb{R} \to \mathbb{C}$, equipped with the essential supremums-norm. For $1 \leq p < \infty$, $L^p(\mathbb{R})$ is the space of functions f for which $|f|^p$ is integrable with respect to the Lebesgue measure:

$$L^p(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ is measurable and } \int_{-\infty}^{\infty} |f(x)|^p\, dx < \infty \right\}.$$

The norm on $L^p(\mathbb{R})$ is

$$||f|| = \left(\int_{-\infty}^{\infty} |f(x)|^p\, dx \right)^{1/p}.$$

To be more precise, $L^p(\mathbb{R})$ consists of equivalence classes of functions that are equal almost everywhere and for which a representative (and hence all) for the equivalence class satisfies the integrability condition. In order not to be too tedious, we adopt the standard terminology and speak about functions in $L^p(\mathbb{R})$ rather than equivalence classes.

The case $p = 2$ plays a special role: in fact, the space

$$L^2(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{C} \mid f \text{ is measurable and } \int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty \right\}$$

is the only one of the $L^p(\mathbb{R})$-spaces that can be equipped with an inner product. Actually, $L^2(\mathbb{R})$ is a Hilbert space with respect to the inner product

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\,dx,\ \ f,g \in L^2(\mathbb{R}).$$

Thus, we can apply all the results in Section 2.3 to the space $L^2(\mathbb{R})$. In particular, Cauchy–Schwarz inequality states that for all $f, g \in L^2(\mathbb{R})$,

$$\left|\int_{-\infty}^{\infty} f(x)g(x)\,dx\right| \leq \left(\int_{-\infty}^{\infty} |f(x)|^2\,dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |g(x)|^2\,dx\right)^{1/2}. \quad (2.15)$$

The spaces $L^2(\Omega)$, where Ω is an open subset of $\mathbb{R}$, are defined similarly. According to the general definition, a sequence of functions $\{g_k\}_{k=1}^{\infty}$ in $L^2(\Omega)$ converges to $g \in L^2(\Omega)$ if

$$||g - g_k|| = \left(\int_{\Omega} |g(x) - g_k(x)|^2\,dx\right)^{1/2} \to 0 \text{ as } k \to \infty.$$

It is essential to be aware that the concept of convergence in L^2-sense is different from pointwise convergence. However, there is a relationship that will play an important role in several proofs: convergence in L^2-sense implies the existence of a subsequence that converges pointwise almost everywhere. This result is known as *Riesz' subsequence theorem;* we state it formally here and refer to page 68 in [565] for a proof.

Theorem 2.7.1 *Let $\Omega \subseteq \mathbb{R}$ be an open set, and let $\{g_k\}$ be a sequence in $L^2(\Omega)$ that converges to $g \in L^2(\Omega)$. Then $\{g_k\}$ has a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ such that*

$$g(x) = \lim_{k\to\infty} g_{n_k}(x)$$

for a.e. $x \in \Omega$.

The result holds no matter how we choose the representatives for the equivalence classes. This is typical for this book, where we rarely deal with a specific representative for a given class. There are, however, a few important exceptions. When we speak about a continuous function, it is clear that we have chosen a specific representative, and the same is the case when we discuss *Lebesgue points.* By definition, a point $y \in \mathbb{R}$ is a Lebesgue point for a measurable function $f : \mathbb{R} \to \mathbb{C}$ if

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{y-\frac{1}{2}\epsilon}^{y+\frac{1}{2}\epsilon} |f(y) - f(x)|\,dx = 0.$$

If f is continuous in y, then y is a Lebesgue point (Exercise 2.8). More generally, one can prove that if $f \in L^1(\mathbb{R})$, then almost every $y \in \mathbb{R}$ is a Lebesgue point; see page 138 in [565] for a proof of this fact, as well as a more detailed discussion.

It is clear from the definition that different representatives for the same equivalence class will have different Lebesgue points. For example, every $y \in \mathbb{R}$ is a Lebesgue point for the function $f = 0$; changing the definition of f in a single point y will not change the equivalence class, but y will no longer be a Lebesgue point. See Exercise 2.8 for some related observations.

The discrete analogue of $L^p(\mathbb{R})$ is $\ell^p(I)$, the space of p-summable scalar-valued sequences with a countable index set I. For $1 \le p < \infty$, let

$$\ell^p(I) := \left\{ \{x_k\}_{k\in I} \;\middle|\; x_k \in \mathbb{C}, \quad \sum_{k\in I} |x_k|^p < \infty \right\}.$$

For $1 \le p < \infty$ the space $\ell^p(\mathbb{R})$ is a Banach space with respect to the norm

$$||\{x_k\}_{k\in I}||_p = \left(\sum_{k\in I} |x_k|^p \right)^{1/p}.$$

In particular, $\ell^2(I)$ is a Hilbert space with respect to the inner product

$$\langle \{x_k\}_{k\in I}, \{y_k\}_{k\in I} \rangle = \sum_{k\in I} x_k \overline{y_k};$$

and Cauchy–Schwarz inequality states that

$$\left| \sum_{k\in I} x_k \overline{y_k} \right|^2 \le \sum_{k\in I} |x_k|^2 \sum_{k\in I} |y_k|^2, \; \{x_k\}_{k\in I}, \{y_k\}_{k\in I} \in \ell^2(I). \tag{2.16}$$

The Banach space $\ell^\infty(I)$ is the set of bounded scalar-valued sequences $\{x_k\}_{k\in I}$, equipped with the norm

$$||\{x_k\}_{k\in I}||_\infty = \sup_{k\in I} |x_k|.$$

We will frequently use the following discrete version of Fatou's lemma (the general version is stated in Lemma A.2.3):

Lemma 2.7.2 *Let I be a countable index set and $f_n : I \to [0, \infty], \; n \in \mathbb{N}$, a sequence of functions. Then*

$$\sum_{k\in I} \liminf_{n\to\infty} f_n(k) \le \liminf_{n\to\infty} \sum_{k\in I} f_n(k).$$

2.8 The Fourier Transform and Convolution

The Fourier transform will be one of the central ingredients in our analysis of structured function systems in Chapters 9–21. In this section, we give a short introduction to the Fourier transforms on $L^2(\mathbb{R})$ and $\ell^2(\mathbb{Z})$. As for most of the other topics in this section, we expect the reader to have a basic knowledge about the subject, so we only collect the main definitions and results here. For more information, we refer to any of the standard texts, e.g., [465, 22], or [629].

For $f \in L^1(\mathbb{R})$, the *Fourier transform* $\widehat{f} : \mathbb{R} \to \mathbb{C}$ is defined by

$$\widehat{f}(\gamma) := \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\gamma}\, dx, \ \gamma \in \mathbb{R}. \tag{2.17}$$

We will also denote the Fourier transform of f by $\mathcal{F}f$. This notation indicates that we will also consider the Fourier transform as an operator; the *Riemann–Lebesgue lemma* says that $\mathcal{F}$ maps $L^1(\mathbb{R})$ into $C_0(\mathbb{R})$, the vector space consisting of continuous functions vanishing at infinity.

The Fourier transform has an extension to a unitary operator on $L^2(\mathbb{R})$. One can show that if $(L^1 \cap L^2)(\mathbb{R})$ is equipped with the $L^2(\mathbb{R})$-norm, the Fourier transform is an isometry from $(L^1 \cap L^2)(\mathbb{R})$ into $L^2(\mathbb{R})$. If $f \in L^2(\mathbb{R})$ and $\{f_k\}_{k=1}^{\infty}$ is a sequence of functions in $(L^1 \cap L^2)(\mathbb{R})$ that converges to f in L^2-sense, then the sequence $\{\widehat{f_k}\}_{k=1}^{\infty}$ is also convergent in $L^2(\mathbb{R})$, with a limit that is independent of the choice of $\{f_k\}_{k=1}^{\infty}$. Defining

$$\widehat{f} := \lim_{k\to\infty} \widehat{f_k}$$

then extends the Fourier transform to a unitary mapping of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$. We will use the same notation to denote this extension.

The above construction and the polarization identity immediately yields *Plancherel's equation,*

$$\langle \widehat{f}, \widehat{g} \rangle = \langle f, g \rangle, \ \forall f, g \in L^2(\mathbb{R}), \text{ and } ||\widehat{f}|| = ||f||. \tag{2.18}$$

If $f \in L^1(\mathbb{R})$, then $\widehat{f}$ is continuous. If the function f as well as $\widehat{f}$ belong to $L^1(\mathbb{R})$, the *inversion formula* describes how to come back to f from the function values $\widehat{f}(\gamma)$, see [22].

Theorem 2.8.1 *Assume that $f, \widehat{f} \in L^1(\mathbb{R})$. Then*

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\gamma)e^{2\pi i x\gamma} d\gamma, \ a.e.\ x \in \mathbb{R}. \tag{2.19}$$

If f is continuous, the pointwise formula (2.19) holds for all $x \in \mathbb{R}$. In general, it holds at least for all Lebesgue points for f.

We note that (2.19) also holds if $f \in L^2(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$.

Given two functions $f, g \in L^1(\mathbb{R})$, the *convolution* $f * g : \mathbb{R} \to \mathbb{C}$ is defined by

$$f * g(y) = \int_{-\infty}^{\infty} f(y - x)g(x)\, dx, \ y \in \mathbb{R}.$$

The function $f * g$ is well defined for all $y \in \mathbb{R}$ and belongs to $L^1(\mathbb{R})$. If $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$ for some $p \in [1, \infty[$, the convolution $f * g(y)$ is well-defined for a.e. $y \in \mathbb{R}$ and defines a function in $L^p(\mathbb{R})$.

The Fourier transform and convolution are related by the following important result:

Theorem 2.8.2 *If $f, g \in L^1(\mathbb{R})$, then $\widehat{f * g}(\gamma) = \widehat{f}(\gamma)\widehat{g}(\gamma)$ for all $\gamma \in \mathbb{R}$; if $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, the formula holds for a.e. $\gamma \in \mathbb{R}$.*

We will also need the Fourier transform on $\ell^2(\mathbb{Z})$. Given a sequence $h = \{h_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z})$, we define its *Fourier transform* as the Fourier series

$$\widehat{h}(\nu) = \sum_{j\in\mathbb{Z}} h_k e^{-2\pi i k\nu}, \ a.e. \ \nu \in \mathbb{R}.$$

Given two scalar-valued sequences $g = \{g_k\}_{k\in\mathbb{Z}}$ and $h = \{h_k\}_{k\in\mathbb{Z}}$, their *convolution* is formally defined as the sequence $g*h$ whose jth coordinate is

$$(g * h)_j = \sum_{k\in\mathbb{Z}} g_k h_{j-k}.$$

If $g \in \ell^1(\mathbb{Z})$ and $h \in \ell^p(\mathbb{Z})$ for some $p \in [1, \infty[$, then the convolution $g * h$ is well-defined and belongs to $\ell^p(\mathbb{Z})$; *Young's inequality* states that

$$||g * h||_p \leq ||g||_1 ||h||_p. \tag{2.20}$$

2.9 Operators on $L^2(\mathbb{R})$

In this section, we consider three classes of operators on $L^2(\mathbb{R})$ that will play a key role in our analysis of Gabor frames and wavelets. Their definitions are as follows:

Definition 2.9.1 *Consider the following classes of linear operators:*

(i) For $a \in \mathbb{R}$, the operator T_a, called translation by a, is defined by

$$T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (T_a f)(x) := f(x - a), \ x \in \mathbb{R}. \tag{2.21}$$

(ii) For $b \in \mathbb{R}$, the operator E_b, called modulation by b, is defined by

$$E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (E_b f)(x) := e^{2\pi i b x} f(x), \ x \in \mathbb{R}. \tag{2.22}$$

(iii) For $a \neq 0$, the operator D_a, called scaling by a, is defined by

$$D_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (D_a f)(x) := \frac{1}{\sqrt{a}} f(\frac{x}{a}), \ x \in \mathbb{R}. \tag{2.23}$$

The operator D_a is also called a dilation operator.

(iv) The dyadic scaling operator is

$$D : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (Df)(x) := D_{1/2} f(x) = 2^{1/2} f(2x). \tag{2.24}$$

A comment about notation: we will usually skip the parentheses and simply write $T_af(x)$ and similarly for the other operators. Frequently, we will also let E_b denote the function $x \mapsto e^{2\pi ibx}$.

All the operators in Definition 2.9.1 indeed map $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ as stated and are bounded (Exercise 2.9). They are even unitary:

Lemma 2.9.2 *The translation operators satisfy the following:*

(i) T_a *is unitary for all* $a \in \mathbb{R}$.

(ii) For each $f \in L^2(\mathbb{R})$, *the mapping* $y \mapsto T_y f$ *is continuous from* $\mathbb{R}$ *to* $L^2(\mathbb{R})$.

Similar statements hold for $E_b, b \in \mathbb{R}$, *and* $D_a,\ a \neq 0$.

Proof. Let us prove that the operators T_a are unitary. Since

$$\begin{aligned} \langle T_a f, g\rangle = \int_{-\infty}^{\infty} f(x-a)\overline{g(x)}\,dx &= \int_{-\infty}^{\infty} f(x)\overline{g(x+a)}\,dx \\ &= \langle f, T_{-a}g\rangle,\ \forall f,g \in L^2(\mathbb{R}), \end{aligned}$$

we see that $T_a^* = T_{-a}$. On the other hand, T_a is clearly an invertible operator with $T_a^{-1} = T_{-a}$, so we conclude that $T_a^{-1} = T_a^*$.

To prove the continuity of the mapping $y \mapsto T_y f$, we first assume that the function f is continuous and has compact support, say, contained in the bounded interval $[c,d]$. For notational convenience, we prove the continuity in $y_0 = 0$. First, for $y \in]-\frac{1}{2}, \frac{1}{2}[$ the function

$$\phi(x) = T_y f(x) - T_{y_0} f(x) = f(x-y) - f(x)$$

has support in the interval $[-\frac{1}{2}+c, d+\frac{1}{2}]$. Since f is uniformly continuous, we can for any given $\epsilon > 0$ find $\delta > 0$ such that

$$|f(x-y) - f(x)| \leq \epsilon \ \text{ for all } x \in \mathbb{R} \text{ whenever } |y| \leq \delta.$$

With this choice of δ, we thus obtain that

$$\begin{aligned} ||T_y f - T_{y_0} f|| &= \left(\int_{-\frac{1}{2}+c}^{\frac{1}{2}+d} |f(x-y) - f(x)|^2\,dx \right)^{1/2} \\ &\leq \epsilon\,\sqrt{d-c+1}. \end{aligned}$$

This proves the continuity in the considered special case. The case of an arbitrary function $f \in L^2(\mathbb{R})$ follows by an approximation argument, using that the continuous functions with compact support are dense in $L^2(\mathbb{R})$ (Exercise 2.10). The proofs of the statements for E_b and D_a are left to the reader (Exercise 2.11). □

Chapters 11–20 will deal with Gabor systems, wavelet systems, and generalized shift-invariant systems in $L^2(\mathbb{R})$; all classes consist of functions that are defined by compositions of some of the operators $T_a, E_b,$

and D_a. For this reason, the following *commutator relations* are important (Exercise 2.12) :

$$T_aE_bf(x) = e^{-2\pi iba}E_bT_af(x) = e^{2\pi ib(x-a)}f(x-a), \quad (2.25)$$

$$T_bD_af(x) = D_aT_{b/a}f(x) = \frac{1}{\sqrt{|a|}}f(\frac{x}{a}-\frac{b}{a}), \quad (2.26)$$

$$D_aE_bf(x) = E_{\frac{b}{a}}D_af(x) = \frac{1}{\sqrt{|a|}}e^{2\pi ixb/a}f(\frac{x}{a}). \quad (2.27)$$

With this notation, the commutator relation (2.26) in particular implies that

$$T_kD^j = D^jT_{2^jk} \text{ and } D^jT_k = T_{2^{-j}k}D^j,\ j,k\in\mathbb{Z}. \quad (2.28)$$

We will often use the Fourier transformation in connection with Gabor systems and wavelet systems. In this context, we need the commutator relations (Exercise 2.13)

$$\mathcal{F}T_a = E_{-a}\mathcal{F},\ \ \mathcal{F}E_a = T_a\mathcal{F},\ \ \mathcal{F}D_a = D_{1/a}\mathcal{F},\ \ \mathcal{F}D = D^{-1}\mathcal{F}. \quad (2.29)$$

2.10 Exercises

2.1 Find a sequence $\{a_k\}_{k=1}^{\infty}$ of real numbers for which $\sum_{k=1}^{\infty} a_k$ is convergent but not unconditionally convergent.

2.2 Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in a Banach space. Prove that absolute convergence of $\sum_{k=1}^{\infty} f_k$ implies unconditional convergence.

2.3 Prove Lemma 2.3.1(ii).

2.4 Prove Lemma 2.3.1(iii).

2.5 Prove Lemma 2.3.1 (iv).

2.6 Prove Lemma 2.3.1 (v).

2.7 Prove that the conditions in (2.11) are equivalent to the construction of the pseudo-inverse in Lemma 2.5.1.

2.8 Here we ask the reader to prove some results concerning Lebesgue points.

(i) Assume that $f:\mathbb{R}\to\mathbb{C}$ is continuous. Prove that every $y\in\mathbb{R}$ is a Lebesgue point.

(ii) Prove that $x = 0$ is not a Lebesgue point for the function $\chi_{[0,1]}$.

(iii) Let $f = \chi_{\mathbb{Q}}$. Prove that every $y \notin \mathbb{Q}$ is a Lebesgue point and that the rational numbers are not Lebesgue points.

2.9 Show that the operators in Definition 2.9.1 map $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$ and are bounded.

2.10 Complete the proof of Lemma 2.9.2 by showing the continuity of the mapping $y \mapsto T_y f$ for $f \in L^2(\mathbb{R})$.

2.11 Prove the statements about E_b and D_a in Lemma 2.9.2.

2.12 Prove the commutator relations (2.25)–(2.27).

2.13 Prove the commutator relations (2.29).

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