From Wind-Blown Sand to Turbulence and Back

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In honor of Ole Barndorff-Nielsen on the occasion of his 80th birthday.

Abstract We describe the recently developed Kolmogorov-Obukhov statistical theories of homogeneous turbulence and its extension to boundary layer turbulence. The theories can be used to describe the size distribution of wind-blown sand but the statistical theory of Lagrangian turbulence is still missing, so this task cannot be completed yet. That this can be done was suggested by Ole-Barndorff Nielsen and we show how his Generalized Hyperbolic Distribution gives the continuous part of the probability distribution functions of the turbulent velocity differences.

Keywords Turbulence · Intermittency · Invariant measure · Kolmogorov-Obukhov scaling · Inertial cascade · Navier-Stokes equation · Large deviations · Poisson processes · Central limit theorem · Structure functions · She-Leveque intermittency corrections · Boundary value turbulence · Lagrangian turbulence · Wind-blown sand

1 Introduction

In his book "The physics of blown-sand and desert dunes", published in 1954, the British military engineer R.A. Bagnold, described the size and mass distribution of wind blown sand [1]. His research was based on years spent in the Saharan Desert studying how the wind moves and forms the desert. He had done careful experiments after his retirement from the military, where he fed sand to the mouth of a wind-tunnel

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and observed the mass distribution as a function of the distance down the tunnel. He discovered that if he plotted this distribution on a log-log plot or he plotted the log of the mass distribution as a function of the log of the distance down the tunnel, then he got the lower-half of a hyperbola. This caught the eye of a brilliant young Danish statistician named Ole Barndorff-Nielsen and he found the distribution [3] that he called the Generalized Hyperbolic Distribution (GHD). What he also noticed [2] was that his distributions looked very similar to the distribution of the velocity differences in turbulence, that people had been able to compute in the sixties and were becoming common in turbulence research in the seventies. It was a remarkable observation and insight and it lead to a conjecture; that it was the turbulence in the wind that was sorting the sand and giving its mass distribution the form of the GHD.

The question I want to ask here is: can we prove this relationship? As you will see the answer is no, not yet, but we have made progress and have probably completed 2/3 of the journey towards this goal.

In 1941 Kolmogorov [16] and Obukhov [18] proposed that there exists a statistical theory of turbulence that should allow the computation of all the statistical quantities that can be computed and measured in turbulent systems. These are quantities such as the moments, the structure functions and the probability density functions (PDFs) of the turbulent velocity field. The Kolmogorov-Obukhov'41 theory predicted that the structure functions of turbulence, that are the moments of the velocity differences at distances separated by a lag variable l, should scale with the lag variable to a power $p/3$ for the pth structure function, multiplied by a universal constant. This was found to be inconsistent with observations and in 1962 Kolomogorov [17] and Obukhov [19] presented a refined scaling hypothesis, where the multiplicative constants are not universal and the scaling exponents are modified to $\zeta_p = p/3 + \tau_p$, by the intermittency correction τ_p that are due to intermittency in the turbulent velocity. It was still not clear what the values of τ_p should be, because the log-normal exponents suggested by Kolmorogov turned out again to be inconsistent with observations. Then in 1994 She and Leveque [22] found the correct (log-Poissonian) formulas for τ_p that are consistent with modern simulations and experiments.

We will outline below how the statistical theory of Kolmogorov and Obukhov is derived from the Navier-Stokes equation without getting into any of the technical details. We start with the classical Reynolds decomposition of the velocity into the mean (large scale) flow and the fluctuations or small scale flow. Then we develop a stochastic Navier-Stokes equation [8], for the small scale flow. If we assume that dissipation takes place on all scales in the inertial range (defined below) then it turns out that the noise in this stochastic Navier-Stokes equation is determined by well-known theorems in probability. The additive noise in the stochastic Navier-Stokes equation is generic noise given by the central limit theorem and the large deviation principle. The multiplicative noise consists of jumps multiplying the velocity, modeling jumps in the velocity gradient. We will explain how this form of the noise follows from a very general hypothesis.

Once the form of the noise in the stochastic Navier-Stokes equation for the small scales is determined, we can estimate the structure functions of turbulence and establish the Kolmogorov-Obukhov'62 scaling hypothesis with the She-Leveque intermittency corrections [7]. Then one can compute the invariant measure of turbulence writing the stochastic Navier-Stokes equation as an infinite-dimensional Ito process and solving the linear Kolmogorov-Hopf [14] functional differential equation for the invariant measure. Finally the invariant measure can be projected onto the PDF. The PDFs turn out to be the GHD of Barndorff-Nielsen [2, 4], and compare well with PDFs from simulations and experiments, as was shown by Barndorff-Nielsen et al. [4] in 2004. The details of the proofs can be found in [7] and the background material can be found in [8].

The problem with sand is that it is always blown along a boundary, but not in homogeneous turbulence as described above. So although we have been able to show that the distributions of the moments of the velocity differences in homogeneous turbulence are indeed given by the GHD, as predicted by Ole's remarkable insight, we have not dealt with turbulent boundary layers. The sand is carried by the turbulent boundary layer, so we have only come 1/3 of the way.

The famous log-law of Prandtl (1925) and von Kármán (1930) is the most distinguished characteristic of turbulent boundary flows. It says that the mean-velocity profile in the inertial region satisfies the formula:

$$\langle u \rangle / u_\tau = \kappa^{-1} \ln(z u_\tau / \nu) + B, \tag{1}$$

where $u_\tau = \sqrt{\tau_w/\rho}$ is the friction velocity based on the wall stress τ_w, ρ is the fluid density, ν is the kinematic viscosity, κ is the von Kármán constant and B is also a constant. $\langle \cdot \rangle$ denotes an ensemble average, this can be an average over many experiments or many computer simulations. The log-law is well established both experimentally and numerically, see reviews by Smits, McKeon and Marusic (2011) and Jimenez (2012). It has proven to be an invaluable tool in the theory of turbulent boundary layers.

It was suggested by Meneveau and Marusic (2013) that the log-law of the fluctuations could by generalized to any moment $p \geq 2$ of the fluctuations ($p = 2$ is due to Townsend) by the law

$$\langle (u^+)^{2p} \rangle^{1/p} = B_p - A_p \ln(z/\delta) = D_p(Re_\tau) - A_p \ln(z u_\tau / \nu) \tag{2}$$

where $z u_\tau / \nu$ are the viscous units and $D_p = B_p + A_p \ln Re_\tau$ is a Reynolds number Re_τ dependent offset.

Recently Birnir et al. [11], extended the theory of Birnir [7, 8] developed for homogeneous turbulence to boundary layers to prove (2) with a more physically-based normalization in the inertial range. They showed that the universal constants satisfy the relationship

$$A_p = \left(\frac{1}{l^*}\right)^{\zeta_1 - \zeta_p} \frac{C_p^{1/p}}{C_1} A_1,$$

where $\zeta_p = p/3 + \tau_p = p/9 + 2(1 - (2/3)^{p/3})$ are the Kolmogorov-Obukhov-She-Leveque, scaling exponents of the structure functions of turbulence, see [8], and l^*

is a small constant. The $C_p s$ are the Kolmogorov-Obukhov scaling coefficients, computed below. The sub-Gaussian (they lie below the values expected from a Gaussian distribution) behavior of the A_ps is caused by the KOSL scaling. These authors also computed the probability density functions of the fluctuations in the inertial range. These PDFs turn out to be Generalized Hyperbolic Distributions multiplied by a discrete measure, see the forthcoming article [10] and Sect. 5. In the viscous range the PDFs are skewed Gaussians. The boundary layer theory allows us to determine the statistical theory in boundary layer flow, we will not give the details below but refer to the forthcoming article [11].

However, even if we know the statistics of the flow of the air in a boundary layer, we still do not know how the sand is carried with the turbulent flow. To do this we have to develop the Lagrangian Theory of turbulence for the boundary layer. The reason is that the grains of sand have their own characteristics and are carried with the airflow and interact with it. Some grains creep along the surface others fly and when the land they can send the grains they land on flying. This is called saltation, see [24]. In Lagrangian flow you move along the air flow and observe the sand as it is carried along this flow. The details are complicated and it is this theory that we have to work out before, we can say that Ole's conjecture that the turbulence in the air is carrying and sorting the sand can be proven. This is the missing 1/3 of the story.

2 The Deterministic Navier-Stokes Equations

A general incompressible fluid flow satisfies the Navier-Stokes Equation

$$u_t + u \cdot \nabla u = \nu \Delta u - \nabla p, \qquad u(x,0) = u_0(x)$$

with the incompressibility condition $\nabla \cdot u = 0$.[1] Eliminating the pressure using the incompressibility condition gives

$$u_t + u \cdot \nabla u = \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2, \qquad u(x,0) = u_0(x).$$

The turbulence is quantified by the dimensionless Taylor-Reynolds number $Re_\lambda = \frac{U\lambda}{\nu}$ [20]. We will impose periodic boundary conditions on the small scales below.

2.1 Reynolds Decomposition

Following the classical Reynolds decomposition [21], we decompose the velocity into mean flow U and the fluctuations u. Then the velocity is written as $U + u$, where U describes the large scale flow and u describes the small scale turbulence. We must

[1] $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$ and $\nabla \cdot u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$.

also decompose the pressure into mean pressure P and the fluctuations p, then the equation for the large scale flow can be written as

$$U_t + U \cdot \nabla U = \nu \Delta U - \nabla P - \nabla \cdot (\overline{u \otimes u}), \tag{3}$$

where in coordinates $\nabla \cdot (\overline{u \otimes u}) = \frac{\partial \overline{u_i u_j}}{\partial x_j}$, that is ∇ is dotted with the rows of $\overline{u_i u_j}$ and $R_{ij} = \overline{u \otimes u}$ is the Reynolds stress, see [5]. The Reynolds stress has the interpretation of a turbulent momentum flux and the last term in (3) is also know as the eddy viscosity. It describes how the small scales influence the large scales. In addition we get divergence free conditions for U, and u

$$\nabla \cdot U = 0, \qquad \nabla \cdot u = 0.$$

Together, (3) and the divergence free condition on U give Reynolds Averaged Navier-Stokes (RANS) that forms the basis for most contemporary simulations of turbulent flow.

Finding a constitutive law for the Reynolds stress $\overline{u \otimes u}$ is the famous closure problem in turbulence and we will solve that by writing down a stochastic equation for the small scale velocity u. The hypothesis is that the large scales influence the small scales directly, through the fluid instabilities and the noise in fully developed turbulence. An example of these mechanics, how the instabilities magnify the tiny ambient noise to produce large noise, in given in [6], see also Chap. 1 in [8].

The consequence of the above hypothesis is that the small scale velocity u in turbulence is a stochastic process that is determined by a stochastic partial differential equation (SPDE). This is the Eq. (4) and it is the Navier-Stokes equation driven by noise. This is the point of view taken by Kolmogorov in [15–17], but the question we have to answer is: what is the form of the noise? There is a large literature on this question, trying to trace the form of the noise back to the fluid instabilities, but these attempts have proven to be unsuccessful. Any memory of the fluid instabilities is quickly forgotten in fully-developed turbulence and the noise seems to be of a general form. Thus it makes sense to try to put generic noise into the Navier-Stokes equations and see how the Navier-Stokes evolution colors generic noise. Below we will answer what generic noise in the Navier-Stokes equation must look like, see [8] for more details.

Now consider the inertial range in turbulence. In Fourier space this is the range of wave numbers k: $\frac{1}{L} \leq |k| \leq \frac{1}{\eta}$, where $\eta = (\nu^3/\varepsilon)^{1/4}$ is the Kolmogorov length scale, ε is the energy dissipation and L the size of the largest eddies, see [8]. If we assume that dissipation takes place on all length scales in the inertial range then the form of the dissipation processes are determined by the fundamental theorems of probability. Namely, if we impose periodic boundary conditions (different boundary conditions correspond to different basis vectors), then the central limit theorem and the large deviation principle stipulate that the additive noise in the Navier-Stokes equation for the small scale must be of the form:

$$\sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0} d_k |k|^{1/3} dt\ e_k(x),$$

where $e_k(x) = e^{2\pi i k\cdot x}$ are the Fourier coefficient and $c_k^{\frac{1}{2}}$ and d_k are coefficients that ensure the series converge in 3 dimensions. The first term describes the mean of weakly coupled dissipation processes given by the central limit theorem and the second term describes the large deviations of that mean, given by the large deviation principle, see [8]. Thus together the two terms give a complete description of the mean of the dissipation process similar to the mean of many processes in probability. The factor $|k|^{1/3}$ implies that the mean dissipation has only one scaling. Notice that we do not impose any convergence rate on the coefficients $c_k^{1/2}$ and d_k beyond that the above term must make sense in $L^2(\mathbb{T}^3)$, the function space of square integrable functions were we want the SPDE to make sense. $\mathbb{T}^3$ is the three-dimensional torus because of the periodic boundary conditions on u. Thus any scaling must come from the Navier-Stokes evolutions acting on the noise. The Fourier coefficients of the first series contain independent Brownian motions b_t^k and thus the noise is white in time in the infinitely many directions in function space. The noise cannot be white in space, hence the decaying coefficients $c_k^{1/2}$ and d_k, because if it was the small scale velocity u would be discontinuous in 3 dimension, see [7]. This is contrary to what is observed in nature.

However, the noise must also have another multiplicative term. This part of the noise, in fully developed turbulence, models the noise associated with the excursion (jumps) in the velocity gradient or vorticity concentrations. It is known from simulations that such jumps in the velocity gradient of the fluid are present in fully-developed turbulence. If we let N_t^k denote the integer number of velocity excursion, associated with kth wavenumber, that have occurred at time t, so that the differential $dN^k(t) = N^k(t+dt) - N^k(t)$ denotes the number of excursions in the time interval $(t, t+dt]$, then the process $df_t^3 = \sum_{k\neq 0}^M \int_{\mathbb{R}} h_k(t,z)\bar{N}^k(dt,dz)$, gives the multiplicative noise term. One can show that any noise corresponding to jumps in the velocity gradients must have this multiplicative noise to leading order, see [7]. A detailed derivation of both the noise terms can be found in [7, 8].

Adding the additive noise and the multiplicative noise we get the stochastic Navier-Stokes equations describing the small scales in fully developed turbulence

$$du = (\nu\Delta u - u\cdot\nabla u + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2)dt + \sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0} d_k|k|^{1/3} dt\ e_k(x)$$
$$+ u(\sum_{k\neq 0}^{M}\int_{\mathbb{R}} h_k \bar{N}^k(dt,dz)), \quad u(x,0) = u_0(x), \tag{4}$$

where we have used the divergence free condition $\nabla\cdot u = 0$ to eliminate the small scale pressure p. Each Fourier component e_k comes with its own Brownian motion b_t^k and a deterministic bound $|k|^{1/3}dt$.

2.2 Solution of the Stochastic Navier-Stokes

The next step is to figure out how the generic noise interacts with the Navier-Stokes evolution. This is determined by the integral form of the Eq. (4),

$$u = e^{Kt} e^{\int_0^t dq} M_t u^0 + \sum_{k \neq 0} \int_0^t e^{K(t-s)} e^{\int_s^t dq} M_{t-s} (c_k^{1/2} d\beta_s^k + d_k |k|^{1/3} ds) e_k(x), \quad (5)$$

where K is the operator $K = \nu\Delta + \nabla\Delta^{-1} tr(\nabla u \nabla)$, and we have omitted the terms $-U \cdot \nabla u - u \cdot \nabla U$ in (4), to simplify the exposition. We solve (4) using the Feynmann-Kac formula, and the Cameron-Martin formula (or Girsanov's Theorem) from probability theory, see [8], to get (5). The Cameron-Martin formula gives the Martingale $M_t = exp\{-\int_0^t u(B_s, s) \cdot dB_s - \frac{1}{2}\int_0^t |u(B_s, s)|^2 ds\}$. The Feynmann-Kac formula gives the exponential of a sum of terms of the form $\int_s^t dq^k = \int_0^t \int_{\mathbb{R}} \ln(1 + h_k) N^k(dt, dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt, dz)$, see [7] or [8] Chap. 2 for details. The form of the processes

$$e^{\int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz)} = e^{N_t^k \ln\beta + \gamma \ln|k|} = |k|^\gamma \beta^{N_t^k} \quad (6)$$

was found by She and Leveque [22], for $h_k = \beta - 1$. It was pointed out by She and Waymire [23] and by Dubrulle [13] that they are log-Poisson processes. The upshot of this computation will be that we see the Navier-Stokes evolution acting on the additive noise to give the Kolmogorov-Obukhov'41 scaling, and the Navier-Stokes evolution acting on the multiplicative noise to produce the intermittency corrections through the Feynmann-Kac formula. Together these two scaling then combine to give the scaling of the structure functions in turbulence. This will become clear when we consider the structure functions and invariant measure of the stochastic Naiver-Stokes equation below.

3 The Kolmogorov-Obukhov-She-Leveque Scaling

The structure functions in turbulence are defined to be

$$S_p(|x-y|, t) = E(|\delta u|^p) = E(|u(x,t) - u(y,t)|^p),$$

$p \in \mathbb{N}$, where E is the expectation, that is substituted by an ensemble average in simulations and experiments. $l = |x-y|$ is called the lag variable. The structure functions are a better probe of turbulence than the usual moments of the velocity, since the latter exhibit (skewed) Gaussian behavior and do not reveal the scaling found in turbulent fluids.

3.1 Computation of the Structure Functions

The structure functions are non-negative so it suffices to estimate them from above, when the lag variable is small the estimates are sharp.

Lemma 3.1 (The Kolmogorov-Obukhov-She-Leveque scaling) *The scaling of the structure functions is*

$$S_p \sim C_p|x-y|^{\zeta_p}, \qquad \zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3}).$$

$\frac{p}{3}$ *being the Kolmogorov scaling and* τ_p *the intermittency corrections, for* $l = |x-y|$ *small. The scaling of the structure functions is consistent with Kolmogorov's* $4/5$ *law,* $S_3 = -\frac{4}{5}\varepsilon|x-y|$*, to leading order, were* $\varepsilon = -\frac{d\mathcal{E}}{dt}$ *is the energy dissipation.*

Here S_3 is the structure function of the velocity differences, without the absolute value.

3.2 The First Few Structure Functions

The first structure functions is estimated by

$$S_1(x,y,\infty) \le \frac{2}{C}\sum_{k\in\mathbb{Z}^3\backslash\{0\}} \frac{|d_k|(1-e^{-\lambda_k t})}{|k|^{\zeta_1}}|\sin(\pi k\cdot(x-y))|.$$

We get a stationary state as $t \to \infty$, and for $|x-y|$ small, $S_1(x,y,\infty) \sim \frac{2\pi^{\zeta_1}}{C}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}|d_k||x-y|^{\zeta_1}$, where $\zeta_1 = 1/3+\tau_1 \approx 0.37$. Similarly, $S_2(x,y,\infty) \sim \frac{4\pi^{\zeta_2}}{C^2}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}[{d_k}^2 + (\frac{C}{2})c_k]|x-y|^{\zeta_2}$, when $|x-y|$ is small, where $\zeta_2 = 2/3 + \tau_2 \approx 0.696$, and $S_3(x,y,\infty) \sim \frac{2^3\pi}{C^3}\sum_{k\in\mathbb{Z}^3\backslash\{0\}}[|d_k|^3 + 3(C/2)c_k|d_k|]|x-y|$. For the pth structure functions, we get that S_p is estimated by

$$S_p \le \frac{2^p}{C^p}\sum_{k\in\mathbb{Z}^3\backslash\{0\}} \frac{\sigma^p 2^{p/2}\Gamma((1+p)/2)\ {}_1F_1\left(-\frac{1}{2}p,\ \frac{1}{2},\ -\frac{1}{2}(M/\sigma)^2\right)}{\sqrt{\pi}|k|^{\zeta_p}}|\sin^p(\pi k\cdot(x-y))|,$$

where ${}_1F_1$ is the hypergeometric function, $M = |d_k|(1-e^{-\lambda_k t})$ and $\sigma = \sqrt{(C/2)c_k(1-e^{-2\lambda_k t})}$. The details of these estimates are given in [7, 8]. These values are consistent with the values found in high-Reynolds number experiments, see [12, 25].

The integral equation can be considered to be an infinite-dimensional Ito process, see [8]. This means that we can find the associated Kolmogorov backward equation for the Ito diffusion associated with the Eq. (5) and this equation that determines the invariant measure of turbulence, see [7], is linear. This was first attempted by Hopf

[14] wrote down a functional differential equation for the characteristic function of the invariant measure of the deterministic Navier-Stokes equation. The Kolmogorov-Hopf (backward) equation for (4) is

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{tr}[P_t C P_t^* \Delta \phi] + \mathrm{tr}[P_t \bar{D} \nabla \phi] + \langle K(z) P_t, \nabla \phi \rangle, \tag{7}$$

see [7, 8] Chap. 3, where $\bar{D} = (|k|^{1/3} D_k)$, $\phi(z)$ is a bounded function of z, $P_t = e^{-\int_0^t \nabla u \, dr} M_t \prod_k^m |k|^{2/3} (2/3)^{N_t^k}$. The variance and drift are defined to be

$$Q_t = \int_0^t e^{K(s)} P_s C P_s^* e^{K^*(s)} ds, \qquad E_t = \int_0^t e^{K(s)} P_s \bar{D} ds. \tag{8}$$

3.3 The Invariant Measure of the Stochastic Navier-Stokes

In distinction to the nonlinear Navier-Stokes equation (4) that cannot be solved explicitly, the linear equation (7) can be solved. The solution of the Kolmogorov-Hopf equation (7) is

$$R_t \phi(z) = \int_H \phi(e^{Kt} P_t z + E_t I + y) \mathcal{N}_{(0,Q_t)} * \mathbb{P}_{N_t}(dy),$$

$\mathbb{P}_{N_t}$ being the law of the log-Poisson process (6). $\mathcal{N}_{(E,Q)}$ is the law of an infinite-dimensional Gaussian process with mean E and variance Q and $*$ denotes convolution. The invariant measure of turbulence that appears in the last equation can now be expressed explicitly,

Theorem 3.1 *The invariant measure of the Navier-Stokes equation on $H_c = H^{3/2^+}$ $(\mathbb{T}^3)$ is,*

$$\mu(dx) = e^{<Q^{-1/2}EI,\ Q^{-1/2}x> - \frac{1}{2}|Q^{-1/2}EI|^2} \mathcal{N}_{(0,Q)}(dx) \sum_k \delta_{k,l} \prod_{j \neq l}^m \delta_{N_t^j} \sum_{j=0}^\infty p_{m_l}^j \delta_{(N_l - j)}$$

where $Q = Q_\infty$, $E = E_\infty$, $m_k = \ln |k|^{2/3}$ is the mean of the log-Poisson processes (6) and $p_{m_k}^j = \frac{(m_k)^j e^{-m_k}}{j!}$ is the probability of $N_\infty^k = N_k$ having exactly j jumps, $\delta_{k,l}$ is the Kroncker delta function and $\delta_{N_t^j}$ is the Dirac delta function concentrated at N_t^j.

This shows that the invariant measure of turbulence is simply a product of two measures, one an infinite-dimensional Gaussian that gives the Kolmogorov-Obukhov scaling and the other a discrete Poisson measure that gives the She-Leveque intermittency corrections. Together they produce the scaling of the structure functions in Lemma 1. This makes it clear how the Navier-Stokes evolution operating on the additive and the multiplicative noise in Eq. (4) produces the scaling. Miraculously, the infinite dimensional Gaussian part (Kolmogorov-Obukhov) and the intermittency part (She-Leveque) separate in the invariant measure on the infinite-dimensional functions space, one represented by a continuous measure the other one by a discrete measure. It is when we project this measure onto measures of experimental or simulated quantities, see below, that these two parts get mixed.

3.4 The Differential Equation for the PDF

A quantity that can be compared directly to experiments and simulations is the probability density function (PDF) for the velocity differences δu. We take the trace of the Kolmogorov-Hopf equation (7), see [8] Chap. 3, to compute the differential equation satisfied by the PDF for δu. First we do this ignoring the intermittency corrections τ_p in Lemma 1, see [9] for details. The stationary equation satisfied by the PDF without intermittency corrections is

$$\frac{1}{2}\phi_{rr} + \frac{1+|c|}{r}\phi_r = \frac{1}{2}\phi. \tag{9}$$

The probability density functions (PDF) is going to be a solution of this equation, if the intermittency corrections τ_p are ignored, the details of the computation are given in [7, 8].

4 The Probability Density Function

4.1 The Generalized Hyperbolic Distributions

Lemma 4.1 *The PDF, without intermittency corrections, is a Generalized Hyperbolic Distribution (GHD) of Barndorff-Nielsen [3]:*

$$f(x) = \frac{(\gamma/\delta)^\lambda}{\sqrt{2\pi}K_\lambda(\delta\gamma)} \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)e^{\beta(x-\mu)}}{\left(\sqrt{\delta^2+(x-\mu)^2}/\alpha\right)^{\lambda-\frac{1}{2}}} \tag{10}$$

where K_λ is the modified Bessel's function of the second kind with index λ, $\gamma = \sqrt{\alpha^2-\beta^2}$. α, β, δ and μ are parameters.

Equation (10) is the solution of (9), see [7, 8] for details of the proof, and the PDF that can be compared a large class of experimental data.

5 The PDF of Turbulence

The PDF of δu becomes more complicated when the intermittency is included. Then it becomes impossible to have a single continuous PDF for all the different moments and instead one has to have a distribution that is a product of a discrete and continuous distributions, just as the invariant measure of turbulence Theorem 3.1 itself. One can put in the intermittency correction in the Eq. (9) defining the PDF and get a different continuous PDF for each moment, this is done in [9]. In this paper we will take a different approach and project the invariant measure Theorem 3.1 to a PDF that is a product of a continuous and a discrete measure analogous to the invariant measure itself. The continuous part of the PDF will be the Generalized Hyperbolic Distribution (10).

We start with the log-Poisson process $|x|\left(\frac{2}{3}\right)^{N_t^k}$ and the mean $m_k = \ln(|x|^{-6})$ of the associated Poisson distribution. Now the mixed continuous and discrete distribution is given by:

$$\bar{\mu}(\cdot) = \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(\ln(|x|^{-6}))^j}{j!} |x|^6 \delta_{N_t^k - j}(\cdot) f(x) dx, \tag{11}$$

where $\bar{\mu}$ denotes the projection of the measure. We assume that the velocity is a Hölder continuous function of Hölder index $1/3$, see [8]. Then evaluating the measure on the pth moment of the velocity differences gives,

$$\begin{aligned} \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(\ln(|x|^{-6}))^j}{j!} |x|^6 \delta_{N_t^k - j}(|x| \left(\frac{2}{3}\right)^{N_t^k})^{\frac{p}{3}} f(x) dx &= \int_{-\infty}^{\infty} |x|^{\frac{p}{3}} |x|^{6(1-(2/3)^{\frac{p}{3}})} f(x) dx \\ &= \int_{-\infty}^{\infty} |x|^{p+3\tau_p} f(x) dx = \int_{-\infty}^{\infty} |x|^{3\zeta_p} f(x) dx, \end{aligned}$$

where

$$\zeta_p = \frac{p}{3} + \tau_p$$

is the scaling exponent of the pth structure function, with the intermittency correction τ_p. The upshot is that the discrete part of the PDF adds the intermittency correction $|x|^{3\tau_p}$ to the pth moment and

$$\bar{\mu}(|\delta u|^p) = \int_{-\infty}^{\infty} |x|^{p+3\tau_p} f(x) dx, \tag{12}$$

where δu are the velocity differences and the intermittency corrections are $\tau_p = -\frac{2p}{9} + 2(1 - (2/3)^{\frac{p}{3}})$.

We have shown that the PDF (11) of the velocity differences in turbulence consists of a continuous distribution convolved with a discrete one. This is a direct consequence of the similar structure of the invariant measure Theorem 3.1 of turbulence. The continuous part of the PDF corresponds to the Kolmogorov-Obukhov'41 theory represented by the infinite-dimensional Gaussian measure in Theorem 3.1, whereas the discrete part corresponds to the She-Leveque intermittency correction just as it does in Theorem 3.1. Remarkably, all of this follows from the Navier-Stokes equations when generic noise is added to it (4).

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