Chapter 2
Algebraic Approach to Quantum Theory

2.1 Algebraic Quantum Mechanics

Before entering the realm of the quantum theory of fields, let's have a look at something simpler and better understood, namely *quantum mechanics* (QM). To prepare the ground for what follows, we will present an abstract formulation of QM and discuss how it relates to the more standard Dirac–von Neumann axioms [Dir30, vN32]. The exposition presented in this chapter is based on [BF09b, Mor13, Fre13, Str08].

2.1.1 Functional Analytic Preliminaries

Let us start by recalling some basic definitions from functional analysis. For more information see [Rud91, RS80, BR87, BR97, Kad83]. Readers familiar with basic functional analysis can skip this subsection.

Definition 2.1 An *algebra* $\mathfrak{A}$ over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a $\mathbb{K}$-vector space with an operation $\cdot : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ called the *product* with the following properties:

1. $(A \cdot B) \cdot C = A \cdot (B \cdot C), \quad \forall A, B, C \in \mathfrak{A}$ (associativity),
2. $A \cdot (B + C) = A \cdot B + A \cdot C$, $(B + C) \cdot A = B \cdot A + C \cdot A$,
   $\alpha(A \cdot B) = (\alpha A) \cdot B = A \cdot (\alpha B)$, for all $A, B, C \in \mathfrak{A}$, $\alpha \in \mathbb{K}$ (distributivity).

We will usually denote the algebra product $\cdot$ simply by juxtaposition, i.e. $A \cdot B \equiv AB$.

Definition 2.2 An algebra $\mathfrak{A}$ is said to have a unit (i.e. $\mathfrak{A}$ is unital) if there exists an element $\mathbb{1} \in \mathfrak{A}$ such that $\mathbb{1}A = A\mathbb{1} = A$, for all $A \in \mathfrak{A}$.

Definition 2.3 An *involutive complex algebra* (a $*$-*algebra*) $\mathfrak{A}$ is an algebra over the field of complex numbers, together with a map, $^* : \mathfrak{A} \to \mathfrak{A}$, called an *involution*. The image of an element A of $\mathfrak{A}$ under the involution is written A^*. Involution is required to have the following properties:

K. Rejzner, *Perturbative Algebraic Quantum Field Theory*,
Mathematical Physics Studies, DOI 10.1007/978-3-319-25901-7_2

1. for all $A, B \in \mathfrak{A}$: $(A+B)^* = A^* + B^*$, $(AB)^* = B^* A^*$,
2. for every $\lambda \in \mathbb{C}$ and every $A \in \mathfrak{A}$: $(\lambda A)^* = \overline{\lambda} A^*$,
3. for all $A \in \mathfrak{A}$: $(A^*)^* = A$.

Definition 2.4 A $*$-morphism is a map $\varphi : \mathfrak{A} \to \mathfrak{B}$ between $*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$, which is an algebra morphism compatible with the involution, i.e.:

1. $\varphi(AB) = \varphi(A)\varphi(B)$, for all $A, B \in \mathfrak{A}$,
2. $\varphi(\lambda A + B) = \lambda\varphi(A) + \varphi(B)$, for all $A, B \in \mathfrak{A}$, $\lambda \in \mathbb{C}$,
3. $\varphi(A^*) = \varphi(A)^*$ for every $A \in \mathfrak{A}$.

Up to now all the properties we have considered are purely algebraic. In order to quantify the notion of distance between the elements of the algebra we need some topology.

Let us start with some basic definitions and notation.

Definition 2.5 *A topological space* $\mathcal{X}$ is a pair (X, τ), where X is a set X and τ is a collection of subsets of X (called open sets), with the following properties:

- $X \in \tau$
- $\varnothing \in \tau$
- the intersection of any two open sets is open: $U \cap V \in \tau$ for $U, V \in \tau$
- the union of every collection of open sets is open: $\bigcup_{\alpha \in A} U_\alpha \in \tau$ for $U_\alpha \in \tau$ $\forall \alpha \in A$, where A is some index set.

Consider mappings between topological spaces. A topology tells us something about the regularity of those mappings, since it contains already a notion of "being close to something" and we can ask ourselves to what extend a given map preserves this notion.

Definition 2.6 A function $f : \mathcal{X} \to \mathcal{Y}$, where $\mathcal{X}$ and $\mathcal{Y}$ are topological spaces, is *continuous* if and only if for every open set $V \subseteq Y$, the inverse image:

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\} \tag{2.1}$$

is open.

Given a collection of topological spaces, one can define a new topological space by taking their *Cartesian product*. This is a very commonly used operation, so we recall here the definition of a natural topology on such product.

Definition 2.7 Let X be a set such that

$$X = \prod_{i \in I} X_i$$

is the *Cartesian product of topological spaces* X_i, indexed by i in some set I. Let $p_i : X \to X_i$ be the canonical projections. The product topology on X is defined as the coarsest topology (i.e. the topology with the fewest open sets) for which all the projections p_i are continuous.

In our applications the topology will not be enough to capture all the structure we need. In the physics context it is common that we want to add certain quantities and scale them. This leads in a natural way to a vector space structure. We want this structure to be compatible also with the topology.

Definition 2.8 A *Topological vector space* (TVS) over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ (with their standard topologies) is a pair $(X, \tau) \equiv \mathcal{X}$, where τ is a topology such that:

- every point of X is a closed set (i.e. its complement is an open set),
- vector addition $\mathcal{X} \times \mathcal{X} \to \mathcal{X}$ and scalar multiplication $\mathbb{K} \times \mathcal{X} \to \mathcal{X}$ are continuous functions with respect to the product topology on the respective domains.

Definition 2.9 Let $\mathcal{X}$, $\mathcal{Y}$ be topological vector spaces over the field $\mathbb{K}$. We denote by $L(\mathcal{X}, \mathcal{Y})$ the space of continuous linear maps from $\mathcal{X}$ to $\mathcal{Y}$ and by $\mathcal{X}'$ the topological dual of $\mathcal{X}$, i.e. the space of continuous linear maps from $\mathcal{X}$ to $\mathbb{K}$.

A topology can be introduced for example by means of a norm. This leads to the concept of a normed space.

Definition 2.10 A complex normed space is a vector space $\mathcal{X}$ over $\mathbb{C}$, equipped with a map $\|.\| : \mathcal{X} \to \mathbb{R}$, which satisfies:

1. $\|\lambda A\| = |\lambda|\|A\|$ (scaling),
2. $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality also called subadditivity),
3. If $\|A\| = 0$, then A is the zero vector (separates points).

One of the nice features of normed spaces is that the continuity of maps between such spaces can be probed by convergent sequences. Recall that in general:

Definition 2.11 A point x of the topological space $\mathcal{X}$ is the limit of the sequence (x_n) in $\mathcal{X}$ if, for every neighbourhood U of x, there is an N such that, for every $n \geq N$, $x_n \in U$.

In particular, for normed spaces:

Definition 2.12 A point x of a normed space $(X, \|.\|)$ is the limit of the sequence (x_n) if, for all $\varepsilon > 0$, there is an N such that, for every $n \geq N$, $\|x_n - x\| < \varepsilon$. A sequence that has a limit is called convergent.

Definition 2.13 Let $\mathcal{X}$, $\mathcal{Y}$ be topological spaces. Then a function $f : \mathcal{X} \to \mathcal{Y}$ is said to be *sequentially continuous* if for every convergent sequence (x_n) in $\mathcal{X}$ with the limit x we have $f(x_n) \to f(x)$ in $\mathcal{Y}$.

An elementary result from analysis states that if $\mathcal{X}$, $\mathcal{Y}$ are normed spaces equipped with topologies induced by the respective norms then $f : \mathcal{X} \to \mathcal{Y}$ is continuous if and only if it is sequentially continuous. However, in Sect. 2.4.1 we will consider spaces where these two notions do not coincide.

Having defined the notion of convergence of sequences, we are now ready to introduce the notion of *completeness*. First we define:

Definition 2.14 A sequence (x_n) in a normed space $\mathcal{X}$ is called a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all integers m, n such that $m, n > N$ we have $\|x_n - x_m\| < \varepsilon$.

Definition 2.15 A normed space $\mathcal{X}$ in which every Cauchy sequence converges to an element of $\mathcal{X}$ is called *complete*.

Given a normed space $\mathcal{X}$ that is not complete one can always construct its *completion*,[1] i.e. a complete normed space that contains $\mathcal{X}$ as a dense subspace.

Let us now come back to our algebras. If an algebra $\mathfrak{A}$ is equipped with a norm, we can ask for the continuity of the algebraic relations with respect to the norm topology and for some notion of completeness. This leads to the following definitions.

Definition 2.16 A *normed algebra* $\mathfrak{A}$ is a normed vector space whose norm $\|.\|$ satisfies

$$\|AB\| \leq \|A\|\|B\|.$$

If $\mathfrak{A}$ is unital, then it is a normed unital algebra if in addition $\|\mathbb{1}\| = 1$.

Definition 2.17 A *Banach space* is a normed vector space equipped with the norm-induced topology that is complete with respect to this topology. A Banach (unital) algebra is a Banach space and a normed (unital) algebra with respect to the same norm.

A particularly important class of Banach algebras with involution is distinguished by the C^*-property. We will see in this chapter that such algebras can be used to describe spaces of observables in quantum systems.

Definition 2.18 A *C^*-algebra* is a Banach involutive algebra (Banach algebra with involution satisfying $\|A^*\| = \|A\|$), such that the norm has the C^*-property:

$$\|A^*A\| = \|A\|\|A^*\|, \quad \forall A \in \mathfrak{A}.$$

2.1.2 Observables and States

In this section we will see how the structures introduced in the previous section are used in quantum physics. First note that in order to describe a physical system we need to specify a collection of physical quantities, which we want to measure (we call them *observables*) and a collection of *states* in which the system can be prepared. Now we want to deduce what kind of mathematical structure is suitable to describe observable and states. Operationally, each observable corresponds to some measurement apparatus, which measures given properties of the system. An example of such an apparatus is a particle detector localized in some region of space.

[1] The completion of $\mathcal{X}$ can be constructed as a set of equivalence classes of Cauchy sequences in $\mathcal{X}$.

Next, one considers operations that can be performed on observables. Scaling of the measurement apparatus means multiplying the corresponding observable A by a real number. One can also consider other functions of the observables, which can be operationally realized as "repainting the scale". The simplest examples are monomials A^n, interpreted as measuring the observable A and taking the nth power of the result.

Now we discuss the notion of states. We need to assume that we are able to repeat experiments, so that we can measure a given observable repeatedly in the same state (i.e. for the same preparation of the system). This statistical interpretation presupposes that each experiment comes with a protocol that allows us to obtain the same initial condition each time it is repeated. Under this assumption, a state ω associates to an observable A a real number $\omega(A)$ obtained by averaging the results of measurements of A for the system prepared to be in the state ω. It is natural to assume that $\omega(\lambda A) = \lambda\omega(A)$ for $\lambda \in \mathbb{R}_+$ (scaling). Let $\mathbb{1}$ be the observable, which always takes value 1. For this observable we require that $\omega(\mathbb{1}) = 1$. One can also deduce the positivity of states from the fact that the average of positive numbers is positive, so $\omega(A^2) \geq 0$.

If we assume that physical properties of observables can be measured only by looking at expectation values in various states of the system, it is natural to identify the observables that give the same expectation values in all the states. Now let $\mathcal{A}$ be the space of equivalence classes of observables, where $A \sim B$ if $\omega(A) = \omega(B)$ for all states ω of the system. A notion of a norm can be introduced by assigning to each observable $A \in \mathcal{A}$ a finite positive number defined by

$$\|A\| \doteq \sup_{\omega} |\omega(A)|$$

The operational properties of states imply that $\|\lambda A\| = |\lambda|\|A\|$ for $\lambda \in \mathbb{R}$ and $\|A\| = 0$ implies that $A = 0$ (states separate observables). What is still missing is the linear structure on $\mathcal{A}$ and the product. Let us start with the linear structure. We want to be able to construct measuring devices that measure the sum of any two observables A and B, i.e. we need the operation "$A + B$". This operation has to satisfy

$$\omega(A + B) = \omega(A) + \omega(B),$$

for all states of the system. It is, however, not clear if an element "$A + B$" exists in $\mathcal{A}$, so one needs to embed the initial space of observables in a larger structure in such a way that states will remain positive linear functionals on this enlarged space. Further considerations (see for example [Str08]) lead to the notion of Jordan algebras [Jor33, JvNW34] and finally, by bringing in a complex structure, to C^*-algebras, introduced in [Gel43] and discussed in [Seg47a, Seg47b] in the context of quantum mechanics.

We can summarize the basic axioms in the algebraic approach to QM as follows:

1. A physical system is defined by its unital C^*-algebra $\mathfrak{A}$.
2. States are identified with positive, normalized linear functionals on $\mathfrak{A}$, i.e. $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$ and $\omega(\mathbb{1}) = 1$.

Note that on a unital C^*-algebra a positive, normalized linear functional is automatically continuous with respect to the topology induced by the C^*-norm. More generally, we can define states also on involutive topological algebras.

Definition 2.19 A *state* on an involutive algebra $\mathfrak{A}$ is a linear functional ω, such that:

$$\omega(A^*A) \geq 0, \ \omega(\mathbb{1}) = 1.$$

Observables are self-adjoint elements of $\mathfrak{A}$ and possible measurement results for an observable A are characterized by its spectrum $\sigma(A)$. Recall that an element A of a C^*-algebra is called self-adjoint if $A^* = A$.

Definition 2.20 The *spectrum* $\mathrm{spec}(A)$ of $A \in \mathfrak{A}$ is the set of all $\lambda \in \mathbb{C}$ such that $A - \lambda\mathbb{1}$ has no inverse in $\mathfrak{A}$.

A standard result from functional analysis states that a spectrum of self-adjoint element is a subset of the real line and this agrees with the physical intuition, as outcomes of measurements have to be real.

2.1.3 Hilbert Space Representations

Having defined the abstract setup we can proceed to a more concrete description that provides a way to recover the Dirac–von Neumann axioms. The crucial observation is that abstract elements of an involutive algebra $\mathfrak{A}$ can be realized as operators on some Hilbert space by a choice of a *representation*. Definitions introduced in this section follow closely [Mor13, RS80]. First let us recall the definition of a Hilbert space.

Definition 2.21 Let $\mathcal{H}$ be a complex vector space. A map $\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a *Hermitian inner product* if

1. $\langle u, v \rangle = \overline{\langle u, v \rangle}, \forall u, v \in \mathcal{H}$,
2. $\langle u, \alpha v + \beta w \rangle = \alpha \langle u, v \rangle + \beta \langle u, w \rangle$ (linear in the second argument),
3. $\langle v, v \rangle \geq 0$ where the case of equality holds precisely when $v = 0$ (positive definite).

Properties 1 and 2 imply that $\langle ., . \rangle$ is antilinear in the first argument. One can define a norm on $\mathcal{H}$ by setting

$$\|v\| \doteq \sqrt{\langle v, v \rangle}.$$

Definition 2.22 A *Hilbert space* $\mathcal{H}$ is a complex vector space with a Hermitian inner product $\langle ., . \rangle$ such that the norm induced by this product makes $\mathcal{H}$ into a Banach space.

In physics *separable Hilbert spaces* play an important role.

Definition 2.23 A Hilbert space $\mathcal{H}$ is called *separable* if it admits a countable subset whose linear span is dense in $\mathcal{H}$. In fact a Hilbert space is separable if it is either finite dimensional or has a countable basis.

We are ready to define the notion of linear operators on Hilbert spaces, which is important in the context of C^*-algebras and physical observables.

Definition 2.24 An *operator* A on a Hilbert space $\mathcal{H}$ is a linear map from a subspace $D \subset \mathcal{H}$ into $\mathcal{H}$. In particular, if $D = \mathcal{H}$ and A satisfies $||A|| \doteq \sup_{||x||=1}\{||Ax||\} < \infty$, it is called *bounded*.

We will always assume that D is dense in $\mathcal{H}$ (i.e. A is *denesly defined*).

Definition 2.25 Let A be a densly defined linear operator on a Hilbert space $\mathcal{H}$. Let $D(A^*)$ be the set of all $v \in \mathcal{H}$ such that there exists $u \in \mathcal{H}$ with

$$\langle Aw, v \rangle = \langle w, u \rangle, \quad \forall w \in D(A).$$

For each such $v \in D(A^*)$ we define $A^*v = u$. A^* is called the adjoint of A.

An important class of bounded operators is provided by the unitary ones.

Definition 2.26 A bounded linear operator $U : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is called a *unitary operator* if it satisfies $U^*U = UU^* = \mathbb{1}$.

Note that the space $\mathscr{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$ forms a C^*-algebra. We will see later on that one can argue the other way and realize any abstract C^*-algebra as the algebra of bounded operators on some $\mathcal{H}$. If A is a bounded operator on a Hilbert space then the self-adjointness is the same as hermiticity, i.e. is the condition that $A^* = A$. In general this is not sufficient.

Definition 2.27 An operator A on a Hilbert space $\mathcal{H}$ with a dense domain $D(A) \subset \mathcal{H}$ is called *symmetric* if for any vectors $u, v \in D(A)$ we have $\langle u, Av \rangle = \langle Au, v \rangle$. This implies that $D(A) \subseteq D(A^*)$. A symmetric operator A is *self-adjoint* if in addition $D(A^*) \subset D(A)$.

Definition 2.28 Let A be an operator on a Hilbert space $\mathcal{H}$ with a dense domain $D(A) \subset \mathcal{H}$. A self-adjoint operator A' is called a *self-adjoint extension* of A if $D(A) \subseteq D(A')$ and if $A'v = Av$ for any $v \in D(A)$.

A is called *essentially self-adjoint* if it admits a unique self-adjoint extension.

Abstract elements of an involutive algebra $\mathfrak{A}$ are realized as operators on some Hilbert space by a choice of a *representation*.

Definition 2.29 A *representation* of an involutive unital algebra $\mathfrak{A}$ is a unital *-homomorphism π into the algebra of linear operators on a dense subspace $\mathcal{K}$ of a Hilbert space $\mathcal{H}$. In particular, a representation of a C^*-algebra $\mathfrak{A}$ is a unital *-homomorphism $\pi : \mathfrak{A} \to \mathscr{B}(\mathcal{H})$.

A representation π is called *faithful* if $\operatorname{Ker} \pi = \{0\}$. It is called *irreducible* if there are no non-trivial subspaces of $\mathcal{H}$ invariant under $\pi(\mathfrak{A})$.

Definition 2.30 Two representations $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ of a C^*-algebra $\mathfrak{A}$ are called *unitarily equivalent*, if $U\pi_1(A) = \pi_2(A)U$ holds for all $A \in \mathfrak{A}$ with some unitary map $U : \mathcal{H}_1 \to \mathcal{H}_2$.

In the Dirac–von Neumann axiomatic framework, one postulates that physical observables are self-adjoint operators acting on some Hilbert space. The connection between the algebraic formulation and the Hilbert space picture is provided by means of the famous GNS (Gelfand–Naimark–Segal) theorem.

Theorem 2.1 *Let ω be a state on an involutive unital algebra $\mathfrak{A}$. Then there exists a representation π of the algebra by linear operators on a dense subspace $\mathcal{K}$ of some Hilbert space $\mathcal{H}$ and a unit vector $\Omega \in \mathcal{K}$, such that*

$$\omega(A) = (\Omega, \pi(A)\Omega),$$

and $\mathcal{K} = \{\pi(A)\Omega, A \in \mathfrak{A}\}$.

Proof First we introduce a scalar product on the algebra $\mathfrak{A}$ using the state ω:

$$\langle A, B\rangle \doteq \omega(A^* B).$$

Linearity for the right and antilinearity for the left argument are easy to prove. Hermiticity $\langle A, B\rangle = \overline{\langle B, A\rangle}$ follows from the positivity of ω and the fact that we can write A^*B and B^*A as linear combinations of positive elements:

$$\begin{aligned} 2(A^*B + B^*A) &= (A + B)^*(A + B) - (A - B)^*(A - B), \\ 2(A^*B - B^*A) &= -i(A + iB)^*(A + iB) + i(A - iB)^*(A - iB). \end{aligned}$$

From the positivity of ω, it also follows that the scalar product is positive semidefinite, i.e. $\langle A, A\rangle \geq 0$ for all $A \in \mathfrak{A}$. We now study the set

$$\mathfrak{N} \doteq \{A \in \mathfrak{A} | \omega(A^*A) = 0\}.$$

We show that $\mathfrak{N}$ is a left ideal of $\mathfrak{A}$. Because of the Cauchy–Schwarz inequality $\mathfrak{N}$ is a subspace of $\mathfrak{A}$. The same inequality implies that for $A \in \mathfrak{N}$ and $B \in \mathfrak{A}$ we obtain

$$\begin{aligned} \omega((BA)^*BA) = \omega(A^*B^*BA) = &\big\langle B^*BA, A\big\rangle \\ &\leq \sqrt{\langle B^*BA, B^*BA\rangle}\sqrt{\langle A, A\rangle} = 0, \end{aligned}$$

hence $BA \in \mathfrak{N}$. Let us define $\mathcal{K}$ as the quotient $\mathfrak{A}/\mathfrak{N}$. Clearly, the scalar product is positive definite on $\mathcal{K}$ and we complete it to obtain a Hilbert space $\mathcal{H}$. The representation π is induced by the operation of left multiplication on $\mathfrak{A}$, i.e.

$$\pi(A)(B + \mathfrak{N}) \doteq AB + \mathfrak{N},$$

and we set $\Omega = \mathbb{1} + \mathfrak{N}$. If $\mathfrak{A}$ is a C*-algebra, one can show that the operators $\pi(A)$ are bounded, hence admitting unique continuous extensions to bounded operators on $\mathcal{H}$.

We now show that the construction is unique up to unitary equivalence. Let $(\pi', \mathcal{K}', \mathcal{H}', \Omega')$ be another quadruple satisfying the conditions of the theorem. Then we define an operator $U : \mathcal{K} \to \mathcal{K}'$ by

$$U\pi(A)\Omega \doteq \pi'(A)\Omega'.$$

U is well defined, since $\pi(A)\Omega = 0$ if and only if $\omega(A^*A) = 0$, but then also $\pi'(A)\Omega' = 0$. Moreover U preserves the scalar product and is invertible, hence it has a unique extension to a unitary operator from $\mathcal{H}$ to $\mathcal{H}'$. It follows that π and π' are unitarily equivalent. $\square$

The representation π is in general not irreducible, i.e. there may exist a nontrivial closed invariant subspace. In this case, the state ω is not pure, which means that it is a convex combination of other states,

$$\omega = \lambda\omega_1 + (1 - \lambda)\omega_2 \ , \ 0 < \lambda < 1 \ , \ \omega_1 \neq \omega_2.$$

We have seen that the algebraic formulation of quantum mechanics (QM) allows us to characterize a physical system purely in terms of its observable C^*-algebra $\mathfrak{A}$ and states on it. The Hilbert space representations can then be obtained from states by means of the GNS theorem. One can also obtain the probabilistic interpretation of QM as follows. Given an observable A and a state ω on a C*-algebra $\mathfrak{A}$ we reconstruct the full probability distribution $\mu_{A,\omega}$ of measured values of A in the state ω from its moments, i.e. the expectation values of powers of A,

$$\int \lambda^n d\mu_{A,\omega}(\lambda) = \omega(A^n).$$

We can now apply these methods to some simple physical situations. The first example is related to the *canonical commutation relations*.

Example 2.1 Let L be a real vector space with a symplectic form σ, i.e. a bilinear form σ on L which is antisymmetric,

$$\sigma(x, y) = -\sigma(y, x),$$

and nondegenerate,

$$\sigma(x, y) = 0 \; \forall \; y \in L \text{ implies } x = 0.$$

We consider the unital *-algebra $\mathcal{W}(L, \sigma)$ over $\mathbb{C}$ generated by abstract symbols $W(x)$ (the Weyl generators), satisfying the relation

$$W(x)W(y) = e^{i\sigma(x,y)}W(x + y).$$

The involution is defined by

$$W(x)^* = W(-x)$$

and the unit is $\mathbb{1} = W(0)$.

We define a norm on $\mathcal{W}(L, \sigma)$ by

$$\left\| \sum_{i=1}^{n} \lambda_i W(x_i) \right\|_1 = \sum_{i=1}^{n} |\lambda_i|.$$

This norm satisfies the condition $\|AB\|_1 \leq \|A\|_1 \|B\|_1$ of an algebra norm. Moreover, the involution is isometric, $\|A^*\|_1 = \|A\|_1$ and we obtain an involutive normed algebra $\mathcal{W}(L, \sigma)$.

After [Mor13] we recall known facts about the existence of the unique C^*-norm on $\mathcal{W}(L, \sigma)$.

Proposition 2.1 *The following hold true:*

1. *There exists a norm $\|.\|_0$ on $\mathcal{W}(L, \sigma)$ satisfying the C^*-property,*
2. *In any C^*-norm Weyl generators have norm 1.*
3. *If we set*

$$\|A\|_c \doteq \sup\{\|A\|_0, \;\; \textit{such that} \;\; \|.\|_0 : \mathcal{W}(L, \sigma) \to [0, \infty) \textit{ is a } C^*\textit{-norm}\},$$

then $\|.\|_c$ is a C^*-norm.

4. *Let $\mathfrak{W}(L, \sigma)$ be the completion of $\mathcal{W}(L, \sigma)$ with respect to $\|.\|_c$, then $\mathfrak{W}(L, \sigma)$ is a C^*-algebra, associated to (L, σ) uniquely up to isomorphism.*
5. *$\mathfrak{W}(L, \sigma)$ is simple, i.e. there are no non-trivial closed, *-invariant two-sided ideals.*

Proof For proof see [BGP07] as well as [Mor13]. To see that the supremum defining $\|.\|_c$ is finite, note that generators $W(x)$ are of norm 1 with respect to every C^*-norm, so if $A = \sum_i a_i W(x_i)$, then $\|A\| \leq \sum_i |a_i| = \|A\|_1$, which provides the upper bound for the supremum. □

Let's consider a particular example of a symplectic space (L, σ), which realizes canonical commutation relations for a free quantum particle in d dimensional space. In this case $L = \mathbb{R}^{2d}$ and we write elements of L in the form $\boldsymbol{X} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$, where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$, $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_d) \in \mathbb{R}^d$. We define

$$\sigma\left((\boldsymbol{\alpha},\boldsymbol{\beta}),(\boldsymbol{\alpha}',\boldsymbol{\beta}')\right)=-\frac{1}{2}\hbar(\boldsymbol{\alpha}\cdot\boldsymbol{\beta}'-\boldsymbol{\alpha}'\cdot\boldsymbol{\beta}),$$

where $\cdot$ is the scalar product on $\mathbb{R}^d$. If the generators of the resulting Weyl C^*-algebra $\mathfrak{W}(L,\sigma)$ are represented by operators on a Hilbert space in such a way that they depend strongly continuously[2] on the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, then such a representation is called regular. It was proven by von Neumann that all the regular reducible representations of the resulting Weyl algebra are unitary equivalent. Another theorem important in this context is due to Stone [Sto30]:

Theorem 2.2 *Let $(U_t)_{t\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group. Then there exists a unique (not necessarily bounded) self-adjoint operator A such that*

$$U_t=e^{itA},\quad \forall t\in\mathbb{R}.$$

Conversely, if A is a (not necessarily bounded) self-adjoint operator on a Hilbert space $\mathcal{H}$, then the one-parameter family $(U_t)_{t\in\mathbb{R}}$ of unitary operators defined by means of the Spectral Theorem for Self-Adjoint Operators (see for example Chap. 9 [Mor13]) as

$$t\mapsto U_t:=e^{itA}$$

is strongly continuous.

For $\mathfrak{W}(L,\sigma)$ this implies that there exist self-adjoint generators $q^1,\dots,q^d$, $p^1,\dots,p^d$ of 1-parameter groups of unitary operators

$$W(0,\dots,\alpha_k,\dots,0)=e^{i\alpha_k p^k},\quad W(0,\dots,\beta_k,\dots,0)=e^{i\beta_k q^k},$$

We denote $\boldsymbol{p}\doteq(p^1,\dots,p^d)$, $\boldsymbol{q}\doteq(q^1,\dots,q^d)$. Generators $\boldsymbol{p}$ and $\boldsymbol{q}$ satisfy the canonical commutation relations

$$[q^k,p^j]=\delta_{kj},\quad [q^k,q^j]=0,\quad [p^k,p^j]=0$$

and one can write an arbitrary generator $W(\boldsymbol{\alpha},\boldsymbol{\beta})$ in the form

$$W(\boldsymbol{\alpha},\boldsymbol{\beta})=e^{-\frac{i\hbar\alpha\cdot\beta}{2}}e^{i\boldsymbol{\alpha}\cdot\boldsymbol{p}}e^{i\boldsymbol{\beta}\cdot\boldsymbol{q}}=e^{\frac{i\hbar\alpha\cdot\beta}{2}}e^{i\boldsymbol{\beta}\cdot\boldsymbol{q}}e^{i\boldsymbol{\alpha}\cdot\boldsymbol{p}}.$$

The Schrödinger representation of this Weyl algebra is defined on the Hilbert space of square integrable functions $\mathcal{L}_2(\mathbb{R}^d)$ with

$$\left(\pi(W(\boldsymbol{\alpha},\boldsymbol{\beta}))\Phi\right)(\boldsymbol{X})=e^{\frac{i\hbar\alpha\cdot\beta}{2}}e^{i\boldsymbol{\beta}\cdot\boldsymbol{X}}\Phi(\boldsymbol{X}+\hbar\boldsymbol{\alpha})\,,\tag{2.2}$$

[2] A net $\{T_\alpha\}$ of operators on a Hilbert space $\mathcal{H}$ converges strongly to an operator T if and only if $||T_\alpha x-Tx||\to 0$ for all $x\in\mathcal{H}$. The definition of a net is at p. 22 in Footnote 5.

for $\Phi \in \mathcal{L}_2(\mathbb{R}^d)$. As mentioned before, all the regular irreducible representations are unitary equivalent to this one. If one does not require continuity there are many more representations. In quantum field theory this uniqueness result does not apply, and one has to deal with a huge class of inequivalent representations. Note, however, that the construction of the Weyl algebra makes sense also for L infinite dimensional, so it can be applied in the context of field theory.

A particularly interesting class of states on $\mathfrak{W}(L, \sigma)$ is provided by *quasi-free states*.

Definition 2.31 A state ω on $\mathfrak{W}(L, \sigma)$ is called *quasi-free* if there exists $\eta : L \times L \to \mathbb{R}$, a symmetric form such that

$$\omega\left(W(x)\right) = e^{-\frac{1}{2}\eta(x,x)}.$$

The form η is then called the covariance of the quasi-free state ω.

The following theorem provides a way to easily find quasi-free states.

Theorem 2.3 *Let $\eta : L \times L \to \mathbb{R}$ be a symmetric form. The following are equivalent:*

1. $\eta_{\mathbb{C}} + \frac{i}{2}\sigma_{\mathbb{C}} \geq 0$ *on $L^{\mathbb{C}}$, the complexification of L, where $\eta_{\mathbb{C}}, \sigma_{\mathbb{C}} : L^{\mathbb{C}} \times L^{\mathbb{C}} \to \mathbb{C}$ are canonical sesquilinear extensions of η, σ.*
2. $|\sigma(x_1, x_2)| \leq 2\sqrt{\eta(x_1, x_1)}\sqrt{\eta(x_2, x_2)}$, *for all $x_1, x_2 \in L$.*
3. *There exists a quasi-free state ω_η on $\mathfrak{W}(L, \sigma)$ with covariance η.*

Proof For proof see for example [AS71, DG13a]. □

This result holds also if L is infinite dimensional and will be used later in Sect. 5.3. We define a complex scalar product on the complex vector space $L^{\mathbb{C}}$ by

$$\langle x, y\rangle = \eta_{\mathbb{C}}(x, y) + \frac{i}{2}\sigma_{\mathbb{C}}(x, y). \tag{2.3}$$

The GNS Hilbert space representation corresponding to ω_η turns out to be the bosonic Fock space:

$$\mathcal{H} = \bigoplus_{n=0}^{\infty}(\mathcal{H}_1^{\otimes n})_{\text{symm}}\ ;\ \mathcal{H}_1 = \overline{L^{\mathbb{C}}/\text{Ker}(\langle .,.\rangle)}$$

The state ω_η is pure (i.e. the associated GNS representation is irreducible) if and only if the map $L \to L^{\mathbb{C}}/\text{Ker}(\langle .,.\rangle)$ is surjective. The latter holds if and only if the pair $(2\eta, \sigma)$ is Kähler.

Definition 2.32 A pair $(2\eta, \sigma)$ consisting of a symmetric form 2η and symplectic form σ on L is called *Kähler* if the ranges of the two coincide $\text{Ran}(2\eta) = \text{Ran}(2\sigma)$, 2η is positive definite and $J \doteq \sigma^{-1}2\eta$ satisfies $J^2 = -\mathbb{1}$ (i.e. J is an anti-involution).

If $(2\eta, \sigma)$ is Kähler, then the quadruple $(L, 2\eta, \sigma, J)$ is a *Kähler space*. We will come back to this structure in the context of QFT in Sect. 5.3.

2.1.4 Dynamics and the Interaction Picture

If we want to model a physical system that evolves with time, we need to introduce the notion of dynamics. A very detailed discussion of quantum dynamics can be found in [BR87, BR97]. Here we only sketch the main ideas. Let $\mathfrak{A}$ be a C^*-algebra of observables and let $A_t \in \mathfrak{A}$ be some observable corresponding to the measurement apparatus A at time t.[3] We postulate that the algebra of observables $\mathfrak{A}$ doesn't change with time, so the assignment $t \mapsto A_t$ can be described by a 1-parameter group of automorphisms α_t, such that $A_t = \alpha_t(A)$ and we assume that α_t is strongly continuous.

For a given state ω we consider the family of states that are related to it by time-translations and it is natural to require some stability properties from the GNS-associated representation π_ω. If π_ω is irreducible, this stability requirement is realized as the condition that α_t has to be implemented by some unitary operator $U(t)$, i.e.

$$\pi_\omega(\alpha_t(A)) = U(t)^{-1}\pi_\omega(A)U(t), \quad \forall A \in \mathfrak{A}. \tag{2.4}$$

Now we apply Stone's Theorem 2.2 to deduce the existence of a self-adjoint generator H, called the *Hamiltonian* and we write

$$U(t) = e^{-itH} \quad \forall t \in \mathbb{R}.$$

By differentiating (2.4) we obtain the known evolution equation in the Heisenberg picture,

$$\frac{d}{dt}A(t) = i[H, A(t)], \tag{2.5}$$

where we have put $A(t) = U(t)^*AU(t)$ and we have omitted the symbol π_ω. To get the Schrödinger picture, we consider $\psi \in D(H)$ a Hilbert space vector in the domain of essential selfadjointness (see Definition 2.28) of H, and define $\psi_S(t) \doteq U(t)\psi$. We can now rewrite (2.5) in the form

$$i\frac{d}{dt}\psi_S(t) = H\psi_S(t). \tag{2.6}$$

This is time-evolution in the Schrödinger picture. If we want to construct a model of a quantum dynamical system, we usually start with a Hamiltonian H which is an operator on $\mathcal{H}_\pi$ that solves (2.6) for some initial data $\psi_S(0)$, within the domain $D(H)$. A solution to the initial value problem then defines the *propagator* $U(t, 0)$, i.e.

$$\psi_S(t) = U(t, 0)\psi_S(0).$$

[3] As sharp localization is physically impossible, operationally we can think of A_t as the average over some interval $[t - \epsilon, t + \epsilon]$ centered at t, for a fixed value of $\epsilon > 0$.

Note that the main difference between (2.5) and (2.6) is that in the Heisenberg picture states remain stationary and operators evolve with time, while in the Schrödinger picture it is the other way round. Often, solving the initial value problem of the form (2.6) is very difficult and it is convenient to split the Hamiltonian into two terms

$$H = H_0 + H_{int},$$

where the propagator for H_0 can be found relatively easily and then we try to solve the problem perturbatively. This point of view is something in-between the Heisenberg and Schrödinger pictures and we call it the *interaction picture*. H_{int} is called the *interaction Hamiltonian*. Let $U_0 \doteq e^{-itH_0}$. In the interaction picture the states are represented by

$$\psi_I(t) = U_0^* \psi_S(t) = e^{itH_0} \psi_S(t) = e^{itH_0} e^{-itH} \psi,$$

where ψ_S is a state in the Schrödinger picture and ψ is a state in the Heisenberg picture. Observables of the interaction picture evolve according to

$$A(t) = U_0(t)^* A_S U_0(t),$$

where A_S denotes the Schrödinger picture observable. In particular

$$H_{int} = U_0(t)^* H_{int} U_0(t)$$

for the interaction Hamiltonian H_{int}. Now the evolution Eq. (2.6) implies that

$$i \frac{d}{dt} \psi_I = H_{int} \psi_I. \tag{2.7}$$

Given initial data $\psi_I(t_0)$, we want to find the solution to this equation in terms of a propagator $U_I(t, t_0)$, so that

$$\psi_I(t) = U_I(t, t_0) \psi_I(t_0).$$

By definition we have

$$U_I(t, t_0) = e^{itH_0} e^{-i(t-t_0)H} e^{-it_0 H_0},$$

and from (2.7) it follows that the propagator has to satisfy

$$i \frac{d}{dt} U_I(t, t_0) = H_{int}(t) U_I(t, t_0), \qquad U_I(t_0, t_0) = \mathbb{1}. \tag{2.8}$$

A formal solution to the above equation is then given by the Dyson series

$$U_I(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} H_{int}(t_1)dt_1 - \int_{t_0}^{t}\int_{t_0}^{t_2} H_{int}(t_2)H_{int}(t_1)dt_1dt_2 + \dots$$
$$= \mathbb{1} + \sum_{n=1}^{\infty}(-i)^n \int_{t_0<t_1<\cdots<t_n<t}\cdots\int H_{int}(t_n)\dots H_{int}(t_1)dt_1\dots dt_n. \tag{2.9}$$

We can simplify the notation by introducing the *time-ordering operator* T defined on operators $A(t)$ and $B(t)$ by

$$T(A(t)B(t')) = \begin{cases} A(t)B(t'), \text{ if } t < t' \\ B(t')A(t), \text{ if } t' < t \end{cases}. \tag{2.10}$$

We can now rewrite the formula (2.9) as a time-ordered exponential, i.e.

$$U_I(t,t_0) = \mathbb{1} + T\left[\sum_{n=1}^{\infty}\frac{(-i)^n}{n!}\left(\int_{t_0}^{t} H_{int}(t')dt'\right)^n\right]$$
$$= T\left[\exp\left(-i\int_{t_0}^{t} H_{int}(t')dt'\right)\right]. \tag{2.11}$$

We define the Møller operators $\mathcal{S}^{\pm}$ as the strong limits of $U_I(t,t_0)$ as t_0 approaches $\pm\infty$, as long as these limits exist.

$$\mathcal{S}_{\pm} \doteq \operatorname*{s-lim}_{t\to\pm\infty} U_I(0,t).$$

The scattering operator $\mathcal{S}$ (the S-matrix) is then defined by

$$\mathcal{S} \doteq \mathcal{S}_+^*\mathcal{S}_-. \tag{2.12}$$

We will use these ideas later on, in Sect. 6.1 to perturbatively construct QFT models.

2.2 Causality

After introducing basic notions from quantum mechanics, the next step towards quantum field theory leads through *spacetime geometry*. Historically, QFT was conceived as a framework that allows us to combine quantum mechanics with special relativity. The latter is based on concepts such as *Minkowski spacetime* and *causality*. In fact, the algebraic approach to QFT can be generalized beyond Minkowski spacetime and one can apply it to construct models on a wide class of Lorenzian manifolds. In this section we will review some basic concepts from Lorentzian geometry that are relevant for our framework.

Definition 2.33 *A spacetime* is a pair $\mathcal{M} = (M, g)$, where M is a smooth (4-dimensional) manifold (we assume it to be Hausdorff, paracompact, connected) and g is a smooth Lorentzian metric, i.e. a smooth tensor field $g \in \Gamma(T^*M \otimes T^*M)$, s.t. for every $p \in M$, g_p is a symmetric non-degenerate bilinear form. We require the metric g to be of the Lorentz signature $(+, -, -, -)$.

Remark 2.1 Let us make a few remarks concerning the above definition:

1. The assumption for a manifold to be *Hausdorff* means that points can be separated (for every pair of points x, y, there exists a neighbourhood U of x and a neighbourhood V of y such that U and V are disjoint ($U \cap V = \varnothing$)). In general topology one can drop this assumption and an example of a non-Hausdorff manifold is a line with two origins, i.e. the quotient space of two copies of the real line $\mathbb{R} \times \{a\}$ and $\mathbb{R} \times \{b\}$, with the equivalence relation $(x, a) \sim (x, b)$ if $x \neq 0$.
2. The *paracompactness* is needed as a sufficient condition for the existence of partitions of unity. It means that for every open cover $(U_\alpha)_{\alpha \in A}$, there exists a refinement[4] $(V_\beta)_{\beta \in B}$ that is locally finite, i.e. each $x \in M$ has a neighborhood that intersects only finitely many sets of $(V_\beta)_{\beta \in B}$.
3. We assumed also that M is connected, i.e. it cannot be represented as a disjoint union of two or more non-empty sets. Later on we will see that in a more general context one has to drop this assumption and consider manifolds that are not connected.

Definition 2.34 A spacetime $\mathcal{M}$ is said to be *oriented* if it is equipped with a differential form of maximal degree (a volume form) that does not vanish anywhere. We say that $\mathcal{M}$ is *time-oriented* if it is equipped with a smooth vector field u on M such that for every $p \in M$, $g(u, u) > 0$ holds.

We will always assume that our spacetimes are orientable and time-orientable. We fix the orientation and choose the time-orientation by selecting a specific vector field u with the above property.

Example 2.2 A standard example is 4 dimensional Minkowski spacetime $\mathbb{M}$, which is $\mathbb{R}^4$ with the diagonal metric $\eta = \mathrm{diag}(1, -1, -1, -1)$.

An important feature of the Lorentzian signature, which distinguishes it from the Euclidean signature, is that it allows to introduce some important classes of smooth curves.

Definition 2.35 Let $\gamma : \mathbb{R} \supset I \to M$ be a smooth curve in M, for I an interval in $\mathbb{R}$ and let $\dot{\gamma}$ be the vector tangent to the curve. We say that γ is

[4] An open cover $(V_\beta)_{\beta \in B}$ is a refinement of an open cover $(U_\alpha)_{\alpha \in A}$, if $\forall \beta \in B$, $\exists \alpha$ such that $V_\beta \subseteq U_\alpha$.

- timelike, if $g(\dot{\gamma}, \dot{\gamma}) > 0$,
- spacelike, if $g(\dot{\gamma}, \dot{\gamma}) < 0$,
- lighlike (null), if $g(\dot{\gamma}, \dot{\gamma}) = 0$,
- causal, if $g(\dot{\gamma}, \dot{\gamma}) \geq 0$.

The classification of curves defined above is referred to as the *causal structure*. The presence of time orientation allows for a further refinement of this classification.

Definition 2.36 Given the global timelike vector field u (the time orientation) on M, a causal curve γ is called future-directed if $g(u, \dot{\gamma}) > 0$ all along γ. It is past-directed if $g(u, \dot{\gamma}) < 0$.

Using the causal structure one can distinguish points of spacetime that are in the future or in the past of a given point $p \in \mathcal{M}$.

Definition 2.37 Let $p \in \mathcal{M}$ be a point in a time-oriented spacetime.

(i) $J^{\pm}(p)$ is defined to be the set of all points in M which can be connected to p by a future(+)/past(−)-directed causal curve $\gamma : I \to M$ so that $x = \gamma(\inf\ I)$.
(ii) The set $J^{+}(p)$ is called the causal future and $J^{-}(p)$ the causal past of p. The boundaries $\partial J^{\pm}(p)$ of these regions are called respectively: the *future/past lightcone*.
(iii) The future (past) of a subset $B \subset M$ is defined by

$$J^{\pm}(B) = \bigcup_{p \in B} J^{\pm}(p) \ .$$

The physical importance of the structures presented above becomes clear in the context of general relativity (GR). One of the postulates of GR states that *massive particles can move only on time-like curves and light travels following null curves*, i.e. *nothing travels faster than light*. Consequently, one of the fundamental principles of physics, *the principle of causality*, states that an event happening at a point p can be influenced only by events in $J^{-}(p)$ and that the consequences of this event can influence only the events in $J^{+}(p)$.

Definition 2.38 A subset $A \subset M$ is called *past-(future-) compact* if $A \cap J^{\mp}(p)$ is compact for all $p \in M$.

Definition 2.39 Two subsets O_1 and O_2 in M are called *causally separated* (or spacetime separated) if they cannot be connected by a causal curve, i.e. if for all $x \in \overline{O_1}$, $J^{\pm}(x)$ has empty intersection with $\overline{O_2}$.

Another important definition is that of the *causal complement* of a given region O.

Definition 2.40 The *causal complement* $O^{\perp}$ is defined as the largest open set in M that is causally separated from O.

It follows from the principle of causality that events happening at spacelike separated points cannot influence each other. In classical physics this property is realized as a consequence of some properties of normally hyperbolic partial differential equations. In Sect. 2.3 we will see how these ideas can be implemented into the framework of quantum theory.

Example 2.3 Consider Minkowski spacetime $\mathbb{M} = (\mathbb{R}^4, \eta)$. The set of points that are causally separated from a given point $P \in \mathbb{M}$ is called the *lighcone* with apex P. It is easy to verify that a point $Q \in \mathbb{M}$

- lies on the lightcone with apex P if and only if the vector $\overrightarrow{PQ}$ is lightlike,
- is in the future (past) of P if and only if the vector $\overrightarrow{PQ}$ is time-like and its 0th component is positive (negative),
- is spacelike to P if and only if $\overrightarrow{PQ}$ is spacelike.

These concepts are illustrated in Fig. 2.1.

Definition 2.41 Motivated by Example 2.3 we introduce the following notation:

- $\overline{V}_+ \doteq \{v \in \mathbb{R}^4 | \eta(v, v) \geq 0,\ v_0 > 0\}$ is called the *closed future lightcone*.
- $\overline{V}_- \doteq \{v \in \mathbb{R}^4 | \eta(v, v) \geq 0,\ v_0 \leq 0\}$ is called the *closed past lightcone*.

These definitions can also be applied to subsets of tangent and cotangent spaces $T_x M$ and $T_x^* M$, as these spaces can be mapped to $\mathbb{R}^4$ with the use of appropriate charts.

Not all Lorentzian spacetimes are equally convenient for constructing quantum field theory models. For example, several conceptual and technical problems appear when we consider spacetimes with closed time-like curves. To exclude such situations, we will restrict ourselves to spacetimes that are *globally hyperbolic*.

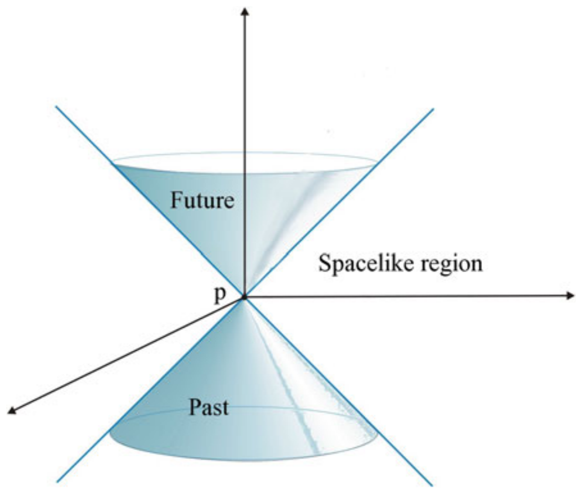

Fig. 2.1 A lightcone in Minkowski spacetime

Definition 2.42 (*after* [BS03]) A spacetime is called *globally hyperbolic* if it does not contain closed causal curves and if for any two points x and y the set $J_+(x) \cap J_-(y)$ is compact.

It was shown in [BS03] that globally hyperbolic spacetimes have many important features. To understand them better we need to introduce some further definitions.

Definition 2.43 A causal curve is *future inextendible* if there is no $p \in M$ such that:

$$\forall U \subset M \text{ open neighborhoods of } p,\ \exists t' \text{ s.t. } \gamma(t) \in U\ \forall t > t'.$$

Definition 2.44 *A Cauchy hypersurface* in M is a smooth subspace of M such that every inextendible causal curve intersects it exactly once.

The significance of Cauchy hypersurfaces lies in the fact that one can use them to formulate the initial value problem for partial differential equations and for some classes of such equations this problem has a unique solution. The fundamental theorem relating different equivalent notions of global hyperbolicity has been proven in [BS03].

Theorem 2.4 (after [BS03]) *The following definitions of global hyperbolicty of a Lorentzian manifold $\mathcal{M}$ are equivalent:*

- $\mathcal{M}$ *does not contain closed causal curves and for any two points x and y the set $J_+(x) \cap J_-(y)$ is compact.*
- $\mathcal{M}$ *contains a Cauchy surface.*
- $\mathcal{M}$ *admits a foliation by Cauchy surfaces.*

2.3 Haag–Kastler Axioms

In Sect. 2.1 we introduced such fundamental notions of quantum theory as states and observables. Now we want to make these compatible with the ideas of special and general relativity, reviewed in Sect. 2.2, where the causal structure plays an important role. The main conceptual difficulty is to find a way to implement the idea that "nothing travels faster than light" in such a way that it doesn't contradict the existence of quantum correlations in the theory. The groundbreaking idea of Rudolf Haag was to combine these notions using the principle of *locality* (*Nahwirkungsprincip*). In this framework, locality is the feature of observables, while states might exhibit correlations, i.e. they carry global information. One defines a QFT model by assigning to each bounded region $\mathcal{O} \subset \mathbb{M}$ of Minkowski spacetime the C^*-algebra of observables $\mathfrak{A}(\mathcal{O})$ that can be measured in this region. The notion of subsystem is realized by the requirement that if $\mathcal{O} \subset \mathcal{O}'$, then $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')$. This condition is called *isotony* and it guarantees that one doesn't lose observables when considering a larger region of spacetime. The complete set of axioms for algebraic quantum field theory (AQFT) can be found in [HK64, Haa93, Ara99]; we will recall them briefly in this section.

The Haag–Kastler axioms (also called Araki-Haag–Kastler axioms) for a net[5] of C^*-algebras $\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O})$ are:

- **Isotony**. For $\mathcal{O} \subset \mathcal{O}'$ we have $\mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{O}')$, see Fig. 2.2.
- **Locality (Einstein causality)**. Algebras associated to spacelike separated regions commute: if $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$, then $[A, B] = 0$, $\forall A \in \mathfrak{A}(\mathcal{O}_1)$, $B \in \mathfrak{A}(\mathcal{O}_2)$, where the commutator is taken in the sense of the inductive limit algebra $\mathfrak{A}$ (see the Definition 2.45 below). This expresses the "independence" of physical systems associated to regions $\mathcal{O}_1$ and $\mathcal{O}_2$.
- **Covariance**. Minkowski spacetime has a large group of isometries, namely the connected component of the Poincaré group $\mathcal{P}$. We require that for each $L \in \mathcal{P}$ there exists an isomorphism $\alpha_L^{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(L\mathcal{O})$, and that for $\mathcal{O}_1 \subset \mathcal{O}_2$ the restriction of $\alpha_L^{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\alpha_L^{\mathcal{O}_1}$ and $\alpha_{L'}^{L\mathcal{O}} \circ \alpha_L^{\mathcal{O}} = \alpha_{L'L}^{\mathcal{O}}$.
- **Time slice axiom**: The algebra of a neighborhood of a Cauchy surface of a given region coincides with the algebra of the full region. Physically this correspond to the well-posedness of an initial value problem, i.e. we only need to determine our observables in some small time interval $(t_0 - \epsilon, t_0 + \epsilon)$ to reconstruct the full algebra.
- **Spectrum condition**. Physically this condition is interpreted as the positivity of energy. One assumes that there exists a compatible family of faithful representations $\pi_{\mathcal{O}}$ of $\mathfrak{A}(\mathcal{O})$ on a fixed Hilbert space (i.e. the restriction of $\pi_{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\pi_{\mathcal{O}_1}$ for $\mathcal{O}_1 \subset \mathcal{O}_2$) such that translations are unitarily implemented, i.e. there is a unitary representation U of the translation group satisfying

$$U(a)\pi_{\mathcal{O}}(A)U(a)^{-1} = \pi_{\mathcal{O}+a}(\alpha_a(A)) \ , \ A \in \mathfrak{A}(\mathcal{O}),$$

  and such that the joint spectrum of the generators P_μ of translations $e^{a \cdot P} = U(a)$, $a \cdot P = \sum_{\mu=0}^{3} a^\mu P_\mu$, is contained in the closed future lightcone: $\sigma(P) \subset \overline{V}_+$.

Definition 2.45 The inductive limit of local algebras $\mathfrak{A}(\mathcal{O})$ defines the *quasilocal algebra* $\mathfrak{A} \doteq \overline{\bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})}$ (the bar means taking the completion in the norm topology).

All these axioms, apart from the **Spectrum condition**, can be generalized to QFT's on general globally hyperbolic spacetimes. We will discuss this in more detail in the next section. There are many important conceptual results that have been proven in the AQFT framework. The first major success was the development of the Haag–Ruelle scattering theory [Haa58, Rue62], which provided an explanation why quantum field theory yields a theory of interacting particles. It is, however, an open question, whether all states in the vacuum representation admit a particle interpretation (the problem of asymptotic completeness). For recent works on that topic see [DT11, DG14b, DG14a]. Another remarkable result of AQFT is the Reeh–Schlieder

[5] A net in a topological space $\mathcal{X}$ is a function from some directed set (nonempty set with a reflexive and transitive binary relation) A to $\mathcal{X}$.

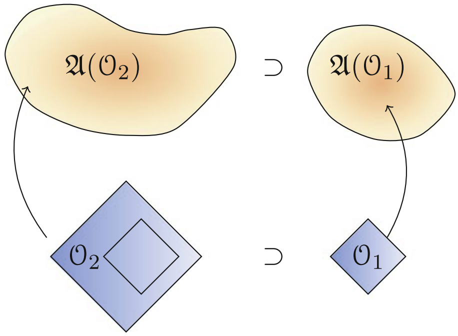

Fig. 2.2 Diagram representing inclusion of spacetime regions and corresponding C^*-algebras

Theorem [Haa93, RS61], see [BS14] for a recent discussion. Another known result achieved with the AQFT methods is the analysis of the superselection structure of QFT models [DHR71, DHR74]. Despite all this insight into the general structure of QFT, there remains the difficulty of constructing 4 dimensional interacting models that fulfil the Haag–Kastler axioms. For models in 2 dimensions see [Lec08, Tan12, BT13, Ala13, BC13] and references therein.

2.4 pAQFT Axioms

In this book we explore the possibility of dropping some of the assumptions of the Haag–Kastler framework, in order to allow for models that exist only in the formal, perturbative sense. The resulting framework is called *perturbative Algebraic Quantum Field Theory* (pAQFT). The generalization of the HK axioms to the perturbative context has been developed in a series of papers [DF01a, DF02, DF04, DF07, DF01b, BD08, Boa00, DB01, BDF09, Rej11b].

The generalization of the HK framework to curved spacetime has been for a long time an independent development. Some important early contributions include [Kay78, Dim80, KW91, Dim92]. Later these two generalizations met as the pAQFT on curved spacetimes after a seminal series of papers [BFK96, BF97, BF00, BFV03, HW01, HW02a, HW02b, HW05].

Abelian gauge theories were later treated in [DF98], while the Yang–Mills theories are the subject of [Hol08]. At the same time the mathematical foundations of pAQFT became better understood, mainly with the use of the functional approach, which is also the approach we take in this book. In [FR12b, FR12a, Rej11a] this framework was used to add the Batalin–Vilkovisky (BV) formalism to the pAQFT toolbox, which allows us to treat very general theories possessing local symmetries, including the bosonic string [BRZ14] and effective quantum gravity [BFR13].

2.4.1 More Functional Analysis

On the functional analytic side, we leave the realm of Banach spaces and allow for structures that have more general topologies. This involves some technical complications, but gives more flexibility in terms of model building. The most general class of topological vector spaces that we will use is the class of locally convex ones.

Definition 2.46 A topological vector space $\mathcal{X} \equiv (X, \tau)$ is called a *locally convex topological vector space* (LCVS) if there is a local base $\mathcal{T}$ whose members are convex.

Here by a *local base* we mean a collection $\mathcal{T}$, of neighborhoods of 0 such that every neighborhood of 0 contains a member of $\mathcal{T}$. The open sets of $\mathcal{X}$ are then precisely those that are unions of translates of members of $\mathcal{T}$.

There is another way to characterize locally convex vector spaces, which allows us to make a connection with normed spaces, introduced in Definition 2.10. Instead of having one norm that characterizes the topology, we have a family of *seminorms*. A seminorm differs from a norm by not fulfilling property 3 in Definition 2.10. More precisely:

Definition 2.47 A *seminorm* on a vector space X is a real-valued function p on X such that:

1. $p(x + y) < p(x) + p(y)$ for all $x, y \in X$.
2. $p(\lambda x) = |\lambda| p(x)$ for all $x \in X$ and all scalars $\lambda \in \mathbb{K}$.

We see that a seminorm already provides us with a lot of information, but it doesn't separate points. However, it is possible that a certain family of seminorms is separating.

Definition 2.48 A family $\mathcal{P}$ of seminorms on X is said to be *separating* if for each $x \neq 0$ there exists at least one $p \in \mathcal{P}$ with $p(x) \neq 0$.

Note that a separating family of seminorms already allows us to distinguish two elements of X.

Theorem 2.5 *To each separating family of seminorms on X we can associate a locally convex topology τ on X and vice versa: every locally convex topology is generated by some family of separating seminorms.*

Proof See [Rud91]. □

In the pAQFT framework a LCVS is usually the best that one can expect. Unfortunately it doesn't share many of the nice properties of a Banach space, but there are some distinguished classes of LCVS that are relatively well behaved and good for defining calculus on them. The "nicest" ones are *Fréchet* spaces. They are distinguished by the fact that their topology can be described in terms of a *metric*.

Theorem 2.6 *A locally convex topological vector space $\mathfrak{X} = (X, \tau)$ is metrizable if and only if τ can be defined by $\mathscr{P} = \{p_n : n \in \mathbb{N}\}$ a countable separating family of seminorms on X. One can equip X with a metric which is compatible with τ and which provides a family of convex balls.*

Proof See [Köt69, Rud91]. □

Usually a Fréchet space topology is defined explicitly by providing a countable separating family of seminorms. A LCVS from Theorem 2.6 can be equipped with the metric:

$$d(x, y) := \sum_{n \in \mathbb{N}} 2^{-n} \frac{p_n(x - y)}{1 + p_n(x - y)} \tag{2.13}$$

This metric is compatible with τ but in general it doesn't provide convex balls (see the discussion in [Rud91] after Theorem 1.24 and Exercise 18). Nevertheless it is good to know that you have a metric that can actually be written down in a closed form.

Definition 2.49 If X is complete with respect to the metric from Theorem 2.6, it is a *Fréchet space*.

In locally convex topological vector spaces which are not Fréchet, using sequences to probe continuity of maps is not enough and some important properties like for example completeness have to be formulated in terms of nets (for definition of a net, see Footnote 5 in Sect. 2.3).

Definition 2.50 A Cauchy net in a locally convex space is a net $\{x_\alpha\}_\alpha$ such that for every $\epsilon > 0$ and every seminorm p, there exists an α such that for all $\lambda, \mu > \alpha$, $p(x_\lambda - x_\mu) < \epsilon$. A locally convex space is complete if and only if every Cauchy net converges.

Compare this with Definitions 2.14 and 2.15 that are valid in normed space. For a locally convex topological vector space that is not complete, one can always construct a completion.

To end this section we remark on one more important aspect of LCVS, namely the definition of tensor products. In quantum theory tensor products are used to model systems that consist of independent subsystems. This is closely related to the notion of causality and we will come back to this issue in Sect. 2.5.

Definition 2.51 Let E and F be locally convex topological vector spaces and let $\otimes : E \times F \to E \otimes F$ be the canonical map into the corresponding tensor product. The finest topology on $E \otimes F$ that makes $\otimes$ continuous is called *the projective tensor topology* or the π-topology. The space $E \otimes F$ equipped with this topology is denoted by $E \otimes_\pi F$ and its completion by $E \hat{\otimes}_\pi F$.

It can be shown that the topology π is locally convex. Another possible topology on $E \otimes F$ is the so called *injective tensor topology*. Its definition is a little bit more involved. In some sense it is the weakest well behaved topology one can put on $E \otimes F$, while the projective tensor topology is the strongest.

The idea behind the injective topology is to define it via the topology on the space of continuous linear mappings $L(E'_\gamma, F)$.

Definition 2.52 We equip E' with the finest locally convex topology γ that coincides with the weak one on equicontinuous[6] sets. One can identify $E \otimes F$ with a subspace of $L(E'_\gamma, F)$. Next we equip $L(E'_\gamma, F)$ with the topology of uniform convergence on equicontinuous compact sets in E'. We denote the resulting topological space by $E\varepsilon F$. It is called *the* ε*-product* of E and F. The corresponding topology induced on $E \otimes F$ is called the ε-topology and $E \otimes F$ equipped with it is the injective tensor product $E \otimes_\varepsilon F$. The corresponding completion is denoted by $E\hat{\otimes}_\varepsilon F$

This topology is better behaved if we want to consider vector-valued distributions and was used (in a slightly modified version) by L. Schwartz in [Sch57, Sch58]. Inequivalent notions of tensor products on LCVS can create problems, but there is a large class of spaces where these notions coincide. These are *nuclear* locally convex topological vector spaces, studied by A. Grothendieck in [Gro55].

2.4.2 Axioms

In this section we introduce the generalization of the Haag–Kastler axioms which is the foundation of pAQFT. It is in fact convenient to extend the pAQFT framework also to classical field theory, to keep a uniform language.

Definition 2.53 A classical field theory model on a spacetime $\mathcal{M}$ is a net of locally convex topological Poisson $*$-algebras $\mathfrak{P}(\mathcal{O})$, each with sequentially continuous product and Poisson bracket $\lfloor .,. \rfloor$;

$$\mathcal{O} \mapsto \mathfrak{P}(\mathcal{O}),$$

where $\mathcal{O} \subset \mathcal{M}$ are bounded, simply-connected regions. The global algebra is obtained as the inductive limit

$$\mathfrak{P}(\mathcal{M}) \doteq \lim_{\mathcal{O}\subset\mathcal{M}} \mathfrak{P}(\mathcal{O}).$$

We require that **Locality** holds, i.e. if $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$, then

$$\lfloor A, B \rfloor = 0,$$

$\forall A \in \mathfrak{P}(\mathcal{O}_1)$, $B \in \mathfrak{P}(\mathcal{O}_2)$, where the Poisson bracket $\lfloor , \rfloor$ is taken in $\mathfrak{P}(\mathcal{M})$.

[6] A set A of continuous functions between two topological spaces E and F is equicontinuous at the points $x_0 \in E$ and $y_0 \in F$ if for any open set $\mathcal{O}$ around y_0, there are neighborhoods U of x_0 and V of y_0 such that for every $f \in A$, if the intersection of $f(U)$ and V is nonempty, then $f(U) \subseteq \mathcal{O}$. One says that A is equicontinuous if it is equicontinuous for all points $x_0 \in E$, $y_0 \in F$. The notion of equicontinuity becomes more intuitive, if we choose E and F to be metric spaces. The family A is equicontinuous at a point x_0 if for every $\epsilon > 0$, there exists a $\delta > 0$ such that $d(f(x_0), f(x)) < \epsilon$ for all $f \in A$ and all x such that $d(x_0, x) < \delta$. In other words we require all member of the family A to be continuous and to have equal variation over a given neighbourhood.

In Chap. 4 we show how to construct models of classical field theories in agreement with the above definition. In Chaps. 5–8 we will show how to quantize such classical models perturbatively. The resulting structure is not a net of C^*-algebras, due to the perturbative character of the construction. Nevertheless, many of the features of a Haag–Kastler net are still present.

Definition 2.54 A perturbative algebraic quantum field theory (pAQFT) model on a spacetime $\mathcal{M}$ is a net of topological $*$-algebras with sequentially continuous product

$$\mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}),$$

where $\mathcal{O} \subset \mathcal{M}$ are bounded, simply-connected regions and we require **Locality**. The global algebra is obtained as the inductive limit

$$\mathfrak{A}(\mathcal{M}) \doteq \lim_{\mathcal{O}\subset\mathcal{M}} \mathfrak{A}(\mathcal{O}).$$

The remaining Haag–Kastler axioms from Sect. 2.3, apart from the **Spectrum condition**, can be easily translated to a pAQFT context.

Definition 2.55 Further axioms:

1. A classical/quantum field theory model on a globally hyperbolic spacetime $\mathcal{M}$ satisfies the **Time-slice axiom** if the algebra of a neighborhood of a Cauchy surface of a given region coincides with the algebra of the full region.
2. If the underlying spacetime $\mathcal{M}$ has a non-trivial group of symmetries $\mathcal{G}$, we say that a model is **Covariant** on $\mathcal{M}$, if for $\beta \in \mathcal{G}$ there exists a family of isomorphisms $\alpha_\beta^{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(\beta\mathcal{O})$, such that for $\mathcal{O}_1 \subset \mathcal{O}_2$ the restriction of $\alpha_\beta^{\mathcal{O}_2}$ to $\mathfrak{A}(\mathcal{O}_1)$ coincides with $\alpha_\beta^{\mathcal{O}_1}$ and $\alpha_{\beta'}^{g\mathcal{O}} \circ \alpha_\beta^{\mathcal{O}} = \alpha_{\beta'\beta}^{\mathcal{O}}$.

The spectrum condition cannot be meaningfully defined on an arbitrary globally hyperbolic spacetime, as it relies on the action of translations, which is a special feature of $\mathbb{M}$. We will replace this condition with a requirement we impose on preferred states on our net of algebras. These preferred states are called *Hadamard states* and they realize the idea of positivity of energy. We discuss them in detail in Sect. 5.1.

2.5 Locally Covariant Quantum Field Theory

In the previous section we recalled the Haag–Kastler axioms and reviewed the generalization of these axioms to the situation where we drop some of the regularity conditions on the topology of local algebras and we drop the restriction to Minkowski spacetime, allowing for general globally hyperbolic backgrounds. We can go a step further and see what happens if we replace the embeddings of bounded regions $\mathcal{O}$ into a fixed spacetime $\mathcal{M}$ with arbitrary embeddings between pairs of globally hyperbolic spacetimes $\mathcal{N}$ and $\mathcal{M}$. We formalize this idea by introducing the notion of an *admissible embedding*.

Definition 2.56 We call an embedding $\chi : \mathcal{M} \to \mathcal{N}$ of a globally hyperbolic manifold $\mathcal{M}$ into another one $\mathcal{N}$ *admissible* if it is an isometry and it preserves orientations and the causal structure. The property of *preserving the causal structure* is defined as follows: for any causal curve $\gamma : [a, b] \to N$, if $\gamma(a), \gamma(b) \in \chi(M)$ then for all $t \in]a, b[$ we have: $\gamma(t) \in \chi(M)$.

The generalization of AQFT which we discuss in this section is called *Locally Covariant Quantum Field Theory* (LCQFT). For a recent extensive review of the area, see [FV15].

As in the original AQFT framework, we assign algebras of observables to globally hyperbolic spacetimes and we also want to require that for each such admissible embedding there exists an injective homomorphism

$$\alpha_\chi : \mathfrak{A}(\mathcal{M}) \to \mathfrak{A}(\mathcal{N}) \tag{2.14}$$

of the corresponding algebras of observables assigned to them, moreover if $\chi_1 : \mathcal{M} \to \mathcal{N}$ and $\chi_2 : \mathcal{N} \to \mathcal{L}$ are embeddings as above then we require the covariance relation

$$\alpha_{\chi_2 \circ \chi_1} = \alpha_{\chi_2} \circ \alpha_{\chi_1} \ . \tag{2.15}$$

Such an assignment $\mathfrak{A}$ of algebras to spacetimes and algebra-morphisms to embeddings can be interpreted in the language of category theory as a *covariant functor* between two categories: the category **Loc** which is an appropriate sub-category of the category whose objects are globally hyperbolic spacetimes and arrows are the admissible embeddings; and the category **Obs** of topological $*$-algebras. The precise choice of the category **Loc** depends on the kind of objects we want to study. If the physical theory we consider is sensitive to some topological (hence non-local) features of the underlying manifold, one first restricts the class of objects considered and then studies possible extensions. The detailed analysis of such topological effects has been provided in [BSS14]. In this section we will present the framework in the simplest version, suitable for the study of scalar fields, as introduced in [BFV03]. First we recall some basic notions of category theory, which are relevant for LCQFT.

Definition 2.57 A *category* **C** consists of:

- a class of objects $\mathrm{Obj}(\mathbf{C})$,
- a class of morphisms (arrows) $\mathrm{Hom}(\mathbf{C})$, such that each $f \in \mathrm{Hom}(\mathbf{C})$ has a unique *source object* and *target object* (both are elements of $\mathrm{Obj}(\mathbf{C})$). For a fixed $a, b \in \mathrm{Obj}(\mathbf{C})$, we denote by $\mathrm{Hom}(a, b)$ the set of morphisms with a as a source and b as a target,
- a binary associative operation $\circ : \mathrm{Hom}(a, b) \times \mathrm{Hom}(b, c) \to \mathrm{Hom}(a, c)$, $f, g \mapsto f \circ g$, called composition of morphisms,
- the identity morphism id_c for each $c \in \mathrm{Obj}(\mathbf{C})$.

Definition 2.58 Let $\mathbf{C}, \mathbf{D}$ be categories. A *covariant functor* $\mathfrak{F}$ assigns to each object $c \in \mathbf{C}$ and object $\mathfrak{F}(c)$ of $\mathbf{D}$ and to each morphism $f \in \mathrm{Hom}(\mathbf{C})$, a morphism $\mathfrak{F}(f) \in \mathrm{Hom}(\mathbf{D})$ in such a way that the following two conditions hold:

- $\mathfrak{F}(\mathrm{id}_c) = \mathrm{id}_{\mathfrak{F}(c)}$ for every object $c \in \mathbf{C}$.
- $\mathfrak{F}(g \circ f) = \mathfrak{F}(g) \circ \mathfrak{F}(f)$ for all morphisms $f : a \to b$ and $g : b \to c$.

Definition 2.59 Let $\mathfrak{F}$ and $\mathfrak{G}$ be functors between categories $\mathbf{C}$ and $\mathbf{D}$, then a *natural transformation* η from $\mathfrak{F}$ to $\mathfrak{G}$ associates to every object $a \in \mathbf{C}$ a morphism $\mathrm{Hom}(\mathbf{D}) \ni \eta_a : \mathfrak{F}(a) \to \mathfrak{G}(a)$, such that for every morphism $\mathrm{Hom}(\mathbf{C}) \ni f : a \to b$ we have:

$$\eta_b \circ \mathfrak{F}(f) = \mathfrak{G}(f) \circ \eta_a .$$

We denote the family of natural transformations between $\mathfrak{F}$ and $\mathfrak{G}$ by $\mathrm{Nat}(\mathfrak{F}, \mathcal{G})$.

For more details on categories and functors, see [ML78]. In LCQFT applied to scalar fields we adopt the following definitions of categories **Loc** and **Obs**.

Definition 2.60 The category **Loc** is a category where objects are globally hyperbolic, oriented time-oriented spacetimes and morphisms are admissible embeddings (see Definition 2.56).

Remark 2.2 Note that **Loc** is a large category, i.e. its class of objects Obj(**Loc**) is not a small set. It was shown in [Few07] that one can improve the situation with the use of the Whitney embedding theorem, which states that every smooth manifold of dimension d may be embedded as a smooth submanifold of $\mathbb{R}^{2d+1}$. Hence the collection of isomorphism equivalence classes in Obj(**Loc**) may be identified with a subset of the power set of $\mathbb{R}^{2d+1}$, so it is a small set. This makes **Loc** essentially small.

Definition 2.61 Depending on the context, we have the following choices for the category of observables.

(i) In the non-perturbative setting: **Obs** is the category with unital C^*-algebras as objects and injective unit-preserving $*$-homomorphisms as arrows.
(ii) In classical theory: $\mathbf{Obs}_c$ is the category with locally convex topological Poisson algebras as objects and injective Poisson homomorphism as arrows.
(iii) In the perturbative setting: $\mathbf{Obs}_p$ is the category with locally convex topological unital $*$-algebras as objects and injective unit-preserving $*$-homomorphisms as arrows.

We are now ready to give a definition of a classical/quantum field theory model in the LCQFT setting.

Definition 2.62 In the LCQFT framework, a model is a functor $\mathfrak{A}$ from **Loc** to …

(i) …**Obs** for a non-perturbative locally covariant QFT model,
(ii) …$\mathbf{Obs}_c$ for a locally covariant classical field theory model,
(iii) …$\mathbf{Obs}_p$ for a perturbative locally covariant QFT model.

If we don't want to specify the context, we write $\mathbf{Obs}_*$. Moreover, we often use the notation $\alpha_\chi \equiv \mathfrak{A}\chi$, where $\chi \in \mathrm{Hom}(\mathbf{Loc})$.

Another useful category is the category of locally convex topological vector spaces.

Definition 2.63 Define **Vec** to be the category whose objects are locally convex topological vector spaces (LCVS) and whose morphisms are injective homomorphisms of LCVS.

The requirement that $\mathfrak{A}$ is a covariant functor already generalizes the Haag–Kastler axioms of *Isotony* and *Covariance*. We can impose further requirements:

- **Einstein causality**: let $\chi_i : \mathcal{M}_i \to \mathcal{M}$, $i = 1, 2$ be morphisms of **Loc** such that $\chi_1(M_1)$ is causally disjoint from $\chi_2(M_2)$, then we require that:

$$[\alpha_{\chi_1}(\mathfrak{A}(\mathcal{M}_1)), \alpha_{\chi_2}(\mathfrak{A}(\mathcal{M}_2))] = \{0\},$$

- **Time-slice axiom**: let $\chi : \mathcal{N} \to \mathcal{M}$, if $\chi(\mathcal{N})$ contains a neighborhood of a Cauchy surface $\Sigma \subset \mathcal{M}$, then α_χ is an isomorphism.

The **Einstein causality** requirement reflects the commutativity of observables localized in spacelike separated regions. From the point of view of category theory, this property is encoded in the tensor structure of the functor $\mathfrak{A}$. In order to make this statement precise, we need to equip our categories **Loc** and $\mathbf{Obs}_*$ with tensor structures (for a precise definition of a tensor category, see [ML78]).

Definition 2.64 We call a category **C** *strictly monoidal (tensor category)* if there exists a bifunctor $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which is associative, i.e. $\otimes(\otimes \times 1) = \otimes(1 \times \otimes)$ and there exists an object e which is a left and right unit for $\otimes$. If $\otimes$ is associative up to a natural isomorphism, then **C** is called *monoidal*.

The category of globally hyperbolic manifolds **Loc** can be extended to a monoidal category $\mathbf{Loc}^{\otimes}$, if we extend the class of objects with finite disjoint unions of elements of $\mathrm{Obj}(\mathbf{Loc})$,

$$\mathcal{M} = \mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_k,$$

where $\mathcal{M}_i \in \mathrm{Obj}(\mathbf{Loc})$. Morphisms of $\mathbf{Loc}^{\otimes}$ are isometric embeddings, preserving orientations and causality. More precisely, they are maps $\chi : \mathcal{M}_1 \sqcup \ldots \sqcup \mathcal{M}_k \to \mathcal{M}$ such that each component satisfies the requirements for a morphism of **Loc** and additionally all images are spacelike to each other, i.e., $\chi(M_i) \perp \chi(M_j)$, for $i \neq j$. $\mathbf{Loc}^{\otimes}$ has the disjoint union as a tensor product, and the empty set as unit object. It is a monoidal category and, using the results of [JS93], it is tensor equivalent to a strict monoidal category, which we will denote by the same symbol $\mathbf{Loc}^{\otimes}$.

On the level of C*-algebras the choice of a tensor structure is less obvious, since, in general, the algebraic tensor product $\mathfrak{A}_1 \odot \mathfrak{A}_2$ of two C^*-algebras can be completed to a C^*-algebra with respect to many non-equivalent tensor norms. The choice of an appropriate norm has to be based on some further physical indications. This problem was discussed in [BFIR14], where it is shown that a physically justified tensor norm is the minimal C^*-norm $\|.\|_{\min}$ defined by

$$\|A\|_{\min} \doteq \sup\{\|(\pi_1 \otimes \pi_2)(A)\|_{\mathscr{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}\}\ , \quad A \in \mathfrak{A}_1 \otimes \mathfrak{A}_2,$$

where π_1 and π_2 run through all representations of $\mathfrak{A}_1$ and of $\mathfrak{A}_2$ on Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ respectively. $\mathscr{B}$ denotes the algebra of bounded operators. If we choose π_1 and π_2 to be faithful, then the supremum is achieved, i.e. $\|A\|_{\min} = \|(\pi_1 \otimes \pi_2)(A)\|_{\mathscr{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)}$. The completion of the algebraic tensor product $\mathfrak{A}_1 \odot \mathfrak{A}_2$ with respect to the minimal norm $\|A\|_{\min}$ is denoted by $\mathfrak{A}_1 \underset{\min}{\otimes} \mathfrak{A}_2$. It was shown in [BFIR14] that, under some technical assumptions, a functor $\mathfrak{A} : \mathbf{Loc} \to \mathbf{Obs}$ satisfies the axiom of **Einstein causality** if and only if it can be extended to a tensor functor $\mathfrak{A}^{\otimes} : \mathbf{Loc}^{\otimes} \to \mathbf{Obs}^{\otimes}$, which means that

$$\mathfrak{A}^{\otimes}(\mathcal{M}_1 \sqcup \mathcal{M}_2) = \mathfrak{A}^{\otimes}(\mathcal{M}_1) \otimes_{\min} \mathfrak{A}^{\otimes}(\mathcal{M}_2), \tag{2.16}$$

$$\mathfrak{A}^{\otimes}(\chi \otimes \chi') = \mathfrak{A}^{\otimes}(\chi) \otimes \mathfrak{A}^{\otimes}(\chi'), \tag{2.17}$$

$$\mathfrak{A}^{\otimes}(\varnothing) = \mathbb{C}. \tag{2.18}$$

In the perturbative setting, we face a similar problem with extending $\mathbf{Obs}_p$ to a tensor category, as there are many possibilities to chose a tensor product. The most natural choices are the injective tensor product (Definition 2.52) and the projective tensor product (Definition 2.51). A way out is to restrict $\mathbf{Obs}_p$ to the category of *nuclear* topological algebras, where these two notions coincide.

Let us now discuss the **Time slice axiom**. We use it to describe the evolution between different Cauchy surfaces. We fix a spacetime $\mathcal{M} = (M, g)$. Let N, K be subsets of M. We denote by ι_{KN} the inclusion of $\mathcal{N} \doteq (N, g\restriction_N)$ into $\mathcal{K} \doteq (K, g\restriction_K)$ and by $\alpha_{KN} \doteq \mathfrak{A}\iota_{KN}$, the corresponding morphism in $\mathrm{Hom}(\mathbf{Obs})$. These morphisms allow us to associate to each Cauchy surface Σ the inverse limit

$$\mathfrak{A}(\Sigma) = \varprojlim_{\mathcal{N} \supset \Sigma} \mathfrak{A}(\mathcal{N}), \tag{2.19}$$

which comes with natural projections $\alpha_{M\Sigma}$ from the algebra $\mathfrak{A}(\Sigma)$ into $\mathfrak{A}(\mathcal{M})$.

From the time slice axiom it follows that each homomorphism α_{KN} is an isomorphism. Hence $\alpha_{M\Sigma}$ is also an isomorphism, and we obtain the "propagator" between two Cauchy surfaces Σ_1 and Σ_2 by

$$\alpha^{M}_{\Sigma_1\Sigma_2} = \alpha^{-1}_{M\Sigma_1} \circ \alpha_{M\Sigma_2}. \tag{2.20}$$

This construction resembles constructions in topological field theory [Seg].

Another important notion in LCQFT is that of a *local quantum field*. In the Haag–Kastler framework on Minkowski spacetime an essential ingredient was the translation symmetry. This symmetry allowed the comparison of observables in different regions of spacetime. This is not possible in the generally covariant framework we describe here, because on a generic spacetime the isometry group might be trivial. It follows that there is a priori no natural way to say what it means to have the same observable in a different region. We need to introduce some extra labels for the observables, which make such a comparison possible. This is where locally covariant quantum fields come into the game. We can think of them as operator-valued distributions assigned to all the objects of **Loc** in a coherent way. Before we give the precise definition, we need to make clear what we mean by test function spaces.

Definition 2.65 Let $\mathfrak{D}$ denote the functor from **Loc** to **Vec** that associates to every spacetime M its space of compactly supported $\mathcal{C}^\infty$-functions,

$$\mathfrak{D}(\mathcal{M}) = \mathscr{D}(M) \doteq \mathcal{C}^\infty_c(M, \mathbb{R}) \,, \tag{2.21}$$

and to every embedding $\chi : M \to N$ of spacetimes the pushforward of test functions $f \in \mathfrak{D}(\mathcal{M})$

$$\mathfrak{D}\chi \equiv \chi_* \; \chi_* f(x) = \begin{cases} f(\chi^{-1}(x)) & x \in \chi(M) \\ 0 & \text{else} \end{cases} \,. \tag{2.22}$$

Note that $\mathfrak{D}$ is a covariant functor. We are now ready to state the definition of a locally covariant quantum field.

Definition 2.66 A locally covariant quantum/classical field Φ is defined as a natural transformation from the functor $\mathfrak{D}$ of test function spaces to the functor $\mathfrak{A}$ of field theory composed with the forgetful functor from $\mathbf{Obs}_*$ to **Vec**.

More concretely, Φ is defined by a family of morphisms $\Phi_{\mathcal{M}} : \mathfrak{D}(\mathcal{M}) \to \mathfrak{A}(\mathcal{M})$, $\mathcal{M} \in \mathrm{Obj}(\mathbf{Loc})$ such that

$$\mathfrak{A}\chi \circ \Phi_{\mathcal{M}} = \Phi_{\mathcal{N}} \circ \mathfrak{D}\chi \tag{2.23}$$

The category theory language, which is used to formulate the axioms of LCQFT, is not only a convenient way to phrase known results, but also leads to new insights. For example, one can use it to formulate what it means to have the same physics in all spacetimes. This property, called SPASS, is a property of the QFT functor and it has been extensively studied in [FV11a, FV11b]. Further study of structures appearing in LCQFT led recently to construction of new theories by using symmetries of the QFT functor [Few13].

The structure of **Loc** introduced above is suitable for scalar fields, but things get more complicated if we want to consider Dirac fields or 1-forms (like in electrodynamics). A convenient and operationally motivated way to do this is to extend the LCQFT framework to the situation where **Loc** is replaced by the category of framed manifolds. This idea has been proposed in [FV15] to prove the locally covariant version of the spin-statistics theorem and presented in more detail in [Few15]. In this book, we apply these concepts in Sect. 6.5.1 to describe the construction of time-ordered products of local functionals that involve derivatives of field configurations. Let us recall after [Few15] some basic definitions.

Definition 2.67 Define the objects of the category **FLoc** to be pairs $\mathcal{M} \doteq (M, \boldsymbol{e})$, where M is a smooth manifold of a fixed dimension (in our context equal to 4), $\boldsymbol{e} = (e^a)_{a=0,\dots,3}$ is a co-tetrad, (a collection of four smooth linearly independent 1-forms on M) and M, equipped with the metric, orientation and time-orientation induced by $\boldsymbol{e}$ is an object in **Loc**.

The metric induced by $\boldsymbol{e}$ is defined by

$$g = \sum_{a,b=0}^{3} \eta_{ab} e^a e^b, \tag{2.24}$$

where η is the Minkowski metric in four dimensions. The existence of orientation and time-orientation is guaranteed if we require that $e^0 \wedge \cdots \wedge e^3$ is everywhere positive and that e^0 is future-directed.

Definition 2.68 Given $(\mathcal{M}, \boldsymbol{e}), (\mathcal{M}', \boldsymbol{e}') \in \mathrm{Obj}(\mathbf{FLoc})$, a morphism ψ in $\mathrm{Hom}((M, \boldsymbol{e}), (M', \boldsymbol{e}'))$ is a smooth map between the underlying manifolds inducing a **Loc**-morphism $(\mathcal{M}, e) \to (\mathcal{M}', e')$ and obeying $\psi^* \boldsymbol{e}' = \boldsymbol{e}$, where $\mathcal{M} = (M, g)$, $\mathcal{M}' = (M', g')$ and g, g' are defined by (2.24).

Given a co-tetrad we obtain its dual tetrad (a set of four independent vector fields) $(e_a)_{a=0,\dots,3}$ by requiring that

$$e^a(e_b) = e^a_\mu e^\mu_b = \delta^a_b,$$

where δ^a_b is the Kronecker delta.

Geometrically, the four vector fields $(e_a)_{a=0,\dots,3}$ define a global section of the frame bundle (a parallelization of M), i.e. they provide an isomorphism $TM \cong M \times \mathbb{R}^4$.

The operational interpretation for elements of Obj(**FLoc**) is provided in terms of *rods and clocks*, but this description is redundant. This corresponds to the freedom to make global frame rotations by elements of the proper orthochronous Lorentz group $\Lambda \in \mathcal{L}^\uparrow_+$ (the group of isometries of Minkowski spacetime that leave the origin fixed). There is a representation of this group as automorphisms of **FLoc** and given a locally covariant QFT functor $\mathfrak{A}$ one obtains a family of theories by applying such frame rotations. More precisely, to each $\Lambda \in \mathcal{L}^\uparrow_+$, there is a functor $\mathscr{T}(\Lambda) : \mathbf{FLoc} \to \mathbf{FLoc}$

$$\mathscr{T}(\Lambda)(M, \boldsymbol{e}) = (M, \Lambda \boldsymbol{e}), \qquad \text{where} \qquad (\Lambda e)^a = \Lambda^a{}_b e^b, \qquad \Lambda \in \mathcal{L}^\uparrow_+. \tag{2.25}$$

Physically, theories defined by $\mathfrak{A} \circ \mathscr{T}(\Lambda)$ for different $\Lambda \in \mathcal{L}^\uparrow_+$ have to be equivalent, so one needs to impose an additional condition on $\mathfrak{A}$ that guarantees that this is indeed the case.

- **Independence of global frame rotations** To each $\Lambda \in \mathcal{L}^\uparrow_+$, there exists a natural transformation $\eta(\Lambda) : \mathfrak{A} \to \mathfrak{A} \circ \mathscr{T}(\Lambda)$, such that

$$\eta(\Lambda)_{(M,\boldsymbol{e})} \alpha_{(M,\boldsymbol{e})} = \alpha_{(M,\Lambda \boldsymbol{e})} \eta(\Lambda)_{(M,\boldsymbol{e})}, \qquad \forall \alpha \in \mathrm{Aut}(\mathfrak{A}), \tag{2.26}$$

where Aut($\mathfrak{A}$) is the group of natural transformations that are automorphisms of the functor $\mathfrak{A}$, see [Few13] for more detail.

The next step in LCQFT research is the proper understanding of the structures of gauge theories, where the topological features lead to new difficulties [DL12, SDH14, BSS14]. It would be desirable to obtain for local symmetries a framework similar to the DHR analysis done for global symmetries [DHR71, DHR74].

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http://www.springer.com/978-3-319-25899-7

Perturbative Algebraic Quantum Field Theory
An Introduction for Mathematicians
Rejzner, K.
2016, XI, 180 p. 4 illus., Hardcover
ISBN: 978-3-319-25899-7