Chapter 1
Interaction of a Charged Particle with Strong Plane Electromagnetic Wave in Vacuum

Abstract What can we expect from particle–strong wave interaction in vacuum? It is well known that the radiation or absorption of photons by a free electron in vacuum is forbidden by the energy and momentum conservation laws, which means that the real energy exchange between a free electron and plane monochromatic wave in vacuum is impossible, isn't it? Then, is it worth considering the interaction of a free electron with strong monochromatic wave in vacuum? In other words, what can we expect from the strong wave fields in nonlinear theory with respect to the weak ones described by the linear theory? For example, what are the changes in cross section of the major electrodynamic process of electron–photon interaction, that is, Compton effect (which in the one-photon approximation within quantum electrodynamics is described by the Klein–Nishen formula) at a high density of incident photons? Lastly, how strong should a wave field be for revelation of nonlinear effects in vacuum? What are the criteria of the strong field? To answer these questions one must first study the dynamics of a charged particle in the field of a plane electromagnetic wave of arbitrary high intensity in vacuum on the basis of the classical and quantum equations of motion. Then, with the help of the classical trajectory of the particle and dynamic wave function in the quantum description, the nonlinear radiation in the scope of the classical and quantum theories—the Compton effect in the field of electromagnetic wave of arbitrary high intensity—will be treated. We will start from the relativistic equations, because in the field of a strong wave even a particle initially at rest becomes relativistic. Then, the amplitude of a strong wave will be assumed invariable, i.e., the radiation effects do not influence the magnitude of a given strong wave field.

1.1 Classical Dynamics of a Particle in the Field of Strong Plane Electromagnetic Wave

Let a particle with a mass m and a charge e (let $e > 0$) interact with a plane electromagnetic (EM) wave of arbitrary form and intensity propagating in vacuum along a direction $\boldsymbol{\nu}_0$ ($|\boldsymbol{\nu}_0| = 1$). Then, for the electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) field strengths we have

H.K. Avetissian, *Relativistic Nonlinear Electrodynamics*,
Springer Series on Atomic, Optical, and Plasma Physics 88,
DOI 10.1007/978-3-319-26384-7_1

$$\mathbf{E}(t,\mathbf{r}) = \mathbf{E}(t - \boldsymbol{\nu}_0\mathbf{r}/c); \quad \mathbf{H}(t,\mathbf{r}) = \mathbf{H}(t - \boldsymbol{\nu}_0\mathbf{r}/c); \quad \mathbf{H} = [\boldsymbol{\nu}_0\mathbf{E}]\,. \tag{1.1}$$

Relativistic classical equation of motion of the particle in the field (1.1) will be written in the form

$$\frac{d\mathbf{p}}{dt} = e\mathbf{E} + \frac{e}{c}[\mathbf{vH}]\,, \tag{1.2}$$

where $\mathbf{p}$ and $\mathbf{v}$ are the particle momentum and velocity in the field and c is the light speed in vacuum.

For integration of the equation of motion (1.2) the latter should be written in components:

$$\boldsymbol{\nu}_0\frac{d\mathbf{p}}{dt} = \frac{e}{c}(\mathbf{vE}), \tag{1.3}$$

$$\frac{d\mathbf{p}_\perp}{dt} = e\left(1 - \frac{\mathbf{v}\boldsymbol{\nu}_0}{c}\right)\mathbf{E}. \tag{1.4}$$

Then the integration of (1.4) is very simple if one takes into account that $\mathbf{E}$ is the function of the variable $\tau = t - \boldsymbol{\nu}_0\mathbf{r}/c$ and passes on the left-hand side of (1.4) from the variable t to τ. So, for the transverse components of the particle momentum we will have

$$\mathbf{p}_\perp = \mathbf{p}_{0\perp} + e\int\limits_{\tau_0}^{\tau}\mathbf{E}(\tau)d\tau, \tag{1.5}$$

where $\mathbf{p}_{0\perp}$ is the particle initial transverse momentum at $\tau = \tau_0$ when $\mathbf{E}(\tau)\mid_{\tau=\tau_0} = \mathbf{H}(\tau)\mid_{\tau=\tau_0} = 0$ corresponding to the free particle state before the interaction. Such definition of the particle free state at the finite moment τ_0 at the interaction with the EM wave is justified when we consider the general case of a plane wave of arbitrary form, which actually corresponds to wave pulses of finite duration; let here $\tau_f - \tau_0$. Then, the interaction will be automatically turned on at $\tau = \tau_0$ and turned off at $\tau = \tau_f$, when $\mathbf{E}(\tau)\mid_{\tau=\tau_f} = \mathbf{H}(\tau)\mid_{\tau=\tau_f} = 0$ too, and the free particle states before the interaction will correspond to $\tau \leq \tau_0$ and after the interaction to $\tau \geq \tau_f$. Such approach also allows passing from the wave pulses of finite duration to quasi-monochromatic or monochromatic waves by extending $\tau_0 \to -\infty$ and $\tau_f \to +\infty$.

The expressions (1.5) can be written in a simpler form through the vector potential ($\mathbf{A}$) of the field according to known relations with the electric and magnetic field strengths for radiation field in the Lorentz gauge

$$\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}; \quad \mathbf{H} = \mathrm{rot}\mathbf{A}; \quad \mathrm{div}\mathbf{A} = 0, \tag{1.6}$$

consequently

$$\mathbf{A}(\tau) = -c \int_{\tau_0}^{\tau} \mathbf{E}(\tau) d\tau. \tag{1.7}$$

The condition $\operatorname{div}\mathbf{A} = 0$ in (1.6) is the condition of transversality of a plane wave: $\boldsymbol{\nu}_0 \mathbf{A}(\tau) = 0$.

So, the particle transverse momentum (1.5) can be represented in the form

$$\mathbf{p}_{\perp} = \mathbf{p}_{0\perp} - \frac{e}{c}\mathbf{A}(\tau), \tag{1.8}$$

where $\mathbf{A}(\tau) \mid_{\tau=\tau_0} = 0$ according to (1.7) ($\mathbf{A}(\tau) \mid_{\tau=\tau_f} = 0$ as well because of $\mathbf{E}(\tau) \mid_{\tau=\tau_f} = \mathbf{H}(\tau) \mid_{\tau=\tau_f} = 0$).

Note that (1.8) may be written without integration of the equation of motion taking into account the space properties in this issue. Thus, the existence of a plane wave does not violate the homogeneity of the space in the plane of the wave polarization. Consequently, the corresponding transverse components of generalized momentum are conserved: $\mathbf{p}_{\perp} + (e/c)\mathbf{A}(\tau) = \text{const}$ and we come at once to (1.8).

For the integration of (1.3) for the longitudinal component of the particle momentum we will use the additional equation for the particle energy variation in the field

$$\frac{d\mathcal{E}}{dt} = e\,(\mathbf{vE})\,. \tag{1.9}$$

From (1.3) and (1.9) follows the integral of motion for the charged particle in the field of a plane EM wave:

$$\mathcal{E} - c\mathbf{p}\boldsymbol{\nu}_0 = \text{const} \equiv \Lambda. \tag{1.10}$$

Now we can define the particle momentum and energy in the field with the help of (1.8) and (1.10), utilizing the dispersion law of the particle energy-momentum as well:

$$\mathcal{E}^2 = \mathbf{p}^2 c^2 + m^2 c^4. \tag{1.11}$$

The following formulas in the field of a plane EM wave of arbitrary form and polarization are obtained:

$$\mathbf{p} = \mathbf{p}_0 - \frac{e}{c}\mathbf{A}(\tau) + \boldsymbol{\nu}_0 \frac{e^2 A^2(\tau) - 2ec\,(\mathbf{p}_0 \mathbf{A}(\tau))}{2c(\mathcal{E}_0 - c\mathbf{p}_0 \boldsymbol{\nu}_0)}, \tag{1.12}$$

$$\mathcal{E} = \mathcal{E}_0 + \frac{e^2 A^2(\tau) - 2ec\,(\mathbf{p}_0 \mathbf{A}(\tau))}{2(\mathcal{E}_0 - c\mathbf{p}_0 \boldsymbol{\nu}_0)}, \tag{1.13}$$

where $\mathbf{p}_0$ and $\mathcal{E}_0$ are the initial momentum and energy of a free particle ($\Lambda = \mathcal{E}_0 - c\mathbf{p}_0\boldsymbol{\nu}_0$).

Then, to obtain the law of the particle motion $\mathbf{r} = \mathbf{r}(t)$ one must integrate the equation

$$\frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(t) = \frac{c^2\mathbf{p}(t)}{\mathcal{E}(t)} . \tag{1.14}$$

However, since the general expressions of particle momentum and energy in the field of a plane EM wave depend only on retarding time τ, the last equation allows exact analytical solution in the parametric form $\mathbf{r} = \mathbf{r}(\tau)$. Thus, passing in (1.14) from the variable t to τ and taking into account the integral of motion (1.10) we obtain

$$\frac{d\mathbf{r}(\tau)}{d\tau} = \frac{c^2\mathbf{p}(\tau)}{\mathcal{E}_0 - c\mathbf{p}_0\boldsymbol{\nu}_0}. \tag{1.15}$$

Integration of (1.15) with the help of (1.12) gives

$$\mathbf{r}(\tau) = \mathbf{r}_0 + \frac{c^2\mathbf{p}_0}{(\mathcal{E}_0 - c\mathbf{p}_0\boldsymbol{\nu}_0)}(\tau - \tau_0) + \frac{c}{(\mathcal{E}_0 - c\mathbf{p}_0\boldsymbol{\nu}_0)}$$
$$\times \int_{\tau_0}^{\tau} \left\{ \frac{\boldsymbol{\nu}_0}{2(\mathcal{E}_0 - c\mathbf{p}_0\boldsymbol{\nu}_0)} \left(e^2 A^2(\tau') - 2ec\mathbf{p}_0\mathbf{A}(\tau') \right) - e\mathbf{A}(\tau') \right\} d\tau', \tag{1.16}$$

where $\mathbf{r}_0(x_0,\ y_0,\ z_0)$ is the particle initial position at $t = t_0$ ($\tau = \tau_0$).

1.2 Intensity Effect. Mass Renormalization

Equations (1.12), (1.13), and (1.16) describe the particle motion in the field of a strong plane EM wave of arbitrary form and polarization. They show that after the interaction ($\tau \geq \tau_f$) $\mathbf{p} = \mathbf{p}_0$, $\mathcal{E} = \mathcal{E}_0$, i.e., the particle remains with the initial energy-momentum, which means that real energy exchange between a free charged particle and a plane EM wave in vacuum is impossible. This result is in congruence with the fact that the real absorption or emission of photons by a free electron in vacuum is forbidden by the energy and momentum conservation laws, which will be discussed in regard to the quantum consideration of this process. Nevertheless, in vacuum the wave intensity effect in the field exists, for revealing of which it should be taken into account the oscillating character of periodic wave field, for which $\overline{\mathbf{A}}(\tau) = 0$. Then, averaging the expressions in (1.12) and (1.13) over time we obtain the following formulas for the particle average momentum and energy in the field:

$$\overline{\mathbf{p}} = \mathbf{p}_0 + \boldsymbol{\nu}_0 \frac{e^2\overline{\mathbf{A}^2}(\tau)}{2c(\mathcal{E}_0 - c\mathbf{p}_0\boldsymbol{\nu}_0)}; \quad \overline{\mathcal{E}} = \mathcal{E}_0 + \frac{e^2\overline{\mathbf{A}^2}(\tau)}{2(\mathcal{E}_0 - c\mathbf{p}_0\boldsymbol{\nu}_0)}. \tag{1.17}$$

Taking into account the dispersion law of the particle energy-momentum (1.11) for these average values we can introduce the "effective mass" of the particle due to the intensity effect of strong wave:

$$m^* = m\sqrt{1 + \overline{\xi^2}(\tau)}. \tag{1.18}$$

This formula describes the renormalization of the particle mass in the field. Here we introduced a relativistic invariant dimensionless parameter of a plane EM wave intensity

$$\xi^2(\tau) = \left(\frac{e\mathbf{A}(\tau)}{mc^2}\right)^2. \tag{1.19}$$

The parameter ξ is the basic characteristic of a strong radiation field at the interaction with the charged particles, which represents the work of the field on the one wavelength in the units of the particle rest energy, i.e., it is the energy (normalized) acquired by the particle on a wavelength of a coherent radiation field.

As strong radiation fields actually relate to laser sources of high coherency, we will consider the case of quasi-monochromatic or monochromatic wave fields (we look aside from the actual intensity profiles of laser beams over space coordinates—deviation from a plane wave because of their finite sizes).

Let us consider the case of a monochromatic wave. Without loss of generality we will direct vector $\boldsymbol{\nu}_0$ along the OX axis of a Cartesian coordinate system: $\boldsymbol{\nu}_0 = \{1, 0, 0\}$, then retarding wave coordinate: $\tau = t - x/c$. In the general case of elliptic polarization the vector potential of a monochromatic wave with a frequency ω_0 and amplitude A_0 may be presented in the form

$$\mathbf{A}(\tau) = \{0, A_0 \cos(\omega_0\tau), gA_0 \sin \omega_0\tau\}, \tag{1.20}$$

where g is the parameter of ellipticity; $g = 0$ corresponds to a linear polarization, while $g = \pm 1$ describes a wave of a circular polarization (right or left). Let $g = 1$ and the initial velocity of the particle is parallel to the wave propagation direction ($\mathrm{v}_0 = \mathrm{v}_{0x}$). In such geometry and circular polarization of the wave the intensity effect becomes apparent (only the latter exists with invariable magnitude, because $\mathbf{p}_0\mathbf{A}(\tau) = 0$). In the future we will mainly consider this case of interaction at which the energy and longitudinal velocity of the particle in the field are invariable, which allows, first, a simpler picture of a particle–wave nonlinear interaction, and second, exact solutions in many processes where the existence of the particle initial transverse momentum prevents obtaining exact analytical solutions.

Concerning the definition of the particle initial and final free states at the interaction with a monochromatic wave of infinite duration we will assume an arbitrarily small damping for the amplitude A_0 to switch on adiabatically the wave at $\tau = -\infty$ and switch off at $\tau = +\infty$, i.e., $\mathbf{A}(\tau)\,|_{\tau=\pm\infty} = 0$ (according to the above-mentioned conditions for a plane wave of finite duration $\tau_f - \tau_0$ it should be extended to $\tau_0 \to -\infty$ and $\tau_f \to +\infty$). For a quasi-monochromatic wave (spectral width

$\Delta\omega \ll \omega_0$) it should be $A_0 \Rightarrow A_0(\tau)$, where $A_0(\tau)$ is a slowly varying amplitude with respect to the phase oscillations over the $\omega_0\tau$ and the conditions of adiabatic switching on and switching off will take place automatically.

Hence from (1.12) and (1.13) we have simple formulas for the particle momentum and energy in the field of a monochromatic wave of circular polarization:

$$p_x = p_0\left[1 + \frac{1}{2}\frac{c}{\mathrm{v}_0}\left(1 + \frac{\mathrm{v}_0}{c}\right)\xi_0^2\right], \tag{1.21}$$

$$p_y = -mc\xi_0 \cos\omega_0\tau, \tag{1.22}$$

$$p_z = -mc\xi_0 \sin\omega_0\tau, \tag{1.23}$$

$$\mathcal{E} = \mathcal{E}_0\left[1 + \frac{1}{2}\left(1 + \frac{\mathrm{v}_0}{c}\right)\xi_0^2\right], \tag{1.24}$$

where the relativistic parameter of the wave intensity (1.19) $\xi^2(\tau) = \xi_0^2 = \mathrm{const}$ and, consequently, one can represent it by the amplitude of the vector potential A_0 or electric field strength E_0:

$$\xi_0 = \frac{eA_0}{mc^2} = \frac{eE_0}{mc\omega_0}. \tag{1.25}$$

Equation (1.24) shows that for the significant energy change of a particle in the field of a plane wave in vacuum the superpower laser beams of relativistic intensities $\xi_0 \gg 1$ are necessary. Such intensities corresponding to gigantic femtosecond laser pulses became available in recent years.

To elucidate the law of particle motion in the field of a monochromatic wave we will choose the frame of reference for the free particle initial position, in which the coordinates $\mathbf{r}_0$ at the moment $t = t_0$ correspond to $\mathbf{r}_0 = \mathbf{v}_0 t_0$. By that we exclude the infinities in the expression $\mathbf{r} = \mathbf{r}(\tau)$ connected with the initial infinity values of the parameters t_0 and $\mathbf{r}_0$, which have no physical meaning. Then one can extend $t_0 \to -\infty$ and, consequently, $\tau_0 = (1 - \mathrm{v}_{0x}/c)t_0 \to -\infty$ in (1.16) providing the particle free state before the interaction ($t_0 \to -\infty$) at infinity ($\mathbf{r}_0 \to -\infty$) with the adiabatic switching on the monochromatic (quasi-monochromatic) wave due to $A_0(-\infty) = 0$. Hence, from (1.16) follows the particle law of motion in the field (1.20) in parametric form. However, considering special cases it is analytically available to represent directly the law of motion $\mathbf{r} = \mathbf{r}(t)$ because of the invariability of longitudinal velocity of the particle in the field

$$\mathrm{v}_x = \mathrm{v}_0 \frac{1 + \frac{1}{2}\frac{c}{\mathrm{v}_0}\left(1 + \frac{\mathrm{v}_0}{c}\right)\xi_0^2}{1 + \frac{1}{2}\left(1 + \frac{\mathrm{v}_0}{c}\right)\xi_0^2}, \tag{1.26}$$

which is exposed only to permanent renormalization due to the intensity effect of the strong wave. Then, with the help of (1.26) we have the following formulas for the particle law of motion:

$$x(t) = \mathrm{v}_x t, \tag{1.27}$$

$$y(t) = -\frac{mc^3\xi_0}{\mathcal{E}_0\omega_0\left(1-\frac{\mathrm{v}_0}{c}\right)}\sin\omega_0\left(1-\frac{\mathrm{v}_x}{c}\right)t, \tag{1.28}$$

$$z(t) = \frac{mc^3\xi_0}{\mathcal{E}_0\omega_0\left(1-\frac{\mathrm{v}_0}{c}\right)}\cos\omega_0\left(1-\frac{\mathrm{v}_x}{c}\right)t. \tag{1.29}$$

Equations (1.27)–(1.29) show that the particle performs circular motion

$$y^2(t) + z^2(t) = \mathrm{const} \tag{1.30}$$

in the plane of the wave polarization (yz) with the radius

$$\rho_\perp = \frac{mc^3\xi_0}{\mathcal{E}_0\omega_0(1-\frac{\mathrm{v}_0}{c})} \tag{1.31}$$

and translational uniform motion along the wave propagation direction (OX axis), i.e., performs a helical motion (Fig. 1.1). Consider now the case of linear polarization of the wave

$$\mathbf{A}(\tau) = \{0, A_0\cos(\omega_0\tau), 0\}. \tag{1.32}$$

From (1.12) and (1.13) for the particle momentum and energy in the field (1.32) we have

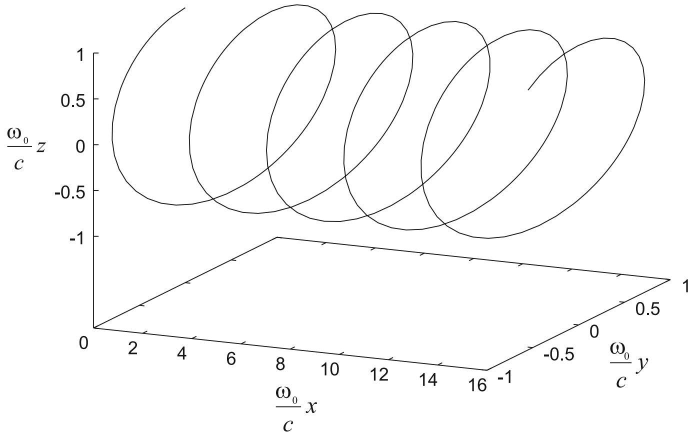

Fig. 1.1 Trajectory of the particle (initially at rest) in the field of circularly polarized EM wave. The relativistic parameter of intensity is taken to be $\xi_0 = 1$

$$p_x = p_0 \left[1 + \frac{1}{2} \frac{c}{\mathrm{v}_0} \left(1 + \frac{\mathrm{v}_0}{c} \right) \xi_0^2 \cos^2 (\omega_0 \tau) \right], \tag{1.33}$$

$$p_y = -mc\xi_0 \cos \omega_0 \tau, \tag{1.34}$$

$$p_z = 0, \tag{1.35}$$

$$\mathcal{E} = \mathcal{E}_0 \left[1 + \frac{1}{2} \left(1 + \frac{\mathrm{v}_0}{c} \right) \xi_0^2 \cos^2 (\omega_0 \tau) \right]. \tag{1.36}$$

In contrast to the case of circular polarization, in the field of linearly polarized wave the intensity effect has the oscillating character (at the second harmonic $2\omega_0$, as follows from (1.33) and (1.36)) and the representation of the particle trajectory analytically is unavailable. The latter may be performed in parametric form with the help of the particle law of motion $\mathbf{r} = \mathbf{r}(\tau)$, which in the field (1.32) has the following form:

$$x(\tau) = \left[1 + \frac{1}{4} \frac{c}{\mathrm{v}_0} \left(1 + \frac{\mathrm{v}_0}{c} \right) \xi_0^2 \right] \frac{\mathrm{v}_0 \tau}{(1 - \frac{\mathrm{v}_0}{c})} + \rho_{\shortparallel} \sin(2\omega_0 \tau), \tag{1.37}$$

$$y(\tau) = -\rho_\perp \sin(\omega_0 \tau), \tag{1.38}$$

$$z = 0, \tag{1.39}$$

where

$$\rho_{\shortparallel} = \frac{1}{8} \frac{c}{\omega_0} \frac{1 + \frac{\mathrm{v}_0}{c}}{1 - \frac{\mathrm{v}_0}{c}} \xi_0^2 \tag{1.40}$$

is the amplitude of longitudinal oscillations of the particle along the wave propagation direction and $\rho_\perp$ is given by the formula (1.31).

To determine the particle trajectory we pass to an inertial system of coordinates connected with the uniform motion of the particle along the axis *OX* with the velocity

$$V = \mathrm{v}_0 \frac{1 + \frac{1}{4} \frac{c}{\mathrm{v}_0} \left(1 + \frac{\mathrm{v}_0}{c} \right) \xi_0^2}{1 + \frac{1}{4} \left(1 + \frac{\mathrm{v}_0}{c} \right) \xi_0^2}, \tag{1.41}$$

to exclude the uniform part of translational movement in the direction of the wave propagation. After the Lorentz transformations for coordinates and wave frequency we have the following law of motion in this system:

$$x^{'}(\tau') = \frac{1}{8} \frac{c}{\omega'} \frac{\xi_0^2}{1 + \frac{\xi_0^2}{2}} \sin(2\omega' \tau'), \tag{1.42}$$

$$y'(\tau') = y(\tau) = -\frac{c}{\omega'}\frac{\xi_0}{\sqrt{1+\frac{\xi_0^2}{2}}}\sin(\omega'\tau'), \tag{1.43}$$

$$z' = 0, \tag{1.44}$$

where

$$\omega' = \frac{\omega_0}{\sqrt{1+\frac{\xi_0^2}{2}}}\sqrt{\frac{1-\frac{v_0}{c}}{1+\frac{v_0}{c}}} \tag{1.45}$$

is the Doppler-shifted frequency of the wave in the system moving with the velocity (1.41).

Now from (1.42) and (1.43) one can obtain the trajectory of the particle in the plane XY

$$\left(\frac{x'}{2\rho'_{\shortparallel}}\right)^2 = \left(\frac{y'}{\rho_\perp}\right)^2 - \left(\frac{y'}{\rho_\perp}\right)^4 \tag{1.46}$$

with the parameters $\rho'_{\shortparallel}$ and $\rho_\perp$:

$$\rho'_{\shortparallel} = \frac{c}{8\omega'}\frac{\xi_0^2}{1+\frac{\xi_0^2}{2}}; \quad \rho'_\perp = \rho_\perp = \frac{c}{\omega'}\frac{\xi_0}{\sqrt{1+\frac{\xi_0^2}{2}}}. \tag{1.47}$$

Equation (1.46) performs a symmetric 8-form figure with the longitudinal axis along the OY (Fig. 1.2).

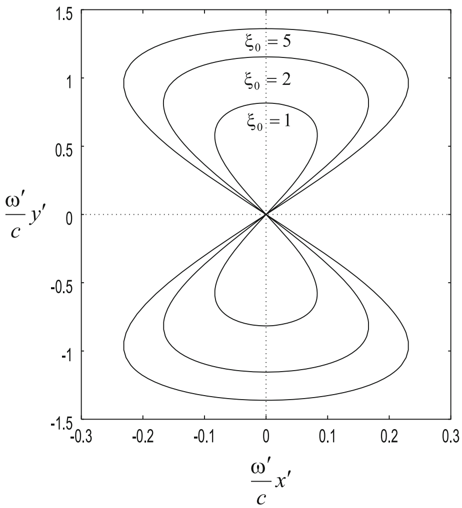

Fig. 1.2 Trajectory of the particle in the field of linearly polarized EM wave (excluding the uniform part of translational movement in the direction of the wave propagation) for the various ξ_0

1.3 Radiation of a Particle in the Field of Strong Monochromatic Wave

Let us now consider the radiation of a charged particle in the specified wave field (1.20) of arbitrary high intensity in the scope of the classical theory. In the strong wave field the radiation of a particle is of nonlinear nature—radiation of high harmonics—which in quantum terminology means that the multiphoton absorption by the particle from the incident wave takes place with subsequent radiation of the corresponding photon. Taking into account certain dependence of harmonics radiation on the direction of particle motion with respect to the initial strong wave propagation and its polarization we will consider the general case of a particle–wave interaction geometry and arbitrary polarization of monochromatic wave (elliptic)

$$\mathbf{A}(\tau) = A_0\{\mathbf{e}_1 \cos\omega_0\tau + \mathbf{e}_2 g \sin\omega_0\tau\}; \tag{1.48}$$

$$\tau = t - \frac{\boldsymbol{\nu}_0\mathbf{r}}{c}; \quad \mathbf{e}_1\boldsymbol{\nu}_0 = \mathbf{e}_2\boldsymbol{\nu}_0 = \mathbf{e}_1\mathbf{e}_2 = 0,$$

where $\mathbf{e}_{1,2}$ are the unit polarization vectors.

The energy radiated by a charged particle in the domain of solid angle dO and interval of frequencies $d\omega$ in the direction of the wave vector $\mathbf{k}$ (summed by all possible polarizations) is given by the formula

$$d\varepsilon_{\mathbf{k}} = \frac{e^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} [\mathbf{kv}]\, e^{i(\mathbf{kr}-\omega t)} dt\right|^2 d\omega dO, \tag{1.49}$$

where $\mathbf{v} = \mathbf{v}(t)$ and $\mathbf{r} = \mathbf{r}(t)$ are the particle velocity and law of motion in the wave field (1.20), which are determined by (1.12), (1.13), and (1.16) in parametric form. The latter requires passing in (1.49) from the variable t to the wave coordinate τ. Then the equation for the radiation energy will be written in the form

$$d\varepsilon_{\mathbf{k}} = \frac{e^2 c^3}{4\pi^2 \Lambda^2}\left|\int_{-\infty}^{\infty} \left[\mathbf{kp}\,(\tau)\right] e^{i\psi(\tau)} d\tau\right|^2 d\omega dO, \tag{1.50}$$

where

$$\psi(\tau) = \omega\tau + k(\boldsymbol{\nu}_0 - \boldsymbol{\nu})\mathbf{r}(\tau) \tag{1.51}$$

is the phase of radiated wave $(\mathbf{kr} - \omega t)$ as a function of the incident strong wave coordinate τ and the unit vector $\boldsymbol{\nu}$ in (1.49) is $\boldsymbol{\nu} = \mathbf{k}/k$.

Using (1.12), (1.13) and introducing the functions

$$G_0 = \int_{-\infty}^{\infty} e^{i\psi(\tau)} d\tau,$$

$$\mathbf{G}_1 = \int_{-\infty}^{\infty} \mathbf{A}(\tau) e^{i\psi(\tau)} d\tau, \tag{1.52}$$

$$G_2 = \int_{-\infty}^{\infty} \mathbf{A}^2(\tau) e^{i\psi(\tau)} d\tau,$$

after the long but straightforward transformations for the radiation energy we obtain

$$d\varepsilon_{\mathbf{k}} = \frac{e^2 m^2 c^3 \omega^2}{4\pi^2 \Lambda^2} \left(\frac{e^2}{m^2 c^4} \left(|\mathbf{G}_1|^2 - Re\left(G_0 G_2^*\right)\right) - |G_0|^2 \right) d\omega dO. \tag{1.53}$$

This is the general formula of the spectral-angular distribution of radiation energy for the arbitrary plane EM wave field. Considering the case of monochromatic wave (1.48) with the corresponding law of motion (1.16) for the phase of radiated wave (1.51), which determines the functions (1.52) and, consequently, the energy of radiation (1.53), we have

$$\psi(\tau) = \left(\frac{\overline{\mathcal{E}} - c\boldsymbol{\nu}\overline{\mathbf{p}}}{\Lambda} \right) \omega\tau + \alpha \sin(\omega_0 \tau - \varphi) - \beta \sin 2\omega_0 \tau, \tag{1.54}$$

where the parameters α, β, and φ are

$$\alpha = \rho_\perp k \sqrt{\left(\boldsymbol{\nu}\mathbf{e}_1 + (\boldsymbol{\nu}^\circ{}_{\mathbf{0}} - 1) \frac{c\mathbf{p}_0\mathbf{e}_1}{\Lambda} \right)^2 + g^2 \left(\boldsymbol{\nu}\mathbf{e}_2 + (\boldsymbol{\nu}\boldsymbol{\nu}_{\mathbf{0}} - 1) \frac{c\mathbf{p}_0\mathbf{e}_2}{\Lambda} \right)^2},$$

$$\beta = (\boldsymbol{\nu}\boldsymbol{\nu}_{\mathbf{0}} - 1)\rho_{\shortparallel} k, \tag{1.55}$$

$$\tan\varphi = \frac{g\left(\boldsymbol{\nu}\mathbf{e}_2 + (\boldsymbol{\nu}\boldsymbol{\nu}_{\mathbf{0}} - 1)\frac{c\mathbf{p}_0\mathbf{e}_2}{\Lambda} \right)}{\boldsymbol{\nu}\mathbf{e}_1 + (\boldsymbol{\nu}\boldsymbol{\nu}_{\mathbf{0}} - 1)\frac{c\mathbf{p}_0\mathbf{e}_1}{\Lambda}}.$$

In these expressions the quantities $\rho_\perp$ and $\rho_{\shortparallel}$ are determined by the (1.31) and (1.40). Here we have omitted the terms with $\mathbf{r}_0$ and τ_0 as these terms (constant phase factor) do not contribute to the single-particle radiation energy. All functions in (1.53) can be expressed by the series of Bessel function production using the following expansion:

$$e^{i\alpha\sin(\omega_0\tau-\varphi)-i\beta\sin 2\omega_0\tau}=\sum_{n,k=-\infty}^{\infty}J_n(\alpha)J_k(\beta)e^{-in\varphi}e^{i(n-2k)\omega_0\tau}.$$

The latter in turn can be expressed by the so-called generalized Bessel function $G_s(\alpha,\beta,\varphi)$:

$$G_s(\alpha,\beta,\varphi)=\sum_{k=-\infty}^{\infty}J_{2k-s}(\alpha)J_k(\beta)e^{i(s-2k)\varphi}. \tag{1.56}$$

Then the functions (1.52) will be written by the function $G_s(\alpha,\beta,\varphi)$ as follows:

$$G_0=2\pi\sum_{s=-\infty}^{\infty}G_s(\alpha,\beta,\varphi)\delta\left(\frac{\overline{\mathcal{E}}-c\boldsymbol{\nu}\overline{\mathbf{p}}}{\Lambda}\omega-s\omega_0\right),$$

$$\begin{aligned}\mathbf{G}_1=\pi A_0\sum_{s=-\infty}^{\infty}&\{\mathbf{e}_1\left(G_{s-1}(\alpha,\beta,\varphi)+G_{s+1}(\alpha,\beta,\varphi)\right)\\&+\mathbf{e}_2 ig\left(G_{s-1}(\alpha,\beta,\varphi)-G_{s+1}(\alpha,\beta,\varphi)\right)\}\,\delta\left(\frac{\overline{\mathcal{E}}-c\boldsymbol{\nu}\overline{\mathbf{p}}}{\Lambda}\omega-s\omega_0\right),\end{aligned} \tag{1.57}$$

$$\begin{aligned}G_2=&\frac{A_0^2}{2}(1+g^2)G_0+\pi A_0^2(1-g^2)\\&\times\sum_{s=-\infty}^{\infty}\left(G_{s-2}(\alpha,\beta,\varphi)+G_{s+2}(\alpha,\beta,\varphi)\right)\delta\left(\frac{\overline{\mathcal{E}}-c\boldsymbol{\nu}\overline{\mathbf{p}}}{\Lambda}\omega-s\omega_0\right).\end{aligned}$$

The function $\delta(x)$ in (1.57) is the Dirac δ-function expressing the resonance condition between the particle oscillation frequency in the incident strong wave field and radiation frequency (conservation law of the Compton effect in quantum terminology). According to (1.57) the radiation energy (1.53) is proportional to the δ^2-function, which should be represented via particle–strong wave interaction time Δt (in the wave coordinate $\Delta\tau=\Delta t\Lambda/\overline{\mathcal{E}}$)

$$\begin{aligned}&\delta\left(\frac{\overline{\mathcal{E}}-c\boldsymbol{\nu}\overline{\mathbf{p}}}{\Lambda}\omega-s\omega_0\right)\delta\left(\frac{\overline{\mathcal{E}}-c\boldsymbol{\nu}\overline{\mathbf{p}}}{\Lambda}\omega-s'\omega_0\right)\\&=\left[\begin{array}{l}0,\ \text{if } s\neq s',\\ \frac{\Delta\tau}{2\pi}\delta\left(\frac{\overline{\mathcal{E}}-c\boldsymbol{\nu}\overline{\mathbf{p}}}{\Lambda}\omega-s\omega_0\right),\ \text{if } s=s'.\end{array}\right.\end{aligned} \tag{1.58}$$

Then instead of the radiation energy (1.53) one can determine the radiation power

$$dP_{\mathbf{k}} = \frac{d\varepsilon_{\mathbf{k}}}{\Delta t}.$$

Substituting (1.57) into (1.53) taking into account (1.58) for the radiation power we obtain (from $\omega > 0$ follows $s > 0$)

$$\begin{aligned} dP_{\mathbf{k}} = \frac{e^2 m^2 c^3 \omega^2}{2\pi \Lambda \overline{\mathcal{E}}} \sum_{s=1}^{\infty} \Bigg\{ \frac{\xi_0^2}{4} \Big[(1+g^2) \left(|G_{s-1}|^2 + |G_{s+1}|^2 \right) \\ + 2(1-g^2) Re \left(G_{s-1}^* G_{s+1} - \frac{1}{2} G_s^* \left(G_{s-2} + G_{s+2} \right) \right) \Big] \\ - \left(1 + \frac{\xi_0^2}{2}(1+g^2) \right) |G_s|^2 \Bigg\} \delta \left(\frac{\overline{\mathcal{E}} - c\boldsymbol{\nu}\overline{\mathbf{p}}}{\Lambda} \omega - s\omega_0 \right) d\omega dO. \end{aligned} \tag{1.59}$$

In the case of the circular polarization of an incident strong wave ($g = \pm 1$) the second argument of the generalized Bessel function $G_s(\alpha, \beta, \varphi)$ is zero and $|G_s|^2 = J_s^2(\alpha)$, so that for the radiation power we have

$$\begin{aligned} dP_{\mathbf{k}} = \frac{e^2 m^2 c^3 \omega^2}{2\pi \Lambda \overline{\mathcal{E}}} \sum_{s=1}^{\infty} \left[\frac{\xi_0^2}{2} \left(J_{s-1}^2(\alpha) + J_{s+1}^2(\alpha) \right) - \left(1 + \xi_0^2 \right) J_s^2(\alpha) \right] \\ \times \delta \left(\frac{\overline{\mathcal{E}} - c\boldsymbol{\nu}\overline{\mathbf{p}}}{\Lambda} \omega - s\omega_0 \right) d\omega dO. \end{aligned} \tag{1.60}$$

Using the known recurrent relations for the Bessel functions

$$J_{s-1}(\alpha) + J_{s+1}(\alpha) = \frac{2s}{\alpha} J_s(\alpha),$$

$$J_{s-1}(\alpha) - J_{s+1}(\alpha) = 2J_s'(\alpha),$$

Equation (1.60) can be represented in the following form:

$$\begin{aligned} dP_{\mathbf{k}} = \frac{e^2 m^2 c^3 \omega^2}{2\pi \overline{\mathcal{E}} (\overline{\mathcal{E}} - c\boldsymbol{\nu}\overline{\mathbf{p}})} \xi_0^2 \sum_{s=1}^{\infty} \left[\left(\frac{s^2}{\alpha^2} - 1 - \xi_0^{-2} \right) J_s^2(\alpha) + J_s'^2(\alpha) \right] \\ \times \delta \left(\omega - \frac{s\omega_0 (\overline{\mathcal{E}} - c\boldsymbol{\nu}_0 \overline{\mathbf{p}})}{\overline{\mathcal{E}} - c\boldsymbol{\nu}\overline{\mathbf{p}}} \right) d\omega dO. \end{aligned} \tag{1.61}$$

For the linear polarization of an incident strong wave ($g = 0$) the third argument of the generalized Bessel function $G_s(\alpha, \beta, \varphi)$ is zero and G_s functions become real. Then for the radiation power in this case we have

$$dP_{\mathbf{k}} = \frac{e^2 m^2 c^3 \omega^2}{2\pi \overline{\mathcal{E}}(\overline{\mathcal{E}} - c\boldsymbol{\nu}\overline{\mathbf{p}})} \sum_{s=1}^{\infty} \left[\frac{\xi_0^2}{4} \left((G_{s-1} + G_{s+1})^2 - G_s\,(G_{s-2} + G_{s+2}) \right) \right.$$
$$\left. - \left(1 + \frac{\xi_0^2}{2}\right) G_s^2 \right] \delta\left(\omega - \frac{s\omega_0(\overline{\mathcal{E}} - c\boldsymbol{\nu}_0\overline{\mathbf{p}})}{\overline{\mathcal{E}} - c\boldsymbol{\nu}\overline{\mathbf{p}}} \right) d\omega dO. \qquad (1.62)$$

1.4 Nonlinear Radiation Effects in Superstrong Wave Fields

Equations (1.59)–(1.62) for the radiation power of a charged particle show that as a result of the particle–strong wave nonlinear interaction in vacuum, numerous harmonics in the radiation spectrum arise, i.e., the radiation process is also nonlinear. In quantum terminology this means that due to multiphoton absorption by a particle from the strong wave the nonlinear Compton effect takes place. The power of harmonics radiation nonlinearly depends on incident strong wave intensity and for its considerable value, laser fields must have relativistic intensities $\xi > 1$.

Up until the last decade, such intensities were practically unachievable (even then the strongest laser fields were $\xi < 1$) and to expect to reach high harmonics radiation via nonlinear Compton channels in vacuum with laser fields of intensities $\xi < 1$ (or any other nonlinear effect at the charge particle–EM wave interaction in vacuum, particularly laser acceleration,) as will be shown below, was unreal. For this reason, actual interest in the nonlinear Compton effect until recently was only theoretical. However, the rapid development of laser technology in the last decade made available laser sources of supershort duration—femtosecond pulses, the intensity of which today much exceeds its relativistic value in the optical domain: $I_{rel} \sim 10^{18}$ W/cm^2 ($\xi \sim 1$), laser fields with $\xi \gg 1$ became available. The latter has provided the necessary intensities for actual radiation of high harmonics in the Compton process. Therefore, we will analyze the process of high harmonics radiation in the nonlinear interaction of a charged particle with superstrong laser fields ($\xi \gg 1$) on the basis of (1.59)–(1.62).

We will analyze the cases of circular and linear polarizations of the incident wave taking into account the specific dependence of harmonics radiation on the strong wave polarization and when the initial velocity of the particle is parallel to the wave propagation direction. This case of particle–wave parallel propagation is of interest since in this case the interaction length with actual laser beams (or, e.g., wiggler field, which in relation to the relativistic particle is equivalent to a counterpropagating laser field) is maximal, which is especially important for the problem of free electron lasers.

In the case of circular polarization of an incident strong wave ($g = \pm 1$) and $\mathbf{p}_0\mathbf{e}_1 = 0$, $\mathbf{p}_0\mathbf{e}_2 = 0$, carrying out the integration over ω and turning to spherical coordinates in (1.61) (OZ axis directed along the vector $\overline{\mathbf{p}}$) for the angular distribution of the radiation power for the sth harmonic we have

$$\frac{dP^{(s)}}{dO} = \frac{e^2 m^2 c^3 \omega_s^2}{2\pi \overline{\mathcal{E}}^2 (1 - \frac{\overline{\mathrm{v}}}{c}\cos\vartheta)} \xi_0^2 \left[\left(\frac{s^2}{\alpha_s^2} - 1 - \xi_0^{-2} \right) J_s^2(\alpha_s) + J_s'^2(\alpha_s) \right], \quad (1.63)$$

where

$$\omega_s = s\omega_0 \frac{\overline{\mathcal{E}} - c\boldsymbol{\nu}_0 \overline{\mathbf{p}}}{\overline{\mathcal{E}} - c\boldsymbol{\nu}\overline{\mathbf{p}}} = s\omega_0 \frac{1 - \frac{\overline{\mathrm{v}}}{c}\cos\vartheta_0}{1 - \frac{\overline{\mathrm{v}}}{c}\cos\vartheta} \quad (1.64)$$

is the radiated frequency and

$$\alpha_s = \frac{smc^2}{\overline{\mathcal{E}}\left(1 - \frac{\overline{\mathrm{v}}}{c}\cos\vartheta\right)} \xi_0 \sin\vartheta \quad (1.65)$$

is the parameter characterizing nonlinear interaction with the strong EM wave. ϑ_0 and ϑ are the incident and scattering angles of the strong and radiated waves with respect to the direction of the particle mean velocity $\overline{\mathbf{v}} = c^2 \overline{\mathbf{p}}/\overline{\mathcal{E}}$.

For a weak EM wave: $\xi_0 \ll 1$ (linear theory) the argument of the Bessel function $\alpha_s \ll 1$ and as is known for such values of the argument $J_s(\alpha_s) \sim {\alpha_s}^s$ and $P^{(s)} \sim \xi_0^{2s}$. Therefore, in the linear theory the main contribution to the radiation power gives the first harmonic. In this case $J_1^2(\alpha_1) \simeq \alpha_1^2/4$, $J_1'^2(\alpha_1) \simeq 1/4$, $\overline{\mathcal{E}} \simeq \mathcal{E}_0$, $\overline{\mathrm{v}} \simeq \mathrm{v}_0$, and

$$\begin{aligned} \frac{dP^{(1)}}{dO} &= \frac{e^2 m^2 c^3 \omega_1^2}{8\pi \mathcal{E}_0^2 (1 - \frac{\mathrm{v}_0}{c}\cos\vartheta)} \xi_0^2 \left[2 - \frac{\alpha_1^2}{\xi_0^2} \right] \\ &= \frac{e^2 m^2 c^3 \omega_1^2}{8\pi \mathcal{E}_0^2 (1 - \frac{\mathrm{v}_0}{c}\cos\vartheta)} \xi_0^2 \left[2 - \left(\frac{mc^2}{\mathcal{E}_0} \right)^2 \frac{\sin^2\vartheta}{\left(1 - \frac{\mathrm{v}_0}{c}\cos\vartheta\right)^2} \right]. \end{aligned} \quad (1.66)$$

Particularly for the particle initially at rest we have the Thomson formula

$$\frac{dP^{(1)}}{dO} = \frac{e^2 \omega_0^2}{8\pi c} \xi_0^2 \left[1 + \cos^2\vartheta \right],$$

$$P^{(1)} = \frac{e^2 \omega_0^2}{4c} \xi_0^2 \int_{-1}^{1} \left[1 + \cos^2\vartheta \right] d\cos\vartheta = \frac{2e^2 \omega_0^2}{3c} \xi_0^2. \quad (1.67)$$

For the moderate relativistic intensities $\xi_0 \sim 1$ (moderate nonlinearity) the power of the low harmonics ($s \sim 10$) exceeds the radiation power of the fundamental frequency ω_1. To show the dependence of the radiation power on the harmonics number the relative differential power

$$P_{rel}^{(s)} = \frac{dP^{(s)}}{dO} / \frac{dP^{(1)}}{dO} = \frac{s^2\left[\left(\frac{s^2}{\alpha_s^2} - 1 - \xi_0^{-2}\right) J_s^2(\alpha_s) + J_s'^2(\alpha_s)\right]}{\left(\frac{1}{\alpha_1^2} - 1 - \xi_0^{-2}\right) J_1^2(\alpha_1) + J_1'^2(\alpha_1)} \tag{1.68}$$

is displayed in Fig. 1.3 for the different harmonics. In Fig. 1.4 the relative differential power is plotted as a function of radiation angle for various harmonics.

For the superstrong EM waves of relativistic intensities (strict nonlinearity): $\xi_0 \gg 1$ a relatively simple analytic formula for the radiation power can be obtained utilizing the properties of the Bessel function. The argument of the latter in (1.63) reaches its maximal value

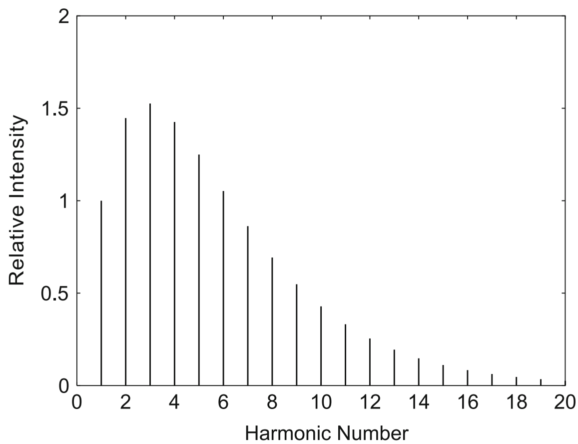

Fig. 1.3 The envelope of the relative differential power of the radiation for the different harmonics is plotted at the $\xi_0 = 1$ and $\vartheta\overline{\gamma} = 1$ ($\overline{\gamma} = \overline{\mathcal{E}}/(m^*c^2) = 10$)

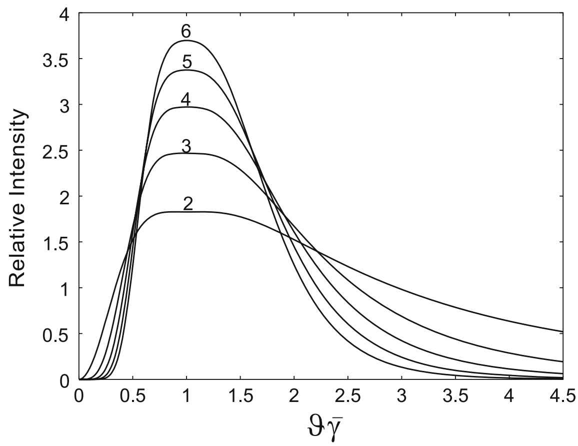

Fig. 1.4 The relative differential power is plotted as a function of radiation angle for various harmonics. The relativistic parameter of intensity is taken to be $\xi_0 = 2$ and $\overline{\gamma} = 10$

$$\alpha_{s\,\max} = \frac{\xi_0}{\sqrt{1+\xi_0^2}} s$$

at the angle $\cos\vartheta_m = \overline{\mathrm{v}}/c$. Therefore, at $\xi_0 \gg 1$ the harmonics with $s \sim \alpha_s \gg 1$ furnish the main contribution to the radiation power. At the angle $\theta = \theta_m$ we have a peak in angular distribution of the radiation power. Besides, in this limit (always $\alpha_s < s$) one can approximate the Bessel function by the Airy one

$$J_s(\alpha_s) \simeq \left(\frac{2}{s}\right)^{1/3} Ai\,(Z)\,;\;\; Z = \left(\frac{s}{2}\right)^{2/3}\left(1 - \frac{\alpha_s^2}{s^2}\right), \tag{1.69}$$

$$J_s' \simeq -\left(\frac{2}{s}\right)^{2/3} Ai'\,(Z),$$

and taking into account that

$$\overline{\mathcal{E}} = \frac{m^* c^2}{\sqrt{1-\frac{\overline{\mathrm{v}}^2}{c^2}}}$$

for the angular distribution of the radiation power we have

$$\frac{dP^{(s)}}{dO} \simeq \frac{e^2\omega_s^2\left(1-\frac{\overline{\mathrm{v}}^2}{c^2}\right)}{2\pi c(1-\frac{\overline{\mathrm{v}}}{c}\cos\vartheta)}\left(\frac{2}{s}\right)^{4/3} \times \left[\left(\frac{s^2}{\alpha_s^2} - 1 - \xi_0^{-2}\right)\left(\frac{s}{2}\right)^{2/3} Ai^2\,(Z) + Ai'^2\,(Z)\right]. \tag{1.70}$$

As far as the Airy function exponentially decreasing with increasing of the argument, one can conclude that the cutoff harmonic s_c is determined from the condition $Z_{\min} \sim 1$, where

$$Z_{\min} = \left(\frac{s}{2}\right)^{2/3}\left(1 - \frac{\alpha_{s\,\max}^2}{s^2}\right) \simeq \left(\frac{s}{2\xi_0^3}\right)^{2/3},$$

which gives $s_c \sim \xi_0^3$.

Consider now the case of linear polarization of the incident strong EM wave. Taking into account the recurrence relation in (1.62)

$$G_{s-2}(\alpha,\beta) + G_{s+2}(\alpha,\beta) = \frac{s}{\beta}G_s(\alpha,\beta) + \frac{\alpha}{2\beta}\left[G_{s-1}(\alpha,\beta) + G_{s+1}(\alpha,\beta)\right],$$

the differential radiation power in this case can be represented in the form

$$\frac{dP^{(s)}}{dO} = \frac{e^2 m^2 c^3 \omega_s^2}{8\pi\overline{\mathcal{E}}^2(1 - \frac{\overline{\mathrm{v}}}{c}\cos\vartheta)}\xi_0^2 \times \left[(G_{s-1} + G_{s+1})\left(G_{s-1} + G_{s+1} - \frac{\alpha}{2\beta}G_s\right) - \left(2 + \frac{4}{\xi_0^2} + \frac{s}{\beta}\right)G_s^2\right]. \tag{1.71}$$

The arguments of the generalized Bessel functions when $\mathbf{p}_0\mathbf{e}_1 = 0$ are

$$\alpha_s = \frac{smc^2}{\overline{\mathcal{E}}\left(1 - \frac{\overline{\mathrm{v}}}{c}\cos\vartheta\right)}\xi_0\left|\boldsymbol{\nu}\mathbf{e}_1\right|,$$

$$\beta_s = \frac{s\xi_0^2}{8 + 4\xi_0^2}\frac{1 - \frac{\overline{\mathrm{v}}^2}{c^2}}{1 - \frac{\overline{\mathrm{v}}}{c}\cos\vartheta_0}\frac{\cos\vartheta_r - 1}{1 - \frac{\overline{\mathrm{v}}}{c}\cos\vartheta}, \tag{1.72}$$

where ϑ_r is the angle between the incident and radiated EM waves.

For the weak EM wave $\xi_0 \ll 1$ the arguments of the generalized Bessel function $\alpha_s, \beta_s \ll 1$ and $P^{(s)} \sim \xi_0^{2s}$, therefore, the main contribution to the radiation power gives the first harmonic. In this case

$$\frac{dP^{(1)}}{dO} = \frac{e^2 m^2 c^3 \omega_1^2}{8\pi\mathcal{E}_0^2(1 - \frac{\mathrm{v}_0}{c}\cos\vartheta)}\xi_0^2\left[1 - \frac{\alpha_1^2}{\xi_0^2}\right]. \tag{1.73}$$

For the particle initially at rest we have the Thomson formula

$$\frac{dP^{(1)}}{dO} = \frac{e^2\omega_0^2}{8\pi c}\xi_0^2\left[1 - (\boldsymbol{\nu}\mathbf{e}_1)^2\right],$$

$$P^{(1)} = \frac{e^2\omega_0^2}{3c}\xi_0^2. \tag{1.74}$$

In contrast to the circular polarization of the strong wave, for the linear polarization there is no azimuthal symmetry and the asymmetry upon the harmonics parity appears. In particular, in the direction opposite to the strong wave propagation ($\boldsymbol{\nu}\mathbf{e}_1 = 0$ and $\vartheta_r = \pi$) only odd harmonics exist. This is a consequence of the particle dynamics in the strong wave field considered in Sect. 1.2. For this case the generalized Bessel function is reduced to the ordinary Bessel function and we have a relatively simple formula. Thus,

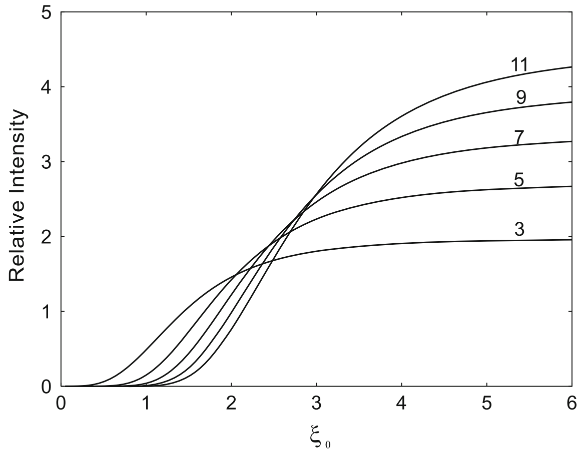

Fig. 1.5 The partial differential power is shown for on axis radiation as a function of ξ_0 for various harmonics ($\overline{\gamma} = 10$)

$$G_s(0, \beta, 0) = \sum_{k=-\infty}^{\infty} J_{2k-s}(0) J_k(\beta)$$
$$= \sum_{k=-\infty}^{\infty} \delta_{2k-s,0} J_k(\beta) = \begin{bmatrix} 0, \text{ if } s \text{ odd} \\ J_{s/2}(\beta), \text{ if } s \text{ even} \end{bmatrix} \tag{1.75}$$

and for the angular distribution of the radiation power we obtain

$$\left.\frac{dP^{(s)}}{dO}\right|_{\vartheta_r=\pi} = \frac{e^2 m^2 c^3 \omega_s^2 \xi_0^2}{8\pi \overline{\mathcal{E}}^2 (1 - \frac{\overline{\mathrm{v}}}{c}\cos\vartheta)} \left[J_{\frac{s+1}{2}}\left(\frac{s\xi_0^2}{4+2\xi_0^2}\right) - J_{\frac{s-1}{2}}\left(\frac{s\xi_0^2}{4+2\xi_0^2}\right)\right]^2 . \tag{1.76}$$

At $\xi_0 \gg 1$ the argument of the Bessel function tends to the value of the index and as in the case of a wave circular polarization the high harmonics $s \gg 1$ give the main contribution to the radiation power and the cutoff harmonic $s_c \sim \xi_0^3$. In Fig. 1.5 the partial differential power is shown for on axis radiation. To show the dependence of the process on the incident wave intensity the relative differential power is plotted as a function of ξ_0 for various harmonics. As we see, with increasing of the wave intensity the power of harmonics well exceeds the power of the fundamental frequency.

1.5 Quantum Description. Volkov Solution of the Dirac Equation

The description of the quantum dynamics of a spinor charged particle (say, electron) in the field of a strong EM wave in vacuum in the scope of relativistic theory requires solution of the Dirac equation, which in the field of arbitrary plane wave allows an exact solution, first obtained by Volkov (1933). This Volkov wave function has the basic role in quantum description of diverse nonlinear electromagnetic processes in superstrong laser fields in vacuum, in particular, major quantum electrodynamic phenomena such as the Compton effect, stimulated bremsstrahlung, and electron–positron pair production, which will be considered in this book. Therefore, this section will be devoted to a description of relativistic wave function of a spinor charged particle in the field of a plane EM wave of arbitrary form and intensity.

The Dirac equation for a spinor particle in a given plane EM wave with arbitrary form of the vector potential $\mathbf{A} = \mathbf{A}(\tau)$ (see (1.7)) is written as follows:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left[c\boldsymbol{\alpha}\widehat{\mathbf{P}} + mc^2\beta \right] \Psi, \tag{1.77}$$

where

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & -\boldsymbol{\sigma} \end{pmatrix}, \ \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.78}$$

are the Dirac matrices in the spinor representation, $\boldsymbol{\sigma} = \left(\sigma_x, \sigma_y, \sigma_z \right)$ are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.79}$$

and

$$\widehat{\mathbf{P}} = \hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}$$

is the operator of the kinetic momentum ($\widehat{\mathbf{p}} = -i\hbar\nabla$ is the operator of the generalized momentum).

Looking for the solution of (1.77) in the form

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}, \tag{1.80}$$

for the spinor functions $\Psi_{1,2}$ we obtain the equations

$$i\hbar \frac{\partial \Psi_1}{\partial t} - c\boldsymbol{\sigma}\widehat{\mathbf{P}}\Psi_1 = mc^2\Psi_2,$$

$$i\hbar \frac{\partial \Psi_2}{\partial t} + c\boldsymbol{\sigma}\widehat{\mathbf{P}}\Psi_2 = mc^2\Psi_1. \tag{1.81}$$

Then acting on the first equation by the operator $i\hbar\partial/\partial t + c\boldsymbol{\sigma}\widehat{\mathbf{P}}$ and taking into account the relation

$$(\boldsymbol{\sigma}\mathbf{a})(\boldsymbol{\sigma}\mathbf{b}) = (\mathbf{ab}) + i\boldsymbol{\sigma}[\mathbf{ab}]$$

we obtain the Dirac equation in quadratic form:

$$\left\{\hbar^2\frac{\partial^2}{\partial t^2} - \hbar^2c^2\left(\boldsymbol{\nu}_0\frac{\partial}{\partial\mathbf{r}}\right)^2 + c^2\widehat{\mathbf{P}}_\perp^2 + m^2c^4 - ec\hbar\boldsymbol{\sigma}(\mathbf{H} - i\mathbf{E})\right\}\Psi_1 = 0. \quad (1.82)$$

A similar equation is obtained for Ψ_2:

$$\left\{\hbar^2\frac{\partial^2}{\partial t^2} - \hbar^2c^2\left(\boldsymbol{\nu}_0\frac{\partial}{\partial\mathbf{r}}\right)^2 + c^2\widehat{\mathbf{P}}_\perp^2 + m^2c^4 - ec\hbar\boldsymbol{\sigma}(\mathbf{H} + i\mathbf{E})\right\}\Psi_2 = 0, \quad (1.83)$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field strengths of the plane EM wave determined by (1.6). The last terms in these equations $\boldsymbol{\sigma}(\mathbf{H} \mp i\mathbf{E})$ describe the spin interaction (for the scalar particles (1.82), (1.83) without which these terms are reduced to the Klein–Gordon equation.) To solve the problem it is more convenient to pass to the retarding and advanced wave coordinates

$$\tau = t - \boldsymbol{\nu}_0\mathbf{r}/c; \;\; \eta = t + \boldsymbol{\nu}_0\mathbf{r}/c,$$

then (1.82) is written as

$$\left\{4\hbar^2\frac{\partial^2}{\partial\tau\partial\eta} + c^2\widehat{\mathbf{P}}_\perp^2 + m^2c^4 - ec\hbar\boldsymbol{\sigma}(\mathbf{H} - i\mathbf{E})\right\}\Psi_1 = 0. \quad (1.84)$$

As the existence of a plane wave does not violate the homogeneity of the space in the plane of the wave polarization ($\mathbf{r}_\perp$) and the interaction Hamiltonian does not depend on the wave advanced coordinate η, i.e., the variables $\mathbf{r}_\perp$, η are cyclic and the corresponding components of generalized momentum $\mathbf{p}_\perp$ and p_η are conserved. Then the solution of (1.84) can be represented in the form

$$\Psi_1\left(\tau, \eta, \mathbf{r}_\perp\right) = F_1(\tau)\exp\left\{\frac{i}{\hbar}(\mathbf{p}_\perp\mathbf{r}_\perp + p_\eta\eta)\right\}. \quad (1.85)$$

From the initial condition $\mathbf{A}(\tau = -\infty) = 0$ it follows that $\mathbf{p}_\perp$ is the free particle initial transverse momentum and the quantity

$$p_\eta = \frac{1}{2}\left(c\mathbf{p}\nu_0 - \mathcal{E}\right), \quad (1.86)$$

where $\mathcal{E}$ and $\mathbf{p}$ are the free particle initial energy and momentum. Note that this quantity coincides with the classical integral of motion (1.10) (with a coefficient).

Substituting (1.85) into (1.84) for the function $F_1(\tau)$ yields the equation

$$\left\{\frac{\partial}{\partial\tau}-\frac{ic^2}{4\hbar p_\eta}\left[\left(\mathbf{p}_\perp-\frac{e}{c}\mathbf{A}\right)^2+m^2c^2-\frac{e\hbar}{c}\boldsymbol{\sigma}(\mathbf{H}-i\mathbf{E})\right]\right\}F_1(\tau)=0. \tag{1.87}$$

The solution of (1.87) can be written in the operator form

$$F_1=\exp\left\{\frac{ic^2}{4\hbar p_\eta}\int_{-\infty}^{\tau}\left[\left(\mathbf{p}_\perp-\frac{e}{c}\mathbf{A}\right)^2+m^2c^2\right]d\tau'\right.$$
$$\left.+\frac{e\left(\boldsymbol{\sigma}\boldsymbol{\nu}_0+1\right)\boldsymbol{\sigma}\boldsymbol{A}}{4p_\eta}\right\}w_1, \tag{1.88}$$

where w_1 is an arbitrary spinor amplitude.

The operator in the exponent should be understood as a expansion into series

$$e^{\widehat{G}}=1+\widehat{G}+\frac{\widehat{G}^2}{2!}+\cdots.$$

Then it is easy to see that all powers greater than 1 of the operator $\left(\boldsymbol{\sigma}\boldsymbol{\nu}_0+1\right)\boldsymbol{\sigma}\boldsymbol{A}$ in (1.88) are zero because

$$\left[\left(\boldsymbol{\sigma}\boldsymbol{\nu}_0+1\right)\boldsymbol{\sigma}\boldsymbol{A}\right]^2=\mathbf{A}^2\left(1-\boldsymbol{\nu}_0^2\right)=0.$$

So, the spinor function (1.88) can be written in the form

$$F_1(\tau)=\exp\left\{\frac{ic^2}{4\hbar p_\eta}\int_{-\infty}^{\tau}\left[\left(\mathbf{p}_\perp-\frac{e}{c}\mathbf{A}\right)^2+m^2c^2\right]d\tau'\right\}$$
$$\times\left[1+\frac{e}{4p_\eta}\left(\boldsymbol{\sigma}\boldsymbol{\nu}_0+1\right)\boldsymbol{\sigma}\mathbf{A}\right]w_1. \tag{1.89}$$

In the same way an analogical expression can be written for the spinor function $F_2(\tau)$.

The spinor components of the bispinor wave function of a particle (1.77) will be written as

$$\Psi_1=\exp\left\{\frac{i}{\hbar}S\left(\mathbf{r},t\right)\right\}\left[1+\frac{e}{4p_\eta}\left(\boldsymbol{\sigma}\boldsymbol{\nu}_0+1\right)\boldsymbol{\sigma}\mathbf{A}\right]w_1,$$

$$\Psi_2=\exp\left\{\frac{i}{\hbar}S\left(\mathbf{r},t\right)\right\}\left[1+\frac{e}{4p_\eta}\left(\boldsymbol{\sigma}\boldsymbol{\nu}_0-1\right)\boldsymbol{\sigma}\mathbf{A}\right]w_2, \tag{1.90}$$

or the ultimate bispinor wave function can be represented via Dirac matrices $\boldsymbol{\alpha}$

$$\Psi(\mathbf{r},t) = \exp\left\{\frac{i}{\hbar}S(\mathbf{r},t)\right\}\left[1+\frac{e}{4p_\eta}(\boldsymbol{\alpha\nu}_0+1)\,\boldsymbol{\alpha}\mathbf{A}\right]w. \tag{1.91}$$

The scalar function $S(\mathbf{r},t)$ in (1.90) and (1.91)

$$S(\mathbf{r},t) = \frac{c^2}{4p_\eta}\int_{-\infty}^{\tau}\left[\frac{e^2}{c^2}\mathbf{A}^2(\tau') - 2\frac{e}{c}\mathbf{pA}(\tau')\right]d\tau' + \mathbf{pr} - \mathcal{E}t \tag{1.92}$$

is the classical action of a charged particle in the plane EM wave field and

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

is a constant bispinor, which should be defined from the condition of the particle wave function normalization according to the above stated initial conditions. Namely, we will demand that at $\tau = -\infty$ this wave function should be reduced to the free Dirac equation solution and for a constant bispinor we will set

$$w = \frac{u_\sigma}{\sqrt{2\mathcal{E}}},$$

where u_σ is the bispinor amplitude of a free Dirac particle with polarization σ. It is assumed that

$$\overline{u}u = 2mc^3,$$

where $\overline{u} = u^\dagger\beta$; $u^\dagger$ denotes the transposition and complex conjugation of u (in what follows we will set the volume of the normalization $V = 1$).

In future consideration of the quantum electrodynamic processes it will be reasonable to use the four-dimensional presentation of the Volkov wave function. Therefore, we will represent the wave function (1.91) in the equivalent four-dimensional form. Here and in what follows for the four-component vectors we choose the metric $a \equiv a^\mu = (a_0, \mathbf{a})$ and $ab \equiv a^\mu b_\mu$ for the relativistic scalar product. The vector potential and the phase of the plane EM wave can be written as

$$A = (0, \mathbf{A}); \quad \tau = t - \boldsymbol{\nu}_0\mathbf{r}/c = \frac{k_\mu x^\mu}{k_0 c},$$

where

$$k = (k_0, \boldsymbol{\nu}_0 k_0)$$

is the four-vector with $k^2 = 0$ and $x = (ct, \mathbf{r})$ is the four-radius vector. Introducing the known $\gamma^\mu = (\gamma_0, \boldsymbol{\gamma})$ matrices

$$\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}, \ \gamma_0 = \beta$$

and taking into account that

$$p_\eta = -\frac{c}{2k_0}pk; \ \ p = \left(\frac{\mathcal{E}}{c}, \mathbf{p}\right),$$

$$\frac{e}{4p_\eta}\left(\boldsymbol{\alpha\nu}_0 + \mathbf{1}\right)\boldsymbol{\alpha}\mathbf{A} = \frac{e}{2c(pk)}\left(\gamma k\right)\left(\gamma A\right),$$

the Volkov wave function may be written as

$$\Psi\left(x\right) = \exp\left\{\frac{i}{\hbar}S\left(x\right)\right\}\left[1 + \frac{e\left(\gamma k\right)\left(\gamma A\right)}{2c(pk)}\right]u,$$

$$S\left(x\right) = -px - \frac{k_0 c}{2pk}\int\limits_{-\infty}^{\tau}\left[2\frac{e}{c}pA(\tau') - \frac{e^2}{c^2}A^2(\tau')\right]d\tau'. \tag{1.93}$$

Consider the Volkov wave function of a spinor particle in the field of the monochromatic wave (1.48). The latter can be presented in the form

$$\Psi_{\mathbf{p}\sigma} = \left[1 + \frac{e\left(\gamma k\right)\left(\gamma A\right)}{2c(kp)}\right]\frac{u_\sigma(p)}{\sqrt{2\mathcal{E}}}\exp\left\{-\frac{i}{\hbar}\left[\Pi x - \frac{eA_0}{c(pk)}\right.\right.$$
$$\left.\left.\times(\mathbf{e}_1\mathbf{p}\sin\omega_0\tau - g\mathbf{e}_2\mathbf{p}\cos\omega_0\tau) + \frac{e^2A_0^2}{8c^2(pk)}(1-g^2)\sin(2\omega_0\tau)\right]\right\}, \tag{1.94}$$

where $k = (\omega_0/c, \mathbf{k}_0)$ is the four-wave vector and $\Pi = (\Pi_0/c, \boldsymbol{\Pi})$ is the average four-kinetic momentum or "quasimomentum" of the particle in the periodic field, which is determined via free particle four-momentum $p = (\mathcal{E}/c, \mathbf{p})$ and relativistic invariant parameter of the wave intensity ξ_0 by the equation

$$\Pi = p + k\frac{m^2c^2}{4kp}(1+g^2)\xi_0^2. \tag{1.95}$$

From this equation it follows that

$$\Pi^2 = m^{*2}c^2; \ m^* = m\left(1 + \frac{1+g^2}{2}\xi_0^2\right)^{1/2}, \tag{1.96}$$

where m^* is the effective mass of the particle in the monochromatic EM wave introduced in Sect. 1.2 (see (1.18)). It is seen that quasimomentum $\mathbf{\Pi} = \overline{\mathbf{p}}$ and quasienergy $\Pi_0 = \overline{\mathcal{E}}$ according to (1.17). The notion of quasimomentum is connected with the space-time translational symmetry-periodicity of the plane wave field as for the electron states in the crystal lattice.

The states (1.94) are normalized by the condition

$$\frac{1}{(2\pi\hbar)^3}\int \Psi^{\dagger}_{\mathbf{p}'\sigma'}\Psi_{\mathbf{p}\sigma}d\mathbf{r} = \delta(\mathbf{p}-\mathbf{p}')\delta_{\sigma,\sigma'},$$

where $\delta_{\sigma,\sigma'}$ is the Kronecker symbol.

By the analogy of the electron states in the crystal lattice the state of a particle in the monochromatic wave can be characterized by the quasimomentum $\mathbf{\Pi}$ and polarization σ as well:

$$\frac{1}{(2\pi\hbar)^3}\int \Psi^{\dagger}_{\mathbf{\Pi}'\sigma'}\Psi_{\mathbf{\Pi}\sigma}d\mathbf{r} = \delta(\mathbf{\Pi}-\mathbf{\Pi}')\delta_{\sigma,\sigma'}.$$

In this case the normalization constant should be changed as follows:

$$\Psi_{\mathbf{\Pi}\sigma} = \sqrt{\frac{\mathcal{E}}{\Pi_0}}\Psi_{\mathbf{p}\sigma}. \tag{1.97}$$

1.6 Nonlinear Compton Effect

With the help of the Volkov wave function (1.94) one can describe the major quantum process of electron scattering in the field of a strong monochromatic wave—nonlinear Compton effect—as a photon radiation by the electron due to the transitions between the "stationary states" of different quasimomentum $\mathbf{\Pi}$ and polarization σ. The spontaneous radiation of a photon by the electron may be considered by the perturbation theory in the scope of quantum electrodynamics (QED). The first-order Feynman diagram (Fig. 1.6) describes the electron–EM wave scattering process, where the electron lines are described via dynamic wave functions in the strong wave field (1.94) (dressed electron). The probability amplitude of transition from the state with a definite quasimomentum and polarization $\Psi_{\mathbf{\Pi}\sigma}$ to the state $\Psi_{\mathbf{\Pi}'\sigma'}$ with the emission of a photon with the frequency ω' and wave vector $\mathbf{k}'$ is given by

$$S_{if} = -\frac{ie}{\hbar c^2}\int j_{if}(x)A^{*}_{ph}(x)\,d^4x, \tag{1.98}$$

where

$$A^{\mu}_{ph}(x) = \sqrt{\frac{2\pi\hbar c^2}{\omega'}}\epsilon^{\mu}e^{-ik'x} \tag{1.99}$$

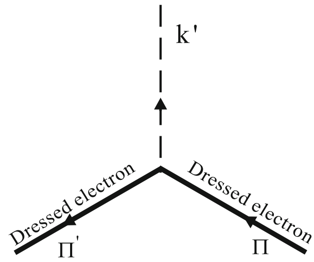

Fig. 1.6 Feynman diagram for nonlinear Compton effect

is the four-dimensional vector potential of quantized photon field (quantization volume $V = 1$), ϵ^{μ} is the four-dimensional polarization vector of the photon, and

$$j_{if}^{\mu} = \overline{\Psi}_{\mathbf{\Pi}'\sigma'}\gamma^{\mu}\Psi_{\mathbf{\Pi}\sigma}$$

is the four-dimensional transition current ($\overline{\Psi}_{\mathbf{\Pi}'\sigma'} = \Psi_{\mathbf{\Pi}'\sigma'}^{\dagger}\gamma_0$ and A^* is the complex conjugate of A).

Hence, for the probability amplitude we have

$$S_{if} = -ie\sqrt{\frac{2\pi}{\hbar\omega' c^2}}\int \overline{\Psi}_{\mathbf{\Pi}'\sigma'}\widehat{\epsilon^*}\Psi_{\mathbf{\Pi}\sigma}e^{ik'x}d^4x. \tag{1.100}$$

Here and in what follows for arbitrary four-component vector $\widehat{a} = \gamma^{\mu}a_{\mu}$. The probability amplitude can be expressed by the generalized Bessel functions $G_s(\alpha, \beta, \varphi)$ introduced in Sect. 1.3. Thus, taking into account the properties of Dirac γ matrices ($\widehat{k}\widehat{k} = 0$ $\widehat{A}\widehat{k} = -\widehat{k}\widehat{A}$) and (1.94) one will obtain

$$S_{if} = -i\frac{e}{c}\sqrt{\frac{\pi}{2\hbar\omega'\Pi_0\Pi_0'}}\int \overline{u}_{\sigma'}(p')\left[\widehat{\epsilon^*} + \left(\frac{e\widehat{A}\widehat{k}\widehat{\epsilon^*}}{2c(kp')} + \frac{e\widehat{\epsilon^*}\widehat{k}\widehat{A}}{2c(kp)}\right)\right.$$
$$\left. - \frac{e^2(k\epsilon^*)A^2}{2c^2(kp')(kp)}\widehat{k}\right]u_{\sigma}(p)e^{i\psi(x)}d^4x. \tag{1.101}$$

Here

$$\psi(x) = \frac{1}{\hbar}\left(\Pi' - \Pi + \hbar k'\right)x + \alpha\sin(kx - \varphi) - \beta\sin 2kx, \tag{1.102}$$

and the parameters α, β, and φ are

$$\alpha = \frac{eA_0}{\hbar c}\left[\left(\frac{\mathbf{e}_1\mathbf{p}}{pk} - \frac{\mathbf{e}_1\mathbf{p}'}{p'k}\right)^2 + g^2\left(\frac{\mathbf{e}_2\mathbf{p}}{pk} - \frac{\mathbf{e}_2\mathbf{p}'}{p'k}\right)^2\right]^{1/2}, \tag{1.103}$$

$$\beta = \frac{e^2 A_0^2}{8\hbar c^2}(1-g^2)\left(\frac{1}{pk} - \frac{1}{p'k}\right), \tag{1.104}$$

$$\tan\varphi = \frac{g\left(\frac{\mathbf{e}_2\mathbf{p}}{pk} - \frac{\mathbf{e}_2\mathbf{p}'}{p'k}\right)}{\left(\frac{\mathbf{e}_1\mathbf{p}}{pk} - \frac{\mathbf{e}_1\mathbf{p}'}{p'k}\right)}. \tag{1.105}$$

After the integration the probability amplitude (1.101) can be represented in the form

$$S_{if} = -i\frac{e}{c}(2\pi\hbar)^4\sqrt{\frac{\pi}{2\hbar\omega'\Pi_0\Pi_0'}}\,\overline{u}_{\sigma'}(p')\widehat{M}_{if}u_\sigma(p), \tag{1.106}$$

where

$$\widehat{M}_{if} = \left[\widehat{\epsilon^*}Q_0 + \left(\frac{e\widehat{Q}_1\widehat{k}\widehat{\epsilon^*}}{2c(kp')} + \frac{e\widehat{\epsilon^*}\widehat{k}\widehat{Q}_1}{2c(kp)}\right) + \frac{e^2(k\epsilon^*)Q_2}{2c^2(kp')(kp)}\widehat{k}\right] \tag{1.107}$$

with the functions Q_0, Q_1^μ, and Q_2:

$$Q_0 = \sum_{s=-\infty}^{\infty} G_s(\alpha,\beta,\varphi)\delta\left(\Pi' - \Pi + \hbar k' - s\hbar k\right), \tag{1.108}$$

$$Q_1^\mu = (0, \mathbf{Q}_1),$$

$$\begin{aligned}\mathbf{Q}_1 = \frac{A_0}{2}\sum_{s=-\infty}^{\infty}\{&\mathbf{e}_1\left(G_{s-1}(\alpha,\beta,\varphi) + G_{s+1}(\alpha,\beta,\varphi)\right)\\ &+ i\mathbf{e}_2 g\left(G_{s-1}(\alpha,\beta,\varphi) - G_{s+1}(\alpha,\beta,\varphi)\right)\}\,\delta\left(\Pi' - \Pi + \hbar k' - s\hbar k\right),\end{aligned} \tag{1.109}$$

$$\begin{aligned}Q_2 = {}&\frac{A_0^2}{2}(1+g^2)Q_0 + \frac{A_0^2}{2}(1-g^2)\\ &\times\sum_{s=-\infty}^{\infty}\left(G_{s-2}(\alpha,\beta,\varphi) + G_{s+2}(\alpha,\beta,\varphi)\right)\delta\left(\Pi' - \Pi + \hbar k' - s\hbar k\right).\end{aligned} \tag{1.110}$$

From the definition of the functions (1.108)–(1.110) follows the useful relation

$$\frac{\mathcal{E}' - \mathcal{E} + \hbar\omega'}{\omega}Q_0 + \frac{e}{c}\left(\frac{p'Q_1}{kp'} - \frac{pQ_1}{kp}\right) + \frac{e^2}{2c^2}\left(\frac{1}{kp'} - \frac{1}{kp}\right)Q_2 = 0 \tag{1.111}$$

We will assume that the Dirac particle is nonpolarized and summation over the final particle polarizations (photon and electron) will be made. Then we need to calculate the sum

$$\frac{1}{2}\sum_{\sigma',\sigma,\epsilon}\left|S_{if}\right|^2=\frac{(2\pi\hbar)^8\,\pi e^2}{4\hbar\omega' c^2\Pi_0\Pi_0'}\sum_{\sigma',\sigma,\epsilon}\left|\overline{u}_{\sigma'}(p')\widehat{M}_{if}u_{\sigma}(p)\right|^2$$
$$=\frac{(2\pi\hbar)^8\,\pi e^2c^2}{4\hbar\omega'\Pi_0\Pi_0'}\sum_{\epsilon}Sp\left[(\widehat{p}'+mc)\widehat{M}_{if}(\widehat{p}+mc)\widehat{\overline{M}}_{if}\right], \quad (1.112)$$

where

$$\widehat{\overline{M}}_{if}=\gamma_0\widehat{M}_{if}^{\dagger}\gamma_0.$$

Taking into account that spur of the product of odd number γ matrices is zero we will obtain

$$\frac{1}{2}\sum_{\sigma',\sigma,\epsilon}\left|S_{if}\right|^2=\frac{(2\pi\hbar)^8\,\pi e^2c^2}{4\hbar\omega'\Pi_0\Pi_0'}\sum_{\epsilon}\left\{Sp\left[\widehat{p}'\widehat{M}_{if}\widehat{p}\widehat{\overline{M}}_{if}\right]+m^2c^2Sp\left[\widehat{M}_{if}\widehat{\overline{M}}_{if}\right]\right\}.$$

The summation over the photon polarizations is equivalent to the replacements

$$\epsilon_{\upsilon}^{*}\epsilon_{\mu}\rightarrow -g_{\upsilon\mu}, \quad \widehat{\epsilon}^{*}\widehat{a}\widehat{\epsilon}\rightarrow 2\widehat{a}, \quad \widehat{\epsilon}^{*}\widehat{a}\widehat{b}\widehat{c}\widehat{\epsilon}\rightarrow 2\widehat{c}\widehat{b}\widehat{a}, \quad (1.113)$$

where $g_{\upsilon\mu}$ is the metric tensor. So,

$$Sp\left[\widehat{M}_{if}\widehat{\overline{M}}_{if}\right]=-16\left|Q_0\right|^2$$

and

$$Sp\left[\widehat{p}'\widehat{M}_{if}\widehat{p}\widehat{\overline{M}}_{if}\right]=8(p'p)\left|Q_0\right|^2$$
$$+\frac{8e}{c}\left(pk-p'k\right)Re\left(\left(\frac{p'Q_1}{kp'}-\frac{pQ_1}{kp}\right)Q_0^{*}\right)$$
$$-\frac{4e^2}{c^2}\left[\frac{kp}{kp'}+\frac{kp'}{kp}\right]\left|Q_1\right|^2-\frac{8e^2}{c^2}Re\left(Q_0Q_2^{*}\right).$$

Then using the relation (1.111) we obtain

$$\frac{1}{2}\sum_{\sigma',\sigma,\epsilon}\left|S_{if}\right|^2=\frac{2\,(2\pi\hbar)^8\,\pi e^2c^2}{\hbar\omega'\Pi_0\Pi_0'}\left[-m^2c^2\left|Q_0\right|^2\right.$$
$$\left.-\frac{e^2}{c^2}\left(1+\frac{\hbar^2\left(kk'\right)^2}{2\,(pk)\,(p'k)}\right)\left(\left|Q_1\right|^2+Re\left(Q_0Q_2^{*}\right)\right)\right]. \quad (1.114)$$

For the differential probability per unit time we have

$$dW = \frac{1}{2T}\sum_{\sigma',\sigma,\epsilon}\left|S_{if}\right|^2 \frac{d\mathbf{\Pi}'}{(2\pi\hbar)^3}\frac{d\mathbf{k}'}{(2\pi)^3}, \tag{1.115}$$

where T is the interaction time. Then taking into account (1.108)–(1.110) and the relation

$$\delta\left(\Pi' - \Pi + \hbar k' - s\hbar k\right)\delta\left(\Pi' - \Pi + \hbar k' - s'\hbar k\right)$$
$$= \begin{bmatrix} 0, \text{ if } s \neq s', \\ \\ \frac{cT}{(2\pi\hbar)^4}\delta\left(\Pi' - \Pi + \hbar k' - s\hbar k\right), \text{ if } s = s', \end{bmatrix} \tag{1.116}$$

for the differential probability of the nonlinear Compton effect we obtain

$$dW = \sum_{s=1}^{\infty} W^{(s)}\delta\left(\Pi' - \Pi + \hbar k' - s\hbar k\right)d\mathbf{\Pi}'d\mathbf{k}', \tag{1.117}$$

$$W^{(s)} = \frac{e^2m^2c^5}{2\pi\omega'\Pi_0\Pi_0'}\left[-\left|G_s\right|^2 + \frac{\xi_0^2}{4}\left(1 + \frac{\hbar^2\left(kk'\right)^2}{2\left(pk\right)\left(p'k\right)}\right)\right.$$
$$\times\left(\left(1+g^2\right)\left(\left|G_{s-1}\right|^2 + \left|G_{s+1}\right|^2 - 2\left|G_s\right|^2\right)\right.$$
$$\left.\left. + (1-g^2)Re\left[2G_{s-1}^*G_{s+1} - G_s^*\left(G_{s-2} + G_{s+2}\right)\right]\right)\right]. \tag{1.118}$$

The four-dimensional δ-functions in (1.117) for differential probability express the conservation laws for quasimomentum and quasienergy of the particle in the nonlinear Compton process. Different s correspond to partial scattering processes with fixed photon numbers and $W^{(s)}$ are the partial probabilities of s-photon absorption by the particle in the strong wave field.

The spectrum of emitted photons is determined from the conservation laws. Taking into account (1.95) and (1.96) we will have the following expression for the radiated frequency:

$$\omega' = s\omega\frac{1 - \frac{\bar{\mathrm{v}}}{c}\cos\vartheta_0}{1 - \frac{\bar{\mathrm{v}}}{c}\cos\vartheta + \frac{s\hbar\omega}{\Pi_0}\left(1 - \cos\vartheta_r\right)}, \tag{1.119}$$

where ϑ_0, ϑ are the incident and scattering angles of incident strong wave and radiated photon with respect to the direction of the particle mean velocity $\bar{\mathbf{v}} = c^2\mathbf{\Pi}/\Pi_0$ and ϑ_r

is the angle between the incident wave and radiated photon propagation directions. The quantum conservation law of nonlinear Compton effect (1.119) differs from the classical formula (1.64) by the last term in the denominator $\sim s\hbar\omega/\Pi_0$, which is the quantum recoil of emitted photon.

Making the integration over $\mathbf{\Pi}'$ in (1.117) and multiplying by the photon energy we obtain the radiation power. In the case of circular polarization of an incident strong wave ($g = \pm 1$) we have $|G_s|^2 = J_s^2(\alpha)$ and the radiation power is

$$dP_{\mathbf{k}'}^{(s)} = \frac{\omega'^2 e^2 m^2 c^3}{2\pi \Pi_0 \Pi_0'} \left[-J_s^2(\alpha) + \xi_0^2 \left(1 + \frac{\hbar^2 \left(kk'\right)^2}{2\left(pk\right)\left(p'k\right)} \right) \right. \\ \left. \times \left[\left(\frac{s^2}{\alpha^2} - 1 \right) J_s^2(\alpha) + J_s'^2(\alpha) \right] \right] \times \delta\left(\frac{\Pi_0' - \Pi_0}{\hbar} + \omega' - s\omega \right) d\omega' dO,$$

where the Bessel function argument

$$\alpha = \frac{eA_0}{\hbar\omega} \left| \left[\mathbf{k} \left(\frac{\mathbf{p}}{pk} - \frac{\mathbf{p}'}{p'k} \right) \right] \right|. \tag{1.120}$$

Taking into account that

$$\delta\left(\frac{\Pi_0' - \Pi_0}{\hbar} + \omega' - s\omega \right) d\omega' \to \left| \frac{\partial}{\partial\omega'} \left(\frac{\Pi_0'}{\hbar} + \omega' \right) \right|^{-1} = \frac{\Pi_0' \omega'}{c^2 \left(\Pi' k'\right)},$$

for the angular distribution of radiation power we obtain

$$\frac{dP^{(s)}}{dO} = \frac{\omega'^3 e^2 m^2 c}{2\pi \Pi_0 \left(\Pi' k'\right)} \left[-J_s^2(\alpha) + \xi_0^2 \left(1 + \frac{\hbar^2 \left(kk'\right)^2}{2\left(pk\right)\left(p'k\right)} \right) \right. \\ \left. \times \left(\left(\frac{s^2}{\alpha^2} - 1 \right) J_s^2(\alpha) + J_s'^2(\alpha) \right) \right]. \tag{1.121}$$

This formula differs from the classical one (1.63) only by the terms of quantum recoil, which are of the order of $\hbar kk'/\left(\Pi' k\right)$. The maximal value of this parameter is $2s\hbar\left(\Pi k\right)/m^{*2}c^2$ and if

$$\frac{2s\hbar\left(\Pi k\right)}{m^{*2}c^2} \ll 1,$$

one can omit the quantum recoil and taking into account that in this case

$$\Pi' k' \simeq \Pi k'; \quad \alpha \simeq \alpha_{classic}; \quad \frac{\hbar^2 \left(kk'\right)^2}{2\left(pk\right)\left(p'k\right)} \ll 1,$$

from (1.121) we obtain the classical formula for radiation power.

In the limit of weak EM wave when $\xi_0 \ll 1$ (linear theory) the argument of the Bessel function $\alpha \ll 1$ and the main contribution to the radiation power gives the first harmonic (as in the classical theory). In this case $J_1^2(\alpha_1) \simeq \alpha_1^2/4$, $J_1'^2(\alpha_1) \simeq 1/4$, $\Pi_0 \simeq \mathcal{E}$, $\Pi_0' \simeq \mathcal{E}'$, and

$$\frac{dP}{dO} = \frac{\omega'^3 e^2 m^2 c}{8\pi\mathcal{E}\,(p'k')}\left[-\alpha^2 + 2\xi_0^2\left(1 + \frac{\hbar^2\left(kk'\right)^2}{2\,(pk)\,(p'k)}\right)\right].$$

Then, using conservation laws, it is easy to see that

$$\left|\left[\mathbf{k}\left(\frac{\mathbf{p}'}{p'k} - \frac{\mathbf{p}}{pk}\right)\right]\right|^2 = 2\hbar\frac{\omega^2}{c^2}\left(\frac{1}{p'k} - \frac{1}{pk}\right) - \omega^2 m^2\left(\frac{1}{pk} - \frac{1}{p'k}\right)^2,$$

$$\left(1 + \frac{\hbar^2\left(kk'\right)^2}{2\,(pk)\,(p'k)}\right) = \frac{1}{2}\left[\frac{pk}{p'k} + \frac{p'k}{pk}\right],$$

and for the one-photon Compton effect we obtain

$$\frac{dP}{dO} = \frac{\omega'^3 e^2 m^2 c}{8\pi\mathcal{E}\,(p'k')}\xi_0^2\left[\left(\frac{m^2c^2}{\hbar\,(p'k)} - \frac{m^2c^2}{\hbar\,(pk)}\right)^2 - 2\left(\frac{m^2c^2}{\hbar\,(p'k)} - \frac{m^2c^2}{\hbar\,(pk)}\right) + \frac{pk}{p'k} + \frac{p'k}{pk}\right]. \tag{1.122}$$

For the differential cross section

$$\frac{d\sigma}{dO} = \frac{1}{\hbar\omega' J}\frac{dP}{dO}$$

one should make the replacement

$$A_0^2 \to \frac{4\pi\hbar c^2}{\omega}, \tag{1.123}$$

corresponding to photon field quantization and

$$J = \frac{c^3\,pk}{\omega\mathcal{E}}$$

is the initial flux density (quantization volume $V = 1$). Hence, for the differential cross section of the one-photon Compton effect we obtain

$$\frac{d\sigma}{dO} = \frac{\omega'^2 e^4}{2c^4 (pk)^2} \left[\left(\frac{m^2c^2}{\hbar (p'k)} - \frac{m^2c^2}{\hbar (pk)} \right)^2 - 2 \left(\frac{m^2c^2}{\hbar (p'k)} - \frac{m^2c^2}{\hbar (pk)} \right) + \frac{pk}{p'k} + \frac{p'k}{pk} \right]. \quad (1.124)$$

For a particle initially at rest

$$pk = m\omega, \quad pk' = m\omega', \quad \frac{mc^2}{\hbar\omega'} - \frac{mc^2}{\hbar\omega} = 1 - \cos\vartheta_r,$$

and the differential cross section of the one-photon Compton effect may be written in the known form of Klein and Nishina formula

$$\frac{d\sigma}{dO} = \frac{r_e^2}{2} \left(\frac{\omega'}{\omega} \right)^2 \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\vartheta_r \right], \quad (1.125)$$

where $r_e = e^2/mc^2$ is the classical radius of the electron.

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http://www.springer.com/978-3-319-26382-3

Relativistic Nonlinear Electrodynamics
The QED Vacuum and Matter in Super-Strong Radiation
Fields
Avetissian, H.K.
2016, XVII, 506 p. 59 illus. in color., Hardcover
ISBN: 978-3-319-26382-3