Chapter 1
Introduction and Survey

Abstract We begin the chapter with some history of the results that form the background of this book. We then define higher matrix factorizations, our main focus. While classical matrix factorizations are factorizations of a single element, higher matrix factorizations deal directly with sequences of elements. In Sect. 1.3, we outline our main results. Throughout the book, we use the notation introduced in Sect. 1.4.

1.1 How We Got Here

Since the days of Cayley [16] and Hilbert [32], minimal free resolutions of finitely generated modules have played many important roles in mathematics. They now appear in fields as diverse as algebraic geometry, invariant theory, commutative algebra, number theory, and topology.

Hilbert showed that the minimal free resolution of any module over a polynomial ring is finite. In the hands of Auslander, Buchsbaum and Serre this property was identified with the geometric property of non-singularity (which had been identified algebraically by Zariski in [58]): local rings for which the minimal free resolution of every module is finite are the *regular local rings*. The Auslander-Buchsbaum formula also identifies the length of the minimal free resolution of a module as complementary to another important invariant, the depth.

The minimal free resolution of the residue field $k = S/\mathbf{m}$, where $\mathbf{m}$ denotes the maximal ideal of a local ring S, plays a special role: in the case of a regular ring this is the Koszul complex (the name is standard, though the idea and the construction were present in the work of Cayley, a century before Koszul).

The condition of minimality is important in these theories. The mere existence of free resolutions suffices for foundational issues such as the definition of Ext and Tor, and there are various methods of producing resolutions uniformly (for example the Bar resolution, in the case of algebras over a field). But without minimality, resolutions are not unique, and the very uniformity of constructions like the Bar resolution implies that they give little insight into the structure of the modules resolved. By contrast, the minimal resolution of a finitely generated module over a local ring is unique and contains a host of subtle invariants of the module. There

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are still many mysteries about minimal free resolutions over regular local rings, and this is an active field of research.

Infinite minimal free resolutions seem first to have come to the fore around 1957 in the work of Tate [57], perhaps motivated by constructions in group cohomology over a field of characteristic $p > 0$ coming from class field theory. The simplest interesting case is that of the group algebra $k[G]$ where G is a group of the form

$$G = \mathbb{Z}/(p^{a_1}) \oplus \cdots \oplus \mathbb{Z}/(p^{a_1})$$

and k is a field of characteristic p. In this case $k[G]$ is a local ring of the special form

$$k[G] \cong k[x_1, \dots, x_c]/(f_1, \dots, f_c).$$

where $f_i(x) = x_i^{p^{a_i}} - 1$ for each i, and the maximal ideal is generated by $(x_1 - 1, \dots, x_c - 1)$. The cohomology of such a group, with coefficients in a $k[G]$-module N is, by definition, $\mathrm{Ext}_{k[G]}(k, N)$, and it is thus governed by a free resolution of the residue field k as a module over $k[G]$. The case of non-commutative groups is of course the one of primary interest; but it turns out that many features of resolutions over non-commutative group algebras are governed by the resolutions over the elementary abelian p-groups related to them, so the commutative case plays a major role in the theory.

Generalizing the example of the group algebras above, Tate gave an elegant description of the minimal free resolution of the residue field k of a ring R of the form

$$R = S/(f_1, \dots, f_c)\,,$$

where S is a regular local ring with residue field k and $f_1, \dots, f_c$ is a *regular sequence*—that is, each f_i is a non-zerodivisor modulo the ideal generated by the preceding ones and $(f_1, \dots, f_c) \neq S$. Such rings are usually called *complete intersections*, the name coming from their role in algebraic geometry. It is with minimal free resolutions of arbitrary finitely generated modules over complete intersections that this book is concerned.

Tate showed that, if R is a complete intersection, then the minimal free resolution of k has a simple structure, just one step removed from that of the Koszul complex. Tate's paper led to a large body of work about the minimal free resolutions of the residue fields of other classes of rings (see for example the surveys by Avramov [5] and McCullough-Peeva [41]). Strong results were achieved for complete intersections, but the structures that emerged in more general cases were far more complex. Still today, our knowledge of infinite minimal free resolutions of modules, or even of the residue class field, over rings that are not regular or members of a few other special classes (Koszul rings, Golod rings) is very slight.

The situation with complete intersections has drawn a lot of interest. Shamash [55] (and later, in a different way, Eisenbud [25]) showed how to generalize Tate's construction to any module ... but, except for the residue class field, this method rarely produces a minimal resolution. A different path was begun in the 1974 paper [31] of Gulliksen, who showed that if N and P are any finitely generated modules over a complete intersection, then $\mathrm{Ext}_R(N, P)$ has a natural structure of a finitely generated graded module over a polynomial ring $\mathcal{R} = k[\chi_1, \dots, \chi_c]$, where c is the codimension of R. He used this to show that the Poincaré series $\sum_i \beta_i^R(N)x^i$, the generating function of the Betti numbers $\beta_i^R(N)$, is rational and that the denominator divides $(1-x^2)^c$. In 1989 Avramov [4] identified the dimension of $\mathrm{Ext}_R(N, k)$, which he called the *complexity* of N, with a correction term in a natural generalization of the Auslander-Buchbaum formula.

Gulliksen's finite generation result implies that the even Betti numbers $\beta_{2i}^R(N)$ are eventually given by a polynomial in i, and similarly for the odd Betti numbers. Avramov [4] proved that the two polynomials have the same degree and leading coefficient. In 1997 Avramov, Gasharov and Peeva [7] gave further restrictions on the Betti numbers, establishing in particular that the Betti sequence $\{\beta_i^R(N)\}$ is eventually either strictly increasing or constant.

The paper [25] of Eisenbud in 1980 brought a somewhat different direction to the field. He took the point of view that what is simple about the minimal free resolution

$$\cdots \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0$$

of a module $N = \mathrm{Coker}(F_1 \longrightarrow F_0)$ over a regular ring S is that the F_i are *eventually* 0 (by Hilbert's Syzygy Theorem): in fact they are 0 for all $i > \dim(S)$ (this would not be true if we did not insist on *minimal* resolutions).

Eisenbud described the eventual behavior of minimal free resolutions of arbitrary finitely generated modules over *hypersurface rings*. These are the rings of the form $R = S/(f)$ where S is a regular local ring, that is, complete intersections of codimension 1. He proved that, in this case, the minimal resolution of every finitely generated R-module eventually becomes periodic, of period at most 2, and that these periodic resolutions correspond to *matrix factorizations* of the defining equation f; that is, to pairs of square matrices (A, B) of the same size such that $AB = BA = f \cdot I$, where I is an identity matrix. As a familiar example, if $f = \det(A)$ then we could take B to be the adjoint matrix of A. As with the case of regular rings, the simple pattern of the matrix factorization starts already after at most $\dim(R)$ steps in the resolution. The theory gives a complete and powerful description of the eventual behavior of minimal resolutions over a hypersurface.

Since there exist infinite minimal resolutions over any singular ring, and the hypersurface ring R is singular if and only if $f \in \mathbf{m}^2$ (where $\mathbf{m}$ is the maximal ideal of S), it follows that matrix factorizations exist for any element $f \in \mathbf{m}^2$—for instance, for any power series of order ≥ 2. Such a power series f has an ordinary factorization—that is, a factorization by 1×1 matrices—if and only if f defines a reducible hypersurface. Factorization by larger matrices seems first to have appeared in the work of P.A.M Dirac, who used it to find a matrix square root—now called the *Dirac operator*—of an irreducible polynomial in partial derivatives

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial z^2}\,.$$

The general theory of matrix factorizations is briefly described in Chap. 2. There are still many open questions there, such as: "*What determines the minimal size of the matrices in the factorizations of a given power series?*"

Matrix factorizations have had many applications. Starting with Kapustin and Li [37], who followed an idea of Kontsevich, physicists discovered amazing connections with string theory—see [1] for a survey. A major advance was made by Orlov [43, 44, 46, 47], who showed that matrix factorizations could be used to study Kontsevich's homological mirror symmetry by giving a new description of singularity categories. Matrix factorizations have also proven useful for the study of cluster tilting [18], Cohen-Macaulay modules and singularity theory [9, 12, 15, 40], Hodge theory [11], Khovanov-Rozansky homology [38, 39], moduli of curves [51], quiver and group representations [2, 4, 36, 52], and other topics, for example, [10, 17, 21–23, 33–35, 50, 53–56].

What about the eventual behavior of minimal free resolutions over other rings? The Auslander-Buchsbaum-Serre Theorem shows that regular rings are characterized by saying that minimal free resolutions are eventually zero, suggesting that the eventual behavior of resolutions over a non-regular ring R corresponds to some feature of the singularity of R. This idea has been extensively developed by Buchweitz, Orlov and others under the name "singularity category"; see [43–47].

The obvious "next" case to study after hypersurface rings is the eventual behavior of minimal resolutions over complete intersections.

One useful method of extending the theory of matrix factorizations to complete intersections was developed by Orlov [45] and subsequent authors, for example [13, 14, 51]. This method regards a complete intersection as a family of hypersurfaces parametrized by a projective space. For example, suppose that $S = k[x_1, \dots, x_n]$ is the coordinate ring of the affine n-space $\mathbb{A}^n_k$ over a field k, and R is the complete intersection $R = S/(f_1, \dots, f_c)$. Consider the element $f = \sum_1^c z_i f_i \in S[z_1, \dots, z_c]$ as defining a hypersurface in the product of $\mathbb{A}^n$ and the projective space $\mathbb{P}^{c-1}$ and consider the category of matrix factorizations (now defined as a pair of maps of vector bundles) of f. This idea has proven useful in string theory and elsewhere, and provides a good definition of a singularity category for complete intersections; but it does not seem to shed any light on the structure of minimal free resolutions over R.

Eisenbud provided easy examples showing that over complete intersections, unlike regular and hypersurface rings, nice patterns may begin only far back in the minimal free resolutions. Even though Eisenbud's paper is entitled "Homological Algebra over a Complete Intersection", it only gives strong results for the case of hypersurfaces and really doesn't get much further than posing questions about minimal free resolutions over complete intersections of higher codimension. In the case of codimension 2, important steps were taken by Avramov and Buchweitz in [6] in 2000 using the classification of modules over the exterior algebra on two variables; in particular they constructed minimal free resolutions of high syzygies over a codimension two complete intersection as quotients. But the general case (of higher codimensions) has remained elusive.

Nevertheless, when one looks at the matrices of the differential in minimal resolutions over a complete intersection, for example in the output of the computer systems Macaulay2 [30] and Singular [19], one feels the presence of repetitive patterns.

The authors of this book have wondered, for many years, how to describe the eventual patterns in the minimal resolutions of modules over complete intersections of higher codimension. With the theory presented here we believe we have found an answer: when M is a sufficiently high syzygy over a complete intersection ring R, our theory describes the minimal free resolutions of M as an S module and as an R-module.

Revealing the Pattern

In the next section we will introduce the notion of a Higher Matrix Factorization. Ordinary matrix factorizations allow one to understand minimal free resolutions of high syzygies over a hypersurface ring in terms of a simple matrix equation; they show in particular that such resolutions are eventually periodic. We introduce higher matrix factorizations to give, in an equational form, the data needed to describe the structure of minimal free resolutions of high syzygies over a complete intersection of arbitrary codimension; like ordinary matrix factorizations, they show that minimal resolutions eventually exhibit stable patterns.

We define higher matrix factorizations in the next section. Here we provide motivation, by sketching how higher matrix factorizations arise in the structure of minimal free resolutions of high syzygies:

Let N be any module over a complete intersection

$$R = S/(f_1, \dots, f_c)\,,$$

where S is a regular local ring and $f_1, \dots, f_c$ is a regular sequence, and let M be a sufficiently high syzygy—a *stable syzygy* in the sense of Chap. 6—over R. The module M is, in particular, a maximal Cohen-Macaulay R-module without free summands. We begin with replacing the sequence $f_1, \dots, f_c$ by a generic choice of generators of the ideal $(f_1, \dots, f_c)$. We will analyze the minimal free resolution of M over R by an induction on the codimension c. The case $c = 0$ is the case of the regular local ring S: the only stable syzygy over S is the module 0.

We thus assume, by induction, that the minimal resolutions of stable syzygies over the ring $R' = S/(f_1, \dots, f_{c-1})$ are understood. As mentioned above, the Shamash construction rarely produces minimal resolutions; but (because of our general position hypothesis on the generators $f_1, \dots, f_c$ and the definition of a stable syzygy) the minimal free resolution of M over R is actually obtained as the Shamash construction starting from a minimal free resolution $\mathbf{U}$ of M as an R' module; thus it suffices to obtain such a resolution. Of course M is *not* a stable syzygy over R'—it is not even a maximal Cohen-Macaulay R' module. One of the main new ideas of this book is that $\mathbf{U}$ can be constructed in a simple way, which we call a *Box complex*, described as follows.

Like any maximal Cohen-Macaulay module over a Gorenstein ring, M is the second syzygy over R of a unique maximal Cohen-Macaulay module L without free summands. We define $M' := \mathrm{Syz}_2^{R'}(L)$ to be the second syzygy of L as a module over R' or, equivalently, as the non-free part of the Cohen-Macaulay approximation of M over R'. We prove that M' is a stable syzygy over R', and thus has a minimal free resolution described by a higher matrix factorization—this will be part of the higher matrix factorization of M.

Let

$$\cdots \xrightarrow{\partial} A'_2 \xrightarrow{\partial} A'_1 \xrightarrow{\partial} A'_0$$

be the minimal free resolution of M' over R', and let

$$\overline{b}:\ \overline{B}_1 \longrightarrow \overline{B}_0 \longrightarrow L \longrightarrow 0$$

be the minimal free presentation of L as an R-module. We prove that any lifting $b' : B'_1 \longrightarrow B'_0$ of b to a map of free R'-modules gives a minimal free presentation of L as an R'-module, and hence the minimal free resolution of L as an R'-module has the form

$$\cdots \xrightarrow{\partial} A'_2 \xrightarrow{\partial} A'_1 \xrightarrow{\partial} A'_0 \longrightarrow B'_1 \xrightarrow{b'} B'_0.$$

Since L is annihilated by f_c there is a homotopy for f_c on this complex. Let ψ' denote the component of that homotopy that maps $B'_1 \longrightarrow A'_0$. We prove that the minimal free resolution of M over R' is

$$\begin{tikzcd}
\cdots \arrow[r] & A'_3 \arrow[r, "\partial'"] & A'_2 \arrow[r, "\partial'"] & A'_1 \arrow[r, "\partial'"] & A'_0 \\
& & & \oplus & \oplus \\
& & & B'_1 \arrow[r, "b'"] \arrow[uur, "\psi'"] & B'_0\,,
\end{tikzcd}$$

which we call the *Box complex*. We show that the minimal free resolution of M over R is the Shamash construction applied to this Box complex.

We will define a Higher Matrix Factorization for M in the next section in a way that captures the information in the "box":

$$\begin{tikzcd} A'_1 \arrow[r, "\partial'"] & A'_0 \\ \oplus & \oplus \\ B'_1 \arrow[r, "b'"] \arrow[uur, "\psi'"] & B'_0 , \end{tikzcd}$$

where A'_1 and A'_0 come inductively from a higher matrix factorization for $f_1, \ldots, f_{c-1}$.

The data in the higher matrix factorization suffice to describe the minimal resolutions of the stable R-syzygy M both as an S-module and as an R-module. These constructions and some consequences are outlined in Sect. 1.3.

1.2 What is a Higher Matrix Factorization?

The main concept we introduce in this book is that of a *higher matrix factorization* (which we sometimes abbreviate HMF) with respect to a sequence of elements in a commutative ring. To emphasize the way in which it generalizes the classical case, we first briefly recall the definition, from [25], of a *matrix factorization* with respect to a single element. More about this case can be found in Chap. 2.

Definition 1.2.1 If $0 \neq f \in S$ is an element in a commutative local ring then a *matrix factorization* with respect to f is a pair of finitely generated free modules A_0, A_1 and a pair of maps

$$A_0 \xrightarrow{h} A_1 \xrightarrow{d} A_0$$

such that the diagram

$$\begin{tikzcd} A_1 \arrow[r, "d"'] \arrow[rr, dashed, bend left, "f"] & A_0 \arrow[r, "h"'] \arrow[rr, dashed, bend left, "f"] & A_1 \arrow[r, "d"'] & A_0 \end{tikzcd}$$

commutes or, equivalently:

$$dh = f \cdot \mathrm{Id}_{A_0} \quad \text{and} \quad hd = f \cdot \mathrm{Id}_{A_1} . \tag{1.1}$$

If f is a non-zerodivisor, S is local with residue field k, and both d and h are *minimal* maps in the sense that $d \otimes k = 0 = h \otimes k$, then the matrix factorization allows us to describe the minimal free resolutions of the module $M := \mathrm{Coker}(d)$

over the rings S and $R := S/(f)$; they are:

$$
\begin{aligned}
&0 \longrightarrow A_1 \xrightarrow{d} A_0 \longrightarrow M \longrightarrow 0 \ \text{ over } S\text{; and} \\
&\cdots \xrightarrow{R\otimes d} R \otimes A_0 \xrightarrow{R\otimes h} R \otimes A_1 \xrightarrow{R\otimes d} R \otimes A_0 \longrightarrow M \longrightarrow 0 \ \text{ over } R.
\end{aligned}
\tag{1.2}
$$

By [25], minimal free resolutions of all sufficiently high syzygies over a hypersurface ring have this form.

Matrix Factorizations of a Sequence of Elements

Definition 1.2.2 Let $f_1, \dots, f_c \in S$ be a sequence of elements of a commutative ring. A *higher matrix factorization* (d, h) *with respect to* $f_1, \dots, f_c$ is:

(1) A pair of finitely generated free S-modules A_0, A_1 with filtrations

$$0 \subseteq A_s(1) \subseteq \cdots \subseteq A_s(c) = A_s, \quad \text{for } s = 0, 1,$$

such that each $A_s(p-1)$ is a free summand of $A_s(p)$;

(2) A pair of maps $d,\ h$ preserving filtrations,

$$\bigoplus_{q=1}^{c} A_0(q) \xrightarrow{h} A_1 \xrightarrow{d} A_0,$$

where we regard $\oplus_q A_0(q)$ as filtered by the submodules $\oplus_{q\le p} A_0(q)$;

such that, writing

$$A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{d_p} A_0(p)$$

for the induced maps, the diagrams

$$
\begin{tikzcd}
A_1(p) \arrow[r, "d_p"'] \arrow[d] & A_0(p) \arrow[r, "h_p"'] \arrow[rr, bend left, dashed, "f_p"] & A_1(p) \arrow[r, "d_p"'] \arrow[d] & A_0(p) \\
A_1(p)\,/\,A_1(p-1) \arrow[rr, bend left, dashed, "f_p"] & & A_1(p)\,/\,A_1(p-1) &
\end{tikzcd}
$$

commute modulo $(f_1,\dots,f_{p-1})$ for all p; or, equivalently,

(a) $d_p h_p \equiv f_p \operatorname{Id}_{A_0(p)} \bmod (f_1,\dots,f_{p-1})A_0(p)$;
(b) $\pi_p h_p d_p \equiv f_p \pi_p \bmod (f_1,\dots,f_{p-1})\big(A_1(p)/A_1(p-1)\big)$, where π_p denotes the projection $A_1(p) \longrightarrow A_1(p)/A_1(p-1)$.

Set $R = S/(f_1,\dots,f_c)$. We define *the module of the higher matrix factorization* (d,h) to be

$$M := \operatorname{Coker}(R \otimes d)\,.$$

We refer to modules of this form as *higher matrix factorization modules* or *HMF modules*.

If S is local, then we call the higher matrix factorization *minimal* if d and h are minimal maps (that is, the image of each map is contained in the maximal ideal times the target).

In Sect. 8.1, we show that a homomorphism of HMF modules induces a morphism of the whole higher matrix factorization structure; see Definition 8.1.1 and Theorem 8.1.2 for details.

For each $1 \le p \le c$, we have a higher matrix factorization $(d_p, (h_1|\cdots|h_p))$ with respect to $f_1,\dots,f_p$, where $(h_1|\cdots|h_p))$ denotes the concatenation of the matrices $h_1,\dots,h_p$ and thus an HMF module

$$M(p) = \operatorname{Coker}(S/(f_1,\dots,f_p) \otimes d_p)\,.$$

This allows us to do induction on p. We will show in Theorem 7.4.1 that if S is Gorenstein then the modules $M(p)$ arise as the essential Cohen-Macaulay approximations of M over the rings $R(p) = S/(f_1,\dots,f_p)$, and on the other hand they arise as syzygies over the rings $R(p)$ of a single R-module.

Definition 1.2.3 Let (d,h) be a higher matrix factorization. Use the notation in Definition 1.2.2 and choose splittings so that

$$A_s(p) = \bigoplus_{q=1}^{p} B_0(q)$$

for all p and $s = 0, 1$. We say that (d,h) is a *strong* matrix factorization if it satisfies

$$\text{(a}'\text{)} \qquad d_p h_p \equiv f_p \operatorname{Id}_{A_0(p)} \bmod \Bigg(\sum_{1\le r<q\le p} f_r B_0(q) \Bigg),$$

which is a stronger condition than condition (a). We shall see in Sect. 5.3 that this property holds if and only if it is possible to extend h to a homotopy on the S-free resolution of the HMF module M constructed in Chap. 3. We will show in Theorem 5.3.1 that if (d,h) is a higher matrix factorization for a regular sequence

$f_1, \ldots, f_c$, then there exists a strong matrix factorization (d, g) with the same HMF module $M = \mathrm{Coker}(R \otimes d)$.

Example 1.2.4 Let $S = k[a, b, x, y]$ over a field k, and consider the complete intersection $R = S/(xa, yb)$. Let $N = R/(x, y)$. The module N is a maximal Cohen-Macaulay R-module. The earliest syzygy of N that is an HMF module is the third syzygy M. We can describe the higher matrix factorization for M as follows. After choosing a splitting

$$A_s(2) = A_s(1) \oplus B_s(2)\,,$$

we can represent the map d as

$$\begin{tikzcd}
A_1(1) = B_1(1) = S^2 \arrow[rr, "\begin{pmatrix} a & 0 \\ y & x \end{pmatrix}"] & & A_0(1) = B_0(1) = S^2 \\
\oplus & & \oplus \\
B_1(2) = S^2 \arrow[rr, "\begin{pmatrix} y & x \end{pmatrix}"] \arrow[uurr, "\begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}"] & & B_0(2) = S\,.
\end{tikzcd}$$

The pair of maps

$$d_1 : A_1(1) \xrightarrow{\begin{pmatrix} a & 0 \\ y & x \end{pmatrix}} A_0(1) \quad \text{and} \quad h_1 : A_0(1) \xrightarrow{\begin{pmatrix} x & 0 \\ -y & a \end{pmatrix}} A_1(1)$$

forms a matrix factorization for the element xa since $d_1 h_1 = h_1 d_1 = xa\,\mathrm{Id}$. The maps

$$h_2 : A_0 = A_0(2) \longrightarrow A_1 = A_1(2)$$

and

$$d_2 : A_1 = A_1(2) \longrightarrow A_0 = A_0(2)$$

are given by the matrices

$$h_2 = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ x & 0 & b \\ -y & a & 0 \end{pmatrix}\,,$$

and

$$d_2 = \begin{pmatrix} a & 0 & 0 & -b \\ y & x & 0 & 0 \\ 0 & 0 & y & x \end{pmatrix}.$$

Hence

$$d_2h_2 = \begin{pmatrix} yb & 0 & 0 \\ 0 & yb & 0 \\ 0 & xa & yb \end{pmatrix} \quad \text{and} \quad h_2d_2 = \left(\begin{array}{cccc} yb & xb & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hdashline xa & 0 & yb & 0 \\ 0 & xa & 0 & yb \end{array}\right).$$

Thus d_2h_2 is congruent, modulo (xa), to $yb\,\mathrm{Id}$. Furthermore, condition (b) of Definition 1.2.2 is the statement that the two bottom rows in the latter matrix are congruent modulo (xa) to $yb\pi_2$. In the context of the diagram in the definition, with $p = 2$, the fact that the lower left (2×2)-matrix is congruent to 0 modulo $f_1 = xa$ is necessary for the map $h_2d_2 : A_1(2) \longrightarrow A_1(2)$ to induce a map

$$A_1(2)/A_1(1) = B_1(2) \longrightarrow A_1(2)/A_1(1) = B_1(2)\,.$$

1.3 What's in This Book?

Extending the case of a single element, if $f_1, \dots, f_c$ is a regular sequence, S is local with residue field k, and both d and h are *minimal* maps, we will show that a higher matrix factorization allows us to describe the *minimal* free resolutions of the HMF module $M := \mathrm{Coker}(d)$ over the rings S and $R := S/(f_1, \dots, f_c)$. Moreover, we will prove that if S is regular, then every sufficiently high syzygy over the complete intersection $R = S/(f_1, \dots, f_c)$ is an HMF module. In the rest of this section we describe our main results more precisely.

We focus on the case when S is a regular local ring and $R = S/(f_1, \dots, f_c)$ is a complete intersection, although most of our results will be proven in greater generality. We will keep the notation of Definition 1.2.2 throughout.

High Syzygies are Higher Matrix Factorization Modules

The next result was the key motivation for our definition of a higher matrix factorization. A more precise version of this result is proved in Corollary 6.4.3.

Theorem 1.3.1 *Let S be a regular local ring with infinite residue field, and let $I \subset S$ be an ideal generated by a regular sequence of length c. Set $R = S/I$, and suppose* *that N is a finitely generated R-module. Let $f_1, \dots, f_c$ be a generic choice of elements minimally generating I. If M is a sufficiently high syzygy of N over R, then M is the HMF module of a minimal higher matrix factorization (d, h) with respect to $f_1, \dots, f_c$. Moreover $d \otimes R$ and $h \otimes R$ are the first two differentials in the minimal free resolution of M over R.*

The meaning of "a sufficiently high syzygy" is explained in Sect. 6.1, where we introduce a class of R-modules that we call *pre-stable syzygies* and show that they have the property given in Theorem 1.3.1. Given an R-module N we give in Corollary 6.4.3 a sufficient condition, in terms of $\operatorname{Ext}_R(N, k)$, for the r-th syzygy module of N to be pre-stable. We also explain more about the genericity condition. Over a local Gorenstein ring, we introduce the concept of a stable syzygy in Sect. 6.1 and discuss it in Sect. 7.2.

Minimal R-Free and S-Free Resolutions

Theorem 1.3.1 shows that in order to understand the eventual behavior of minimal free resolutions over the complete intersection R it suffices to construct the minimal free resolutions of HMF modules. This is accomplished by Construction 5.1.1 and Theorem 5.1.2.

The finite minimal free resolution over S of an HMF module is given by Construction 3.1.3 and Theorem 3.1.4. Here is an outline of the codimension 2 case: Let (d, h) be a codimension 2 higher matrix factorization. We first choose splittings

$$A_s(2) = B_s(1) \oplus B_s(2)\,.$$

Since $d(B_1(1)) \subset B_0(1)$, we can represent the differential d as

$$\begin{tikzcd}
\mathbf{B}(1): & B_1(1) \arrow[rr, "b_1"] & & B_0(1) \\
& \oplus & & \oplus \\
\mathbf{B}(2): & B_1(2) \arrow[rr, "b_2"'] \arrow[rruu, "\psi_2"] & & B_0(2)\,,
\end{tikzcd}$$

which may be thought of as a map of two-term complexes

$$\psi_2 : \mathbf{B}(2)[-1] \longrightarrow \mathbf{B}(1)\,.$$

This extends to a map of complexes

$$\mathbf{K}(f_1) \otimes \mathbf{B}(2)[-1] \longrightarrow \mathbf{B}(1)\,,$$

as in the following diagram:

$$
\begin{tikzcd}
& B_1(1) \arrow[rr, "b_1"] & & B_0(1) \\
& \oplus & & \oplus \\
& B_1(2) \arrow[rr, "b_2"] \arrow[urru, "\psi_2"] & & B_0(2) \\
& \oplus & & \\
B_1(2) \arrow[r, "b_2"'] \arrow[uuuur, "h_1\psi_2"] \arrow[uur, "-f_1"'] & B_0(2) \arrow[uurr, "f_1"'] & &
\end{tikzcd}
$$

Theorem 3.1.4 asserts that this is the minimal S-free resolution of the HMF module $M = \mathrm{Coker}(S/(f_1,f_2) \otimes d)$.

Strong restrictions on the finite minimal S-free resolution of a high syzygy M over the complete intersection $S/(f_1,\ldots,f_c)$ follow from our description: for example, by Corollary 3.2.4 the minimal presentation matrix of M must include $c-1$ columns of the form

$$
\begin{pmatrix}
f_1 & \cdots & f_{c-1} \\
0 & \cdots & 0 \\
\vdots & & \vdots \\
0 & \cdots & 0
\end{pmatrix}
$$

for a generic choice of $f_1,\ldots,f_c$. For instance, in Example 1.2.4, where $c = 2$, the presentation matrix of M is

$$
\begin{pmatrix}
a & 0 & 0 & -b & 0 \\
y & x & 0 & 0 & 0 \\
0 & 0 & y & x & xa
\end{pmatrix},
$$

and the last column is of the desired type. There are numerical restrictions as well; see Corollary 6.5.3 and the remark following it.

Remark 1.3.2 Every maximal Cohen-Macaulay $S/(f_1)$-module is a pre-stable syzygy, but this is not true in higher codimension—one must go further back in the syzygy chain. This is not surprising, since *every* S-module of finite length is a maximal Cohen-Macaulay module over an artinian complete intersection, and it seems hopeless to characterize the minimal free resolutions of all such modules.

In Corollaries 3.2.6 and 5.2.1 we get formulas for the Betti numbers of an HMF module over S and over R respectively. Furthermore, the graded vector spaces

$\mathrm{Ext}_S(M,k) := \oplus_i \mathrm{Ext}_S(M,k)$ and $\mathrm{Ext}_R(M,k) := \oplus_i \mathrm{Ext}_R(M,k)$ can be expressed as follows:

Corollary 1.3.3 *Suppose that $f_1,\ldots,f_c$ is a regular sequence in a regular local ring S with infinite residue field k, so that $R = S/(f_1,\ldots,f_c)$ is a local complete intersection. Let M be the HMF module of a minimal higher matrix factorization (d,h) with respect to $f_1,\ldots,f_c$. Using notation as in Definition 1.2.2, choose splittings $A_s(p) = A_s(p-1) \oplus B_s(p)$ for $s = 0,1$ and $p = 1,\ldots,c$, so that*

$$A_s(p) = \oplus_{1\le q\le p} B_s(q)\,.$$

Set $B(p) = B_1(p) \oplus B_0(p)$, where we think of $B_s(p)$ as placed in homological degree s. There are decompositions

$$\mathrm{Ext}_S(M,k) \cong \bigoplus_{p=1}^{c} k\langle e_1,\ldots,e_{p-1}\rangle \otimes \mathrm{Hom}_S(B(p),k)$$

$$\mathrm{Ext}_R(M,k) \cong \bigoplus_{p=1}^{c} k[\chi_p,\ldots,\chi_c] \otimes \mathrm{Hom}_S(B(p),k),$$

as vector spaces, where $k\langle e_1,\ldots,e_{p-1}\rangle$ denotes the exterior algebra on variables of degree 1 and $k[\chi_p,\ldots,\chi_c]$ denotes the polynomial ring on variables of degree 2.

The former formula in Corollary 1.3.3 follows from Remark 3.1.5 and the latter from Corollary 5.1.6. We explain in [28] and Corollary 5.1.6 how the given decompositions reflect certain natural actions of the exterior and symmetric algebras on the graded modules $\mathrm{Ext}_S(M,k)$ and $\mathrm{Ext}_R(M,k)$.

The package CompleteIntersectionResolutions, available in the Macaulay2 system starting with version 1.8, can compute examples of many of the constructions in this book.

Syzygies over Intermediate Quotient Rings

For each $0 \le p \le c$ set

$$R(p) := S/(f_1,\ldots,f_p)\,.$$

In the case of a codimension 1 matrix factorization (d,h), one can use the data of the matrix factorization to describe two minimal free resolutions, as explained in (1.2). In the case of a codimension c higher matrix factorization we construct the minimal free resolutions of its HMF module over all $c+1$ rings

$$S = R(0),\ S/(f_1) = R(1),\ \ldots,\ S/(f_1,\ldots,f_c) = R(c)\,.$$

See Theorem 5.4.4.

By Definition 1.2.2 an HMF module M with respect to the regular sequence $f_1, \ldots, f_c$ determines, for each $p \leq c$, an HMF $R(p)$-module $M(p)$ with respect to $f_1, \ldots, f_p$. In the notation and hypotheses as in Theorem 1.3.1, we have the following properties of the modules $M(p)$: Proposition 7.2.2 shows that

$$M(p-1) = \mathrm{Syz}_2^{R(p-1)}\Big(\mathrm{Syz}_{-2}^{R(p)}\big(M(p)\big)\Big),$$

where $\mathrm{Syz}_i(-)$ and $\mathrm{Syz}_{-i}(-)$ denote syzygy and cosyzygy, respectively. Theorem 7.4.1(2) expresses the modules $M(p)$ as syzygies of the R-module $P := \mathrm{Syz}_{-c-1}^R(M)$ over the intermediate rings $R(p)$ as

$$M(p) = \mathrm{Syz}_{c+1}^{R(p)}(P).$$

Furthermore, Proposition 7.4.2 says that if we replace M by its first syzygy, then all the modules $M(p)$ are replaced by their first syzygies:

$$\big(\mathrm{Syz}_1^R(M)\big)(p) = \mathrm{Syz}_1^{R(p)}\big(M(p)\big).$$

1.4 Notation and Conventions

Unless otherwise stated, **all rings are assumed commutative and Noetherian, and all modules are assumed finitely generated**.

A map $\phi : A \longrightarrow B$ of S-modules is called *minimal* if S is local and $\phi(A) \subset \mathbf{m}B$, where $\mathbf{m}$ is the maximal ideal of S.

To distinguish a matrix factorization for one element from the general concept, sometimes we will refer to the former as a *codimension* 1 *matrix factorization* or a *hypersurface matrix factorization*.

We will frequently use the following notation about higher matrix factorizations.

Notation 1.4.1 A higher matrix factorization

$$\Big(d:\ A_1 \longrightarrow A_0,\ h : \oplus_{p=1}^{c} A_0(p) \longrightarrow A_1\Big)$$

with respect to $f_1, \ldots, f_c$ as in Definition 1.2.2 involves the following data:

- a ring S over which A_0 and A_1 are free modules;
- for $1 \leq p \leq c$, the rings $R(p) := S/(f_1, \ldots, f_p)$, and in particular $R = R(c)$;
- for $s = 0, 1$, the filtrations

$$0 = A_s(0) \subseteq \cdots \subseteq A_s(c) = A_s,$$

  preserved by d;

- the induced maps

$$A_0(p) \xrightarrow{h_p} A_1(p) \xrightarrow{d_p} A_0(p);$$

- the quotients $B_s(p) = A_s(p)/A_s(p-1)$ and the projections

$$\pi_p : A_1(p) \longrightarrow B_1(p) ;$$

- the two-term complexes induced by d:

$$\mathbf{A}(p) : \ A_1(p) \xrightarrow{d_p} A_0(p)$$

$$\mathbf{B}(p) : \ B_1(p) \xrightarrow{b_p} B_0(p)$$

- the modules

$$M(p) = \operatorname{Coker}\Big(R(p) \otimes d_p : \ R(p) \otimes A_1(p) \longrightarrow R(p) \otimes A_0(p)\Big),$$

  and in particular, the HMF module $M = M(c)$ of (d, h).

We sometimes write $h = (h_1|\cdots|h_c)$. We say that the higher matrix factorization is *trivial* if $A_1 = A_0 = 0$.

If $1 \le p \le c$ then d_p together with the maps h_q for $q \le p$, is a higher matrix factorization with respect to $f_1, \ldots, f_p$; we write it as $(d_p, h(p))$, where $h(p) = (h_1|\cdots|h_p)$. We call (d_1, h_1) the *codimension* 1 *part* of the higher matrix factorization; (d_1, h_1) is a hypersurface matrix factorization for f_1 over S (it could be trivial). If $q \ge 1$ is the smallest number such that $A(q) \ne 0$ and $R' = S/(f_1, \ldots, f_{q-1})$, then writing $-'$ for $R' \otimes -$, the maps

$$b'_q : \ B_1(q)' \longrightarrow B_0(q)' \ \text{ and } \ h'_q : \ B_0(q)' \longrightarrow B_1(q)'$$

form a hypersurface matrix factorization for the element $f_q \in R'$. We call it the *top non-zero part* of the higher matrix factorization (d, h).

For each $0 \le p \le c$ set $R(p) := S/(f_1, \ldots, f_p)$. The HMF module

$$M(p) = \operatorname{Coker}(R(p) \otimes d_p)$$

is an $R(p)$-module.

Next, we make some conventions about complexes which we use throughout.

Conventions on Complexes 1.4.2 We write $\mathbf{U}[-a]$ for the *shifted complex* with

$$\mathbf{U}[-a]_i = \mathbf{U}_{i+a}$$

and differential $(-1)^a d$.

Let $(\mathbf{W}, \partial^W)$ and $(\mathbf{Y}, \partial^Y)$ be complexes. The complex $\mathbf{W} \otimes \mathbf{Y}$ has differential

$$\partial_q^{W \otimes Y} = \sum_{i+j=q} \left((-1)^j \partial_i^W \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_j^Y \right).$$

If $\varphi : \mathbf{W}[-1] \longrightarrow \mathbf{Y}$ is a map of complexes, so that $-\varphi \partial^W = \partial^Y \varphi$, then the *mapping cone* $\mathbf{Cone}(\varphi)$ is the complex $\mathbf{Cone}(\varphi) = \mathbf{Y} \oplus \mathbf{W}$ with modules

$$\mathbf{Cone}(\varphi)_i = Y_i \oplus W_i$$

and differential

$$\begin{array}{cc} & \begin{array}{cc} Y_i & W_i \end{array} \\ \begin{array}{c} Y_{i-1} \\ W_{i-1} \end{array} & \begin{pmatrix} \partial_i^Y & \varphi_{i-1} \\ 0 & \partial_i^W \end{pmatrix} \end{array}.$$

A map of complexes of free modules $\gamma : \mathbf{W}[a] \longrightarrow \mathbf{Y}$ is homotopic to 0 if there exists a map $\alpha : \mathbf{W}[a+1] \longrightarrow \mathbf{Y}$ such that

$$\gamma = \partial^{\mathbf{Y}} \alpha - \alpha \partial^{\mathbf{W}[a+1]} = \partial^{\mathbf{Y}} \alpha - (-1)^{a+1} \alpha \partial^{\mathbf{W}}.$$

We say that a complex $(\mathbf{U}, d)$ is a *left complex* if $U_j = 0$ for $j < 0$; thus for example the free resolution of a module is a left complex.

If f is an element in a ring S then we write $\mathbf{K}(f)$ for the two-term Koszul complex $f : eS \longrightarrow S$, where we think of e as an exterior variable. If $(\mathbf{W}, \partial)$ is any complex of S-modules we write

$$\mathbf{K}(f) \otimes \mathbf{W} = e\mathbf{W} \oplus \mathbf{W};$$

it is the mapping cone of the map $\mathbf{W} \longrightarrow \mathbf{W}$ that is $(-1)^i f : W_i \longrightarrow W_i$.

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