Chapter 2
A Few Basic Concepts

For simplicity, we frame our discussions in terms of unweighted, undirected networks. When such a network is time-independent, it can be represented using a symmetric adjacency matrix $\mathbf{A} = \mathbf{A}^T$ with elements $A_{ij} = A_{ji}$ that are equal to 1 if nodes i and j are connected (or, more properly, "adjacent") and 0 if they are not. We also assume that $A_{ii} = 0$ for all i, so none of our networks include self-edges.[1] We denote the total number of nodes in a network (i.e., a network's "size") by N. The *degree* k_i of node i is the number of edges that are connected to it. For a large network, it is common to examine the distribution of degrees over all of its nodes. The *degree distribution* P_k is defined as the probability that a node—chosen uniformly at random from the set of all nodes—has degree k, and the *degree sequence* is the set of all node degrees (including multiplicities). The *mean degree* z is the mean number of edges per node and is given by $z = \sum_k kP_k$. For example, classical Erdős–Rényi (ER) random graphs have a Poisson degree distribution, $P_k = \frac{z^k e^{-z}}{k!}$, in the $N \to \infty$ limit.[2] However, many real-world networks have right-skewed (i.e., *heavy-tailed*) degree distributions [55], so the mean degree z only provides minimal information about the structure of a network. The most popular type of heavy-tailed distribution is a *power law* [295], for which $P_k \sim k^{-\gamma}$ as $k \to \infty$ (where the parameter γ is called the "power" or "exponent"). Networks with a power-law degree distribution are often called "scale-free networks" (though such networks can still have scales in them, so the monicker is misleading), and many generative mechanisms—such as de Solla Price's model [68] and the Barabási–Albert (BA) model [16]—produce networks with power-law degree distributions.

[1]This is a standard assumption, but it is not always desirable. For example, one may wish to investigate narcissism in people tagging themselves in pictures on Facebook, a set of coupled oscillators can include self-interactions, and so on.

[2]By analogy with statistical physics, the $N \to \infty$ limit is often called a "thermodynamic limit."

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When studying dynamical processes on networks, it can be very insightful to construct networks using convenient random-graph ensembles (i.e., probability distributions on graphs), including both "realistic" ones and patently unrealistic ones.[3] The effects of network structure on dynamics are often studied using a random-graph ensemble known as the *configuration model* [34, 228]. In this ensemble, one specifies the degree distribution P_k (or the degree sequence), but the network *stubs* (i.e., ends of edges) are then connected to each other uniformly at random. In the limit of infinite network size, one expects a network drawn from a configuration-model ensemble to have vanishingly small *degree–degree correlations* and *local clustering*.[4] It is also important to consider computational implementations (and possible associated biases) of the configuration model and its generalizations [21]. Moreover, note that there exist multiple variants of the configuration model.

Degree–degree correlation measures the (Pearson) correlation between the degrees of nodes at each end of a randomly chosen edge of a network. (The edge is chosen uniformly at random from the set of edges.) Degree–degree correlation can be significant, for example, if high-degree nodes are connected preferentially to other high-degree nodes. This is true in a social network if popular people tend to be friends with other popular people, and one would describe the network as "homophilous" by degree. By contrast, a network for which high-degree nodes are connected preferentially to low-degree nodes is "heterophilous" by degree.

The simplest type of local clustering arises as a result of a preponderance of triangle motifs in a network. (More complicated types of clustering—which need not be local—include motifs with more than three nodes, community structure, and core–periphery structure [64, 228, 259].) Triangles are common, for example, in social networks, so the lack of local clustering in configuration-model networks (in the $N \to \infty$ limit) is an important respect in which their structure differs significantly from that in most real networks. Investigations of dynamical systems on networks with different types of clustering is a focus of current research [129, 213, 216].

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[3]Reference [212] gives one illustration of how considering a very unrealistic random-graph ensemble can be crucial for developing understanding of the behavior of a dynamical process on networks.

[4]Strictly speaking, one also needs to ensure appropriate conditions on the moments of P_k as $N \to \infty$. For example, one could demand that the second moment remains finite as $N \to \infty$.

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