Chapter 5
Hartshorne Conjectures and Severi Varieties

5.1 Hartshorne Conjectures

Around the beginning of the 1970s there was a renewed interest in solving concrete problems and in finding applications of the new theories developed during the refoundation of Algebraic Geometry performed during the 1950s by Weil and Zariski and which culminated during the 1960s in the amazing contributions of Serre and of Grothendieck together with his school. To know the state of the art one can consult the beautiful book of Robin Hartshorne, [86]. In [86] several open problems were solved and many outstanding questions were discussed such as the set-theoretic complete intersection of curves in $\mathbb{P}^3$ (still open), the characterization of $\mathbb{P}^N$ among the smooth varieties with ample tangent bundle, which was solved by Mori in [136] and which cleared the path to the foundation of Mori theory in [137]. In related fields we only mention Deligne's proof of the Weil conjectures or later Faltings' proof of the Mordell conjecture.

The interplay between topology and algebraic geometry returned to flourish as seen in Chap. 3. Lefschetz's Theorem and the Barth–Larsen Theorem also suggested that smooth varieties whose codimension is small with respect to their dimension should satisfy very strong restrictions. To get a feeling for this we remark that a codimension two smooth complex subvariety of $\mathbb{P}^N$, $N \geq 5$, has to be simply connected. If $N \geq 6$, the transversal intersections of two hypersurfaces in $\mathbb{P}^N$ are the only known examples of smooth irreducible projective varieties of codimension 2.

Let us recall the definition of complete intersections and some of their notable properties.

Definition 5.1.1 (Complete Intersection) An equidimensional variety of dimension $n \geq 1$,

$$X^n \subset \mathbb{P}^N$$

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is a *complete intersection* if there exist $N - n = \text{codim}(X)$ homogeneous polynomials

$$f_i \in K[X_0, \ldots, X_N]_{d_i}$$

of degree $d_i \geq 1$, generating the homogeneous ideal $I(X) \subset K[X_0, \ldots, X_N]$, that is if

$$I(X) = < f_1, \ldots, f_{N-n} > .$$

Let us recall that since $f_1, \ldots, f_{N-n}$ form a regular sequence in $K[X_0, \ldots, X_N]$, the homogeneous coordinate ring

$$S(X) = \frac{K[X_0, \ldots, X_N]}{I(X)}$$

has depth $n + 1$ so that $X^n \subset \mathbb{P}^N$ is an arithmetically Cohen–Macaulay variety by definition.

Thus a complete intersection $X^n \subset \mathbb{P}^N$ is connected if $n > 0$ and moreover $H^i(\mathcal{O}_X(m)) = 0$ for every i such that $0 < i < n$ and for every $m \in \mathbb{Z}$. Furthermore, a complete intersection $X \subset \mathbb{P}^N$ is projectively normal, meaning that the restriction morphisms

$$H^0(\mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(\mathcal{O}_X(m))$$

are surjective for every $m \geq 0$.

Moreover, by the so-called Grothendieck–Lefschetz Theorem on the Picard group of Complete Intersections we have

$$\text{Pic}(X) \simeq \mathbb{Z} < \mathcal{O}_X(1) >,$$

as soon as $n \geq 3$, see [86, Chap. IV, Corollary 3.2]. This result was proved by Severi for smooth hypersurfaces of dimension $n \geq 3$ in [177] and extended to arbitrary smooth complete intersections by Fano in [58]. Notwithstanding, the previous statement is usually attributed to Lefschetz who stated and proved it in [126], see the historical note in [86, Chap. IV, Sect. 4].

By Lefschetz's Theorem on Hyperplane sections, see Chap. 3, complete intersections defined over $K = \mathbb{C}$ are simply connected, as soon as $n \geq 2$, and have the same cohomology $H^i(X, \mathbb{Z})$ of the projective spaces containing them for $i < n$, see [86].

An equidimensional variety $X^n \subset \mathbb{P}^N$ is a *scheme-theoretic intersection* of $m \geq N - n = \text{codim}(X)$ hypersurfaces $H_i = V(g_i) \subset \mathbb{P}^N$, $g_i \in K[X_0, \ldots, X_N]_{d_i}$ homogeneous of degree $d_i \geq 1$, if, as schemes,

$$X = V(g_1, \ldots, g_m).$$

This means that the ideal sheaf $\mathscr{I}_X \subset \mathscr{O}_{\mathbb{P}^N}$ is locally generated by the $m \geq \operatorname{codim}(X) = N - n$ elements $g_1, \ldots, g_m$. Equivalently, one can say that $X^n \subset \mathbb{P}^N$ is equal to $H_1 \cap \ldots \cap H_m$ and that at each point $p \in X$

$$T_pX = T_pH_1 \cap \ldots \cap T_pH_m$$

holds.

An equidimensional variety $X^n \subset \mathbb{P}^N$ is a *scheme-theoretic complete intersection* if it is the scheme-theoretic intersection of $N - n$ hypersurfaces.

One immediately sees, for example via the Koszul complex, that from the algebraic point of view the last condition is equivalent to the fact that the homogeneous ideal $I(X) \subset K[X_0, \ldots, X_N]$ is generated precisely by $\operatorname{codim}(X) = N - n$ homogeneous elements. Thus a projective variety is a complete intersection if and only if it is a scheme-theoretic complete intersection, a useful observation used repeatedly later in this chapter.

On the basis of some empirical observations, inspired by the Theorem of Barth and Larsen but, according to Fulton and Lazarsfeld, also "*on the basis of few examples*", Hartshorne was led to formulate the following conjectures.

Before stating them we wish to quote from the Introduction of the book [113]:

> *Algebraic geometry is a mixture of the ideas of two Mediterrean cultures. It is the superposition of the Arab science of the lightning calculation of the solutions of equations over the Greek art of position and shape. This tapestry was originally woven by on European soil and is still being refined under the influence of international fashion.*
> *Algebraic geometry studies the delicate balance between the geometrically plausible and the algebraic possible. Whenever one side of this mathematical teeter–tooter outweighs the other, one immediately loses interest and runs off in search of a more exciting amusement.*
> George R. Kempf

Hartshorne's Complete Intersection Conjecture, [87]:

Let $X^n \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety of dimension $n = \dim(X) \geq 1$.

If $N < \frac{3}{2}\dim(X)$, or equivalently if $\operatorname{codim}(X) < \frac{1}{2}\dim(X)$, then X is a complete intersection.

Let us quote R. Hartshorne, see [87]:

> *While I am not convinced of the truth of this statement, I think it is useful to crystallize one's idea, and to have a particular problem in mind.*

Hartshorne immediately remarks that the conjecture is sharp, due to the examples of $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ and of the spinorial variety $S^{10} \subset \mathbb{P}^{15}$, see Examples 4.2.5 and 4.2.6 for the definition/construction of these varieties.

Moreover, the examples of cones over curves in $\mathbb{P}^3$, not complete intersections, reveal the necessity of the non-singularity assumption.

Definition 5.1.2 (Hartshorne Variety) A smooth irreducible projective variety $X^n \subset \mathbb{P}^{\frac{3n}{2}}$ is called a *Hartshorne variety* if it is not a complete intersection.

It is not accidental that the previous examples of Hartshorne varieties are homogeneous. Indeed, a useful technique for constructing varieties of not too high codimension is the theory of algebraic groups, see for example [198, Chap. 3] or the appendix to [125].

One of the main difficulties of Hartshorne's Conjecture is a good translation in geometrical terms of the algebraic condition of being a complete intersection and more generally of dealing with the equations defining a variety.

Here is not the place to stress how many important results originated and still arise today from this open problem in the areas of vector bundles on projective space, of the study of defining equations of a variety, of k-normality and so on. The list of these achievements is so long that we prefer to avoid citations, being confident that everyone has met at some time a problem or a result related to this conjecture.

For a long time the Hartshorne Conjecture has been forgotten and today continues to be out of the interests of the main research groups in Algebraic Geometry around the world, although it remains almost as open as before.

We recall that a quadratic manifold is a non-degenerate manifold $X^n \subset \mathbb{P}^N$ which is the scheme-theoretic intersection of $m \geq N - n$ quadratic hypersurfaces. Let us state our contributions in the subject, whose proofs are postponed in Sect. 5.2.

Theorem 5.1.3 (Hartshorne's Conjecture for Quadratic Manifolds, [105, Theorem 3.8, part (4)]) *Let $X^n \subset \mathbb{P}^N$ be a quadratic manifold. If* $\operatorname{codim}(X) < \frac{\dim(X)}{2}$, *then X is a complete intersection.*

Theorem 5.1.4 (Classification of Quadratic Hartshorne Varieties, [105, Theorem 3.9]) *Let $X^n \subset \mathbb{P}^{\frac{3n}{2}}$ be a quadratic manifold. Then either $X \subset \mathbb{P}^N$ is a complete intersection or it is projectively equivalent to one of the following:*

1. $\mathbb{G}(1,4) \subset \mathbb{P}^9$;
2. $S^{10} \subset \mathbb{P}^{15}$.

In the same survey paper Hartshorne posed another conjecture, motivated by the remark that complete intersections are linearly normal and by the behavior of some examples in low dimension.

Let us recall that linearly normal varieties have been introduced in Definition 3.4.3 as those varieties $X \subset \mathbb{P}^N$ which are not isomorphic external projections of a variety $Y \subset \mathbb{P}^M$ with $M > N$.

Conjecture 5.1.5 (Hartshorne's Linear Normality Conjecture, [87]) Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate projective variety.

$$\text{If } N < \frac{3}{2}\dim(X) + 1, \text{ i.e. if } \operatorname{codim}(X) < \frac{1}{2}\dim(X) + 1, \text{ then } X \text{ is linearly normal.}$$

By Proposition 1.3.5 and by the discussion after Definition 3.4.3 relating linear normality and isomorphic linear projections, we can equivalently reformulate this conjecture via secant varieties. Indeed, putting "$N = N + 1$", we get the following equivalent formulation of the Linear Normality Conjecture:

$$\text{If } N < \frac{3}{2}\dim(X) + 2, \text{ then } SX = \mathbb{P}^N.$$

We quote Hartshorne to understand his point of view on this second problem:

Of course in settling this conjecture, it would be nice also to classify all nonlinearly normal varieties with $N = \frac{3n}{2} + 1$*, so as to have a satisfactory generalization of Severi's theorem. As noted above, a complete intersection is always linearly normal, so this conjecture would be a consequence of our original conjecture, except for the case* $N = \frac{3n}{2}$*. My feeling is that this conjecture should be easier to establish than the original one.*

R. Hartshorne in [87].

Once again the bound is sharp as shown by the projected Veronese surface in $\mathbb{P}^4$. The conjecture on linear normality was proved by Zak at the beginning of the 1980s as an immediate consequence of Terracini's Lemma and of Theorem 3.2.1. We now reproduce Zak's original proof.

Theorem 5.1.6 (Zak's Theorem on Linear Normality) *Let* $X^n \subset \mathbb{P}^N$ *be a smooth non-degenerate projective variety. If* $N < \frac{3}{2}n + 2$*, then* $SX = \mathbb{P}^N$*. Or equivalently if* $SX \subsetneq \mathbb{P}^N$*, then* $\dim(SX) \geq \frac{3}{2}n + 1$ *and hence* $N \geq \frac{3}{2}n + 2$*.*

Proof Suppose $SX \subsetneq \mathbb{P}^N$. Then there exists a hyperplane H containing the general tangent space to SX, let us say $T_z SX$. Then by Corollary 1.4.7, the hyperplane H is tangent to X along $\Sigma_z(X)$, which by the generality of z has pure dimension

$$\delta(X) = 2n + 1 - \dim(SX).$$

Since $T(\Sigma_z(X), X) \subseteq H$, the non-degenerate variety $S(\Sigma_z(X), X) \supseteq X$ is not contained in H, yielding $T(\Sigma_z(X), X) \neq S(\Sigma_z(X), X)$. By Theorem 3.2.1 we get

$$2n + 1 - \dim(SX) + n + 1 = \dim(S(\Sigma_z(X), X)) \leq \dim(SX),$$

yielding

$$3n + 2 \leq 2\dim(SX)$$

and finally

$$N - 1 \geq \dim(SX) \geq \frac{3}{2}n + 1.$$

□

As outlined by Hartshorne, the previous result naturally leads to the notion of a Severi variety, first introduced by F.L. Zak in [198].

Definition 5.1.7 (Severi Variety) A smooth non-degenerate irreducible projective variety $X^n \subset \mathbb{P}^{\frac{3}{2}n+2}$ such that $SX \subsetneq \mathbb{P}^{\frac{3}{2}n+2}$ is called *a Severi variety*.

As we shall see in Corollary 5.4.2, Severi varieties are *LQEL*-manifolds of type $\delta = n/2$ so that their classification follows from Theorem 4.3.3. We shall come back to this in Sect. 5.4 below.

5.2 Proofs of Hartshorne's Conjecture for Quadratic Manifolds and of the Classification of Quadratic Hartshorne Manifolds

We introduce some results which will be crucial for the proof of Hartshorne's Conjecture for quadratic manifolds. First we recall some notation introduced in Sects. 2.3 and 2.3.3.

Let

$$X = V(f_1, \ldots, f_m) \subset \mathbb{P}^N \qquad (**)$$

be a projective manifold, let $x \in X$ be a general point, let $n = \dim(X)$ and let $c = \operatorname{codim}(X) = N - n$.

Thus (**) means precisely that $X \subset \mathbb{P}^N$ is scheme-theoretically the intersection of $m \geq 1$ hypersurfaces of degrees $d_1 \geq d_2 \geq \ldots \geq d_m \geq 2$, where $d_i = \deg(f_i)$. Moreover, it is implicitly assumed that m is minimal, i.e. none of the hypersurfaces contains the intersection of the others. Define, following [105], the integer

$$d := \min\{\sum_{i=1}^{c}(d_i - 1) \text{ for expressions } (**) \text{ as above}\} \geq c.$$

With these definitions $X \subset \mathbb{P}^N$ (or more generally a scheme $Z \subset \mathbb{P}^N$) is called *quadratic* if it is scheme-theoretically an intersection of quadrics, that is we can assume $d_1 = 2$ in (**). Equivalently, $X \subset \mathbb{P}^N$ is quadratic if and only if $d = c$.

5.2.1 The Bertram–Ein–Lazarsfeld Criterion for Complete Intersections

The following characterization of Complete Intersections is stated in a slightly different way in [19, Corollary 4].

Theorem 5.2.1 (Bertram–Ein–Lazarsfeld Criterion, [19, Corollary 4]) *Let $X^n \subset \mathbb{P}^N$ be a manifold of dimension $n \geq 1$ which is scheme-theoretically defined* *by hypersurfaces of degrees* $d_1 \geq d_2 \geq \ldots \geq d_m$. *Let* $c = N - n = \mathrm{codim}(X)$. *Then* $X \subset \mathbb{P}^N$ *is a complete intersection of type* $(d_1, \ldots, d_c)$ *if and only if*

$$\mathscr{O}(-K_X) = \mathscr{O}(N + 1 - \sum_{i=1}^{c} d_i). \tag{5.1}$$

Proof If $X^n \subset \mathbb{P}^N$ is a complete intersection of type $(d_1, \ldots, d_c)$, then (5.1) holds by the Adjunction Formula.

Suppose now that (5.1) holds. Then there exist $g_i \in H^0(\mathbb{P}^N, \mathscr{I}_X(d_i))$, $i = 1, \ldots, c$, such that, letting $Q_i = V(g_i) \subset \mathbb{P}^N$, we obtain the complete intersection scheme

$$Y = Q_1 \cap \ldots \cap Q_c = X \cup X',$$

where X' (if non-empty) is either disjoint from X or meets X in a divisor, see the proof of [19, Corollary 4] for further details.

Since $n \geq 1$ the scheme Y, being a complete intersection, is also connected so that to show that $Y = X$ it is enough to prove that $X \cap X' = \emptyset$. To this end, observe that the g_i's define a morphism

$$\bigoplus_{i=1}^{c} \mathscr{O}_{\mathbb{P}^N}(-d_i) \to \mathscr{I}_X$$

which restricted to X yields a morphism

$$\alpha : \bigoplus_{i=1}^{c} \mathscr{O}_X(-d_i) \to \frac{\mathscr{I}_X}{\mathscr{I}_X^2}$$

of locally free sheaves of rank c. Since the g_i's generically generate $\mathscr{I}_X$ outside a codimension one subset of X, the map α is an injective homomorphism of coherent sheaves. Furthermore, $\mathrm{coker}(\alpha)$ is supported on $X \cap X'$. Computing first Chern classes, it follows that $X \cap X'$ is supported on a divisor D such that

$$\mathscr{O}_X(D) \simeq \det(\frac{\mathscr{I}_X}{\mathscr{I}_X^2}) \otimes \mathscr{O}_X(\sum_{i=1}^{c} d_i) \simeq \mathscr{O}_X(\sum_{i=1}^{c} d_i - N - 1) \otimes \mathscr{O}(-K_X).$$

In conclusion, under the hypothesis (5.1), α is an isomorphism and $X \cap X' = \emptyset$. Thus $X \subset \mathbb{P}^N$ is the scheme-theoretic intersection of c hypersurfaces of degree $d_1, \ldots, d_c$ and hence X is the complete intersection of these hypersurfaces. $\square$

5.2.2 *Faltings' and Netsvetaev's Conditions for Complete Intersections*

We now prove an interesting result due to Faltings, following the ideas and presentation in [144].

Theorem 5.2.2 (Faltings' Condition, [57, Korollar zu Satz 3]) *Let $X^n \subset \mathbb{P}^N$ be a manifold scheme-theoretically defined by hypersurfaces of degrees $d_1 \geq d_2 \geq \ldots \geq d_m$. If $m \leq \frac{N}{2}$, then $X \subset \mathbb{P}^N$ is the complete intersection of c hypersurfaces among the m defining it scheme-theoretically.*

Proof For $i = 1, \ldots, m$ let

$$\pi_i : \bigoplus_{k=1}^{m} \mathscr{O}_X(d_k) \to \mathscr{O}_X(d_i)$$

be the canonical projection and let

$$j_i : \mathscr{O}_X(d_i) \to \bigoplus_{k=1}^{m} \mathscr{O}_X(d_k)$$

be the canonical injection.

From the surjection

$$\beta : \bigoplus_{k=1}^{m} \mathscr{O}_X(-d_k) \to \frac{\mathscr{I}_X}{\mathscr{I}_X^2}$$

we deduce the exact sequence of locally free sheaves

$$0 \longrightarrow N_{X/\mathbb{P}^N} \xrightarrow{a} \bigoplus_{k=1}^{m} \mathscr{O}_X(d_k) \xrightarrow{b} \mathscr{K} \longrightarrow 0, \tag{5.2}$$

with $a = \beta^t$ and with $\mathscr{K}$ locally free sheaf of rank $m - c$.

Let V_1 be the locus of points of X such that $\pi_1 \circ a$ is not surjective and let W_1 be the locus of points of X such that $b \circ j_1$ is not injective. Since (5.2) is an exact sequence we deduce $V_1 \cap W_1 = \emptyset$. Moreover, if $V_1 \neq \emptyset$, then $\operatorname{codim}(V_1) \leq c$. Analogously if $W_1 \neq \emptyset$, then $\operatorname{codim}(W_1) \leq m - c$.

Suppose $W_1 \neq \emptyset$ and let $[W_1] \in \mathrm{H}^i(X, \mathbb{C})$ with

$$i = 2\operatorname{codim}(W_1) \leq 2(m + n - N) \leq 2n - N.$$

By Theorem 3.1.1 we can suppose that $[W_1]$ is cut out by the classes of hyperplanes, yielding $V_1 = \emptyset$. Otherwise $\operatorname{codim}(W_1) + \operatorname{codim}(V_1) \leq m \leq n$ since $m \leq N/2$

implies $n \geq N/2$ and a fortitori $n \geq m$. Thus $W_1 \cap V_1 \neq \emptyset$, yielding a contradiction. In conclusion we deduce that either $V_1 = \emptyset$ or $W_1 = \emptyset$.

If $V_1 = \emptyset$, let $N^{(2)} = \ker(\pi_1 \circ a)$, let $\mathscr{K}^{(2)} = \mathscr{K}$ and let

$$0 \longrightarrow N^{(2)} \xrightarrow{a_2} \bigoplus_{k=2}^{m} \mathscr{O}_X(d_k) \xrightarrow{b_2} \mathscr{K}^{(2)} \longrightarrow 0, \tag{5.3}$$

be the corresponding exact sequence of coherent sheaves.

If $W_1 = \emptyset$, let $N^{(2)} = N$, let $\mathscr{K}^{(2)} = \operatorname{coker}(b \circ j_1)$ and let

$$0 \longrightarrow N^{(2)} \xrightarrow{a_2} \bigoplus_{k=2}^{m} \mathscr{O}_X(d_k) \xrightarrow{b_2} \mathscr{K}^{(2)} \longrightarrow 0, \tag{5.4}$$

be the corresponding exact sequence of coherent sheaves.

We can define analogously V_2, respectively W_2, via $\pi_2 \circ a_2$, respectively $b_2 \circ j_2$. As before from (5.3) we deduce $V_2 \cap W_2 = \emptyset$. In this way we can construct the sets $V_i,\ W_i$ and the coherent sheaves $N^{(i)},\ \mathscr{K}^{(i)}$ for $i = 1, \ldots, m$. The number r of indices $j \in \{1, \ldots, m\}$ such that $V_j = \emptyset$ is greater than or equal to c. Indeed, on the contrary, we would have an injection of a locally free sheaf of rank $m - r$, $r < c$, into $\mathscr{K}$ which has rank $m - c$. In conclusion, $r = c$.

Thus, modulo a renumbering, we can suppose $V_1 = \ldots = V_c = \emptyset$ (and $W_{c+1} = \ldots = W_m = \emptyset$) and that there exists an isomorphism induced by a and the π_i's

$$N_{X/\mathbb{P}^N} \simeq \bigoplus_{k=1}^{c} \mathscr{O}_X(d_k).$$

From this it easily follows that $X^n \subset \mathbb{P}^N$ is the scheme-theoretic intersection of the selected c hypersurfaces among the initial $m \geq c$ and hence that $X^n \subset \mathbb{P}^N$ is the complete intersection of these c hypersurfaces.

We shall immediately state Netsvetaev's improvement of Faltings' condition. The proof follows the same path, see [144, Sect. 4.12], showing that under the previous hypothesis and with the notation introduced above we still have either $V_i = \emptyset$ or $W_i = \emptyset$ for every $i = 1, \ldots, m$. Instead of Barth's Theorem 3.1.1 Netsvetaev applies the following topological result due to Oka.

Theorem 5.2.3 (Oka's Theorem, See [146]) *Let $X \subset \mathbb{P}^N$ be an algebraic set defined set-theoretically by m equations. Then the restriction maps*

$$r : \mathrm{H}^i(\mathbb{P}^N_{\mathbb{C}}, \mathbb{C}) \to \mathrm{H}^i(X, \mathbb{C})$$

are isomorphisms for $i < N - m$.

Netsvetaev's condition has an apparently mysterious formulation but the conditions appearing in its statement are the ideal tools for the proof of Theorem 5.1.4.

Theorem 5.2.4 (Netsvetaev's Condition, see [144, Theorem 3.2]) *Let $X \subset \mathbb{P}^N$ be a manifold scheme-theoretically defined by hypersurfaces of degrees $d_1 \geq d_2 \geq \ldots \geq d_m$. Suppose one of the following conditions holds:*

1. $m < N - \frac{2n}{3}$;
2. $n \geq \frac{3N}{4} - \frac{1}{2}$.

If $m \leq n+1$, then $X \subset \mathbb{P}^N$ is the complete intersection of c hypersurfaces among the m defining it scheme-theoretically.

The last two results are among the most important sufficient conditions to ensure that a smooth manifold $X \subset \mathbb{P}^N$ is a complete intersection. Notwithstanding they did not seem to have any direct or obvious relation with Hartshorne's Conjecture on Complete Intersections in arbitrary codimension. The main applications were obviously to varieties defined by a small number of equations with respect to codimension, yielding restricted forms of the conjecture on Complete Intersections, see [57, Sect. 3]. On the other hand, when these results are applied to $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ for a quadratic manifold of small codimension but without any assumption on the number $m \geq c$ of quadratic equations defining X, they reveal their deepness and powerfulness as we shall see in the next subsection.

We remark that an obvious sufficient condition for $\mathscr{L}_x \neq \emptyset$ is $d \leq n-1$, see Proposition 2.3.8. On the other hand, $c \leq d$ by definition, while the hypothesis in Hartshorne's Conjecture reads as $c \leq \frac{n-1}{2}$ and finally for quadratic manifolds $d = c$. In conclusion quadratic manifolds satisfying the hypothesis in Hartshorne's Conjecture are ruled by a large family of lines and the linear system of quadrics in the second fundamental form at a general point satisfies Faltings' condition in Theorem 5.2.2.

5.2.3 *Proofs of the Main Results*

All the necessary tools to provide simple proofs of Theorem 5.1.3 and of Theorem 5.1.4 have now been introduced and we can proceed quickly.

Proof of Theorem 5.1.3 Since $c \leq \frac{n-1}{2}$, the Barth–Larsen Theorem 3.1.1 yields $\mathrm{Pic}(X) \simeq \mathbb{Z} < O_X(1) >$. Since $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is not empty for $x \in X$ general, then X is a Fano manifold of index $i(X) = \dim(\mathscr{L}_x) + 2$ by Proposition 2.3.9. Since $X \subset \mathbb{P}^N$ is a quadratic manifold we deduce from Proposition 2.3.8 that $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is scheme-theoretically defined by $m \leq c \leq \frac{n-1}{2}$ quadrics in the second fundamental form of X at x. Faltings' Theorem yields that $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is the complete intersection of $s \leq c$ quadrics in the second fundamental form of X at x.

Thus we have

$$\dim(\mathscr{L}_x) \geq n-1-s \geq n-1-c \geq n-1-\frac{n-1}{2} = \frac{n-1}{2}$$

and $i(X) = \dim(\mathscr{L}_x) + 2 \geq \frac{n+3}{2}$. Proposition 2.3.9 implies that $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is non-degenerate while Proposition 2.3.5 assures that there are c linearly independent quadrics vanishing on $\mathscr{L}_x$. Since $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is a non-degenerate complete intersection of s quadrics we deduce $s = \mathrm{h}^0(\mathscr{I}_{\mathscr{L}_x}(2)) \geq c$, yielding $s = c$, $\dim(\mathscr{L}_x) = n - 1 - c$ and $i(X) = n + 1 - c$. In conclusion, $X \subset \mathbb{P}^N$ is a complete intersection by Theorem 5.2.1. $\square$

Proof of Theorem 5.1.4 Suppose $X \subset \mathbb{P}^N$ is not a complete intersection. Reasoning as in the previous proof we know that $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is a manifold scheme-theoretically defined by c linearly independent quadratic equations. Moreover, $\dim(\mathscr{L}_x) \geq n - 1 - c$ with equality holding if and only if X is a complete intersection. Thus we can assume $\dim(\mathscr{L}_x) \geq n - c = \frac{n}{2} > \frac{n-1}{2}$ and that $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is non-degenerate by Proposition 2.3.9.

Netsvetaev's Condition in Theorem 5.2.4 applied to $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ gives the inequalities

$$\frac{3n-6}{4} \leq \dim(\mathscr{L}_x) < \frac{3n-5}{4}.$$

We may assume $n \geq 6$ (otherwise $\mathscr{L}_x$ is a complete intersection), so we deduce that $n = 4r + 2$ and $\dim(\mathscr{L}_x) = 3r$, for a suitable r. If $r > 2$ (equivalently $n > 10$), we would deduce $\dim(\mathscr{L}_x) \geq 2(n - 1 - \dim(\mathscr{L}_x)) + 1$ and $\mathscr{L}_x$ would be a complete intersection by Theorem 5.1.3. In conclusion, $n = 6$ or $n = 10$. In the first case $i(X) = \dim(\mathscr{L}_x) + 2 = 5$ and we get case (1) by the classification of del Pezzo manifolds, see [65]. In the second case, $i(X) = 8$, leading to case (2) by [138]. $\square$

It is worth remarking that there exist Hartshorne varieties different from the ones described in the previous theorem, see for example [52, Proposition 1.9].

5.3 Speculations on Hartshorne's Conjecture

From the point of view of the defining equations quadratic manifolds represent the simplest possible case. This case, although important and meaningful, did not shed new light on the general case of Hartshorne's Conjecture, which remains as intriguing as before. In any case, we would like to point out in this section that the applications of Hartshorne's Conjecture to $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ can be used as a method to produce natural geometrical bounds on X itself and to pose some new natural questions on Fano manifolds of high index.

Before applying this method we shall analyze some of the results obtained until now from this perspective. The classification of *LQEL*-manifolds of type $\delta = \frac{n}{2}$ was well known classically for $n = 2, 4$. Thus suppose $n \geq 6$ and that $X^n \subset \mathbb{P}^N$ is a *LQEL*-manifold of type $\delta = \frac{n}{2} \geq 3$. In particular, $N \geq \dim(SX) = \frac{3n}{2} + 1$.

Then, by Theorem 4.2.3, $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is a non-degenerate *QEL*-manifold of dimension $\frac{3n}{4}-2$ and type $\delta = \frac{n}{2}-2 \geq 1$, which is not a complete intersection since

$$\mathrm{h}^0(\mathscr{I}_{\mathscr{L}_x}(2)) \geq \dim(|II_{x,X}|) + 1 = N - n \geq \frac{n}{2} + 1 > \frac{n}{4} + 1 = \operatorname{codim}(\mathscr{L}_x, \mathbb{P}^{n-1}).$$

Moreover, we have that

$$\frac{3n}{4} - 2 = \dim(\mathscr{L}_x) > \frac{2(n-1)}{3} \text{ if and only if } n > 16.$$

In conclusion the existence of an *LQEL*-manifold of type $\delta = \frac{n}{2}$ and of dimension $n > 16$ would have produced a counterexample to Hartshorne's Conjecture on Complete Intersections!

Obviously the interesting fact in the proof of Theorem 4.3.3 is that we can prove directly that for $n \geq 6$ necessarily $n = 8$ or $n = 16$ without invoking Hartshorne's Conjecture. To the best of our knowledge, these sharp connections have been overlooked before and they were first pointed out in [160, Remark 3.3].

Analogously if $X^n \subset \mathbb{P}^N$ is an *LQEL*-manifold the bound $\delta \leq \frac{n+8}{3}$, which for $\delta = n/2$ implies $n \leq 16$, is equivalent to the condition $\dim(\mathscr{L}_x) \leq \frac{2(n-1)}{3}$. This is precisely the negation of the condition in Hartshorne's Conjecture for $\mathscr{L}_x \subset \mathbb{P}^{n-1}$. Indeed, reasoning as above, one deduces that for an arbitrary *LQEL*-manifold we have

$$\mathrm{h}^0(\mathscr{I}_{\mathscr{L}_x}(2)) > \operatorname{codim}(\mathscr{L}_x, \mathbb{P}^{n-1}),$$

unless $\delta = n$, which implies that $X \subset \mathbb{P}^{n+1}$ is a quadric hypersurface. Thus the variety $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ of an *LQEL*-manifold with $n > \delta > \frac{n+8}{3}$ would have been a counterexample to Hartshorne's Conjecture. Corollary 4.4.11 shows that this is impossible and once again the interesting fact is that one proves directly the previous bound on δ via the Divisibility Theorem without assuming Hartshorne's Conjecture.

Similarly the bound $\operatorname{def}(X) \leq \frac{n+2}{3}$ in Corollary 4.4.12 is equivalent to the hypothesis in Hartshorne's Conjecture for the manifold $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ containing the *contact lines* appearing on a dual defective manifold.

The previous analysis reveals that the application of Hartshorne's Conjecture to the Hilbert scheme of lines passing through a general point of a prime Fano variety of high index is an efficient way to postulate some new problems, which are either true or would produce a counterexample to Hartshorne's Conjecture.

The validity of Hartshorne's Conjecture for quadratic manifolds also supports the possibility that this conjecture could be valid at least for $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ under some hypothesis on $X^n \subset \mathbb{P}^N$ like small codimension or under other geometrical restrictions (prime Fano and/or small degree, etc, etc). Now we let this principle act in practice.

From this point of view, taking into account that the quadrics in the second fundamental form of $X^n \subset \mathbb{P}^N$ at a general point $x \in X$ vanish on $\mathscr{L}_x \subset \mathbb{P}^{n-1}$, one could pose this first basic question, which is a kind of restricted Hartshorne Conjecture. In this context we put a restriction on the variety but not on the defining equations.

Conjecture 5.3.1 (HCL = Hartshorne's Conjecture for $\mathscr{L}_x \subset \mathbb{P}^{n-1}$, [105]) Assume that $X^n \subset \mathbb{P}^N$ is covered by lines with $\dim(\mathscr{L}_x) \geq \frac{n-1}{2}$ and let $T = \langle \mathscr{L}_x \rangle$ be the span of $\mathscr{L}_x$ in $\mathbb{P}^{n-1}$. If $\dim(\mathscr{L}_x) > 2\,\mathrm{codim}(\mathscr{L}_x, T)$, then $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is a complete intersection.

Since for smooth complete intersections $X^n \subset \mathbb{P}^N$, the variety $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is also a smooth complete intersection, see Example 2.3.11, one may ask if for manifolds covered by lines the converse also holds, illustrating a kind of recovery principle from $\mathscr{L}_x$ to X.

Conjecture 5.3.2 If $X \subset \mathbb{P}^N$ is covered by lines and $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is a (say smooth irreducible non-degenerate) complete intersection, then X is a complete intersection.

We now specialize to some *restricted* form of the Complete Intersection Conjecture.

Conjecture 5.3.3 (HCF = Hartshorne's Conjecture for Fano Manifolds, [105]) If $X^n \subset \mathbb{P}^N$ is a Fano manifold and if $n \geq 2c + 1$, then $X^n \subset \mathbb{P}^N$ is a complete intersection.

Manifolds of (very) small degree are known to be complete intersections, cf. [14]. Ionescu proved in [101] that for a manifold $X^n \subset \mathbb{P}^N$ with $\deg(X) \leq n - 1$ there are only the following two possibilities: either $X \simeq \mathbb{G}(1,4) \subset \mathbb{P}^9$ or $c \leq \frac{n-1}{2}$ and $X^n \subset \mathbb{P}^N$ is a prime Fano manifold. Therefore Conjecture 5.3.3 would yield the following optimal result, which obviously can be stated independently.

Conjecture 5.3.4 (Barth–Ionescu Conjecture, [14, 101]) If $\deg(X) \leq n - 1$, then X is a complete intersection, unless it is projectively equivalent to $\mathbb{G}(1,4) \subset \mathbb{P}^9$.

Note that the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ has degree n and it is not a complete intersection if $n \geq 3$.

Hartshorne's Conjecture combined with the previous Conjectures can be applied to $\mathscr{L}_x \subset \mathbb{P}^{n-1}$, with $X^n \subset \mathbb{P}^N$ a prime Fano manifold of high index, to derive some characterizations of Fano Complete Intersections of high index. This is also justified by the remark that prime Fano manifolds of high index tend to be complete intersections, see [65, 138], and also quadratic manifolds. Let us recall these results.

Proposition 5.3.5 ([105, Proposition 3.6])

1. *Let $X \subset \mathbb{P}^N$ be a Fano manifold with* $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle \mathscr{O}(1) \rangle$ *and of index $i(X) \geq \frac{2n+5}{3}$. If the* (HCL) *and the* (HCF) *are true, then X is a complete intersection.*
2. *The same conclusion holds assuming only the* (HCF), *but asking instead that $i(X) \geq \frac{3(n+1)}{4}$.*

Thus we have found two different statements pointing out two explicit bounds on the index of a prime Fano manifold which so far have never been stated, to the best of our knowledge.

We now put our results in the perspective of being natural generalizations of some known properties of quadratic manifolds. Mumford in his seminal series of lectures [140] called attention to the fact that many interesting embedded manifolds are scheme-theoretically defined by quadratic equations.

As an application of the main vanishing theorem in [19, Corollary 2] it is proved that for a quadratic manifold $X \subset \mathbb{P}^N$ the following implications hold:

a) If $n \geq c-1$, then X is projectively normal;
b) If $n \geq c$, then X is projectively Cohen–Macaulay.

The natural continuation is contained in the next result.

Theorem 5.3.6 ([105, Theorem 3.8]) *Assume that $X \subset \mathbb{P}^N$ is a quadratic manifold.*

1. *If $n \geq c$, then X is Fano.*
2. *If $n \geq c+1$, then X is covered by lines. Moreover, $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is scheme-theoretically defined by c independent quadratic equations.*
3. *If $n \geq c+2$, then the following conditions are equivalent:*
   (i) *$X \subset \mathbb{P}^N$ is a complete intersection;*
   (ii) *$\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is a complete intersection;*
   (iii) $\dim(\mathscr{L}_x) = n-1-c$.

Proof We use the notation in (**). Moreover, denote by $a := \dim(\mathscr{L}_x)$. Since X is quadratic, we have $d = c$ and that $N^*_{X/\mathbb{P}^N}(2)$ is spanned by global sections. Therefore, $\det(N^*_{X/\mathbb{P}^N})(2c)$ is also spanned and $\det(N^*_{X/\mathbb{P}^N})(2c+1)$ is ample. From $n \geq c$ it follows that $N+1 \geq 2c+1$. We get the ampleness of $-K_X = \det(N^*_{X/\mathbb{P}^N})(N+1)$, thus proving (1).

From Proposition 2.3.8 we deduce the first part in (2) and that $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is defined scheme-theoretically by at most c quadratic equations. Since $n \geq c+1$, we must have $\delta \geq n-c+1 > 0$. From Proposition 2.3.5 we infer that the quadratic equations defining $\mathscr{L}_x$ are independent, proving (2). This also shows that, when $\mathscr{L}_x$ is a complete intersection, it has codimension c in $\mathbb{P}^{n-1}$. From Barth's Theorem (3.1.1) we deduce that $X^n \subset \mathbb{P}^N$ is a prime Fano manifold. The equivalence between conditions (i), (ii) and (iii) of part (3) is now clear, the implication (iii)$\Rightarrow$ (i) coming from Theorem 5.2.1. □

In Proposition 2.3.8 we saw that if $d \leq n-1$ then $\mathscr{L}_x \subset \mathbb{P}^{n-1}$ is set-theoretically defined by d equations. If one is able to prove that these equations define $\mathscr{L}_x$ scheme-theoretically, Faltings' condition could help to prove that $\mathscr{L}_x$ is the complete intersection of some of these equations. Expressing the hypothesis in Faltings' condition one could pose the following problem, which is a weaker form of Hartshorne's Conjecture since in general $d \geq c$.

Conjecture 5.3.7 (Scheme-Theoretic Hartshorne Conjecture) Let

$$X^n = V(f_1, \dots, f_m) \subset \mathbb{P}^N$$

be a manifold which is scheme-theoretically defined as the intersection of the hypersurfaces $V(f_i)$ of degree d_i with $d_1 \geq d_2 \geq \dots \geq d_m \geq 2$.

Let

$$d := \min\{\sum_{i=1}^{c}(d_i - 1) \text{ for expressions as above}\} \geq c.$$

If

$$d \leq \frac{n-1}{2},$$

then $X \subset \mathbb{P}^N$ is a complete intersection.

5.4 A Refined Linear Normality Bound and Severi Varieties

Now we are in position to provide a slight refinement of Zak's Linear Normality Theorem, [198, Theorem 2.8]. The proof is essentially identical to Zak's original argument, but it reveals the importance of the tangential invariants defined in Sect. 3.3 to which we refer for the notation.

The bound obtained also strengthens the bound of Landsberg for smooth varieties involving $\widetilde{\gamma}(X) = \gamma(X) - \delta(X)$, which equals $\dim(F_v)$ in Landsberg's notation, see [119] and also [107, 3.15].

Theorem 5.4.1 *Let $X \subset \mathbb{P}^N$ be an irreducible non-degenerate variety such that $SX \subsetneq \mathbb{P}^N$. Let $b = \dim(\mathrm{Sing}(X))$, $\xi = \xi(X)$ and $\delta = \delta(X)$. Then:*

$$\dim(SX) \geq \frac{3}{2}n + \frac{1-b}{2} + \frac{\xi - \delta}{2}; \tag{5.5}$$

$$N \geq \frac{3}{2}n + 1 + \frac{1-b}{2} + \frac{\xi - \delta}{2}; \tag{5.6}$$

$$n \leq \frac{1}{3}(2N + b - (\xi - \delta)) - 1. \tag{5.7}$$

In particular, if $X \subset \mathbb{P}^N$ is also smooth, then

$$\dim(SX) \geq \frac{3}{2}n + 1 + \frac{\xi - \delta}{2}; \tag{5.8}$$

$$N \geq \frac{3}{2}n + 2 + \frac{\xi - \delta}{2} \tag{5.9}$$

and

$$n \leq \frac{2(N-2)}{3} - \frac{\xi - \delta}{3}. \tag{5.10}$$

Proof If $\xi \leq b+1$, then

$$\dim(SX) > \frac{3}{2}n + \frac{1-b}{2} + \frac{\xi-\delta}{2},$$

so that we can assume $\xi \geq b+2$ and hence $n \geq b+3$.

Fix a general $p \in SX$ and consider a general hyperplane $H \subset \mathbb{P}^N$ containing T_pSX. By definition H is tangent to X along $\Xi_p \setminus \mathrm{Sing}(X)$. Consider a general $L = \mathbb{P}^{N-b-1}$ and set $\widehat{X} = X \cap L$, $\widehat{\Xi}_p = \Xi_p \cap L$ and $\widehat{H} = H \cap L$. The variety $\widehat{X} \subset \mathbb{P}^{N-b-1} = L$ is smooth, irreducible and non-degenerate of dimension $n-b-1 \geq 2$, while the hyperplane $\widehat{H} = \mathbb{P}^{N-b-2}$ is tangent to $\widehat{X}$ along the variety $\widehat{\Xi}_p$, whose dimension is $\xi - b - 1 \geq 1$.

Since $\widehat{X} \subset L$ is non-degenerate and contained in $S(\widehat{\Xi}_p, \widehat{X})$, we get $S(\widehat{\Xi}_p, \widehat{X}) \neq T(\widehat{\Xi}_p, \widehat{X})$, where $T(\widehat{\Xi}_p, \widehat{X}) := \cup_{y \in \widehat{\Xi}_p} T_y\widehat{X} \subseteq \widehat{H}$. Therefore by applying Theorem 3.2.1 to $S(\widehat{\Xi}_p, \widehat{X})$ we deduce that

$$\begin{aligned}\dim(SX) - b - 1 &= \dim(SX \cap L) \geq \dim(S\widehat{X}) \geq \dim(S(\widehat{\Xi}_p, \widehat{X})) = \\ &= \dim(\widehat{\Xi}_p) + \dim(\widehat{X}) + 1 = (\xi - b - 1) + (n - b - 1) + 1.\end{aligned}$$

Hence

$$2n + 1 - \delta \geq n + \xi - b = n + \delta + \widetilde{\xi} - b,$$

where $\widetilde{\xi} = \xi - \delta$, that is

$$\delta \leq \frac{n + b + 1 - \widetilde{\xi}}{2}.$$

Thus

$$\dim(SX) = 2n + 1 - \delta \geq \frac{3}{2}n + \frac{1-b}{2} + \frac{\widetilde{\xi}}{2} = \frac{3}{2}n + \frac{1-b}{2} + \frac{\xi - \delta}{2}.$$

Since $SX \subsetneq \mathbb{P}^N$, we deduce $N \geq \dim(SX) + 1$, which combined with the above estimates yields $n \leq \frac{1}{3}(2N + b - (\xi - \delta)) - 1$. The other assertions are now obvious. $\square$

Theorem 5.4.1 implies that a Severi variety $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ has $\xi(X) = \gamma(X) = \delta(X) = \frac{n}{2}$ and that $SX \subsetneq \mathbb{P}^{\frac{3}{2}n+2}$ is a hypersurface.

With these powerful tools at hand and via Scorza's Lemma we can immediately prove the next consequence, in a way different from [198, IV.2.1, IV.3.1, IV.2.2].

Corollary 5.4.2 *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety. Then*

1. *$X \subset \mathbb{P}^{\frac{3}{2}n+2}$ is an LQEL-variety of type $\delta = \frac{n}{2}$.*
2. *The image of a general tangential projection of X, $\pi_x(X) = W_x \subset \mathbb{P}^{\frac{n}{2}+1}$, is a smooth quadric hypersurface.*
3. *Given three general points $x, y, z \in X$, let $Q_{x,z}$, respectively $Q_{y,z}$, be the smooth quadrics passing through x and z, respectively y and z. Then $Q_{x,z} \cap Q_{y,z} = z$, the intersection being transversal.*

Proof As we observed above, for a Severi variety we have $\xi(X) = \gamma(X) = \delta(X) = \frac{n}{2}$ due to Theorem 5.4.1. The conclusion of the first part follows from Scorza's Lemma, Theorem 3.3.3.

The proof of Theorem 5.4.1 yields $SX = S(\Sigma_q, X)$ for a general $q \in SX$. From Terracini's Lemma we get $T_xX \cap T_w\Sigma_q = \emptyset$ for a general $x \in X$ and for a general $w \in \Sigma_q$. Thus $\dim(\pi_x(\Sigma_q)) = \frac{n}{2}$ for $q \in SX$ general. Since $\pi_x(X) = W_x$ has dimension $\frac{n}{2}$, we deduce that $\pi_x(\Sigma_q) = W_x$ for $q \in SX$ general. Therefore the variety $W_x \subset \mathbb{P}^{\frac{n}{2}+1}$ is a quadric hypersurface, being a hypersurface and also a non-degenerate linear projection of a quadric hypersurface. The smoothness of W_x follows from $0 = \widetilde{\xi}(X) = \operatorname{def}(W_x)$. In particular, the restriction of π_x to Σ_q is an isomorphism. Scorza's Lemma also yields $\Sigma_p = \pi_x^{-1}(\pi_x(z))$. Take $q \in < y, z >$ general and consider Σ_q. From the previous analysis $\pi_{x|\Sigma_q} : \Sigma_q \to W_x$ is an isomorphism so that Σ_q intersects Σ_p only at z, the intersection being transversal. □

Clearly the dimension n of a Severi variety $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ is even so that the first case to be considered is $n = 2$. These are smooth surfaces in $\mathbb{P}^5$ such that $SX \subsetneq \mathbb{P}^5$. They were completely classified in the classical and well-known theorem of Severi, [176], see Theorem 3.4.1. This justifies the name given by Zak to such varieties. By Theorem 5.1.6, it follows that $SX \subset \mathbb{P}^{\frac{3}{2}n+2}$ is necessarily a hypersurface, that is $\dim(SX) = \frac{3}{2}n + 1$.

In Exercise 1.5.11 we showed that the Segre variety $X = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ is an example of a Severi variety of dimension 4. Indeed, $N = 8 = \frac{3}{2} \cdot 4 + 2$ and SX is a cubic hypersurface, see *loc. cit.* By the classical work of Scorza, last page of [163], it turns out that $\mathbb{P}^2 \times \mathbb{P}^2$ is the only Severi variety of dimension 4. We provided a short, geometrical, self-contained and elementary proof in Theorem 4.3.3.

The realization of the Grassmann variety of lines in $\mathbb{P}^5$ Plücker embedded, $X = \mathbb{G}(1, 5) \subset \mathbb{P}^{14}$, as the variety given by the pfaffians of the general antisymmetric 6×6 matrix, yields that $\mathbb{G}(1, 5)$ is a Severi variety of dimension 8 such that its secant variety is a degree 3 hypersurface whose equation is the pfaffian of the 6×6 antisymmetric matrix, see for example [85, p. 112 and p. 145] for the last assertion.

A less trivial example is a variety studied by Elie Cartan and also by Room. It is a homogeneous complex variety of dimension 16, $X \subset \mathbb{P}^{26}$, associated to the representation of E_6 and for this reason called an *E_6-variety*, or *Cartan variety* by Zak. It has been shown by Lazarsfeld and Zak that its secant variety is a degree 3 hypersurface, see, for example, [125] and [198, Chap. 3].

There is a unitary way to look at these 4 examples, by realizing them as *Veronese surfaces over the composition algebras over* K, $K = \overline{K}$, char(K)=0, [198, Chap. 3]. Let $\mathscr{U}_0 = K$, $\mathscr{U}_1 = K[t]/(t^2 + 1)$, $\mathscr{U}_2 =$ quaternion algebra over K, $\mathscr{U}_3 =$ Cayley algebra over K. Let $\mathscr{J}_i$, $i = 0, \dots, 3$, denote the Jordan algebra of Hermitian (3×3)-matrices over $\mathscr{U}_i$, $i = 0, \dots, 3$. A matrix $A \in \mathscr{J}_i$ is called *Hermitian* if $\overline{A}^t = A$, where the bar denotes the natural involution in $\mathscr{U}_i$ (for $\mathscr{U}_0$ there is no involution and we have symmetric matrices). Let

$$X_i = \{[A] \in \mathbb{P}(\mathscr{J}_i) \,:\, \mathrm{rk}(A) = 1\} \subset \mathbb{P}(\mathscr{J}_i).$$

Then

$$N_i = \dim(\mathbb{P}(\mathscr{J})) = 3 \cdot 2^i + 2, \quad n_i = \dim(X_i) = 2^{i+1} = 2\dim_K(\mathscr{U}_i),$$

and

$$SX = \{[A] \in \mathbb{P}(\mathscr{J}_i) \,:\, \mathrm{rk}(A) \le 2\} = V(\det(A)) \subset \mathbb{P}(\mathscr{J}_i)$$

is a degree 3 hypersurface. By definition $X_i \subset \mathbb{P}(\mathscr{J}_i)$ is a Severi variety of dimension 2^{i+1}, projectively equivalent to one of the above examples.

A theorem of Jacobson states that over a fixed algebraically closed field K, modulo isomorphism, there exist only four simple rank three Jordan algebras, which are exactly the algebras $\mathscr{J}_i$'s. Equivalently there are only three composition algebras over such a field K. Thus this construction gives the four known examples.

A highly non-trivial and notable result is the classification of Severi varieties, which was first proved by Zak in [196] and which now is an immediate consequence of Corollary 4.3.3. Via the previous construction the next theorem is equivalent to Jacobson's Classification Theorem recalled above.

Theorem 5.4.3 (Zak's Classification of Severi Varieties, [27, 119, 125, 196, 198], [160, Corollary 3.2] or Theorem 4.3.3 Here) *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety of dimension n, defined over an algebraically closed field K of characteristic 0. Then X is projectively equivalent to one of the following:*

1. *the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$;*
2. *the Segre fourfold $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;*
3. *the Grassmann variety $\mathbb{G}(1, 5) \subset \mathbb{P}^{14}$;*
4. *the E_6-variety $X \subset \mathbb{P}^{26}$.*

Proof By Corollary 5.4.2 a Severi variety $X^n \subset \mathbb{P}^{\frac{3n}{2}+2}$ is a *QEL*-manifold of type $\delta = \frac{n}{2}$ so that the conclusion follows from Corollary 4.3.3. $\square$

Since in Corollary 4.3.3 we explicitly proved only that the dimension of a Severi variety is equal to 2, 4, 8 or 16 in the next subsection we shall provide a direct proof that a Severi Variety of dimension 2^r, $r \in \{1, 2, 3, 4\}$, is as stated in Theorem 5.4.3.

5.5 Reconstruction of Severi Varieties of Dimension 2, 4, 8 and 16

We propose an elementary approach to the classification of Severi varieties of dimension 2, 4, 8 or 16. At this point it should be clear that the classification of Severi varieties in dimension 2, 4, 8 and 16 is a straightforward consequence of the above results. As we said above for dimension 2 and 4 it is classical and elementary. Due to the relevance of this classification, we shall reproduce here a short argument which will also be an interesting detour into higher-dimensional projective and birational geometry.

Let us recall the following picture of the known Severi varieties in dimension $n = 2, 4, 8$ and 16. Let $Y \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$ be either $\emptyset$, respectively $\mathbb{P}^1 \sqcup \mathbb{P}^1 \subset \mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ Segre embedded, $S^{10} \subset \mathbb{P}^{15}$ the ten dimensional spinor variety. On $\mathbb{P}^{n-1}$ we take coordinates $x_0, \ldots, x_{n-1}$ and on $\mathbb{P}^n$ coordinates $x_0, \ldots, x_n$. Let $Q_1, \ldots, Q_{\frac{n}{2}+2}$ be the quadratic forms in the variables $x_0, \ldots, x_{n-1}$ defining $Y \subset \mathbb{P}^{n-1}$. The subvariety $Y \subset \mathbb{P}^n$ is scheme-theoretically defined by the $\frac{3n}{2} + 3$ quadratic forms: Q_i, $x_n x_j$, $i = 1, \ldots, \frac{n}{2} + 2$, $j = 0, \ldots, n$. More precisely, these quadric hypersurfaces form the linear system of quadrics on $\mathbb{P}^n$ vanishing along Y, that is $|H^0(\mathscr{I}_{Y,\mathbb{P}^n}(2))|$. Let

$$\phi_{|H^0(\mathscr{I}_{Y,\mathbb{P}^n}(2))|} : \mathbb{P}^n \dashrightarrow \mathbb{P}^{\frac{3n}{2}+2}.$$

For $Y = \emptyset$, clearly $\phi(\mathbb{P}^2) = \nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. For $Y = \mathbb{P}^1 \sqcup \mathbb{P}^1 \subset \mathbb{P}^3$ we get $\phi(\mathbb{P}^4) = \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$, a particular form of a result known to C. Segre, see [170], found by him when he first studied what are now called *Segre varieties*. For $Y = \mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, one obtains $\phi(\mathbb{P}^8) = \mathbb{G}(1, 5) \subset \mathbb{P}^{14}$ Plücker embedded, a particular case of a general result of Semple, see [162, 173]. For $Y = S^{10} \subset \mathbb{P}^{15}$, Zak has shown in [125, 198], Chap. III, that $\phi(\mathbb{P}^{16}) = E_6 \subset \mathbb{P}^{26}$ is the Cartan, or E_6, variety.

The birational inverse of ϕ, $\phi^{-1} : X \dashrightarrow \mathbb{P}^n$, with $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ one of the Severi variety described above, is given by the linear projection from the linear space

$$\mathbb{P}_p^{\frac{n}{2}+1} = \langle \Sigma_p \rangle,$$

$p \in SX$ a general point, onto a skew $\mathbb{P}^n$.

We show that, more generally and a priori, a Severi variety of dimension n can be birationally projected from $\mathbb{P}_p^{\frac{n}{2}+1}$, $p \in SX$ general, onto $\mathbb{P}^n$. This was originally proved in [198, IV.2.4 f)] and we provide a proof for the reader's convenience and with the desire of being as self-contained as possible.

Proposition 5.5.1 *Let $X \subset \mathbb{P}^{\frac{3}{2}n+2}$ be a Severi variety. Let $p \in SX$ be a general point, let $\Sigma_p \subset \mathbb{P}_p^{\frac{n}{2}+1}$ be its entry locus and let*

$$\pi = \pi_{L_p} : X \dashrightarrow \mathbb{P}^n$$

be the projection from $L_p = \langle \Sigma_p \rangle = \mathbb{P}_p^{\frac{n}{2}+1}$. Let $\widetilde{X} = Bl_{\Sigma_p} X \xrightarrow{\alpha} X$, let E be the exceptional divisor and let F be the strict transform of $H_p = T_pSX \cap X$ on $\widetilde{X}$. Let $\widetilde{\pi} : \widetilde{X} \to \mathbb{P}^n$ be the resolution of π_L and let $\widetilde{\pi}(F) = T_pSX \cap \mathbb{P}^n = \mathbb{P}^{n-1} \subset \mathbb{P}^n$. By definition of $\widetilde{\pi}$ we have $\widetilde{\pi}^{-1}(\mathbb{P}^{n-1}) = E \cup F$. Then:

i) if $\dim(\widetilde{\pi}^{-1}(z)) > 0$, *$z \in \mathbb{P}^n$, then $\widetilde{\pi}^{-1}(z) \subseteq E \cup F$;*
ii) the morphism $\widetilde{\pi}$ is birational and defines an isomorphism between $\widetilde{X} \setminus (E \cup F)$ and $\mathbb{P}^n \setminus \mathbb{P}^{n-1}$. In particular, the locus of indetermination of $\widetilde{\pi}^{-1}$ is a subscheme $Y \subset \mathbb{P}^{n-1}$.

Proof Let $w \in \mathbb{P}^n \setminus \mathbb{P}^{n-1}$, let $W = \widetilde{\pi}^{-1}(w))$ and suppose $\dim(W) > 0$. Then

$$W \cap (E \cup F) = \emptyset,$$

so that $\alpha(W) = W'$ is positive-dimensional and it does not cut L_p. Thus W' contains a positive-dimensional variety $M \subset X$ such that $L_p \cap M = \emptyset$ and such that $\pi(M) = w$. This contradicts the fact that a linear projection, when it is defined everywhere, is a finite morphism. The first part is proved.

To prove part (ii) let us remark that if $p \in < x, y >$, $x, y \in X$ general points, then

$$L_p = \langle \Sigma_p \rangle = \langle T_xX, y \rangle \cap \langle T_yX, x \rangle, \tag{5.11}$$

by Terracini's Lemma (see also the proof of Scorza's Lemma). The projection from the linear space $\langle T_xX, y \rangle$ can be regarded as the composition of the tangential projection $\pi_x : X \dashrightarrow W_x \subset \mathbb{P}^{\frac{n}{2}+1}$ and the projection of the smooth quadric hypersurface W_x from the point $\pi_x(y)$. Thus the projection from $\langle T_xX, y \rangle$, $\pi_{x,y} : X \dashrightarrow \mathbb{P}^{\frac{n}{2}}$ is dominant and for a general point $z \in X$ we get

$$\langle T_xX, y, z \rangle \cap X \setminus (\langle T_xX, y \rangle \cap X) = \pi_{x,y}^{-1}(\pi_{x,y}(z)) = Q_{x,z} \setminus (Q_{x,z} \cap \langle T_xX, y \rangle), \tag{5.12}$$

where as always $Q_{x,z}$ is the entry locus of a general point on $< x, z >$. Similarly

$$\langle T_yX, x, z \rangle \cap X \setminus (\langle T_yX, x \rangle \cap X) = \pi_{y,x}^{-1}(\pi_{y,x}(z)) = Q_{y,z} \setminus (Q_{y,z} \cap \langle T_xX, y \rangle). \tag{5.13}$$

By definition of projection we have that

$$\pi^{-1}(\pi(z)) = \langle L_p, z \rangle \cap (X \setminus \Sigma_p). \tag{5.14}$$

By the generality of z, we get

$$\pi^{-1}(\pi(z)) = \langle L_p, z\rangle \cap (X \setminus H_p). \tag{5.15}$$

By Terracini's Lemma the linear spaces $\langle T_xX, y\rangle$ and $\langle T_yX, x\rangle$ are contained in T_pSX, so that $\langle T_xX, y\rangle \cap X$ and $\langle T_yX, x\rangle \cap X$ are contained in H_p. By combining (5.11), (5.12), (5.13) and (5.15) we finally get

$$z \subseteq \pi^{-1}(\pi(z)) \subseteq Q_{x,z} \cap Q_{y,z} = z,$$

where the last equality is scheme-theoretical by Corollary 5.4.2 and by the generality of $x, y, z \in X$. □

Proposition 5.5.2 *Let $X^n \subset \mathbb{P}^{\frac{3n}{2}+2}$ be a Severi variety with $n \in \{2, 4, 8, 16\}$. Then $X^n \subset \mathbb{P}^{\frac{3n}{2}+2}$ is projectively equivalent to one of the following:*

1. *the Veronese surface $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$;*
2. *the Segre fourfold $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$;*
3. *the Grassmann variety $\mathbb{G}(1, 5) \subset \mathbb{P}^{14}$;*
4. *the Cartan (or E_6) variety $X \subset \mathbb{P}^{26}$.*

Proof For $n = 2$ one can apply Proposition 4.3.1 (or Corollary 3.4.1) to get case (1).

Assume $n = 4$, so that $\delta = 2$. The base locus scheme of $|II_{x,X}|$, which is a linear system of dimension 3, is a smooth not necessarily irreducible curve in $\mathbb{P}^3$ with one apparent double point by Theorem 4.2.3. It immediately follows that $\mathscr{L}_x \subset \mathbb{P}^3$ is the union of two skew lines and that $\mathscr{L}_x \subset \mathbb{P}^3$ coincides with the base locus scheme B_x of $|II_{x,X}|$.

Suppose $n = 8$ and $\delta = 4$. By Theorem 4.2.3, the variety $\mathscr{L}_x \subset \mathbb{P}^7$ is a smooth, irreducible, non-degenerate, *QEL*-manifold of dimension 4 and such that $S\mathscr{L}_x = \mathbb{P}^7$. Furthermore, by Theorem 4.2.3 part 1), there are at least six quadric hypersurfaces vanishing on $\mathscr{L}_x \subset \mathbb{P}^7$. By restricting to a general $\mathbb{P}^3 \subset \mathbb{P}^7$, the usual Castelnuovo Lemma yields $\deg(\mathscr{L}_x) \leq 4$ and hence $\deg(\mathscr{L}_x) = 4$ since $\mathscr{L}_x \subset \mathbb{P}^7$ is non-degenerate. Thus $\mathscr{L}_x \subset \mathbb{P}^7$ is projectively equivalent to the Segre variety $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$ and clearly $\mathscr{L}_x$ is the base locus scheme B_x of $|II_{x,X}|$.

Suppose $n = 16$. Then by Theorem 4.2.3, $\mathscr{L}_x \subset \mathbb{P}^{15}$ is a Mukai variety of dimension 10 and type $\delta = 6 > n/2 = 5$. Thus by Corollary 4.3.2, $\mathscr{L}_x \subset \mathbb{P}^{15}$ is projectively equivalent to $S^{10} \subset \mathbb{P}^{15}$.

From now on we suppose $n \geq 4$ so that a general entry locus is not a divisor on X. Let $p \in SX \setminus X$ be a general point and let $\mathbb{P}_p^{\frac{n}{2}+1}$ be the locus of secant lines through p. Take a $\mathbb{P}^n$ disjoint from $\mathbb{P}_p^{\frac{n}{2}+1}$ and let $\varphi : X \dashrightarrow \mathbb{P}^n$ be the projection from $\mathbb{P}_p^{\frac{n}{2}+1}$. By Proposition 5.5.1 the map φ is birational and an isomorphism on $X \setminus (T_pSX \cap X)$. Let $Y \subset T_pSX \cap \mathbb{P}^n = \mathbb{P}^{n-1}$ be the base locus scheme of $\varphi^{-1} : \mathbb{P}^n \dashrightarrow X \subset \mathbb{P}^{\frac{3n}{2}+2}$, that is of φ^{-1} composed with the inclusion $i : X \to \mathbb{P}^{\frac{3n}{2}+2}$.

Take a general point $y \in \Sigma_p$. A general smooth conic through y cuts Σ_p transversally so that it is mapped onto a line by φ. By Theorem 4.2.2 there is an irreducible family of dimension $\frac{3n}{2} - 2$ of such conics through y. By varying $y \in \Sigma_p$ the projected lines form a $(2n-2)$-dimensional family of lines on $\mathbb{P}^n$, which is then a part of the whole family of lines in $\mathbb{P}^n$. This means that φ^{-1} is given by a linear system of quadric hypersurfaces vanishing on the subscheme $Y \subset \mathbb{P}^{n-1}$. Moreover, since $X \subset \mathbb{P}^{\frac{3n}{2}+2}$ is linearly normal, φ^{-1} is given by $|H^0(\mathscr{I}_{Y,\mathbb{P}^n}(2))|$.

Consider a general point $q \in \mathbb{P}^n \setminus \mathbb{P}^{n-1}$, which we can write as $\varphi(x)$ with $x \in X$ general. Consider the family of lines through $\varphi(x)$ and parametrized by the not necessarily irreducible variety $Y_{\mathrm{red}} \subset \mathbb{P}^{n-1}$. The image via φ^{-1} of these lines are lines passing through x. Indeed, these lines cannot be contracted by Proposition 5.5.1, they cut Y and the restriction of φ^{-1} to such a line is given by a sublinear system of $|\mathscr{O}_{\mathbb{P}^1}(2)|$ with a base point. Thus we get a morphism $\alpha_x : Y_{\mathrm{red}} \to \mathscr{L}_x$, since $\mathscr{L}_x$ is isomorphic to the Hilbert scheme of lines through x. Moreover, the birational map φ^{-1} is an isomorphism near $\varphi(x)$, so that the morphism α_x is one-to-one.

A general line through a general point $x \in X$ is sent into a line passing through $\varphi(x)$, because it does not cut the center of projection. Since φ^{-1} is given by a linear system of quadric hypersurfaces, a general line through $\varphi(x)$ cuts Y in one point, proving that $\alpha_x : Y_{\mathrm{red}} \to \mathscr{L}_x$ is dominant and hence surjective. Thus $\alpha_x : Y_{\mathrm{red}} \to \mathscr{L}_x$ is an isomorphism by Zariski's Main Theorem. Moreover, the variety $Y_{\mathrm{red}} \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\mathscr{L}_x$. Thus $Y_{\mathrm{red}} \subset \mathbb{P}^{n-1}$ has homogeneous ideal generated by $\frac{n}{2} + 2$ quadratic equations and therefore it coincides with $Y \subset \mathbb{P}^{n-1}$.

Therefore the previous analysis yields that $Y \simeq \mathscr{L}_x \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\mathbb{P}^1 \sqcup \mathbb{P}^1 \subset \mathbb{P}^3$, $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$, respectively $S^{10} \subset \mathbb{P}^{15}$. The conclusion follows from the birational representation of the known Severi varieties recalled above. $\square$

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